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On Minimally Imperfect Graphs with Circular Symmetry[1]

by

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ABSTRACT

Results of Lovász and Padberg entail that the class of so-called partitionable graphs contains all the potential counterexamples to Berge's famous Strong Perfect Graph Conjecture (which asserts that the only minimally imperfect graphs are the odd chordless cycles with at least five vertices and their complements). Only two constructions (due to Chvátal, Graham, Perold and Whitesides) are known for making partitionable graphs. The first one does not produce any counterexample to Berge's Conjecture, as shown by Sebő. Here we prove that the second construction does not produce any counterexample either. We conjecture that every partitionable graph with circular symmetry can be generated by this second construction, and give results in this direction. We also show that, regardless of this conjecture, every minimally imperfect graph with circular symmetry must have an odd number of vertices.

1 Introduction

In 1960, Claude Berge introduced the notion of perfect graphs. A graph is called *perfect* if, for every induced subgraph H, the chromatic number of H does not exceed the maximum number of pairwise adjacent vertices in H. Naturally, a graph is called *minimally imperfect* if it is not perfect but every proper induced subgraph is perfect. It is not difficult to see that odd holes and their complements are minimally imperfect (where a *hole* is a chordless cycle with at least four vertices). Berge conjectured that there are no minimally imperfect graphs other than odd holes and their complements. This conjecture is still open and is called the Strong Perfect Graph Conjecture. A weaker conjecture, i.e. the complement of a perfect graph is perfect, was proposed also by Berge, and proved in 1972 by Lovász (see [11]).

The results of Lovász [11] and Padberg [13] yield certain properties of minimally imperfect graphs. Following the paper by Bland, Huang and Trotter [2], for integers $p, q \geq 2$, we say that a graph G is (p,q)-*partitionable* if, for every vertex v of G, the induced subgraph $G - v$ admits a partition into p cliques of cardinality q and also admits a partition into q stable sets of cardinality p. It is easy to see that p must be equal to the maximum cardinality of a stable set in G, and similarly that q is the maximum cardinality of a clique in G. According to [2] (p,q)-partitionable graphs have the following properties:

(i) G has exactly $n = pq + 1$ vertices;

(ii) G has exactly n stable sets of cardinality p;

(iii) G has exactly n cliques of cardinality q;

(iv) Each maximum clique meets exactly $n - 1$ of the maximum stable sets (and misses exactly one);

(v) Each maximum stable set meets exactly $n - 1$ of the maximum cliques (and misses exactly one);

(vi) Each vertex belongs to exactly q maximum cliques;

(vii) Each vertex belongs to exactly p maximum stable sets.

With this definition, results of Lovász and Padberg simply say that every minimally imperfect graph with maximum clique size q and stability number p is (p,q)-partitionable. The converse is not true; there exists infinitely many partitionable graphs that are not minimally imperfect. One way to attack Berge's Strong Perfect Graph Conjecture would be to explore the class of partitionable graphs and for each such graph to give a certificate that it is not minimally imperfect or that it is. This method would probably use a construction of all partitionable graphs. So far, two constructions are known for subclasses of partitionable graphs [7]. The first of these two constructions yields no counterexample to the Strong Perfect Graph Conjecture as shown by András Sebő [14]. The second construction produces circular graphs, but we will show in the first part of this paper that it does not produce any counterexample to Berge's conjecture, thereby generalizing a result of Grinstead [8]. The second part of the paper deals with all circular graphs. This paper is a blend of several research reports [1, 3, 12] on this topic.

It is important to notice that a given (p, q)-partitionable graph G may admit a pair of vertices that lies neither in a maximum clique nor in a maximum stable set of G. We call such a pair *indifferent*. As observed in [7], adding to or removing from the edge set of G any choice of indifferent pairs yields another (p, q)-partitionable graph having the same maximum cliques and maximum stable sets as G. We call any such graph a *variant* of G. The variant in which no indifferent pair is an edge is called the *normalized* variant. Since the properties studied and used in this paper will always be based not on mere adjacency but on the arrangement of maximum cliques and maximum stable sets in a partitionable graph, the corresponding results will hold true for any variant of the graph.

Now we recall the definition of a subfamily of partitionable graphs introduced in [7].

For any two sets of integers X, Y, let $X+Y$ denote the set $\{x+y \mid x \in X,\ y \in Y\}$. If $X = \{x\}$ we will often write $x + Y$ instead of $\{x\} + Y$.

Let α and ω be two integers greater than or equal to two. Given factorizations $\omega = m_1 m_3 \cdots m_{2r-1}$ and $\alpha = m_2 m_4 \cdots m_{2r}$, where each factor is a positive integer $m_i \geq 2$, we consider the Chvátal-Graham-Perold-Whitesides graph defined as follows. First write:

$$
\begin{aligned}
\mu_i &= m_1 m_2 \cdots m_i \quad (\mu_0 = 1),\\
M_i &= \{0, \mu_{i-1}, 2\mu_{i-1}, \ldots, (m_i - 1)\mu_{i-1}\},\\
C &= M_1 + M_3 + \cdots + M_{2r-1},\\
S &= M_2 + M_4 + \cdots + M_{2r},\\
n &= m_1 m_2 \cdots m_{2r} + 1.
\end{aligned}
$$

Then let x_C (resp. x_S) be the n-dimensional characteristic vector of C (resp. S) with respect to the set $\{0, 1, 2, \ldots, n-1\}$. Let A_C (resp. A_S) be the matrix whose rows are the n possible circular permutations of x_C (resp. x_S). There exists an

(α,ω)-partitionable graph whose ω-clique matrix is A_C and whose α-stable set matrix is A_S. Such a graph is obtained by taking the elements $0,\ldots,n-1$ of the cyclic group $\mathbb{Z}_n$ as vertices, and adding an edge xy whenever the difference $x-y$ modulo n is equal to the difference of two elements in C. This graph will be denoted by $C[m_1,m_2,\ldots,m_{2r}]$. Any such graph and its variants will be called CGPW graphs.

