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# Rectangle Breaking in Grids

by

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## ABSTRACT

Given an $n \times n$ square grid, there are the outlines of $n^2$ $1 \times 1$ squares, $(n-1)^2$ $2 \times 2$ squares, and so on. What is the minimum number of edges that can be removed from the grid such that there is no complete outline of a square remaining? Using techniques from tiling problems and other graph theoretic methods, we solve the problem for all $n$ and prove similar results for rectangular grids for which the side lengths differ by no more than six. We also introduce another version of the problem where we ask for the minimum number of edges to "break" all of the rectangles in a grid.

# 1 Introduction

On the radio show *Sunday Edition* on National Public Radio in March, 1996, New York Times' crossword puzzle editor Will Shortz posed the following problem:

Take 60 matchsticks and make a 5 by 5 square grid. In this grid are the outlines of 25 $1 \times 1$ squares, 16 $2 \times 2$ squares, and so on. What is the fewest matchsticks that can be removed such that there is no complete outline of a square remaining?

This problem has a natural extension. Given an $m \times n$ rectangular grid, consisting of $m(n+1)+n(m+1)$ edges, what is the minimum number of edges that can be removed such that no complete outline of a square remains? We say that a square in a grid is *broken* if an edge has been removed from its outline. Letting $s(m,n)$ represent the minimum number of edges that need to be removed from an $m \times n$ grid to break all of the squares in the grid, we will give upper and lower bounds on the value of $s(m,n)$ for all $m$ and $n$ and also consider the problem of breaking all of the rectangles in a grid.

# 2 Lower Bounds

An $m \times n$-grid contains the outlines of $mn$ $1 \times 1$ squares. Each edge in the grid is contained in the outline of at most two $1 \times 1$ squares, so to break all of the $1 \times 1$ squares, $s(m,n)$ must be at least $\lceil \frac{mn}{2} \rceil$. However, in almost all cases, more than $\lceil \frac{mn}{2} \rceil$ edges must be removed to break all of the squares.

There is a connection between this problem and problems involving the tiling of a rectangle using polyominoes such that no outline of a square is formed by the outlines of the polynominoes. Namely, the removal of an edge between two squares can be viewed as joining the two squares adjacent to it. For example, removing an edge between two adjacent squares will form a $2 \times 1$ tile which we call a *domino*. However, we must also allow tiles in which an edge or a portion of an edge is missing to break the outermost square in the case that we are tiling a square. For instance, we introduce a tile called a *umino*, which is a $1 \times 1$ square with an edge missing. This tile can be placed on the border of a square to break the outermost square, or can be joined with other polynominoes to form larger polynominoes (see Figure 1).

**Lemma 2.1** *If $mn$ is even and $m$ and $n$ are greater than 1, then every tiling of an $m \times n$ rectangle by $\frac{mn}{2}$ dominoes contains the outline of a $2 \times 2$ square.*

**Proof:**

One method to prove this statement involves what we call an *extended grid.* The extended grid consists of the normal rectangular $m \times n$ grid with an additional external vertex added. Edges are added between the external vertex and the vertices on the exterior of the grid such that all vertices in the original grid now have degree four (thus two edges are added between the corners of the grid and the external vertex). Thus, the external vertex has degree $2m + 2n + 4$.

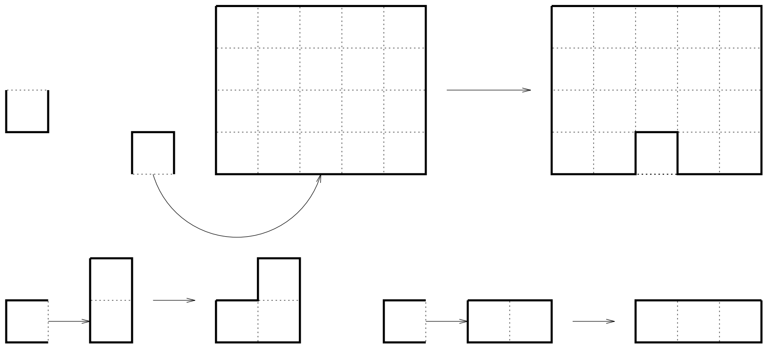


Figure 1: A umino and how it interacts with the border of a rectangle and with other tiles.

Assume there is a tiling using $\frac{mn}{2}$ dominoes, but no outline of a $2 \times 2$ square exists. Consider the subgrid formed by the outlines of these dominoes, and then consider the extended grid for this subgrid. This subgrid has $(m+1)(n+1)+1$ vertices and $m(n+1)+n(m+1)+2m+2n+4-\frac{mn}{2}$ edges.

