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Graph drawings with no k pairwise crossing edges[1]

by

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ABSTRACT

A *geometric graph* is a graph $G = (V, E)$ drawn in the plane so that the vertex set V consists of points in general position and the edge set E consists of straight line segments between points of V. It is known that, for any fixed k, any geometric graph G on n vertices with no k pairwise crossing edges contains at most $O(n \log n)$ edges. In this paper we give a new, simpler proof of this bound, and show that the same bound holds also when the edges of G are represented by x-monotone curves (Jordan arcs).

1 Introduction

A *graph drawing* is a drawing of a graph in the plane such that each vertex is represented by a distinct point and each edge is represented by a Jordan arc connecting the corresponding two points (vertices) so that any two arcs meet in at most one point which is either a common endpoint or a common interior point where the two arcs cross each other. A *geometric graph* is a graph drawing in which all arcs are straight line segments. See [8] for a survey of results about geometric graphs.

Most papers on geometric graphs deal with the crossing number or with extremal questions motivated by extremal graph theory and Ramsey theory. One of the extremal results is due to Pach and Törőcsik [10] who proved that any geometric graph on n vertices with no k pairwise disjoint edges contains at most $(k-1)^4 n$ edges. A related ("dual") question is what is the maximum number of edges that can be contained in a geometric graph (or in a graph drawing) on n vertices with no k pairwise crossing edges. Euler's formula shows that this number is $3n-6$ for $k=2$ (and $n \geq 3$). Pach [7] proved that any geometric graph with no 3 pairwise crossing edges contains at most $13n^{3/2}$ edges, and that any geometric graph with no k pairwise crossing edges contains at most $O(n^{2-1/(25k^2)})$ edges. This result was improved by Pach, Sharokhi, and Szegedy [9] who proved that any graph drawing on n vertices with no k pairwise crossing edges contains at most $O(n\log^{2k-4} n)$ edges ($k \geq 3$). This bound was further improved to $O(n\log^{2k-6} n)$ in [2] ($k \geq 3$) (thus, in particular, to $O(n)$ for $k=3$). In [12] we improved the bound for geometric graphs to $O(n\log n)$ (for $k \geq 4$).

Theorem 1 *Let $k \geq 4$ be a constant. Then any geometric graph on n vertices with no k pairwise crossing edges has at most $O(n\log n)$ edges.*

In this paper we give a simpler proof of Theorem 1, and extend Theorem 1 to graphs drawn in the plane by x-monotone Jordan arcs.

Theorem 2 *Let $k \geq 4$ be a constant, and let G be a graph drawing with n vertices such that all edges are drawn by x-monotone Jordan arcs and no k edges (arcs) are pairwise crossing. Then G has at most $O(n\log n)$ edges.*

In the proof of Theorems 1 and 2 we apply the Erdős-Szekeres theorem, Ramsey's theorem, and results on generalized Davenport–Schinzel sequences. Our paper is organized as follows. In Section 2 we recall generalized Davenport-Schinzel sequences, in Section 3 we give the proof of Theorems 1 and 2 which relies on two lemmas proved in Sections 4 and 5, respectively.

2 Generalized Davenport–Schinzel sequences

For $l \geq 1$, a sequence is called *l-regular*, if any at most l consecutive terms are pairwise different. For $l \geq 2$, a sequence

$$S = s_1, s_2, \ldots, s_{3l-2}$$

of length $3l-2$ is said to be *of type* up-down-up(l), if the first l terms are pairwise different and, for $i = 1, 2, \ldots, l$,

$$s_i = s_{2l-i} = s_{(2l-2)+i}.$$

Thus, a sequence is of type up-down-up(2) if and only if it is an alternating sequence of length 4. It is well–known [3] (and not difficult to prove) that any 2-regular sequence over an n-element alphabet containing no alternating subsequence of length 4 has length at most $2n-1$. In the proof of Theorems 1 and 2 we apply the following more general result.

Theorem 3 (Klazar and Valtr [6]) *Let $l \geq 2$ be a constant. Then the length of any l-regular sequence over an n-element alphabet containing no subsequence of type* up-down-up*(l) has length at most $O(n)$.*

3 Proof of Theorems 1 and 2

Theorem 1 is weaker than Theorem 2. Thus, it suffices to show Theorem 2.