We are interested in proving that no CGPW graph can be a counterexample to the Strong Perfect Graph Conjecture. In this endeavour, the following lemma will be crucial. Let us say that a subset of vertices T is a *transversal* if T meets every maximum-size clique and every maximum-size stable set in G. Let us say that a (transversal) subset of vertices is *small* if its cardinality is at most $\alpha+\omega-1$. Chvátal proved:

Lemma 1.1 (see [6]) *A minimally imperfect graph contains no small transversal.* □

We present a criterion which enables us to find a small transversal in many partitionable graphs. Let us say that a maximum clique K of G is a *mother* of a vertex $x \in K$ if every maximum clique K' containing x satisfies $|K' \cap K| \geq 2$. Similarly a maximum stable set S is a *father* of a vertex $x \in S$ if every maximum stable set S' containing x satisfies $|S' \cap S| \geq 2$.

Lemma 1.2 (The Parents Lemma) *If a vertex of a partitionable graph has a father and a mother then the graph has a small transversal.*

Proof. Let G be a partitionable graph and x be a vertex of G having a mother K and a father S. Let S' be the unique maximum stable set disjoint from K and K' be the unique maximum clique disjoint from S. Obviously $K' \neq K$ and $S' \neq S$. So S' and K' have a common element y, or else S' would be disjoint from two maximum cliques K' and K. Now we claim that $T = S \cup K \cup \{y\} \setminus \{x\}$ is a small transversal. It is clear that T has cardinality $\alpha+\omega-1$. Moreover, $S \setminus \{x\}$ meets every maximum clique of G except K' and the maximum cliques containing x. However, K' is met by y, and a maximum clique containing x is met by $K \setminus \{x\}$, since K is a mother of x. So, T meets every maximum clique of G and similarly every maximum stable set. □

Our main result is:

Theorem 1.1 *No CGPW graph with $\alpha > 2$ and $\omega > 2$ is minimally imperfect.*

The problem of knowing which minimally imperfect graphs can be obtained from the CGPW constructions was already studied by Grinstead [8]. In our terminology, Grinstead's result is that the normalized variant of any CGPW graph contains an odd hole or an odd antihole. Unfortunately this does not carry over to all variants, precisely because of the indifferent pairs. The proof of our theorem, on the other hand, uses only the arrangements of maximum cliques and maximum stable sets and so is true for all variants, but it does not exhibit a minimally imperfect proper subgraph. Instead, it proves the existence of a small transversal.

In order to prove Theorem 1.1, it appears necessary to distinguish between three types of CGPW graphs. More precisely, consider two special properties:

$$m_1 = m_3 = \cdots = m_{2r-1} = 2, \quad (1)$$
$$m_2 = m_4 = \ldots = m_{2r} = 2. \quad (2)$$

Let us say that a CGPW graph is of Type 1 if it satisfies none of (1) and (2), of Type 2 if it satisfies exactly one of them, and of Type 3 if it satisfies both of them.

Proposition 1.1 (Gurvich and Temkin [9]) *If G is a CGPW graph of Type 1 with $\alpha \geq 3$ and $\omega \geq 3$ then G admits a small transversal.* $\square$

Proposition 1.2 *If G is a CGPW graph of Type 2 with $\alpha \geq 3$ and $\omega \geq 3$ then G admits a small transversal.*

Proposition 1.3 *If G is a CGPW graph of Type 3 then G contains an induced C_5 without indifferent pairs of vertices.*

Clearly, Propositions 1.1–1.2–1.3 imply the theorem. Before proving these propositions, it is useful to recall some technical properties of CGPW graphs.

First, note that, with the notation above, if G is any n-vertex CGPW graph with maximum clique size ω and maximum stable set size α, then the n ω-cliques of G are the sets $C + x$ for each $x \in V(G)$; likewise the n α-stable sets are the set $S + x$ for each $x \in V(G)$. (All arithmetic is modulo n.)

Next, let us recall an isomorphism lemma first proved by Grinstead.

Lemma 1.3 ([8]) *The graph $C[m_3, m_4, \ldots, m_{2r}, m_1, m_2]$ is isomorphic to the graph $C[m_1, m_2, \ldots, m_{2r}]$.* $\square$

The next lemma entails that the complement of a CGPW graph is also a CGPW graph. More precisely:

Lemma 1.4 *The graph $C[m_2, m_3, \ldots, m_{2r}, m_1]$ is isomorphic to a variant of the complement of $C[m_1, m_2, \ldots, m_{2r}]$.*

The proof of Lemma 1.4 is similar to that of Lemma 1.3 and we omit it. □

We make three further observations whose proofs are easy and left to the reader.

Claim 1.1 *If i is an even subscript, Q is an ω-clique and x is a vertex of Q, then Q contains $x - \mu_i$ or $x + \mu_i$.* □

Claim 1.2 *If j is an odd subscript, U is an α-stable set and x is a vertex of U, then U contains $x - \mu_j$ or $x + \mu_j$.* □

Remark that Claims 1.1 and 1.2 are equivalent through Lemma 1.4.

