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Note on the Erdős-Szekeres theorem

by

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ABSTRACT

Let $g(n)$ denote the least integer such that among any $g(n)$ points in general position in the plane there are always n in convex position. In 1935 P. Erdős and G. Szekeres showed that $g(n)$ exists and $2^{n-2}+1 \leq g(n) \leq \binom{2n-4}{n-2}+1$. Recently, the upper bound has been slightly improved by Chung and Graham and by Kleitman and Pachter. In this note we further improve the upper bound to

$$g(n) \leq \binom{2n-5}{n-2}+2.$$

1 Introduction

In the 1930's Esther Klein raised the following question. Is it true that for every n there is a least number $g(n)$ such that among any $g(n)$ points in general position in the plane there are always n in convex position?

Her question was answered in the affirmative in a classical paper of Erdős and Szekeres [ES35]. In fact, they showed [ES35, ES60] that

$$2^{n-2}+1 \leq g(n) \leq \binom{2n-4}{n-2}+1.$$

The lower bound, $2^{n-2}+1$, is sharp for $n=2,3,4,5$ and has been conjectured to be sharp for all n. However, the upper bound, $\binom{2n-4}{n-2}+1 \approx c\frac{4^n}{\sqrt{n}}$, was not improved for 60 years. Recently Chung and Graham [CG97] managed to improve it by 1. Then Kleitman and Pachter [KP97] showed that $g(n) \leq \binom{2n-4}{n-2}+7-2n$.

Inspired by these papers, in this note we get a further improvement, roughly by a factor of 2.

Theorem *Any set of* $\binom{2n-5}{n-2}+2$ *points in general position in the plane contains* n *points in convex position.*

In other words, $g(n) \leq \binom{2n-5}{n-2}+2$. Since $2\binom{2n-5}{n-2}=\binom{2n-4}{n-2}$, our upper bound is about a half of the original bound of Erdős and Szekeres. We present the proof in two versions. The first one is based on a related result of Erdős and Szekeres [ES35], the second one is based on its generalization, due to Chvátal and Komlós [CK71].

2 First proof

In the original proof, Erdős and Szekeres were looking for special convex n-gons, namely for n-caps and n-cups.

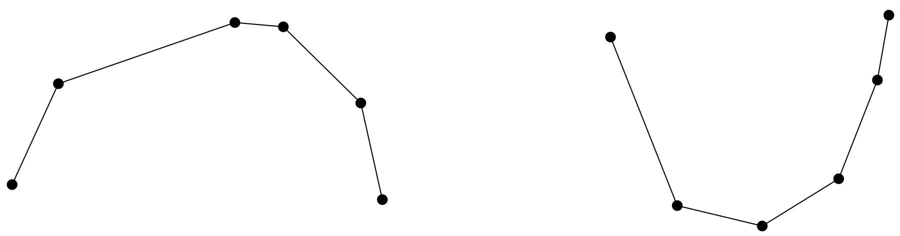

Figure 1: A 6-cap and a 6-cup.

Definition *The points* $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$, $x_1<x_2<\ldots<x_n$, *form an* n*-cap if*

$$\frac{y_2-y_1}{x_2-x_1} > \frac{y_3-y_2}{x_3-x_2} > \ldots > \frac{y_n-y_{n-1}}{x_n-x_{n-1}}.$$

Similarly, they form an n-cup if

$$\frac{y_2 - y_1}{x_2 - x_1} < \frac{y_3 - y_2}{x_3 - x_2} < \ldots < \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

See Fig. 1.

Lemma 1 (Erdős and Szekeres [ES35]) *Let f(n, m) be the least integer such that any set of $f(n,m)$ points in general position in the plane contains either an n-cap or an m-cup. Then*

$$f(n,m) = \binom{n+m-4}{n-2} + 1.$$

Let P be a set of points in general position in the plane and suppose that P does not contain n points in convex position. Let a be a vertex of the convex hull of P. Let b be a point outside the convex hull of P such that none of the lines determined by the points of $P \setminus \{a\}$ intersects the segment $\overline{ab}$. Finally, let ℓ be a line through b which avoids the convex hull of P (see Fig. 2).