Let $k \geq 2$ be a constant, and let $f_k(n)$ be the maximum number of edges in a graph drawing on n vertices with edges represented by x-monotone Jordan arcs and with no k pairwise crossing edges. Let $G = (V, E)$ be a graph drawing on n vertices with edges represented by x-monotone Jordan arcs and with no k pairwise crossing edges. Introduce Cartesian coordinate system so that the y-axis partitions V into two parts which are as equal as possible, thus the sets

$V^- = \{v \in V | \text{the } x\text{-coordinate of } v \text{ is negative}\}$,

$V^+ = \{v \in V | \text{the } x\text{-coordinate of } v \text{ is positive}\}$

have sizes

$$|V^-| = \lfloor n/2 \rfloor , \qquad |V^+| = \lceil n/2 \rceil .$$

Partition E into three subsets E^+, E^-, E' such that E^+ contains the edges with both endpoints in V^+, E^- contains the edges with both endpoints in V^-, and E' contains the edges crossing the y-axis.

Obviously,

$$|E^-| \leq f_k(\lfloor n/2 \rfloor), \qquad |E^+| \leq f_k(\lceil n/2 \rceil).$$

To obtain Theorem 2, it suffices to show that

$$|E'| \;=\; O(n), \tag{1}$$

because this implies

$$f_k(n) \leq f_k(\lfloor n/2 \rfloor) + f_k(\lceil n/2 \rceil) + O(n)$$

and, consequently,

$$f_k(n) = O(n \log n).$$

We may assume without loss of generality that no two edges of E' cross the y-axis in the same point. We label the vertices of V by integers from 1 to n. We do not make any notational distinction between a point and its label. Let

$$e_1, e_2, \ldots, e_r$$

be the edges of E' ordered in the order in which they cross the y-axis (from $-\infty$ to $+\infty$, say). Further, let S_1 and S_2 be the sequences of $r = |E'|$ integers obtained from the sequence

$$e_1, e_2, \ldots, e_r$$

by replacing each edge by (the label of) its left endpoint and by (the label of) its right endpoint, respectively. The proof of (1) is based on a proper handling with the sequences S_1 and S_2. In particular, it relies on the following two lemmas:

Lemma 4 *For each $l \geq 1$, at least one of the sequences S_1, S_2 contains an l-regular subsequence of length at least $|E'|/(4l) = r/(4l)$.*

Lemma 5 *Neither of the sequences S_1 and S_2 contains a subsequence of type* up-down-up*$((r(k))^2)$, where $r(k)$ is the kth Ramsey number (i.e., $r(k)$ is the minimum number R such that any complete graph on at least R vertices with edges colored by 2 colors contains a monochromatic complete subgraph on k vertices).*

We remark that Lemma 5 holds in a stronger form for geometric graphs (with $(r(k))^2$ replaced by $(k-1)^3 + 1$). This is shown in Section 5. Before proving Lemmas 4 and 5 we complete the proof of (1), which, as argued above, implies Theorem 2.

According to Lemma 4, at least one of the sequences S_1, S_2 contains an $(r(k))^2$-regular subsequence S of length at least $|E|/((4r(k))^2)$. It follows from Lemma 5 that the sequence S contains no subsequence of type up-down-up$((r(k))^2)$. Consequently, Theorem 3 implies that the length of S is at most $O(n)$. Thus,

$$|E|/(4(r(k))^2) \leq \text{length of } S \leq O(n).$$

Consequently,

$$|E| \leq 4(r(k))^2 \cdot O(n) = O(n). \quad \square$$

To complete the proof of Theorems 1 and 2, it remains to prove Lemmas 4 and 5.

4 Proof of Lemma 4

We apply a simple greedy algorithm [1] which, for given integer $l \geq 1$ and finite sequence A, returns an l-regular subsequence $B(A,l)$ of A. At the beginning of the algorithm, an auxiliary sequence B is taken empty. Then the terms of A are considered one by one from left to right, and in each step the considered term is placed at the end of B iff this doesn't violate the l-regularity of B (otherwise B is unchanged). Finally, the obtained l-regular subsequence B of A is taken for $B(A,l)$.

E.g., if $A = 1,3,1,3,5,2,2,5,1,5,1,2$ and $l = 3$, then the algorithm returns the sequence $B(A,3) = 1,3,5,2,1,5,2$. We prove Lemma 4 by showing that, given $l \geq 1$, at least one of the sequences $B(S_1,l)$, $B(S_2,l)$ obtained by the algorithm has length at least $|E'|/(4l) = r/(4l)$.

