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Chromatic-Index Critical Graphs of Even Order

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ABSTRACT

A k-critrical graph G has maximum degree $k \geq 0$, chromatic index $\chi'(G) = k+1$ and $\chi'(G-e) < k+1$ for each edge e of G.

The Critical Graph Conjecture, Jakobsen [8] and Beineke, Wilson [1], claims that every k-critical graph is of odd order. Fiorini and Wilson [6] conjectured that every k-critical graph of even order has a 1-factor. Chetwynd and Yap [4] stated the problem whether it is true that if G is a k-critical graph of odd order, then $G - v$ has a 1-factor for every vertex v of minimum degree. These conjectures are disproved and the problem is answered in the negative for $k \in \{3, 4\}$.

We disprove these conjectures and answer the problem in the negative for all $k \geq 3$.

We also construct k-critical graphs on n vertices with degree sequence $23^2 4^{n-3}$, answering a question of Yap [11].

1 Introduction

We consider connected multigraphs $M = (V(M), E(M))$ without loops, where $V(M)$ $(E(M))$ denotes the set of vertices (edges) of M. The *degree* $d_M(v)$ of a vertex $v \in VM$ is the number of edges incident with v. The *maximum* degree of a vertex in M is denoted by $\Delta(M)$. If $m \geq 1$ edges join the same pair of vertices v, w of M then vw is called *multiple edge* with *multiplicity* $\mu(vw) = m$. With $\mu(M)$ we denote the maximum multiplicity in M, and we say that M is a *graph* if $\mu(M) = 1$. Throughout this paper we will use G to denote graphs.

A *k-coloring* of M is a function $\phi : E(M) \to \{1, \dots, k\}$ so that $\phi(e) \neq \phi(f)$ for adjacent edges e and f. The *chromatic index* $\chi'(M)$ of a multigraph M is the smallest number k so that there is a k-coloring of M.

Vizing [10] proved that $\chi'(M) \in \{\Delta(M), \dots, \Delta(M) + \mu(M)\}$, i.e. the chromatic index of a graph is Δ or $\Delta + 1$. For $k \geq 0$ we define a multigraph M to be *k-critical* if $\Delta(M) = k$, $\chi'(M) = k + 1$ and $\chi'(M - e) < \chi'(M)$ for each $e \in E(M)$.

Jakobsen [8] gave a list of all 3-critical graphs with at most 10 vertices and showed that there are 3-critical graphs with 5, 7 and 9 vertices while there are no such graphs of even order. He formulated the following conjecture which was also stated by Beineke and Wilson [1].

Conjecture 1.1 (Critical Graph Conjecture) *For every $k \geq 1$ there is no k-critical graph of even order.*

This conjecture is certainly true for $k = 1, 2$. Goldberg [7] disproved it for $k = 3$, and Chetwynd and Fiol independently disproved it for $k = 4$, see [3]. The following two statements are also known to be false for $k \in \{3, 4\}$, see [5].

Conjecture 1.2 ([6]) *For all $k \geq 1$ every k-critical graph of even order has a 1-factor.*

Problem 1.3 ([4]) *For all $k \geq 1$, is it true that for every k-critical graph G of odd order the graph $G - v$ has a 1-factor for every vertex v of minimum degree?*

This paper is to disprove Conjectures 1.1, 1.2 and to answer Problem 1.3 in the negative for all $k \geq 3$.

We basically give two constructions of k-critical (multi-) graphs, one works for each $k \geq 4$, and the other works for odd $k \geq 3$ and gives smaller graphs than the first. The focus on the order of these graphs is motivated by the fact that every k-critical graph of even order gives rise to an infinite family of such graphs. Our main results are the following theorems.

Theorem 1.4 *For any $k \geq 4$ there is a k-critical graph of order $4(3k^2 - 6k + 1)$. For any odd $k \geq 3$ there is a k-critical graph of order $\frac{(k+1)(7k-1)}{2} + 2$.*

By switching between multigraphs and graphs we obtain

Theorem 1.5 *For any $k \geq 4$ there is a k-critical multigraph of order $12k - 20$. For any odd $k \geq 3$ there is a k-critical multigraph of order $7k + 1$.*

Applying results of Choudum [5] we derive:

Theorem 1.6 *For each $k \geq 3$ there is a k-critical graph of even order without 1-factor.*

Theorem 1.7 *For any $k \geq 3$ and any $2 \leq n \leq k-1$ there is a k-critical graph G of odd order and minimum degree n, which has a vertex v of minimum degree such that $G - v$ has no 1-factor.*

As a consequence of our investigations we state the following corollary which answers a question of Yap [11].