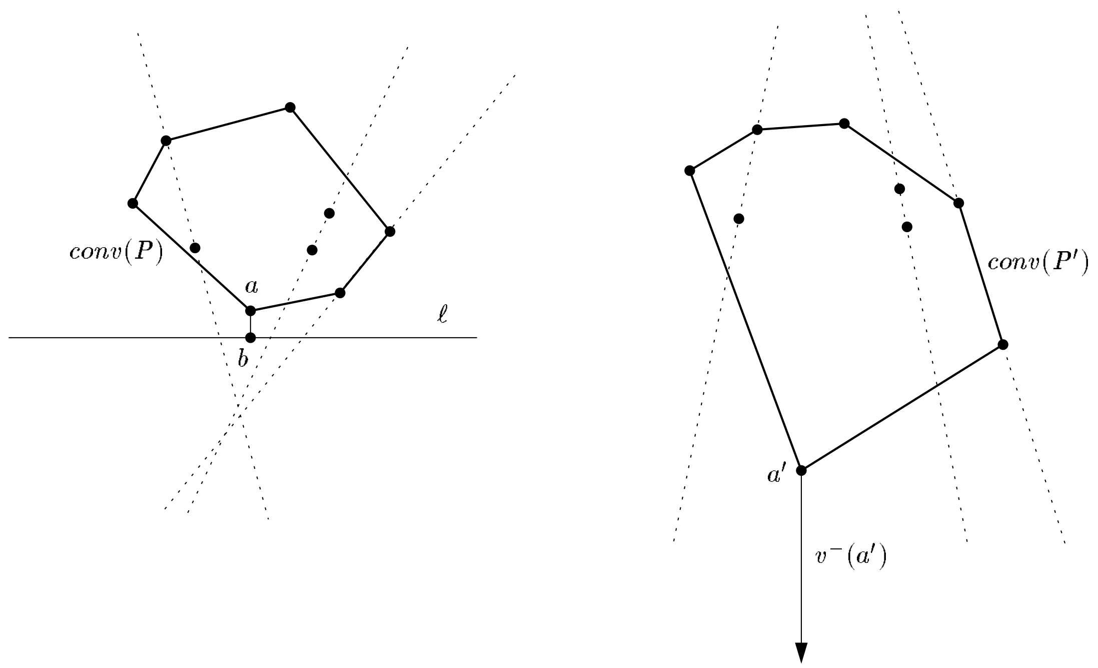

Figure 2: The set P and its image $P' = T(P)$.

Consider a projective transformation T which maps the line ℓ to the line at infinity, and maps the segment $\overline{ab}$ to the vertical half-line $v^-(a')$, emanating downwards from $a' = T(a)$. We get a point set $P' = T(P)$ from P. Since ℓ avoided the convex hull of P, the transformation T does not change convexity on the points of P, that is, any subset of P is in convex position if and only if the corresponding points of P' are in convex position. So the assumption holds also for P', no n points of P' are in convex position. By the choice of the point b, none of the lines determined by the points of $P' \setminus \{a'\}$ intersects $v^-(a')$. Therefore, any m-cap in the set $Q' = P' \setminus \{a'\}$ can be extended by a' to a convex $(m+1)$-gon.

Since no n points of P' are in convex position, Q' cannot contain any n-cup or $(n-1)$-cap. Therefore, by Lemma 1,

$$|Q'| \leq f(n, n-1) - 1 = \binom{2n-5}{n-2}, \quad |P| \leq \binom{2n-5}{n-2} + 1,$$

and the theorem follows. □

3 Second proof

The combinatorial seed of Lemma 1 was isolated and generalized by Chvátal and Komlós. A complete graph, whose edges are arbitrarily oriented, is called a *tournament*. A tournament is said to be *transitive* if it does not contain directed cycles.

Definition *Let G be a directed graph and f any real-valued function on its edge set. A directed path $v_1, v_2, \ldots, v_m$ in G is called f-decreasing if*

$$f(\overrightarrow{v_1v_2}) > f(\overrightarrow{v_2v_3}) > \ldots > f(\overrightarrow{v_{m-1}v_m}).$$

Similarly, it is called f-nondecreasing if

$$f(\overrightarrow{v_1v_2}) \leq f(\overrightarrow{v_2v_3}) \leq \ldots \leq f(\overrightarrow{v_{m-1}v_m}).$$

Lemma 2 (Chvátal and Komlós [CK71]) *Let G be a transitive tournament with more than $\binom{p+q-4}{p-2}$ vertices, and let f be any real-valued function defined on its edge set.*

Then G contains an f-decreasing directed path with p vertices or an f-nondecreasing directed path with q vertices.

Let P be a set of $r = \binom{2n-5}{n-2} + 2$ points in general position in the plane and let p be its leftmost point. Denote the points of $P \setminus \{p\}$ by $p_1, p_2, \ldots, p_{r-1}$, in clockwise order of visibility from p.

Consider a transitive tournament on the vertices $p_1, p_2, \ldots, p_{r-1}$ such that each edge is directed toward its endpoint of larger index. For any $i < j$, let $f(\overrightarrow{p_ip_j})$ denote the size of the clockwise angle from the direction of the y-axis to the direction of $\overrightarrow{p_ip_j}$.

By Lemma 2, there is an f-decreasing directed path $p_{i_1}p_{i_2}\cdots p_{i_n}$ or an f-nondecreasing directed path $p_{j_1}p_{j_2}\cdots p_{j_{n-1}}$. In the first case $p_{i_1}, p_{i_2}, \cdots, p_{i_n}$ are in convex position, in the second case $p, p_{j_1}, p_{j_2}, \cdots, p_{j_{n-1}}$ are in convex position (see Fig. 3). □

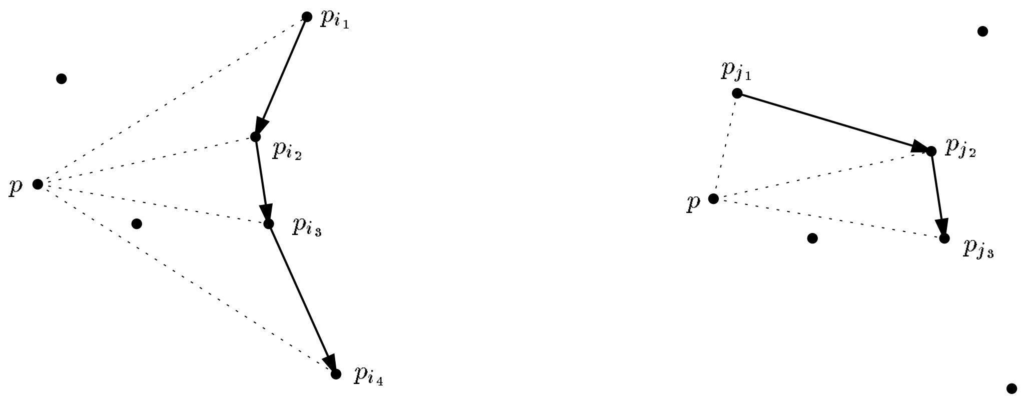

Figure 3: An f-decreasing and an f-nondecreasing path in G.

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