Let $l \geq 1$ be given. For $i = 1,2$ and for $1 \leq j_1 \leq j_2 \leq m$, let $S_{i,[j_1,j_2]}$ be the subsequence of S_i obtained from S_i by removing all terms preceding the j_1th term and following the j_2th term. Thus, $S_{i,[j_1,j_2]}$ is a continuous part of S_i of length $j_2 - j_1 + 1$.

Let $|T|$ denote the length of a sequence T, and let $I(T)$ denote the set of integers appearing in T.

Claim 6 *For each* $j = 1,2,\ldots,m$,

$$\sum_{i=1}^{2} |B(S_{i,[1,j]},l)| \geq j/(2l).$$

Proof. First, consider two integers j_1, j_2 such that $1 \leq j_1 \leq j_2 \leq m$. Obviously,

$$\{e_{j_1}, e_{j_1+1}, \ldots, e_{j_2}\} \subseteq \{(v_a, v_b) | a \in I(S_{1,[j_1,j_2]}), b \in I(S_{2,[j_1,j_2]})\}.$$

Thus,

$$|\{e_{j_1}, e_{j_1+1}, \ldots, e_{j_2}\}| \leq |\{(v_a, v_b) | a \in I(S_{1,[j_1,j_2]}), b \in I(S_{2,[j_1,j_2]})\}|.$$

Consequently,

$$j_2 - j_1 + 1 \leq |I(S_{1,[j_1,j_2]})| \cdot |I(S_{2,[j_1,j_2]})|.$$

By the inequality between algebraic and geometric means,

$$\frac{\sum_{i=1}^{2} |I(S_{i,[j_1,j_2]})|}{2} \geq \sqrt{j_2 - j_1 + 1}. \tag{2}$$

We now can prove the claim by induction on j. If $j \leq \min\{16l^2, m\}$, then by (2) and by $j \leq 16l^2$

$$\begin{aligned} \sum_{i=1}^{2} |B(S_{i,[1,j]},l)| &\geq \sum_{i=1}^{2} |I(S_{i,[1,j]})| \\ &\geq 2\sqrt{j} \\ &\geq j/(2l). \end{aligned}$$

Suppose now that $16l^2 < j_0 \leq m$ and that Claim 6 holds for $j = 1, 2, \ldots, j_0 - 1$. Since for $i = 1, 2$ each integer of $I(S_{i,[j_0-4l^2+1,j_0]})$ not appearing among the last $l-1$ terms in $B(S_{i,[1,j_0-4l^2]}, l)$ appears more times in $B(S_{i,[1,j_0]}, l)$ than in $B(S_{i,[1,j_0-4l^2]}, l)$, we have

$$|B(S_{i,[1,j_0]}, l)| \geq |B(S_{i,[1,j_0-4l^2]}, l)| + |I(S_{i,[j_0-4l^2+1,j_0]})| - (l-1).$$

Consequently, by the inductive hypothesis and by (2),

$$\begin{aligned} \sum_{i=1}^{2} |B(S_{i,[1,j_0]}, l)| &\geq (j_0 - 4l^2)/(2l) + 2\sqrt{4l^2} - 2(l-1) \\ &> j_0/(2l). \qquad \square \end{aligned}$$

Proof of Lemma 4. Lemma 4 easily follows from Claim 6 (with $j = m$) and from the pigeon-hole principle. □