Claim 1.3 *If i is an even subscript, R is an ω-clique and x is a vertex of R, then R contains $x + \mu_i$ or $x - (m_{i+1} - 1)\mu_i$.* □

We will use some further notation. The "largest" elements of C and S, respectively, are:

$$c_{max} = (m_1 - 1)\mu_0 + (m_3 - 1)\mu_2 + \cdots + (m_{2r-1} - 1)\mu_{2r-2}$$

and

$$s_{max} = (m_2 - 1)\mu_1 + (m_4 - 1)\mu_3 + \cdots (m_{2r} - 1)\mu_{2r-1}.$$

Clearly,

$$c_{\max} + s_{\max} = n - 2.$$

Observe that the unique maximum stable set disjoint from C is $S + c_{\max} + 1$. More generally,

$$\text{The maximum clique } C + u \text{ is disjoint from the maximum stable set } S + v \text{ if and only if } v = u + c_{\max} + 1. \tag{3}$$

Let us also observe that for every $c \in C$, the "conjugate" element $\overline{c} = c_{\max} - c$ is again an element of C, and analogously, for every $s \in S$, the element $s_{\max} - s$ belongs to S.

Let us present a useful characterization of the CGPW graphs where a vertex has at least one parent. A subset of three vertices of C that can be written as $c, c + x, c - x$ for some x is called an isoceles triangle.

Lemma 1.5 *The following conditions are equivalent:*

(a) $m_1 = m_3 = \cdots = m_{2r-1} = 2$*, i.e., condition (1);*

(b) C *contains no isoceles triangle;*

(c) There do not exist distinct vertices c, v, w *of* C *such that* $c - \overline{c} = v - w \bmod n$*;*

(d) No vertex has a mother.

Proof. (a)⇒(b). Assume C contains an isoceles triangle $c, c-x, c+x$. Write $u = c - x$ and $v = c + x$. So $2c = u + v$. By (a), we have $\mu_{2i+2} \geq 4\mu_{2i}$, and every vertex in C can be written as $\sum_{i=0}^{r-1} \varepsilon_i \mu_{2i}$ with $\varepsilon_i = 0, 1$. Under these conditions it is clear that $2c = u + v$ is possible only if $c = u = v$, a contradiction.

(b)⇒(c). Assume there is a triangle $c, v, w \in C$ such that $c - \overline{c} = v - w$. If $w = \overline{v}$, then we get $c - (c_{\max} - c) = v - (c_{\max} - v)$, so $2c = 2v$. Since $c \neq v$, this implies that n must be even, so $|C|$ is odd. Hence there is a vertex $z \in C$ with $\overline{z} = z$, and then $0, z, c_{\max}$ is an isoceles triangle in C, a contradiction. If $w \neq \overline{v}$, then $c - \overline{w} = c - \overline{c} + \overline{c} - \overline{w} = v - w + c_{\max} - c - c_{\max} + w = v - c$, so $c, v, \overline{w}$ is isoceles in C.

(c)⇒(d). Assume without loss of generality that C is the mother of a vertex $v \in C$. Consider the clique $C + v - \overline{v}$, which contains v since C contains $\overline{v}$. So $|C \cap (C + v - \overline{v})| \geq 2$ and there is a u different from v in that intersection. We can then write $u = w + v - \overline{v}$ for some $w \in C$. Now $u - w = v - \overline{v}$, contradicting (c).

(d)⇒(a). Assume $m_i \geq 3$ for some odd i. Consider the vertex $\mu_{i-1} \in C$. Since $m_i \geq 3$, the vertices 0 and $2\mu_{i-1}$ are also in C. By Claim 1.1, every ω-clique containing μ_{i-1} contains 0 or $2\mu_{i-1}$. Hence C is a mother of μ_{i-1}. In fact we obtain that for every vertex v, the clique $C + (v - \mu_{i-1})$ is a mother of v. □

Proof of Proposition 1.1

Assume that G is any variant of $C[m_1, m_2, \ldots, m_{2r}]$ of Type 1. Hence there exists an odd subscript i such that $m_i \geq 3$, and by the preceding lemma, there is a vertex which has a mother, and thus, by the circular symmetry of vertices, every vertex has a mother. Likewise, there exists an even subscript j such that $m_j \geq 3$ and so every vertex has a father. Now we are done by the Parents Lemma. □

Proof of Proposition 1.2

By Lemmas 1.3 and 1.4, and the trivial fact that a transversal of a partitionable graph is also a transversal of its complement, we can assume in this proof that $m_2 > 2$.

Let us remark that:

$$\text{For each } c \text{ in } C,\ n-1 \text{ is in } C + c + s_{\max} + 1. \tag{4}$$

Indeed, $n - 1 = \overline{c} + c + s_{\max} + 1$. Likewise:

$$\text{For each } s \text{ in } S,\ n-1 \text{ is in } S + s + c_{\max} + 1. \tag{5}$$

Let us introduce the subset $D = m_1 + M_{2r}$ of S and define:

$$T = ((C + m_1) \cup S \cup (D + \mu_{2r-2}) \cup \{n-1\}) \setminus D. \tag{6}$$

We claim that T is a small transversal. To establish this, first let us check that $|T| \leq \alpha + \omega - 1$. The maximum clique $C + m_1$ intersects the maximum stable set S in m_1, so $|(C+m_1) \cup S| = \alpha + \omega - 1$. Since $(D + \mu_{2r-2}) \cap (C + m_1) = \{m_1 + \mu_{2r-2}\}$ and $D \subset S$, it follows that adding the vertices of $(D + \mu_{2r-2}) \cup \{n-1\}$ to $(C + m_1) \cup S$ and removing those of D does not increase the cardinality.