Corollary 1.8 *There are 4-critical graphs of odd order n with degree sequence $23^2 4^{n-3}$.*

2 Basic Results and Constructions

The Parity Lemma [2] is a useful tool in the theory of edge-colorings of multigraphs. We state it as follows.

Lemma 2.1 (Parity Lemma) *Let M be a graph whose edges are colored with colors $1, \ldots, \Delta(M)$, and let a_i be the number of vertices v in M such that no edge incident to v is colored i. Then for all $i = 1, \ldots, \Delta(M) : a_i \equiv |V(G)| \pmod 2$.*

Proof. For $i = 1, \ldots \Delta(M)$ let E_i be the set of edges colored i. Then $a_i = |V(M)| - 2|E_i|$, and hence $a_i \equiv |V(G)| \pmod 2$.

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The *Hajós-union* of two multigraphs M and M' is the multigraph N obtained by

1. choosing vertices $v \in V(M)$ and $v' \in V(M')$,
2. removing edges vw and $v'w'$ from M and M', respectively,
3. identifying v and v',
4. adding the edge ww'.

We also say that N is obtained from M and M' by Hajós-union where v and v' are identified. It is well known that any Hajós-union of two k-critical graphs is k-critical whenever the sum of the degrees of the two vertices used in the construction is smaller or equal to $k+2$, cf. [11] Thm. 4.6. With almost the same prove this statement holds true for k-critical multigraphs. We state:

Lemma 2.2 *Let M and M' be two k-critical multigraphs and $v \in V(M)$, $v' \in V(M')$ two vertices such that $d_G(v) + d_{G'}(v') \leq k+2$. Then any multigraph obtained from M and M' by Hajós-union where v and v' are identified is k-critical.*

Next we prove some statements which allow us to switch from multigraphs to graphs and vice versa. Let M be a multigraph on vertices $v_1, \ldots, v_n$ and with maximum vertex-degree k. We call $s(M) = kn - \sum_{i=1}^{n} d_M(v_i)$ the *deficiency* of M. Let $X \subset V(M)$ then $\partial(X)$ denotes the set of edges with exactly one end in X.

Lemma 2.3 *Let M be a k-critical multigraph, $k \geq 2$, and N an induced proper submultigraph of odd order. If $s(N) = k$ then the multigraph M^- obtained from M by contracting N to a single vertex v is k-critical.*

Proof. Clearly M^- is not k-colorable. On the other hand N is k-colorable, and for each $e \in E(M) - E(N)$ there is a k-coloring of $M - e$. It follows with Lemma 2.1 that such a coloring simply can be transformed to a coloring of $M^- - e$, where edges wv correspond to edges wx of $\partial(V(N))$.

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We construct k-critical graphs from k-critical multigraphs by replacing vertices by the bipartite graph $K_{k,k-1}$. This idea is basically due to Meredith [9].

Theorem 2.4 *For $k \geq 2$, let M be a multigraph with $\Delta(M) = k$, and $v \in V(M)$ with $d_M(v) = d \leq k$. Let $v_1, \ldots, v_d$ be the not necessarily distinct neighbors of v, and $u_1, \ldots, u_k$ the vertices of degree $k-1$ in the complete bipartite graph $K_{k,k-1}$. Let M' be the multigraph obtained from $M - v$ and $K_{k,k-1}$ by adding edges $v_i u_i$ for $i = 1, \ldots, d$. Then M is k-critical if and only if M' is k-critical.*

Proof of the Theorem. The "if-part" follows from Lemma 2.3.

To prove the "only-if-part" we define a path between two vertices v and w whose edges are colored with two colors alternately, say with colors 1 and 2, to be a $(1,2)$-*path* from v to w.

Claim 2.4.1 *Let $k \geq 2$ and $u_1, \ldots, u_k$ be the vertices of degree $k-1$ in $K_{k,k-1}$. For each edge $e \in K_{k,k-1}$ there is a k-coloring of $K_{k,k-1}$ such that color i is missing at u_i, and there is a $(1,2)$-path from u_1 to u_2 which contains e.*

Proof. Choose $e \in E(K_{k,k-1})$, and assume w.l.o.g. that the vertices of degree k in $K_{k,k-1}$ are labeled with $w_1, \ldots, w_{k-1}$ such that $e = u_1 w_{k-1}$ or $e = u_{i+1} w_i$ for an $i \in \{1, \ldots k-1\}$.