5 Proof of Lemma 5

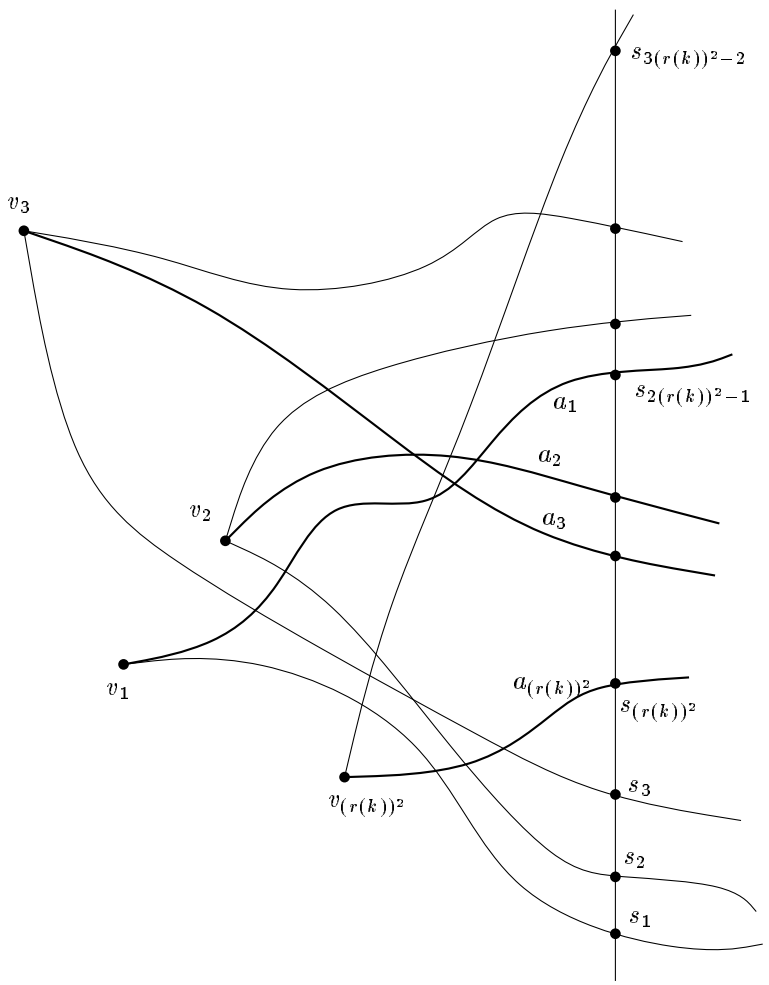

Figure 1: Arcs a_i.

Because of symmetry, it suffices to prove Lemma 5 for the sequence S_1. Suppose to the contrary that S_1 contains a subsequence of type up-down-up$((r(k))^2)$. Thus, there is a subsequence

$$S = s_1, s_2, \ldots, s_{3(r(k))^2-2}$$

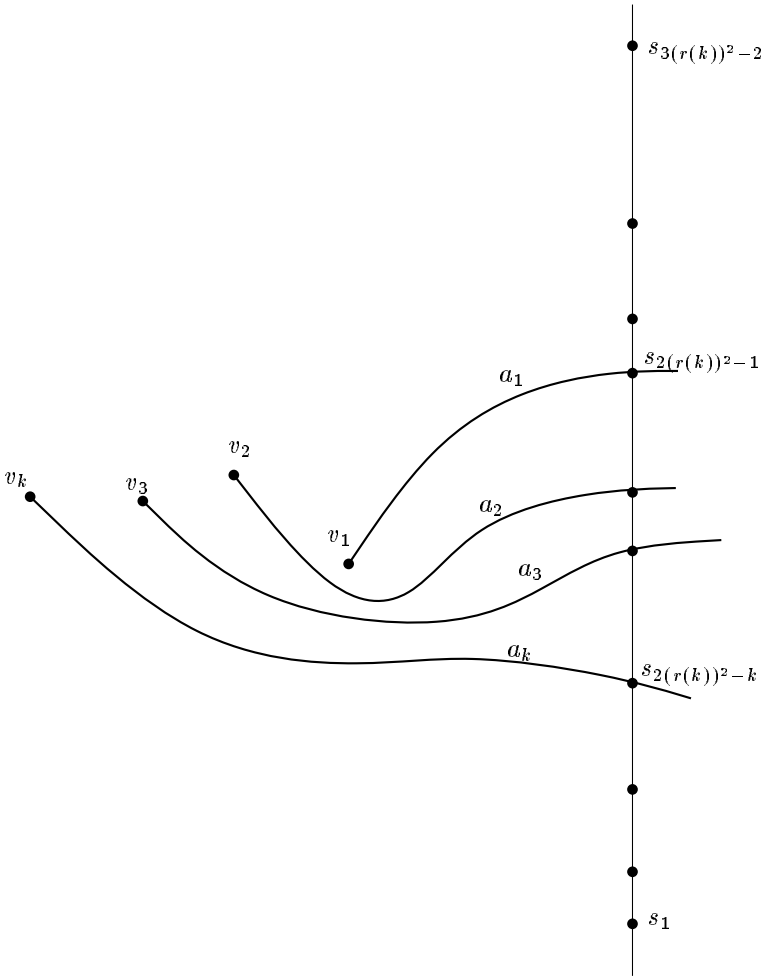

Figure 2: Case 1.