Since there is no outline of a $2 \times 2$ square, the subgrid cannot have any vertices of degree 2. It also has no vertices of degree 0 or 1. Thus, its external vertex has degree $2m+2n+4$, and all other vertices have degree 3 or 4. Let $v_i$ denote the number of vertices of degree $i$. Thus, $v_3+v_4=(m+1)(n+1)$, and $3v_3+4v_4+2m+2n+4 = 2\#$ edges $= 3mn+6m+6n+8$. Some calculation shows $v_4 = m+n+1$ and $v_3 = mn$.

Consider the *contracted border* of the $m \times n$ grid. This is the border of the grid, except that each corner vertex has been removed and replaced by an edge joining the two vertices on the border adjacent to that vertex. Thus, the contracted border is a cycle with $2m+2n-4$ vertices.

There cannot be two consecutive vertices of degree 3 on the contracted border, as that would mean that two edges have been removed from the same $1 \times 1$ square in the grid, which is impossible since exactly one edge has been removed from each $1 \times 1$ square.

Therefore, by the pigeonhole principle, there are at most $m+n-2$ vertices of degree 3 on the contracted border. The corner vertices are vertices of degree 4, therefore the border contains at least $2m+2n-(m+n-2) = m+n+2$ vertices of degree 4. This is a contradiction, since $v_4 = m+n+1$. □

We are left to consider the case where $m$ and $n$ are both odd.

**Lemma 2.2** *If $m$ and $n$ are odd and greater than 1, then every tiling of an $m \times n$ rectangle with $\frac{mn-1}{2}$ dominoes and one $1 \times 1$ tile, such that the $1 \times 1$ tile is incident with the border of the rectangle contains the outline of a $2 \times 2$ square.*

**Proof:**

Given a tiling of an $m \times n$ rectangle using $\frac{mn-1}{2}$ dominoes and one $1 \times 1$ tile, assume the subgrid formed by the outline of these tiles does not contain the outline of a $2 \times 2$ square. The extended grid for this subgrid has $2mn+3m+3n+4-\frac{mn-1}{2}$ edges. Again,

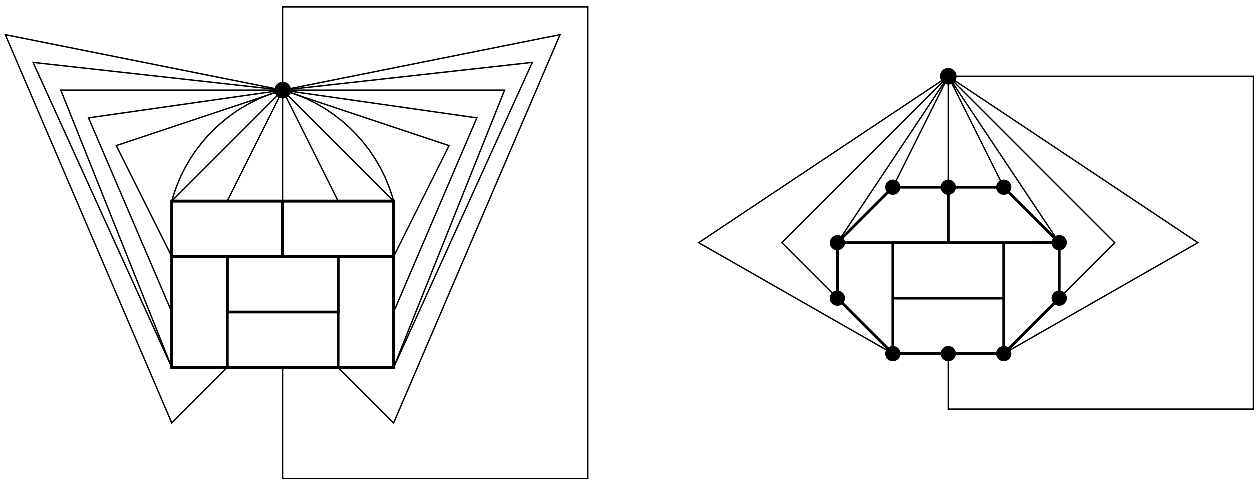


Figure 2: An extended grid for a tiling of a $3 \times 4$ rectangle, and the contracted border associated with it. In the extended grid, the only vertex of degree two is in the center of the $2 \times 2$ square, and there are no two consecutive vertices of degree 3 in the contracted border.

$v_3 + v_4 = (m+1)(n+1)$, and $3v_3 + 4v_4 + 2m + 2n + 4 = 2\#\text{edges} = 3mn + 6m + 6n + 9$. Therefore, $v_4 = m + n + 2$ and $v_3 = mn$.