Let $\phi : E(K_{k,k-1}) \rightarrow \mathcal{Z}_k$ be the function with $\phi(u_i w_j) = n$ if and only if $n \equiv (i-j) \pmod{k}$. This function is well defined and it is a k-coloring of $K_{k,k-1}$.

Then color i is missing at u_i for $i = 1, \ldots, k$, and $\phi^{-1}(1) = \{u_{i+1} w_i | i = 1, \ldots, k-1\}$, $\phi^{-1}(2) = \{u_{i+2} w_i | i = 1, \ldots, k-2\} \cup \{u_1 w_{k-1}\}$. Hence e is contained in the $(1,2)$-path $u_1, w_{k-1}, u_k, w_{k-2}, u_{k-1}, w_{k-3}, u_{k-2}, \ldots, w_2, u_3, w_1, u_2$. Thus ϕ is a coloring whith the required properties.

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Assume that M' is k-colored. Since $K_{k,k-1}$ is of odd order and has deficiency k it follows with Lemma 2.1 that the edges $v_1 u_1, \ldots, v_d u_d$ are colored pairwise differently. Henceforth contracting $K_{k,k-1}$ to a single vertex yields a k-coloring of M, a contradiction. Thus $\chi'(M') \geq k+1$

Next we show that $\chi'(M'-e) = k$ for each $e \in E(M')$. This implies $\chi'(M') = k+1$, and hence M' is k-critical.

Case 1: $e \in M'[V(M)]$ or $e = u_jv_j$ for a $j \in \{1, \dots, d\}$

Let ϕ and ψ be a k-coloring of $M - e$ and $K_{k,k-1}$, respectively. By Claim 2.4.1 this can be done such that $\phi(v_iv)$ is missing at u_i, for $i = 1, \dots, d$. These two partial k-colorings can be combined to a k-coloring of $M' - e$.

Case 2: $e \in M'[V(K_{k,k-1})]$

There is a k-coloring ϕ of $M - v_1v$ such that color 1 is missing at v_1 and color 2 is missing at v. Clearly, $\phi(v_lv) = 1$ for an $l \in \{1, \dots, d\}$, say $l = 2$. Claim 2.4.1 implies that there is k-coloring of $K_{k,k-1}$ such that color 1 is missing at u_1, color 2 is missing at u_2, color $\phi(v_iv)$ is missing at u_i for $3 \leq i \leq d$, and there is a $(1,2)$-path from u_1 to u_2 which contains e.

Thus interchanging colors 1 and 2 in the $(1,2)$-path beginning in u_2 yields a k-coloring ψ of $K_{k,k-1} - e$ where color 1 is missing at both of v_1 and v_2.

These two partial colorings ϕ and ψ can simply be extended to a k-coloring of $M' - e$.

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Theorem 2.5 *Let $k \geq 3$ be odd, M a multigraph and $v \in V(M)$ with $d_M(v) = k = \Delta(M)$. Let $v_1, \dots, v_k$ be the not necessarily distinct neighbors of v, and $u_1, \dots, u_k$ the vertices of the complete graph K_k. Let M' be the multigraph obtained from $G - v$ and K_k by adding edges v_iu_i for $i = 1, \dots, k$. Then M is k-critical if and only if M' is k-critical.*

Proof. Again, the "if-part" follows from Lemma 2.3.

With Lemma 2.1 follows that $\chi'(M') = k+1$. We show that M' is k-critical. Let $e \in E(M') \setminus E(K_k)$ and ψ be a k-coloring of K_k with $\psi^{-1}(i) = u_{i-1}u_{i+1}, u_{i-2}u_{i+2}, \dots$ $\dots, u_{i-\frac{k-1}{2}}u_{i+\frac{k-1}{2}}$ for $i = 1, \dots, k$, where the indices are added modulo (k). Color i is missing at u_i, and hence an appropriate labelling of the vertices of K_k yields a k-coloring of $M' - e$.

Let $e = u_nu_m \in E(K_k)$. For $1 \leq i < j \leq k$ let $\phi_{i,j}$ be a (non-proper) k-coloring of M which colors incident edges differently except v_iv and v_jv, say $\phi_{i,j}(v_iv) = \phi_{i,j}(v_jv) = 1$, and w.l.o.g. color 2 is missing at v. Call $\phi_{i,j}$ a *critical k-coloring* for i,j. As seen above there is a k-coloring $\psi_{i,j}$ of K_k such that color 1 is missing at u_i, color 2 is missing at u_j and color $\phi_{i,j}(v_tv)$ is missing at u_t, for $1 \leq t \leq k$, $t \neq i,j$. Call $\psi_{i,j}$ the k-coloring *corresponding* to $\phi_{i,j}$. There is $(1,2)$-path $p_{i,j}$ from u_i to u_j of length $k-1$ in K_k.