of S_1 such that the integers $s_1, s_2, \ldots, s_{(r(k))^2}$ are pairwise different and that, for $i = 1, 2, \ldots, (r(k))^2$, $s_i = s_{2(r(k))^2-i} = s_{(2(r(k))^2-2)+i}$. For each i, we denote by v_i the vertex of V^- labeled by s_i, and we denote by x_i the x-coordinate of v_i. Further, let a_i be the arc emanating from the vertex v_i and passing through the point on the y-axis corresponding to $s_{2(r(k))^2-i}$ (see Fig. 1). We may assume that the coordinates x_i are pairwise different. By the Erdős-Szekeres theorem [5], the sequence

$$x_1, x_2, \ldots, x_{(r(k))^2}$$

contains a monotone subsequence of length $r(k)$. For simplicity of notation (and without loss of generality), we assume that the sequence

$$x_1, x_2, \ldots, x_{r(k)}$$

is monotone. Let $T = \{v_1, \ldots, v_{r(k)}\}$. We color the edges of a complete graph on the vertex set T by 2 colors as follows. An edge v_iv_j with $x_i < x_j$ is colored blue if v_j lies below the arc a_i, otherwise its color is red. By Ramsey's theorem [11], our graph contains a monochromatic complete subgraph on k vertices. For simplicity of notation and without loss of generality, we assume that the complete subgraph on vertices $v_1, v_2, \cdots, v_k$ is monochromatic. Then, one of the four cases in Figures 2–5, respectively, applies. In cases shown in Figs. 3 and 4, the arcs $a_i, i = 1, \ldots, k$, are pairwise crossing. In Fig. 2 the k arcs passing through the points on the y-axis corresponding to $s_1, s_2, \ldots, s_k$ are pairwise crossing (see Fig. 6), in Fig. 5 the k arcs passing through the points on the y-axis corresponding to $s_{2(r(k))^2-1}, s_{2(r(k))^2}, \ldots, s_{2(r(k))^2+k-2}$ are pairwise crossing (see Fig. 7). Thus, we have obtained a contradiction by showing that k of the arcs in our graph drawing are pairwise crossing. □

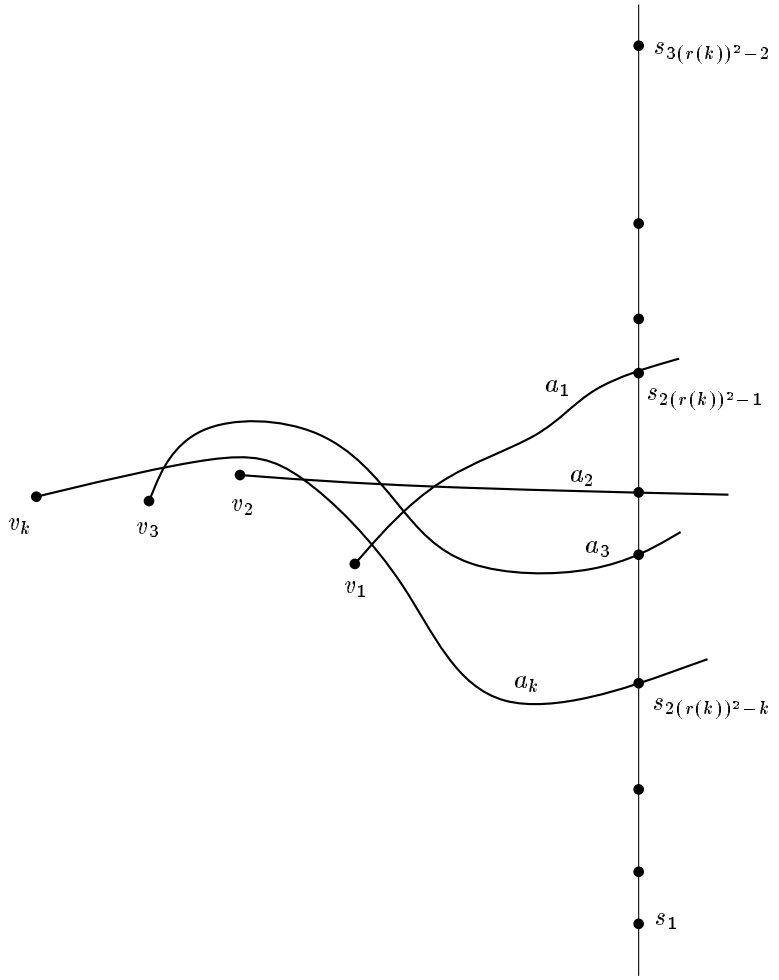

Figure 3: Case 2.