Since $v_4 = m + n + 2$, the argument from the previous lemma fails. However, since the $1 \times 1$ tile is incident to the border, we have two consecutive vertices on the contracted border of degree 4. Therefore, there can be at most $m + n - 3$ vertices of degree 3 on the border. Consequently, there are at least $m+n+3$ vertices of degree 4 on the border, a contradiction. □

**Lemma 2.3** *Assume $n$ is odd. A square-free tiling of the an $n \times n$ square removes at least $\frac{n^2+1}{2} + 1$ edges.*

**Proof:** We have already shown that we need to remove at least $\frac{n^2+1}{2}$ edges. Assume there is a square-free tiling $\mathcal{T}$ where exactly $\frac{n^2+1}{2}$ edges have been removed. This implies that each removed edge is incident to exactly two $1 \times 1$ squares, with the exception of one edge, say $e'$. Also, at least one edge incident to the border must have been removed. Since such an edge is incident to at most one $1 \times 1$ square, it must be $e'$. So $e'$ is incident to one $1 \times 1$ square and to the border.

Create a new tiling $\mathcal{T}'$ by re-inserting $e'$. This tiling consists of $\frac{n^2-1}{2}$ dominoes, and one $1 \times 1$ tile which shares an edge with the border. By Lemma 2.2, there must be the outline of a $2 \times 2$ square in $\mathcal{T}'$. However, $\mathcal{T}$ was square-free, so the outline of the $2 \times 2$ square must contain $e'$. But this implies that $e'$ is incident to a domino, an impossibility since $e'$ is incident only to the $1 \times 1$ tile. □

The lower bound of $\lceil \frac{mn}{2} \rceil + 1$ does not necessarily hold when $m \neq n$ because nothing forces the $1 \times 1$ tile to be incident to the border of the grid. In fact, the only two cases for which this lower bound does not hold are for a $1 \times 2k$ rectangle (which has a trivial tiling by $k$ dominoes removing $k$ edges) and for an $n \times (n+2)$ rectangle, where $n$ is odd. There is a square-free tiling of this rectangle that removes exactly $\lceil \frac{n(n+2)}{2} \rceil$ edges (see Figure 3).

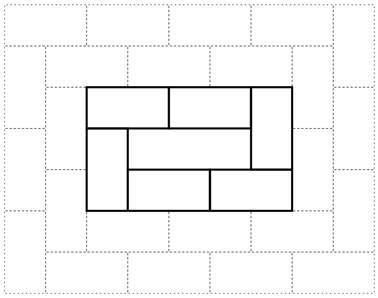


Figure 3: A square-free tiling of a $3 \times 5$ rectangle with only eight edges removed. This tiling can be extended to a $n \times (n+2)$ grid by adding "walls" around the rectangle as shown.

We use a different technique, which we call the *diagonal argument* to prove that $s(m,n) > \lceil \frac{mn}{2} \rceil$ in the case where $m$ and $n$ are odd and greater than 1, $|m-n| \neq 2$. To motivate our argument, we reprove Lemma 2.1.

Assume $mn$ is even and $m < n$. Orient an $m \times n$ rectangle such that it is wider than it is tall and assume that it is tiled by $\frac{mn}{2}$ dominoes such that no outline of a $2 \times 2$ square occurs. Consider the domino which covers the lower left corner square. It is either horizontal, as in Figure 4(a) or vertical, as in Figure 4(b). We label the lower left corner square 1. To achieve a square-free tiling, there is only one way to cover the square marked 2 with a domino, which forces how the square marked 3 must be covered, and so on. If $m < n$, this method continues until another vertical domino cannot be placed, which occurs after covering square $2m-2$ in (a), $2m-3$ in (b). Then, to cover the square marked **, a horizontal domino must be placed which creates the outline of a $2 \times 2$ square. If $m = n$, the argument is similar, as can be seen in Figure 4 (c) and (d).

The only difference between this proof of Lemma 2.1 and the proof for the case where both $m$ and $n$ are odd is that a umino is added to the tiling, but the diagonal argument is unaffected by the addition of this umino in almost all cases.

**Lemma 2.4** *Let $m$ and $n$ be odd numbers greater than 1 such that $n - m \geq 4$. Then $s(m,n) > \lceil \frac{mn}{2} \rceil$.*

**Proof:** We omit the case where $m = n$, since the umino must lie on the border of the square grid, and we have proven the result for this case. We also omit the case $|m-n| = 2$, since there are counterexamples where $s(m, m+2) = \lceil \frac{m(m+2)}{2} \rceil$.