Secondly we prove that T intersects all maximum stable sets. Remark that T contains $(C + m_1) \setminus \{m_1\}$ which, by Property (iv) of partitionable graphs, meets every maximum stable set except for the unique stable set disjoint from $C + m_1$ and for the stable sets containing m_1. The stable set disjoint from $C + m_1$ is $S + m_1 + c_{\max} + 1$ by (3), and it contains $n - 1$ by (5), so it meets T. The maximum stable sets containing m_1 all contain 0 or $2m_1$ by Claim 1.2. However, 0 and $2m_1$ are both in $S \setminus D$ (notice that $m_2 > 2$ is used here), and so these stable sets also meet T.

Thirdly we show that T intersects all maximum cliques. Remark that T contains $S \setminus D$ which, by Property (v) of partitionable graphs, meets every maximum clique except for the unique clique disjoint from S and for any clique containing a vertex of D. The maximum clique disjoint from S is $C + s_{\max} + 1$ by (3), and it contains $n - 1$ by (4), so it meets T.

Finally let us consider a maximum clique $C + u$ containing a vertex x of D and prove that this clique meets T. So $x = c + u = m_1 + \delta\mu_{2r-1}$ for some $c \in C$ and for some integer δ with $0 \leq \delta \leq \mu_{2r} - 1$.

First, let us make the assumption that $\delta > 0$. Claim 1.3 tells us that $C + u$ contains at least one of the vertices $x + \mu_{2r-2}$ and $x - (m_{2r-1} - 1)\mu_{2r-2}$, so it suffices to prove that these two vertices are in T. The first vertex $x + \mu_{2r-2}$ is trivially in T since x is in D and T contains $D + \mu_{2r-2}$. For the second vertex, we have:

$$\begin{aligned} x - (m_{2r-1} - 1)\mu_{2r-2} &= m_1 + \delta\mu_{2r-1} - m_{2r-1}\mu_{2r-2} + \mu_{2r-2} \\ &= m_1 + (\delta - 1)\mu_{2r-1} + \mu_{2r-2}, \end{aligned}$$

which clearly is the sum of μ_{2r-2} and of an element of D, since $\delta - 1 \geq 0$.

Now, let us assume $\delta = 0$. Then we have $x = c+u = m_1$, hence $c_{\max}+u = m_1+\overline{c}$. However $c_{\max} + u$ is in $C + u$, and $m_1 + \overline{c}$ is in $C + m_1$, hence in T unless $\overline{c} = 0$. So $C + u$ meets T except maybe if $c = c_{\max}$. If $c = c_{\max}$, we consider the vertex $y = m_1 - 1 - (m_{2r-1} - 1)\mu_{2r-2}$. This vertex is in $C + u$ because it is equal to $c_{\max} - [1 + (m_{2r-1} - 1)\mu_{2r-2}] + u$, where the square bracket is in C (notice $r > 1$ is used here). Vertex y is also in T because:

$$\begin{aligned} y &= y + \mu_{2r} + 1 \\ &= m_1 - 1 - (m_{2r-1} - 1)\mu_{2r-2} + m_{2r}\mu_{2r-1} + 1 \\ &= m_1 + (m_{2r} - 1)\mu_{2r-1} + \mu_{2r-2} \end{aligned}$$

showing clearly that y is in $D + \mu_{2r-2}$, but not in D. Hence $T \cap (C + u) \neq \emptyset$ as desired. □

The reader might remark that the transversal built in this proof also works for CGPW graphs of Type 1. But for Type 1 we preferred to include above the other (simpler) proof for Proposition 1.1.

For the sake of exhaustivity, let us mention that in [12] another transversal was defined, as follows. Consider a graph G of Type 2 with condition (2) and with $m_1 > 2$ and $r > 1$. With the notation above, write

$$W = (m_1 - 2) + M_3 + \cdots + M_{2r-1}$$

and set

$$T' = C \cup (S + c_{\max} + 1) \cup (W + m_1) \cup \{n - 2\} \setminus (W \cup \{c_{\max} + 1, n - 1\}).$$

Then it can be proved that T' is a small transversal of G. It is not difficult to check on the graph $C[3, 2, 2, 2]$ that, even up to the isomorphisms mentioned in Lemmas 1.3 and 1.4, this second transversal is not the same as the one given in the preceding proof.

Proof of Proposition 1.3

Let G be an n-vertex CGPW graph of Type 3 (hence $n = 2^{2r} + 1$), i.e., a variant of $C[2, 2, \cdots, 2, 2]$. The basic clique is

$$C = \{\epsilon_0 + 2^2\epsilon_2 + 2^4\epsilon_4 + \cdots + 2^{2r-2}\epsilon_{2r-2} \mid \epsilon_{2i} = 0, 1\},$$

and the basic stable set is

$$S = \{2\epsilon_1 + 2^3\epsilon_3 + \cdots + 2^{2r-1}\epsilon_{2r-1} \mid \epsilon_{2i+1} = 0, 1\} = 2C.$$

Set $a = \lfloor n/5 \rfloor$. We claim that the vertices $0, a, 2a, 3a, 4a$ induce a C_5 in G. To establish this, it is convenient to distinguish between the case r odd and the case r even.