We now show that Lemma 5 holds in a stronger form for geometric graphs. The strengthening is based on an application of Dilworth's theorem instead of Ramsey's theorem.

Lemma 7 *If G is a geometric graph, then neither of the sequences S_1 and S_2 contains a subsequence of type* up-down-up *$((k-1)^3+1)$.*

Proof. Because of symmetry, it suffices to prove Lemma 7 for the sequence S_1. Suppose to the contrary that S_1 contains a subsequence of type up-down-up(l), where $l = (k-1)^3+1$. Thus, there is a subsequence

$$S = s_1, s_2, \ldots, s_{3l-2}$$

of S_1 such that the integers $s_1, s_2, \ldots, s_l$ are pairwise different and that, for $i = 1, 2, \ldots, l$, $s_i = s_{2l-i} = s_{(2l-2)+i}$. For each i, we denote by v_i the vertex of V^- labeled by s_i, and we denote by x_i the x-coordinate of v_i. We define three relations on the set $L = \{1, 2, \ldots, l\}$ as follows:

$$\begin{aligned}
\prec_1 &= \{(i,j) : i < j \text{ and the ray } \overrightarrow{v_i v_j} \text{ intersects the } y\text{-axis below the point corresponding to } s_{2l-1}\},\\
\prec_2 &= \{(i,j) : i < j \text{ and the ray } \overrightarrow{v_j v_i} \text{ intersects the } y\text{-axis above the point corresponding to } s_l\},\\
\prec_3 &= \{(i,j) : i < j \text{ and the pair } (i,j) \text{ lies neither in } \prec_1 \text{ nor in } \prec_2\}.
\end{aligned}$$

It is not difficult to see that each of the relations $\prec_i$ is a partial ordering. It follows from Dilworth's theorem [4] that either there are two elements in L which are not comparable by any of the partial orders

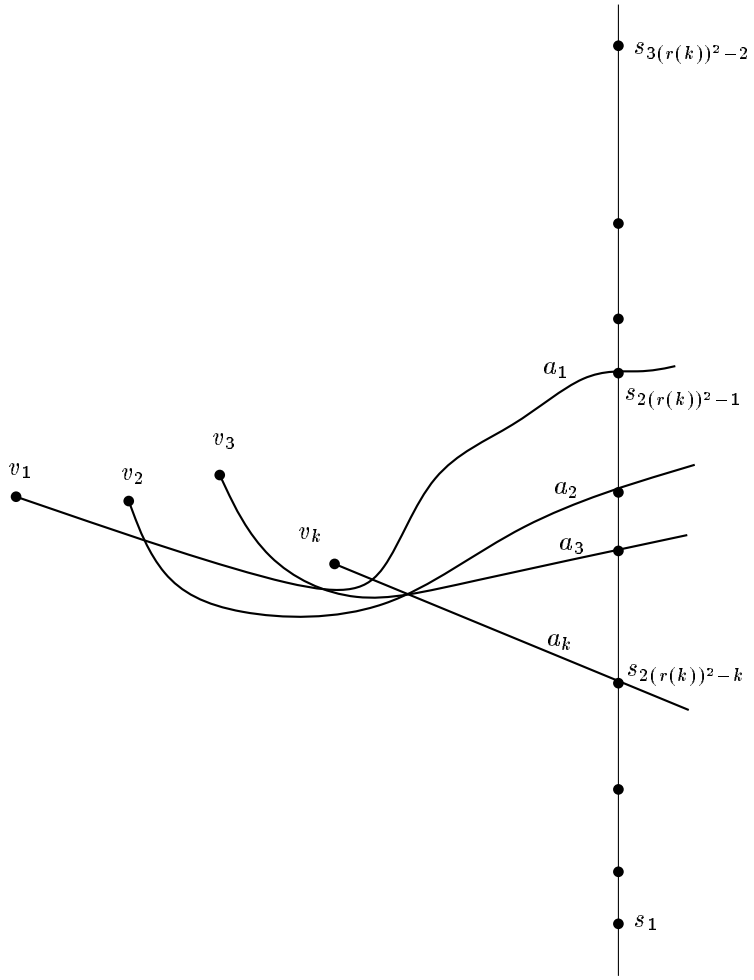

Figure 4: Case 3.