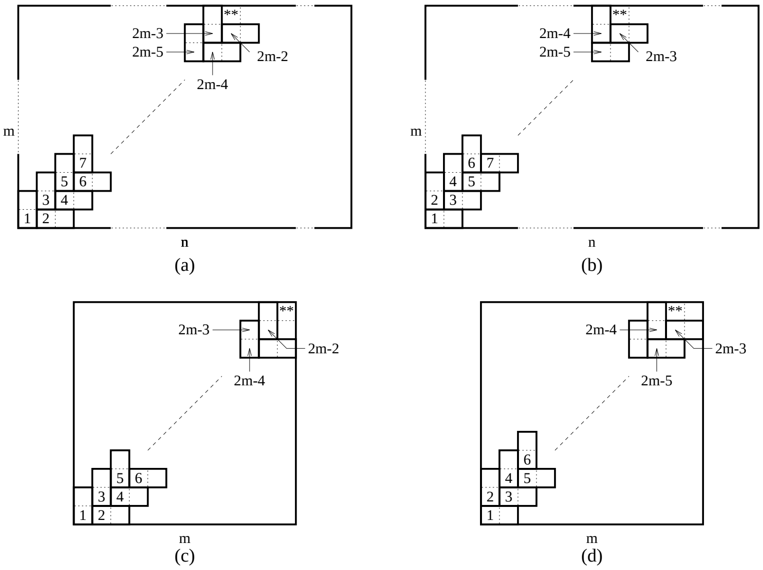



Figure 4: The diagonal argument, for $m < n$ and $m = n$, where the corner square is covered by (a) and (c) a horizontal domino, (b) and (d) a vertical domino.

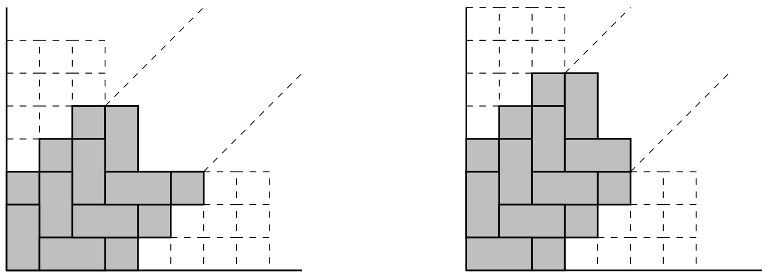


Figure 5: The shaded squares represent where a umino could disrupt the diagonal argument for that particular diagonal.

Given an $m \times n$ rectangular grid, enumerate the $1 \times 1$ squares $(1,1)\ldots(m,n)$, such that square $(i,j)$ is the $j$th square in the $i$th row of squares. Let $U_{1,1}$ represent the set of squares $(i,j)$ such that if the umino is placed on any of the squares in $U_{1,1}$, it will disrupt the diagonal argument starting at $(1,1)$ for either of the orientations of the domino covering $(1,1)$. $U_{m,1}$, $U_{1,n}$, and $U_{m,n}$ are defined similarly. By superimposing the two images in Figure 5 together, we see that $U_{1,1} = \{(i,j) : |i-j| \leq 3\}$. The other $U$ sets are represented similarly:

$$
\begin{aligned}
U_{1,1} &= \{(i,j) : |i-j| \leq 3\} \\
U_{m,1} &= \{(i,j) : |i+j-(1+m)| \leq 3\} \\
U_{1,n} &= \{(i,j) : |i+j-(1+n)| \leq 3\} \\
U_{m,n} &= \{(i,j) : |i-j-(m-n)| \leq 3\} \\
U_{1,1} \cap U_{m,n} &= \{(i,j) : n-2 \leq i+j \leq m+4\}
\end{aligned}
$$

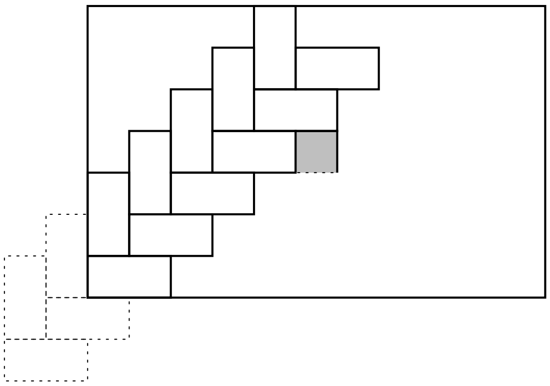
$$U_{m,1} \cap U_{1,n} = \{(i,j) : -3 \le i-j \le 3-(n-m)\}$$

If $U_{1,1} \cap U_{1,n} \cap U_{m,1} \cap U_{m,n} = \emptyset$, then there is some diagonal for which the diagonal argument will work regardless of the placement of the umino. If $n-m > 6$, then both $U_{1,1} \cap U_{m,n}$ and $U_{m,1} \cap U_{1,n}$ are empty, so there are only two cases to consider.