Clearly, there is an $1 \leq r \leq k$ and an $1 \leq l \leq \frac{k-1}{2}$ such that $n = r + l$ and $m = r - l$.

Since M is k-critical and $d_M(v) = k$ it has a critical k-coloring $\phi_{r,s}$, $1 \leq s \leq k$, $s \neq r$. The corresponding k-coloring $\psi_{r,s}$ colors e with color 1, and therefore e is an edge of the $(1,2)$-path $p_{s,r}$ between u_s and u_r in K_k. Remove e, then interchanging colors 1 and 2 in the remaining $(1,2)$-path to u_s yields a k-coloring ψ' of $K_k - e$ such that 1 is missing at u_s, too. Combining $\phi_{r,s}$ and ψ' yields a k-coloring of $M' - e$.

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Our strategy is to construct k-critical graphs from k-critical multigraphs and then we will apply the following theorem of Yap [11].

Theorem 2.6 (Yap's Construction) *For $k \geq 4$ let G be a k-critical graph with two vertices v, w of degree 2. Let G' be a copy of G with the corresponding vertices v', w'. Let H' be obtained from G and G' by Hajós-union where v and v' are identified. Then the graph H obtained from H' by identifying w and w' is k-critical.*

3 Construction of k-critical Graphs of Even Order

Let M be a multigraph and F be a set of independent edges in M. Let F' be a copy of F. We say that $H = M + F'$ is the multigraph obtained from M *by adding* F'. Let P denote the Petersen graph.

Theorem 3.1 *Let $k \geq 0$ and M be a multigraph obtained from P by adding k 1-factors of P. Then $M - v$ is a $(k+3)$-critical multigraph for any $v \in V(G)$.*

Proof. Clearly, M is $(k+3)$-regular multigraph and $\chi'(M) \leq k+4$.

Since P has no 1-factor which contains exactly one edge of an induced 5-circuit, any subgraph induced by the vertices of a 5-circuit in P has odd size. Hence M is not 1-factorizable and therefore $\chi'(M) = k+4$. With the Parity Lemma follows $\chi'(M-v) = k+4$ for each $v \in V(M)$.

Any edge $e \in E(M-v)$ can be interpreted as an edge of $P - v$. Since $(P-v) - e$ is 3-colorable it follows that $(M-v) - e$ is $(k+3)$-colorable, i.e. $M - v$ is $(k+3)$-critical.

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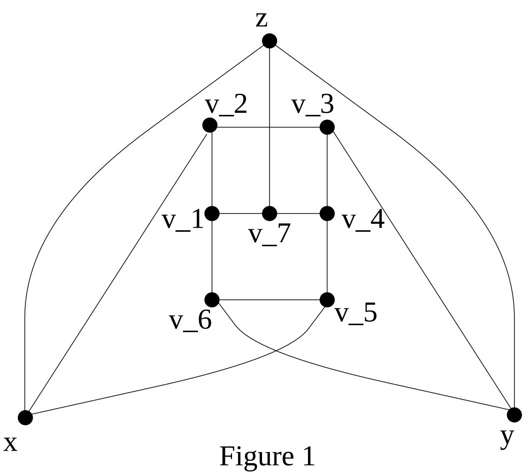

Figure 1

Theorem 3.2 *For each $k \geq 4$ there is a k-critical multigraph of order $12k - 20$, and there is a k-critical graph of order $4(3k^2 - 6k + 1)$. Both have a divalent vertex.*

Proof. Let the vertices and edges of P be labeled as in Figure 1.

Then $F_1 = \{v_2v_3, v_1v_6, v_4v_7, v_5x, zy\}$ and $F_2 = \{v_2v_3, v_4v_5, v_1v_7, v_6y, zx\}$ are 1-factors of P.

For $k \geq 3$ and $1 \leq j \leq k-2$ let P_k^j be the multigraph obtained from P by adding $j-1$ copies of F_1 and $k-j-2$ copies of F_2. It follows with Lemma 3.1 that $M_k^j = P_k^j - z$ is a k-critical multigraph. Furthermore $d_{M_k^j}(v_7) = k-1$, $d_{M_k^j}(x) = j+1$, $d_{M_k^j}(y) = k-j$, and all the other vertices of M_k^j have degree k.

