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Asymptotics of the total chromatic number for multigraphs*

by

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ABSTRACT

For loopless multigraphs, the total chromatic number is asymptotically its fractional counterpart as the latter invariant increases without bound. This extends DIMACS Technical Report 96-20, where the same assertion, restricted to simple graphs, was proved. The proof for multigraphs — presented in this note — is based on a recent theorem, due to Kahn, establishing the analogous asymptotic behaviour of the list-chromatic index.

The result

This note extends the main result of [3], an assertion about simple graphs, to loopless multigraphs (henceforth *multigraphs*). We prove:

Theorem 1 *For multigraphs,*

$$\chi_t \sim \chi_t^* \quad \textit{as} \quad \chi_t^* \to \infty. \tag{1}$$

That is, for each $\varepsilon > 0$ there exists $D = D(\varepsilon)$ such that every multigraph G with $\chi_t^*(G) > D$ satisfies

$$(1+\varepsilon)^{-1} < \frac{\chi_t(G)}{\chi_t^*(G)} < 1 + \varepsilon. \tag{2}$$

Disclaimer

In the interest of brevity, we rely on [3] for background, notation, terminology and motivation. Here we provide only the bare essentials needed for proving Theorem 1.

Preliminaries

The key ingredient is the following result, proved by Kahn [2]:

Theorem 2 *For multigraphs,*

$$\chi'_\ell \sim \chi'^* \quad \textit{as} \quad \chi'^* \to \infty.$$

The convergence here is in the same sense as that in (1), but we again spell out the quantifiers for reference in the proof of Theorem 1: for each $\gamma > 0$ there exists $C = C(\gamma)$ such that every multigraph G with $\chi'^*(G) > C$ satisfies $\chi'_\ell(G) < (1+\gamma)\chi'^*(G)$.

Other essentials

We will also need several elementary connections between certain multigraph colouring invariants (in (4), k is a positive constant and $\Delta \geq 1$):

$$\begin{aligned} \chi_t^* &\leq \chi_t; & (3)\\ \chi_t^* &\leq k\chi'^*; & (4)\\ \chi_t &\leq \chi'_\ell + 2; & (5)\\ \chi'^* &\leq \chi_t^*. & (6) \end{aligned}$$

Proof of (3). The left side is the optimal value of the linear relaxation of the IP defining the right. ∎

Proof of (4). Kostochka proved (see, *e.g.*, [1, p. 86]) that $\chi_t \leq \lfloor 3\Delta/2 \rfloor$, but, for our needs, this is using a sledge for a finishing nail; greedy colouring yields $\chi_t \leq 2\Delta+1$. Either of these bounds together with (3) and the obvious $\Delta \leq \chi'^*$ gives (4). ∎

Proof of (5). See, *e.g.*, [1, p. 87] or [3, Lemma 2.2]. ∎

Proof of (6). From an optimal fractional total colouring $f : \mathcal{T} \to [0,1]$, one may obtain a fractional edge-colouring $h : \mathcal{M} \to [0,1]$ by shifting the weight $f(T)$ from each total stable set $T = M \cup S$ to the matching M in the natural way. This yields an h with $h(G) = f(G) = \chi_t^*(G)$, and (6) follows since $\chi'^*(G) \leq h(G)$. ∎

The proof

In light of (3), it remains only to establish the right-hand inequality in (2) for arbitrary $\varepsilon > 0$ and sufficiently large χ_t^*. Given $\varepsilon > 0$, let $\gamma = \varepsilon/2$, and choose C so large (according to Theorem 2) that

$$\chi'^* > C \quad \text{implies} \quad \chi'_\ell < (1+\gamma)\chi'^*. \tag{7}$$

Let k be as in (4). If $\chi_t^* > D := \max\{kC, 4k/\varepsilon\}$, then, since $\chi'^* \geq \chi_t^*/k$ (by (4)), we see that χ'^* exceeds both C and $4/\varepsilon = 2/\gamma$. Thus, provided $\chi_t^* > D$, we have

$$\chi_t \leq \chi'_\ell + 2 < (1+\gamma)\chi'^* + \gamma\chi'^* = (1+\varepsilon)\chi'^* \leq (1+\varepsilon)\chi_t^*$$

(justifying the inequalities, respectively, by: (5); the preceding sentence and (7); and (6)), as desired. ∎

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References

[1] T.R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley, New York, 1995.

[2] J. Kahn, Asymptotics of the list-chromatic index for multigraphs, submitted.

[3] P.M. Kayll, Asymptotics of the total chromatic number for simple graphs, Technical Report 96-20, DIMACS, Piscataway, NJ, 1996 [available via anonymous ftp at: `ftp://dimacs.rutgers.edu/pub/dimacs/TechnicalReports/TechReports/1996/96-20.ps.gz`].