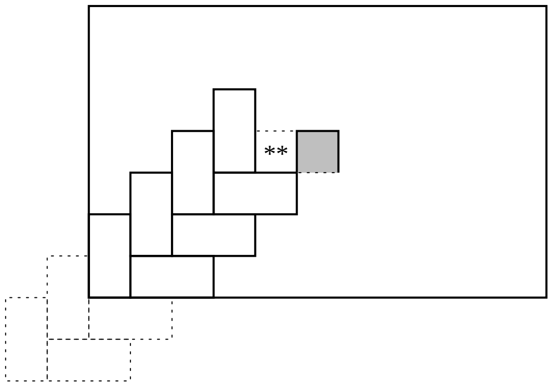
Case 1: $n-m=6$. There is a unique square in both $U_{1,1} \cap U_{m,n}$ and $U_{m,1} \cap U_{1,n}$, it is $((m+1)/2,(m+7)/2)$. However, the number of squares $(i,j)$ where $i+j$ is even is one more than the number of squares where $i+j$ is odd, therefore, the umino must be placed on a square where $i+j$ is even, since every domino covers exactly one square where $i+j$ is even and one square where $i+j$ is odd. $(m+1)/2+(m+7)/2 = m+4$, which is odd, so the umino cannot be placed on the square $((m+1)/2,(m+7)/2)$.

Case 2: $n-m=4$. There are five squares in $U_{1,1} \cap U_{1,n} \cap U_{m,1} \cap U_{m,n}$, but the only one where the sum of the coordinates is even is the square $((m+1)/2,(n+1)/2)$. Consider the placement of the umino on the center square. Because of symmetry, we may assume the open end of the umino faces downward or to the right. As can be seen in Figure 6, the diagonal argument for $(1,1)$ is either unaffected by the umino or it forces a $2\times 2$ square to be created, depending on the placement of the initial domino covering $(1,1)$.

□



(a)




(b)

Figure 6: When $m$ and $n$ differ by 4, depending on the orientation of the domino covering $(1,1)$, the umino either (a) does not affect the diagonal argument for that corner or (b) forces a vertical domino to be placed to cover the square marked **, forming the outline of a $2\times 2$ square. A $7\times 11$ rectangle is shown, but the argument can be extended easily.

# 3 Upper Bounds

**Theorem 3.1** *If $m = n$, $s(m,n) \le \lceil \frac{mn}{2} \rceil + 1$. Otherwise, $s(m,n) \le \lceil \frac{mn}{2} \rceil + \lceil \frac{|m-n|}{6} \rceil$.*

**Proof:** For all $m$ and $n$ we will present a construction that generates a tiling of an $m \times n$ rectangle where at most the number of edges given above is removed. For $k \ge 1$ consider the tiling of a $1 \times 3k$ rectangle by $k$ $1 \times 3$ tiles. This tiling is square-free and the number of edges removed is $2k = \frac{3k}{2} + \frac{k}{2} \le \lceil \frac{3k}{2} \rceil + \lceil \frac{3k-1}{6} \rceil$, thus satisfying the upper bound given above. As in Figure 3, where we build walls around a single $1 \times 3$ tile to get optimal tilings for $n \times n+2$ rectangles when $n$ is odd, we also build walls around this string of $3 \times 1$ tiles using dominoes to get a tiling of a $(2j+1) \times (2j+3k)$ rectangle for all $j \ge 0, k > 1$. Because the layers of the wall are staggered such that no $2 \times 2$ square occurs between two layers, the tiling is square-free.

**Lemma 3.1** *For all $j \ge 0, k > 1$, the number of edges removed in the tiling of a $(2j+1) \times (2j+3k))$ rectangle as above is less than or equal to $\lceil \frac{(2j+1)(2j+3k)}{2} \rceil + \lceil \frac{3k-1}{6} \rceil$.*

**Proof:** The proof is by induction on $j$ for a fixed $k$. We showed above that the statement is true for $j = 0$. Given a tiling of a $(2j+1) \times (2j+3k)$ rectangle as above, by our inductive assumption, we know that the tiling of the inner $(2j-1) \times (2(j-1)+3k)$ rectangle removes at most $\lceil \frac{(2j-1)(2(j-1)+3k)}{2} \rceil + \lceil \frac{3k-1}{6} \rceil$ edges. The outermost layer of the wall has area $((2j+1)(2j+3k) - (2j-1)(2(j-1)+3k) = 8j+6k-2$ and is therefore covered by $4j+3k-1$ dominoes, so the number of edges removed for the entire $(2j+1) \times (2j+3k)$ rectangle is at most:

$$\begin{aligned}
\lceil \frac{(2j-1)(2(j-1)+3k)}{2} \rceil + \lceil \frac{3k-1}{6} \rceil + 4j + 3k - 1 &= \\
\lceil \frac{(2j-1)(2(j-1)+3k)}{2} + 4j + 3k - 1 \rceil + \lceil \frac{3k-1}{6} \rceil &= \\
\lceil \frac{(2j+1)(2j+3k)}{2} \rceil + \lceil \frac{3k-1}{6} \rceil &
\end{aligned}$$

□

Given an initial center of $k$ $1 \times 3$ tiles, each time a wall of dominoes is built around a rectangle in the proof above, the length and width of the rectangle increases by two. However, the wall can be built one side at a time, and depending on in what order the sides are added, square-free tilings of rectangles can be generated where the side lengths differ by $3k-2$, $3k-1$, or $3k$. Looking at Figure 7, starting with a single $1 \times 3$ tile, building the wall above this tile gets a tiling of a $2 \times 3$ rectangle with four edges removed, and then building the wall to the left of this rectangle gets a tiling of a $2 \times 4$ rectangle, and so on. It remains to show that the number of edges removed in tilings by this method is less than or equal to the desired upper bound.