We are going to apply the Hajós-union on the graphs M_k^j for a fixed $k \geq 3$. To avoid confusion we indicate the involved vertices of M_k^j with an upper index j.

By definition $d_{M_k^j}(y^j) + d_{M_k^{j+1}}(x^{j+1}) = k+2$. Identify y^j and x^{j+1} to a new vertex which will be labelled y^j in the new graph and let all the other labellings unchanged. Substitute $v_3^j v_2^{j+1}$ for $y^j v_3^j$ and $x^{j+1} v_2^{j+1}$, and call this graph M. Note that the degree of vertices y^j and x^{j+1} is unchanged in M. Clearly M is the Hajós-union of M_k^j and M_k^{j+1}, and we write $M = M_k^j +_{(y^j, x^{j+1})} M_k^{j+1}$. Define

$$M_k = (\ldots(M_k^1 +_{(y^1,x^2)} M_k^2) +_{(y^2,x^3)} M_k^3) \ldots +_{(y^{k-3},x^{k-2})} M_k^{k-2})$$

For $k \geq 4$ multigraph M_k is the Hajós-union of k-critical multigraphs and hence it is k-critical. It has $8k-15$ vertices and two of them are divalent, namely x^1 and y^{k-2}.

For $j = 1, \ldots, k-3$ the multigraphs T_j induced by vertices y^j, v_6^j, v_5^{j+1} have deficiency k. Contracting each T_j to a single vertex t_j yields a new multigraph N_k of order $6k-9$, where $d_{N_k}(t_j) = k$ and $d_{N_k}(x^1) = d_{N_k}(y^{k-2}) = 2$. Lemma 2.3 implies that N_k is k-critical.

For $j = 1, \ldots, k-2$ substitute vertices v_1^j, v_3^j, v_4^j by the complete bipartite graph $K_{k,k-1}$ as described in Theorem 2.4. We obtain a simple graph G_k having $6k^2 - 12k + 3$ vertices, which is k-critical by Theorem 2.4.

Applying Yap's construction (Thm. 2.6) on G_k yields a k-critical graph G_k^* of order $4(3k^2 - 6k + 1)$. Surely G_k^* has a vertex of degree 2.

Contracting all induced complete bipartite graphs $K_{k,k-1}$ used in the construction of G_k from N_k yields a multigraph M^k of order $12k-20$, which is k-critical by Lemma 2.3.

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Remarks: **1.** In case of odd $k \geq 5$ applying Theorem 2.5 in the above construction yields k-critical graphs of order $2(3k^2 - 3k - 4)$.

2. Let M_4^1 be as in the proof of the above theorem. This multigraph has degree sequence $23^2 4^6$. Substituting v_3, v_4, v_6 and v_7 by $K_{3,4}$ yields a 4-critical graph graph with degree sequence $23^2 4^{30}$. This proves Corollary 1.8.

3. Theorems 1.6 and 1.7 are simple consequences from Theorem 3.2 and the following results of Coudum.

Theorem 3.3 ([5]) *If there is a k-critical graph ($k \geq 3$) of even order with one vertex of degree 2, then*

1. *there is a k-critical graph of even order without a 1-factor.*

2. *for any n ($2 \leq n \leq k-1$) there is a k-critical graph G of odd order and minimum degree n that has a vertex v of minimum degree for which $G - v$ has no 1-factor.*

4 Goldberg-Type Construction of k-Critical Graphs of Even Order, k Odd

Theorem 4.1 *For each odd $k \geq 3$ there is a k-critical multigraph of order $7k+1$, and there is a k-critical graph of order $\frac{(k+1)(7k-1)}{2}+2$.*

Proof. Let P be labeled as in Figure 1. Then $F_1 = \{v_2v_3, v_1v_6, v_4v_7, v_5x, zy\}$ and $F_2 = \{v_2v_3, v_1v_7, v_4v_5, v_6y, zx\}$ are 1-factors of P. Let N_k be the multigraph obtained from P by adding $\frac{k-3}{2}$ ($k \geq 3$ and odd) copies of each of F_1 and F_2. By Theorem 3.1 $N_k^- = N_k - z$ is a k-critical multigraph.

Let $M_k = N_k^- - \{x,y\}$, $K_1 = \{v_3, v_6\}$, $K_2 = \{v_2, v_5\}$ and $K_3 = \{v_7\}$.