If r is odd, then trivial induction on r shows that 5 divides n, so $n = 5a$. More precisely,

$$\begin{aligned} a &= 2^{2r-2} - 2^{2r-4} + \cdots - 4 + 1 \\ &= (2^{2r-2} + 2^{2r-6} + \ldots + 1) - (2^{2r-4} + \cdots + 4) \end{aligned}$$

where it appears clearly that a is the difference of two elements of C. In consequence $2a$ is the difference of two elements of S. Therefore, among the vertices $0, a, 2a, 3a, 4a$, any pair $\{x, x + a\}$ is in a maximum clique and any pair $\{x, x + 2a\}$ is in a maximum stable set, thus they induce a C_5. If r is even, it is again trivial to show that $n = 2 \bmod 5$, so $n = 5a + 2$. Writing $x_i = ia$ (i being understood modulo 5), it follows that $x_{i+1} - x_i = a$ or $= a + 2 \bmod n$, and that $x_{i+2} - x_i = 2a$ or $= 2a + 2 \bmod n$. However,

$$\begin{aligned} a &= 2^{2r-2} - 2^{2r-4} + \cdots + 4 - 1 \\ &= (2^{2r-2} + 2^{2r-6} + \cdots + 4) - (2^{2r-4} + \cdots + 1), \end{aligned}$$

from which it follows again that each of a, $a + 1$ and $a + 2$ is the difference of two elements of C. Consequently, $2a$ and $2a + 2$ are differences of two elements of S. Therefore, the vertices $0, a, 2a, 3a, 4a$ form a C_5 whose five edges are in maximum cliques and whose five non-edges are in maximum stable sets as above. □

2 Near-factorizations of the cyclic group

Let us say that a partitionable graph with n vertices is *circular* if there exists a cyclic numbering of its vertices (modulo n) such that, for every vertex x, for every maximum clique C and for every maximum stable set S, the set $C + x$ is a maximum clique and the set $S + x$ is a maximum stable set. Clearly, every odd hole and odd antihole is circular. More generally and as pointed out in the previous section, every CGPW graph is circular. The converse statement is not established, but we believe that it is also true:

Conjecture 1 *Every circular partitionable graph is a CGPW graph.*

This conjecture, together with the result of the previous section, would imply that no counterexample to the Strong Perfect Graph Conjecture is a circular graph. We will present some results in this direction.

Given a circular partitionable graph G, let $S = \{s_1, \ldots, s_\alpha\}$ be a maximum stable set and C be a maximum clique of G. We claim that the cliques $C + s_1, \ldots, C + s_\alpha$ are disjoint maximum cliques of G. Indeed, if not there would exist distinct elements c, c' in C and s, s' in S with $c + s = c' + s'$, so $c - c' = s' - s$. Consequently the two elements $0, c - c'$ would be in the clique $C - c'$ and the same two elements $0, s' - s$ would be in the stable set $S - s$, which is absurd. Therefore:

$$C + S = \mathbb{Z}_n \setminus \{u\} \text{ for some } u. \tag{7}$$

This property relates circular partitionable graphs to the following definition of group theory. In a finite, not necessarily Abelian, group H with multiplicative notation, two subsets A, B are said to form a *near-factor* of H if $|A| \times |B| = |H| - 1$ and $AB = H \setminus \{x\}$ for some $x \in H$. In other words every element of $H \setminus \{x\}$ can be written in a unique way as the product of an element from A with an element from B. In this terminology (7) means that every circular partitionable graph with n vertices yields a near-factorization of the cyclic group $(\mathbb{Z}_n, +)$. The reverse is also true: given a near-factorization A, B of $\mathbb{Z}_n$, build a graph whose vertices are the elements of $\mathbb{Z}_n$ and where ij is an edge whenever $i - j$ is equal to the difference of two elements of A. It is then easy to see that the resulting graph is partitionable and circular, with maximum cliques $A + x$ ($x \in \mathbb{Z}_n$) and maximum stable sets $B + x$ ($x \in \mathbb{Z}_n$). Any such graph and its variants will be denoted by $G(A, B)$.

Various properties of near-factors of groups were studied in [5]. Here we will be concerned only with near-factors in the cyclic group $\mathbb{Z}_n$. Given such a near factorization (A, B), it is possible to obtain another one through various tranformations:

- *Shifting.* Consider $(A + x, B + y)$ for any $x, y \in \mathbb{Z}_n$.
- *Scaling.* Consider $(\lambda A, \lambda B)$ where λ is an inversible element of $\mathbb{Z}_n$.
- *Swapping.* Consider $(-A, B)$.

(Actually shifting and swapping can be defined for any group.)

It is clear that any pair obtained from a near-factor of $\mathbb{Z}_n$ by shifting or scaling is a near-factor. Grinstead [8] showed that this is also true for swapping. Any such pair obtained through shifting, scaling or swapping will be considered isomorphic to (A, B). These isomorphisms induce an equivalence relation among the near-factors of $\mathbb{Z}_n$. A result of De Caen, Gregory, Hughes and Kreher [5] entails that every isomorphism class contains a pair (A, B) where both A and B are *symmetric*, i.e., $A = -A$ and $B = -B$. This can also be derived from Grinstead's swapping argument. In the rest of this note we will always assume, unless otherwise stated, that we are given a pair of symmetric sets forming a near-factor of $\mathbb{Z}_n$.

It is possible to build a large family of near-factorizations of $\mathbb{Z}_n$ by considering the type of near-factors introduced in the preceding section. More precisely, write $n = 1 + m_1 m_2 m_3 \cdots m_{2r}$, with integers $m_i \geq 2$ and define as before:

$$\begin{aligned}
\mu_i &= m_1 m_2 \cdots m_i \quad (\mu_0 = 1),\\
M_i &= \{0, \mu_{i-1}, 2\mu_{i-1}, \ldots, (m_i - 1)\mu_{i-1}\},\\
C &= M_1 + M_3 + \cdots + M_{2r-1},\\
S &= M_2 + M_4 + \cdots + M_{2r},
\end{aligned}$$

so that (C, S) is a near-factor of $\mathbb{Z}_n$ as pointed out in (7). Because of the strong similarity of this decomposition with what De Bruijn called "Degenerate British Number Systems" [4], we suggest calling such a pair and any isomorphic pair a *De Bruijn near-factorization* of the cyclic group $\mathbb{Z}_n$. The number r will be called the rank of the near-factorization.