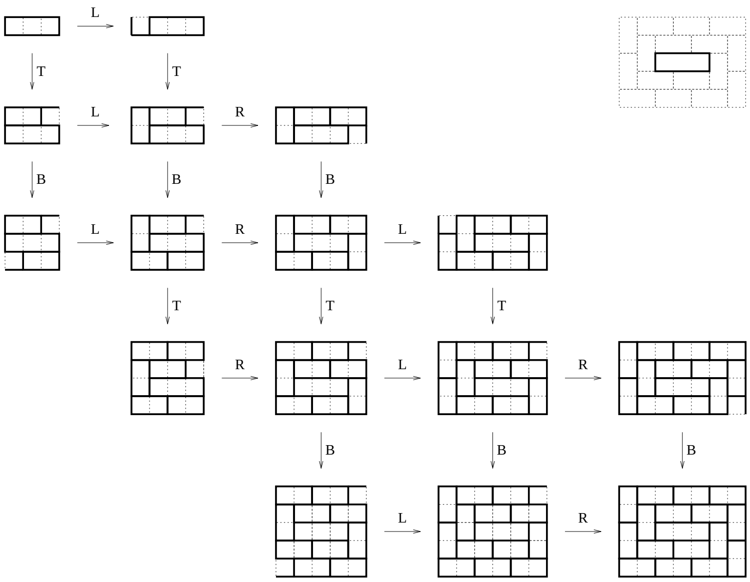



Figure 7: Building walls around a single $1 \times 3$ tile one side at a time allows us to generate square-free tilings of all rectangles where the side lengths differ by no more than 3.

Given a $(2j+1) \times (2j+3k)$ rectangle, where $j \geq 0$ and $k \geq 1$, assume that the rectangle is tiled as described in the above lemma. In particular, if $j > 0$, then the lower left corner square in the rectangle is covered by a horizontal domino. Furthermore, when adding sides of the wall to this rectangle, we will always add the left side of the wall before the right, and the top side of the wall before the bottom. This is mainly to coincide with Figures 7 and 8, but it also reduces the number of cases we have to consider. The cases are summarized in the table below.

| Area of rectangle | Additional Area | Additional edges removed, $k$ odd | Additional edges removed, $k$ even |
|---|---|---|---|
| $(2j+1) \times (2j+3k)$ | $0$ | $0$ | $0$ |
| $(2j+1) \times (2j+3k+1)$ | $2j+1$ | $j+1$ | $j+1$ |
| $(2j+2) \times (2j+3k)$ | $2j+3k$ | $j+\frac{3k+1}{2}$ | $j+\frac{3k}{2}$ |
| $(2j+2) \times (2j+3k+1)$ | $4j+3k+2$ | $2j+\frac{3k+3}{2}$ | $2j+\frac{3k}{2}+1$ |
| $(2j+2) \times (2j+3k+2)$ | $6j+3k+4$ | $3j+\frac{3k+5}{2}$ | $3j+\frac{3k}{2}+2$ |
| $(2j+3) \times (2j+3k+1)$ | $6j+6k+3$ | $3j+3k+2$ | $3j+3k+2$ |
| $(2j+3) \times (2j+3k+2)$ | $8j+6k+6$ | $4j+3k+3$ | $4j+3k+3$ |

The table above describes the number of additional edges removed and the increase in area when building a part of a wall around a tiling of a $(2j+1) \times (2j+3k)$ rectangle. We will show that the number of edges removed is no larger than the desired upper bound for the $(2j+2) \times (2j+3k+1)$ rectangle where $k$ is odd. The calculations for the other cases are quite similar.