Claim 4.1.1 *For each $k \geq 3$ there is a k-coloring of M_k if and only if one color, say 1, is missing at precisely three vertices x_1, x_2, x_3, either $K_1 \subset \{x_1, x_2, x_3\}$ or $K_2 \subset \{x_1, x_2, x_3\}$, and each color different from 1 is missing at precisely one vertex of $K_1 \cup K_2 \cup K_3$.*

Proof. Let M_k be k-colored. The deficiency of M_k is $k+2$, and hence Lemma 2.1 implies that one color, say 1, is missing at three vertices and all the other at precisely one. Since N_k^- is not k-colorable, it follows that 1 is missing either at both vertices of K_1 or at both vertices of K_2. Since all vertices of $V(M_k) \setminus (K_1 \cup K_2 \cup K_3)$ have maximum degree the statement follows.

For $k=3$ it is easy to see that the assumption, that one color is missing at both vertices of precisely one of K_1 and K_2, allows to 3-color M_3. For all $k > 3$ multigraph M_k is obtained from M_3 by adding $k-3$ independent sets of edges. Therefore M_k is k-colorable.

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We now recursively define multigraphs M_k^*, $k \geq 3$ and odd.

Construction: For $k=3$ take three copies M_3^1, M_3^2, M_3^3 of M_3 (where the vertices of M_3^j are labelled with an upper index j). Join each of v_7^1, v_7^2, v_7^3 to a new vertex u by one edge and add edges $v_5^1v_6^2$, $v_5^2v_6^3$ and $v_2^1v_3^2$, $v_2^2v_3^3$.

For $k > 3$ let M_{k-2}^* be given. To construct M_k^* take M_{k-2}^* and two copies M_k^{k-1}, M_k^k of M_k. Join v_5^{k-2} with v_6^{k-1} and v_5^{k-1} with v_6^k by $\frac{k-1}{2}$ edges, respectively. Add single edges $v_2^{k-2}v_3^{k-1}$, $v_2^{k-1}v_3^k$, $v_7^{k-1}u$ and v_7^ku. Add the set of edges E_k consisiting of two edges between $v_2^jv_3^j$, and single edges $v_6^jv_1^j, v_1^jv_7^j, v_7^jv_4^j, v_4^jv_5^j$ for $j = 1, \dots, k-2$, and single edges $v_5^jv_6^{j+1}$ for $j = 1, \dots, k-3$.

Claim 4.1.2 *For each $k \geq 3$ multigraph M_k^* is k-critical.*

Proof. For $k=3$, M_k^* is the 3-critical graph constructed by Goldberg [7].

Let $k > 3$ and F_k be the multigraph obtained from M_k^* by contracting the components M_k^j to single vertices u_j ($j = 1, \dots, k$), removing u_k and then two edges between u_{2i} and u_{2i+1} for $i = 1, \dots, \frac{k-3}{2}$. With Claim 4.1.1 follows that any k-coloring of M_k^* implies a k-coloring of F_k. This contradicts the fact that F is not k-colorable since $s(F) = k-2$. Therefore M_k^* is not k-colorable.

We have to show that $M_k^* - e$ is k-colorable for each $e \in M_k^*$. Because of the symmetry of M_k^* edge e can be interpreted as an edge of M_{k-2}^*. By induction hypothesis $M_{k-2}^* - e$ has a $(k-2)$-coloring. By construction it can be extended to a k-coloring ϕ of $M_{k-2}^* + E_k$. This can be done such that the colors missing at the vertices of K_2^{k-2} are pairwise different.

By Claim 4.1.1 ϕ can be extended to a k-coloring ϕ' of the induced multigraph on $V(M_{k-2}^* \cup M_k^{k-1})$ such that $\phi'(v_7^{k-1}u) = k-1$ and color $k-1$ is missing at both vertices of K_2^{k-1}. Again by Claim 4.1.1 ϕ' can be extended to a k-coloring of M_k^*.

$\bullet$

Multigraph M_k^* has $7k+1$ vertices. To obtain a simple graph G_k^* substitut K_k for vertices v_1^{2i-1}, v_2^{2i-1}, v_4^{2i-1} for $i = 1, \ldots, \frac{k+1}{2}$, and for vertices v_6^{2i}, v_7^{2i}, v_5^{2i}, v_2^{2i} for $i = 1, \ldots, \frac{k-1}{2}$. Graph G_k^* has $\frac{(k+1)(7k-1)}{2} + 2$ vertices and it is k-critical by Lemma 2.5.

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