It is clear that any CGPW graph with n vertices, as in the preceding section, is exactly a graph $G(A, B)$ for some De Bruijn near-factor (A, B) of $\mathbb{Z}_n$. Hence the above Conjecture can be restated as saying that every near-factorization of the cyclic group is De Bruijn. It is easy to see that this conjecture holds for near-factors with one factor of size two. Indeed this is equivalent to saying that a minimally imperfect triangle-free graph is an odd hole. We will consider here near-factorizations where one of the factors has size at most five, and prove that they satisfy Conjecture 1.

For $X \subseteq \mathbb{Z}_n$ and d a divisor of n, we let $R_d^i(X)$ denote the number of elements of X whose remainder modulo d is i.

Lemma 2.1 *Let (A, B) be a near-factorization of $\mathbb{Z}_n$ with $|A| \geq 2$, and d a divisor of n. Then:*

(1) Among the values of $R_d^0(A+B), \ldots, R_d^{d-1}(A+B)$, $d-1$ of them are equal to n/d, and the remaining one is equal to $n/d - 1$.

(2) $\gcd(A \cup \{n\}) = 1$.

Proof. Part (1) of the lemma is obtained by a trivial counting argument of the residual classes modulo d of $A + B$.

To prove Part (2), suppose $d = \gcd(A \cup \{n\}) \geq 2$. So all elements of A have the same remainder modulo d, and it follows that $R_d^i(A + B)$ is a multiple of $|A|$ for each i. This contradicts Part (1). □

Lemma 2.2 *Let (A, B) be a near-factorization of $\mathbb{Z}_n$. If $A = \{0, 1, \ldots, k-1\}$ for some k, then (A, B) is De Bruijn.*

Proof. Write $B = \{b_1, \ldots, b_q\}$. The fact that (A, B) is a near-factorization means that $A + b_1, \ldots, A + b_q$ form a partition of $\mathbb{Z}_n \setminus \{U\}$ for some U. Since A is an interval, so is each $A + b_i$. It is clear that there is a unique such partition up to the choice of U, and it entails $B = \{0, k, 2k, \ldots, (q-1)k\}$ up to a shift. Hence (A, B) is the De Bruijn near-factor obtained from the factorization $n = 1 + kq$, $m_1 = k$, $m_2 = q$, $r = 1$. □

Proposition 2.1 *Let (A, B) be a symmetric near-factorization of $\mathbb{Z}_n$ with $|A| = 3$. Then (A, B) is De Bruijn.*

Proof. Since A is symmetric we can write $A = \{-a, b, a\}$, with $b = -b$. So we must have either $b = 0$ or $b = n/2$ with n even. In the second case it is easy to check that $A + b$ is a symmetric shift of A, with $0 \in A + b$. So the second case reduces to the first one by isomorphism, and we may assume that $b = 0$. Note that a is inversible in $\mathbb{Z}_n$, or else Part (2) of Lemma 2.1 is contradicted. Now recall that $(a^{-1}A, a^{-1}B)$ is isomorphic to (A, B), and note that $a^{-1}A = \{-1, 0, 1\}$. So A is isomorphic to an interval, and we are done by Lemma 2.2. □

Proposition 2.2 *Let (A, B) be a symmetric near-factorization of $\mathbb{Z}_n$ with $|A| = 4$. Then (A, B) is De Bruijn.*

Proof. Since A is symmetric we can write $A = \{-b, -a, a, b\}$. We claim that:

$$b - a \text{ is inversible in } \mathbb{Z}_n.$$

Suppose on the contrary that $b - a$ is not inversible. So there is a common divisor d of $b - a$ and n with $d \geq 2$. Observe that a and b have the same residuals modulo d, and likewise $-a$ and $-b$ have the same residuals. It follows that:

$$R_d^i(A + B) = 2R_d^{i-a}(B) + 2R_d^{i+a}(B)$$

for all $i = 1, \ldots, d$. Therefore all the $R_d^i(A+B)$'s are even, which contradicts Part (1) of Lemma 2.1.

Now $b-a$ is inversible, with inverse λ. Consider the pair $(\lambda A, \lambda B)$, which is isomorphic to (A, B). We have $\lambda A = \{-\lambda b, -\lambda a, \lambda a, \lambda b\}$ and $\lambda b = \lambda a + 1$. So $\lambda A = \{-\lambda b, \lambda a\} + \{0, 1\}$. By shifting we can write $A = \{0,1\} + \{0, x\}$, with $x \in \mathbb{Z}_n \setminus \{0, 1\}$, and we write $B = \{b_1, \ldots, b_q\}$. Without loss of generality, we may assume that x is even, or else we consider $A - x = \{0,1\} + \{0, n-x\}$, where $n - x$ is even since n is odd. Call U the uncovered element, i.e., $\{U\} = \mathbb{Z}_n \setminus (A+B)$.

If $x = 2$, then $A = \{0,1,2,3\}$ is an interval and we are done, by Lemma 2.2. Now let us assume $x \geq 3$. So any shift $A + b_i$ can be written as $L_i \cup R_i$ with $L_i = \{b_i, b_i+1\}$ and $R_i = \{b_i + x, b_i + x + 1\}$. Notice that $\mathbb{Z}_n \setminus (A + b_i)$ splits in two intervals: the interval $[b_i + x + 2, \ldots, b_i - 1]$ is odd; and the interval $[b_i + 2, \ldots, b_i + x - 1]$ is even. Necessarily U is in the odd one for each i, for otherwise it would not be possible to cover this odd interval with parts of size two.