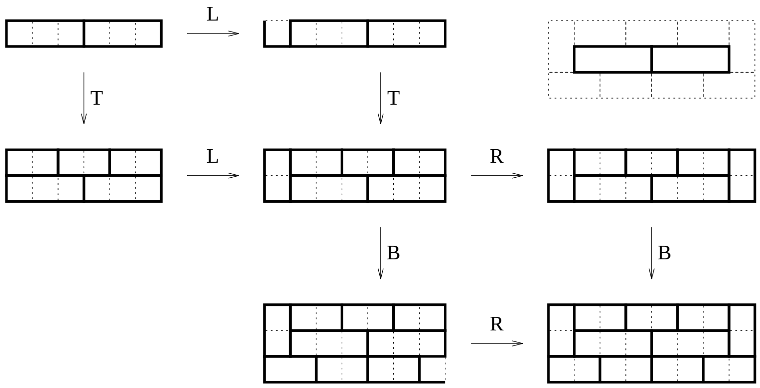



Figure 8: Building walls around two $1 \times 3$ tiles one side at a time allows us to generate square-free tilings of all rectangles where the side lengths differ by 4, 5, or 6.

$$\lceil\frac{(2j-1)(2(j-1)+3k)}{2}\rceil + \lceil\frac{3k-1}{6}\rceil + 2j + \frac{3k+3}{2} =$$
$$\lceil\frac{(2j-1)(2(j-1)+3k)+4j+3k+3}{2}\rceil + \lceil\frac{3k-1}{6}\rceil =$$
$$\lceil\frac{(2j+1)(2j+3k)+4j+3k+2}{2}\rceil + \lceil\frac{3k-1}{6}\rceil$$

The final equality is true because $\frac{(2j+1)(2j+3k)+4j+3k+2}{2}$ is odd, so $\lceil\frac{(2j+1)(2j+3k)+4j+3k+2}{2}\rceil = \lceil\frac{(2j-1)(2(j-1)+3k)+4j+3k+3}{2}\rceil$.

Every case is covered except when $m = n$. In that case, if $m$ is even, a layer of $m/2$ dominoes is added to the top side of the tiling of the $(m-1) \times m$ rectangle described above, or if $m$ is odd, a layer of $(m-1)/2$ dominoes and one umino oriented to break the outermost square can be placed on the bottom side of the tiling of the $(m-1) \times m$ rectangle such that no outlines of $2 \times 2$ squares are formed (See Figure 7). It is a simple calculation to show that the desired upper bound is valid.

$\square$

If $|m-n| \leq 6$, then the upper and lower bounds of $s(m,n)$ coincide and the problem is solved for such rectangles. It is a bit more difficult to show that, for example, $s(4,11) > 23$ (and therefore, equal to 24 by the above theorem). Finding methods for establishing stronger lower bounds remains an open problem.

# 4 Rectangle-free Grids

Given an $m \times n$ rectangular grid, another question that could be asked is what the minimum number of edges that need to be removed in order to break all of the *rectangles* in the grid? Let $r(m,n)$ denote the fewest edges that need to be removed from an $m \times n$ rectangular grid such that no complete outline of a rectangle remains.

**Theorem 4.1** $\frac{2mn}{3} \leq r(m,n) \leq \frac{2mn}{3} + \frac{4(m+n)}{3} + 2.$

**Proof:** Consider the tiles that can be used in a rectangle-free tiling. Uminoes can be used and they break one $1 \times 1$ square with the removal of one edge. Dominoes cannot be used (since they are in themselves rectangles). L-tiles formed by 3 squares (see Figure 9), break three $1 \times 1$ squares with the removal of two edges. Any other tile has a greater ratio of number of edges removed to $1 \times 1$ squares broken, so $r(m,n)$ must be at least $2/3$ of the area of the rectangle.

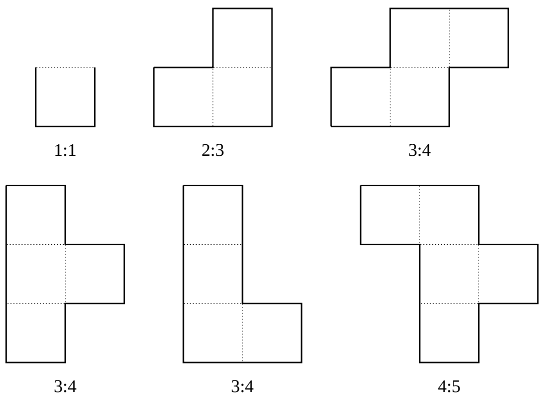



Figure 9: A set of polynominoes that can be used for rectangle-free tilings, with the ratios of edges removed per 1-squares broken. Note that the L-tile has the best such ratio.

To get a tiling where the fewest edges are removed, it would make sense to try to create tilings of rectangles that consist primarily of these L-tiles. One interesting property of these L-tiles is that they can tile the plane in such a way that no outline of a rectangle occurs. Thus, any rectangle can be placed on this infinite tiling, and the tiles (or portions of tiles) that lie within the rectangle form a rectangle-free tiling (See Figure 10). Along any row of the grid, at most $\frac{m}{3}+1$ edges are removed, similarly along any column of the grid, at most $\frac{n}{3}+1$ are removed, thus the number of edges removed is at most $(n+1)(\frac{m}{3}+1)+(m+1)(\frac{n}{3}+1) = \frac{2mn}{3} + \frac{4(m+n)}{3} + 2$.