$\prec_i$, or one of the partial orders $\prec_i$ contains a chain of length k. The first case is in contradiction with the definition of the partial orders $\prec_i$, thus one of the partial orders, $\prec_i$, contains a chain $j_1 \prec_i j_2 \prec_i \ldots \prec_i j_k$ of length k. If $i = 1$ then the k segments corresponding to the terms $s_{(2l-2)+j_t}, t = 1, 2, \ldots, k$, are pairwise crossing. If $i = 2$ then the k segments corresponding to the terms $s_{j_t}, t = 1, 2, \ldots, k$, are pairwise crossing. Finally, if $i = 3$ then the k segments corresponding to the terms $s_{2l-j_t}, t = 1, 2, \ldots, k$, are pairwise crossing. Thus, we have obtained a contradiction by showing that k of the edges in our geometric graph are pairwise crossing. □

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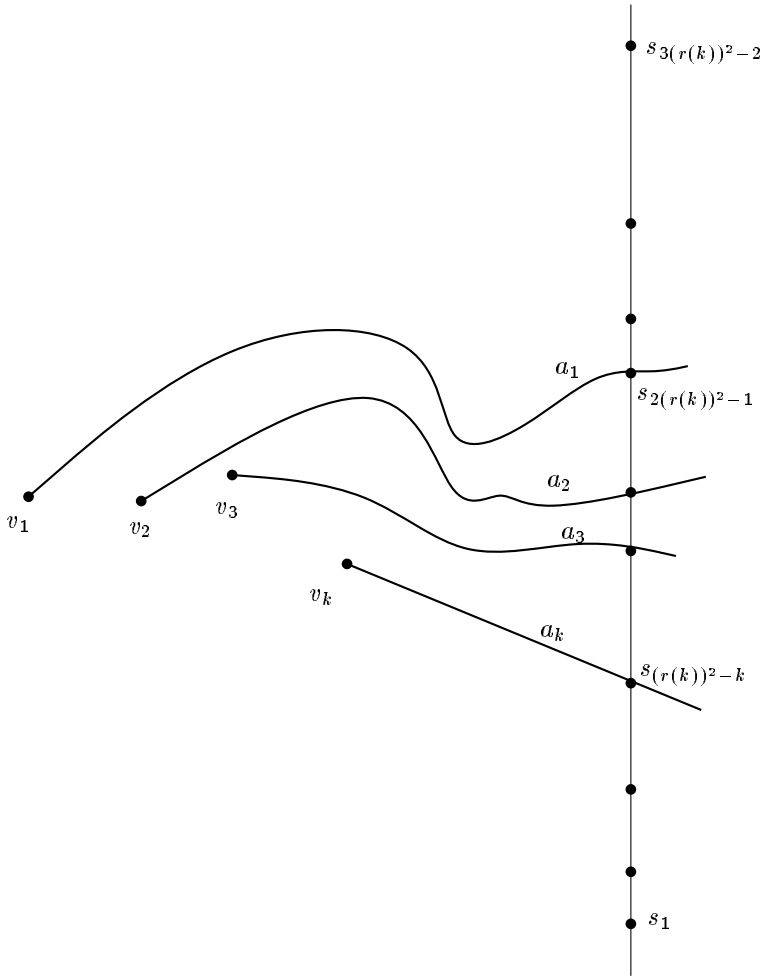

Figure 5: Case 4.