Now, starting from U in increasing order, we first encounter a part L_1 or R_1. In fact R_1 is excluded for otherwise there would be no room for L_1 on the cycle. Let k be the largest integer for which there exists a segment:

$$U,\ L_1,\ \cdots,\ L_k.$$

So the part following L_k must be R_j for some $j = 1, \ldots, q$. In fact we must have $1 \leq j \leq k$ or else there would be no room for L_j. In fact $j = 1$ or else there would be no room for R_1. Now we have a segment:

$$U,\ L_1,\ \cdots,\ L_k, R_1,\ \cdots, R_k,$$

which implies $2k = x$. We denote this by U, L_1—R_k. We can repeat this argument starting from the part that follows R_k. Finally, we obtain that all of $\mathbb{Z}_n$ is covered by the segment:

$$U, L_1 \text{—} R_k,\ L_{k+1} \text{—} R_{2k},\ \cdots, L_{q-k+1} \text{—} R_q.$$

In particular, $2k$ must divide q. It follows easily that we can write, up to shift, $B = \{0, 2, 4, \ldots, 2(k-1)\} + \{0, 4k, 8k, \ldots, (q/k - 1)4k\}$. Hence (A, B) is a De Bruijn near-factorization of rank 2, with $m_1 = 2$, $m_2 = k$, $m_3 = 2$, $m_4 = q/k$. □

Proposition 2.3 *Let (A, B) be a symmetric near-factorization of $\mathbb{Z}_n$ with $|A| = 5$. Then (A, B) is De Bruijn.*

Proof. Since A is symmetric we may write $A = \{-b, -a, u, a, b\}$ and, as in the proof of Proposition 2.1 we may assume $u = 0$. Let $d_1 = \gcd(a, n)$ and $d_2 = \gcd(b, n)$. Note that:

$$\gcd(d_1, d_2) = 1, \tag{8}$$

or else Part (2) of Lemma 2.1 would be contradicted.

Case 1: $\max\{d_1, d_2\} > 2$.

We may assume $d_1 > 2$ by symmetry. Write $x_i = R^i_{d_1}(B)$ and $t_i = R^i_{d_1}(A+B)$. Also let r be the remainder of b modulo d_1. Since $\gcd(d_1, d_2) = 1$, the elements b and $-b$ are not congruent modulo d_1. So we have:

$$x_{j-r} + 3x_j + x_{j+r} = t_j \quad (j = 0, ..., d_1 - 1), \tag{9}$$

where indices are modulo d_1. Recall from Part (1) of Lemma 2.1 that $t_j = n/d_1$ except for exactly one subscript j for which $t_j = n/d_1 - 1$. Define $x = \min\{x_0, x_1, \ldots, x_{d_1-1}\}$, and introduce new variables $x'_j = x_j - x$ and $t'_j = t_j - 5x$. Notice that (9) implies $t_j \geq 5x$ for all j. The equations (9) then become:

$$x'_{j-r} + 3x'_j + x'_{j+r} = t'_j \quad (j = 0, ..., d_1 - 1). \tag{10}$$

Notice that $x'_j \geq 0$ and $t'_j \geq 0$. All t'_j's are equal except for one which is one unit smaller, hence all t'_j's except maybe one are strictly positive. Now consider any subscript i such that $x_i = x$. The $(i-r)$-th, i-th and $(i+r)$-th equations are:

$$\begin{array}{ccccl} x'_{i-2r} & +3x'_{i-r} & & & = t'_{i-r} \\ & x'_{i-r} & +x'_{i+r} & & = t'_i \\ & & 3x'_{i+r} & +x'_{i+2r} & = t'_{i+r} \end{array} \tag{11}$$

¿From these three equations we can obtain:

$$x'_{i-2r} + x'_{i+2r} = t'_{i-r} + t'_{i+r} - 3t'_i. \tag{12}$$

Here the left-hand side is non-negative. It is not hard to check that the right-hand side can be non-negative only if t'_i is the special one that is one unit smaller than the other and

$$t'_i \in \{0, 1, 2\}. \tag{13}$$

So we can conclude that there is only one subscript i such that $x_i = x$, and that for this i we must have

$$t'_i = t'_j - 1 \quad \text{ for each } j \neq i.$$

¿From the above three equations, we can also get:

$$x'_{i-2r} + x'_{i-r} + x'_{i+r} + x'_{i+2r} = t'_{i-r} + t'_{i+r} - 2t'_i = 2. \tag{14}$$

implying that there are at least two zeros among x'_{i-2r}, x'_{i-r}, x'_{i+r} and x'_{i+2r}. This will contradict the unicity of i if the four subscripts $i-2r, i-r, i+r, i+2r$ are

distinct modulo d_1. So it must be that either $i-2r=i+2r$ or $i-r=i+2r$. If $i-2r=i+2r$ then $4r=0 \bmod d_1$, so $d_1=4r$. Hence r is a common divisor of a and b and so, by (8), $r=1$, whence $d_1=4$. Now $n=d_1(1+t_i)=4(1+t'_i+5x)$, and so (13) entails that $n=2,3,4 \bmod 5$, which contradicts $n=5|B|+1$.
If $i-r=i+2r$ then $3r=0 \bmod d_1$, so $d_1=3r$. Again r is a common divisor of a and b and so $r=1$, whence $d_1=3$. Now the equalities $n=3(1+t'_i+5x)$ and $n=5|B|+1$ are compatible modulo 5, but only if $t'_i=1$. So $|B|=3x+1$. However, the definition of the x_j's implies $|B|=x_0+x_1+x_2$, and so it must be that two of the x_j's are equal to x (and the third one is equal to $x+1$), which again contradicts the unicity of i.

Case 2: $d_1 \leq 2$ and $d_2 \leq 2$.