□

A stronger lower bound for $r(m,n)$ is $2\lfloor\frac{mn}{3}\rfloor + r_{m,n}$, where $r_{m,n} = 1$ if $mn \equiv 0$ or $1$ modulo 3, and $r_{m,n} = 2$ if $mn \equiv 2$ modulo 3. If $mn$ is divisible by three, then even if there

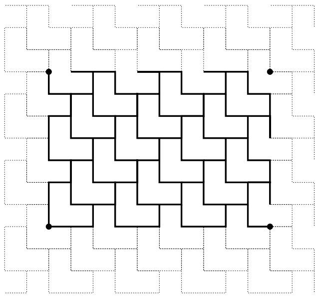


Figure 10: An infinite tiling of the plane by L-tiles, and a rectangle-free tiling of a rectangle using this infinite tiling. The corners of the tiled rectangle are highlighted.

is a rectangle-free tiling of the rectangle by $\frac{mn}{3}$ L-tiles, one edge must be removed from the perimeter to break the outermost rectangle. For the other two cases, the number of edges removed from an $m \times n$ rectangle must be at least as many as if the rectangle was tiled by $\lfloor \frac{mn}{3} \rfloor$ L-tiles and 1 or 2 uminoes, depending on the residue of $mn$ modulo 3. We present some rectangles with their optimal rectangle-free tilings in Figure 11.

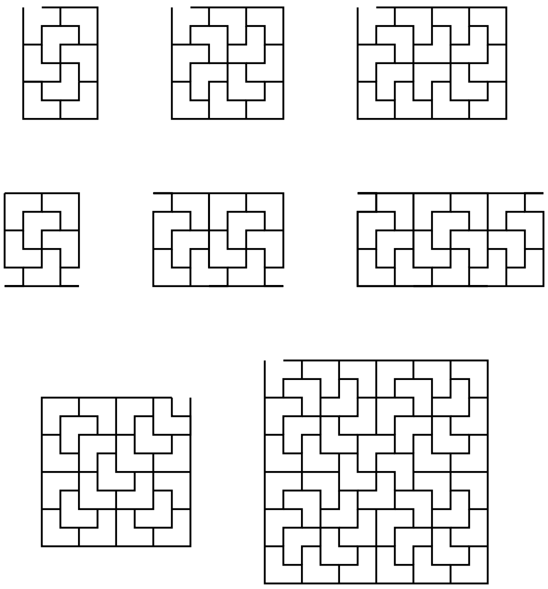


Figure 11: Optimal tilings of some rectangles.

# 5 Conclusion

In this paper we have presented lower and upper bounds for the fewest number of edges that need to be removed from a rectangular grid such that no outline of a square (or a rectangle) remains. This problem is actually a special case of the minimum hitting set problem, which is NP-complete.[2] In that problem, the input is a collection $C$ of subsets of some set $X$ and the problem is to find the size of the smallest *hitting set* $H \subset X$ such that every set in the collection $C$ has a non-empty intersection with $H$. If we consider the outlines of the squares (or rectangles) in the grid to be a collection of subsets of the edge set of the grid, then the removal of a hitting set breaks all of the squares (or rectangles) in the grid. Although our particular problem has not been appeared in the literature, it is related to other problems involving tilings of the plane and patterns that arise from the tilings. The authors recommend a book by Grünbaum and Shephard for further information on this topic.[1]

Proving lower bounds for the values of both $s(m,n)$ and $r(m,n)$ beyond the trivial ones given in this paper proves to be quite difficult. For example, simply showing $s(4,11) > 23$ required us to consider several cases, all of which were generally unpleasant. As far as upper bounds are concerned, we believe that the upper bound for $r(m,n)$ may be improved by embedding rectangles in other infinite tilings of the plane. For example, using the fact that four L-tiles can be used to make a larger L-tile, which can be used to make larger and larger L-tiles provides an excellent way to produce an optimal rectangle free tiling for a $2^n \times 2^n$ square (see Figure 11), perhaps embedding a rectangle in a tiling of an sufficiently large L will produce a better upper bound. Also, there may be methods for finding upper bounds for $r(m,n)$ that do not involve embedding rectangles in infinite tilings. There are still several interesting problems to be solved related to this topic; how about parallelpiped breaking? We hope this paper entices people to explore some of these problems.

# References

[1] B. Grünbaum and G.C. Shephard. *Tilings and patterns.* W.H. Freeman and Company, New York, 1987.

[2] R.M. Karp. Reducibility among combinatorial problems. In R.E. Miller and J.W. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. Plenum Press, New York, 1972.