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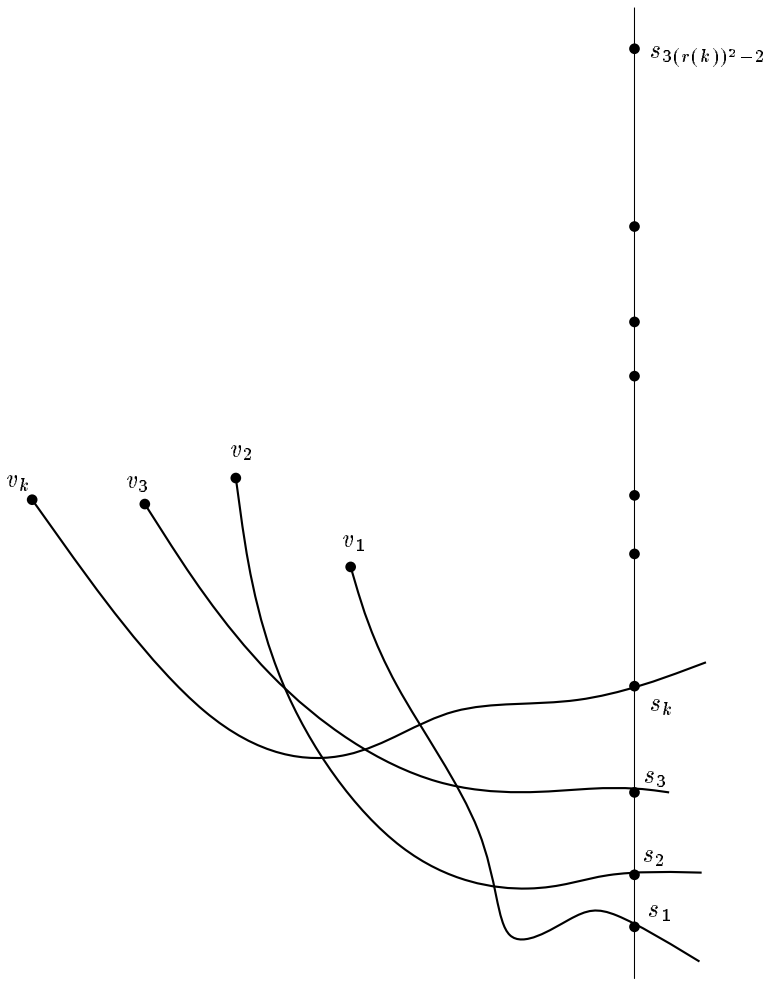

Figure 6: k pairwise crossing arcs in case 1.

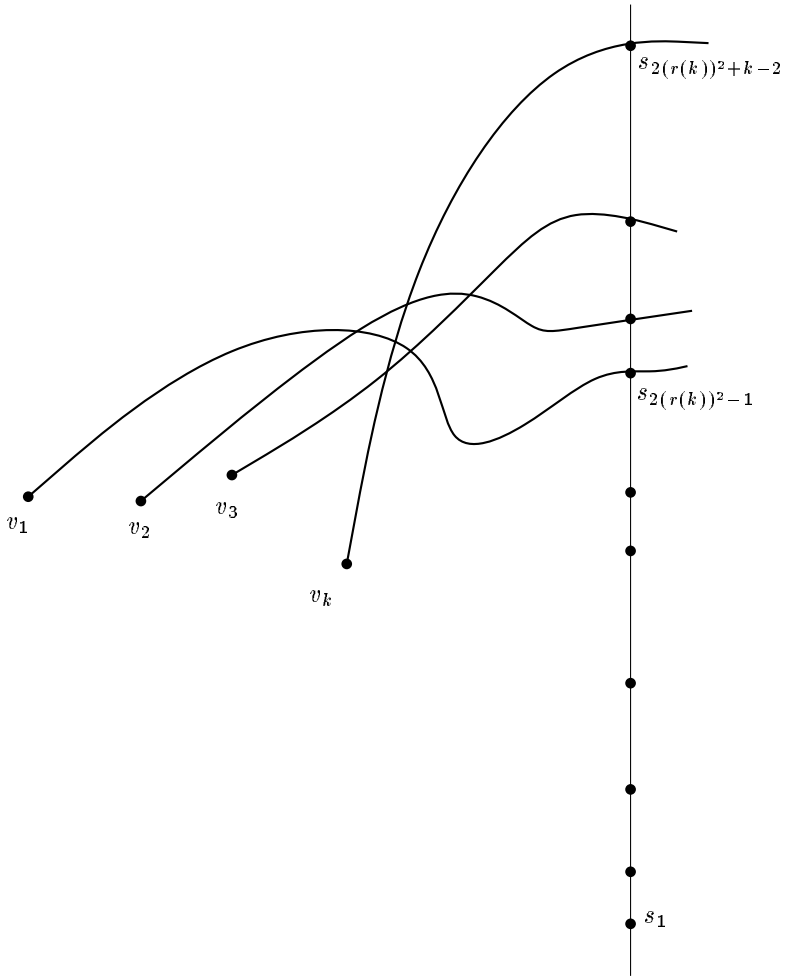

Figure 7: k pairwise crossing arcs in case 4.