Since $\gcd(d_1,d_2)=1$, we can assume that at least one of d_1,d_2 is equal to 1, i.e., one of a,b is inversible, with inverse λ. Using scaling isomorphism, we can multiply by λ and consequently assume that

$$A=\{-y,-1,0,1,y\}.$$

Since (A,B) is a near-factorization, the union of the shifts $A+u$ of A, for $u \in B$, covers disjointly all of $\mathbb{Z}_n \setminus \{U\}$ for some uncovered element U. The middle interval $\{u-1,u,u+1\}$ of $A+u$ will be called the *head* of the shift $A+u$, while $\{-y+u\}$ and $\{y+u\}$ will be called its *left wing* and *right wing*, respectively.

Call two left wings (resp. right wings, heads) "consecutive" if one of the two intervals between them contains no other left wing (resp. right wing, head).

Claim 2.1 *Any two consecutive left-wings are at least 3 apart, and analogously for the right-wings.*

This follows clearly from the fact that the corresponding heads must not overlap.

Claim 2.2 *It is impossible to have all consecutive left-wings (and consecutive right-wings) at least 4 apart.*

Proof. Let us assume, on the contrary that all consecutive left-wings (and right-wings) are at least 4 apart. Then, it is impossible to fill the gap between consecutive left-wings using only right-wings (and maybe once the uncovered element), therefore there must be a head between any two consecutive left-wings. The same obviously holds for consecutive right-wings. Since there is the same number of left-wings, right-wings and heads, there must be exactly one head between any two consecutive left-wings, and consecutive right-wings. Thus, there must be exactly one left-wing and one right-wing between any two consecutive heads, and in the same order. In particular, there must be at least two spaces between any two consecutive heads. More precisely there are exactly two spaces, except once, when there are three spaces including U. But then, for the two consecutive heads with U between them, the distances to their left wings is different from the distances to their right wings, which is impossible. □

Claim 2.3 *If there is a pair of consecutive left-wings exactly 3 apart, then $2y = n-1$, and (A, B) is a De Bruijn near-factorization.*

To prove this second claim, let (L_i, H_i, R_i) and (L_j, H_j, R_j) be two shifts of A such that the left-wings L_i and L_j are consecutive along $\mathbb{Z}_n$ and are exactly at distance 3 apart. In this case of course the right-wings are also consecutive and exactly 3 apart, i.e. they form the following sequences:

$$\cdots,\ L_i,\ *,\ *,\ L_j,\ \cdots,\ R_i,\ *,\ *,\ R_j,\ \cdots$$

The two intervals of length two between L_i and L_j and between R_i and R_j must be covered (maybe except for the uncovered element U). However, this is almost impossible since heads do not fit, and two left- or right-wings cannot get that close by the first claim. The only possibility is that these two intervals overlap, so that there is only one uncovered position, which then must be U, i.e.

$$\cdots,\ L_i,\ R_i,\ *,\ L_j,\ R_j,\ \cdots$$

implying that the left and right wings of A are only one unit apart. In this case $-y = y+1$ follows, and so $2y = n-1$. Moreover,

$$A = \{1, -y, 0, y, -1\} = \{-1\} + \{0, y+1, 2(y+1), 3(y+1), 4(y+1)\}.$$

Since $y+1$ is inversible, because $2(y+1) - n = 1$, A is isomorphic to $\{0, 1, 2, 3, 4\}$. Now Lemma 2.2 entails that B must be equal to $\{0, 5, 10, \ldots, 5|B| - 5\}$, and that (A, B) is as desired a De Bruijn near-factorization (of rank 1). □

This concludes the proof of Proposition 2.3.

In summary, we have:

Theorem 2.1 *If (A, B) is a near-factorization of a cyclic group with* $\min\{|A|, |B|\} \leq 5$*, then (A, B) is a De Bruijn near-factorization.* □

Regardless of our conjecture, it is also possible to kill off another set of circular graphs that could be counterexamples to Berge's Strong Perfect Graph Conjecture, as follows.

Theorem 2.2 *Assume (A, B) is near-factorization of the cyclic group $\mathbb{Z}_n$ with n even. Then any graph $G(A, B)$ has a small transversal, i.e., no such graph is minimally imperfect.*

Proof. We may assume that A and B are symmetric and so, as in the proof of Proposition 2.1, that $0 \in A$ and $0 \in B$. We claim that A is a mother of 0 and B is a father of 0. To see that A is a mother of 0, consider any maximum clique $A + u$ such that $0 \in A + u$. Hence $-u \in A$, and by symmetry $u \in A$ too. However $u \in A + u$, and so $|A \cap (A + u)| \geq 2$. The proof that B is a father of 0 is similar. The desired conclusion now follows from the Parents Lemma. □

This theorem implies that any circular counterexample to Berge's Strong Perfect Graph Conjecture has an odd number of vertices.

It is conjectured in [10] that in any partitionable graph except CGPW graphs of type 2 and 3, there exists a vertex which has both a mother and a father. According to the Parents Lemma, this would imply that all partitionable graphs, other than CGPW graphs of type 2 and 3, have a small transversal, and thus are not minimally imperfect. For graphs of order up to 24 this was verified in [10].

The equivalence of (a) and (d) in Lemma 1.5 shows that in CGPW graphs of type 2 and 3 there cannot be a vertex having both a mother and a father. Proposition 1.2 shows, however, that in CGPW graphs of type 2 there still exist small transversals. We conjecture that there exists a small transversal in any partitionable graph, except maybe in CGPW graphs of type 3. For the special case of $C[2, 2, 2, 2]$ one can check that it has no small transversal at all, see e.g. [7].

It should be noted that any of the above two conjectures together with the results of this paper would imply the Strong Perfect Graph Conjecture.

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