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Threshold of Broadcast in Random Graphs

by

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ABSTRACT

Broadcasting process is to send a piece of information which resides at one node in the graph to all the remaining nodes. At each time step, a node knows about the information can send it to one of its neighbors. The broadcast problem is to find the minimum time step needed. The problem is NP hard in general. For a random graph $G_{n,p}$, we are interested in at what value of p there exists a broadcast tree of depth exactly $\lceil \log_2 n \rceil$. Frieze and Molloy [8] show that p is of magnitude $\Theta(\ln n/n)$. In this paper, we give the exact threshold of the edge probability for the existence of such a tree.

1 Introduction

The broadcast problem can be stated as follow:

Given graph $G = (V, E)$, and for $v \in V$, let $N(v)$ denote the set of v's neighbors in G. Assuming initially a piece of information I is residing at one given vertex v_0, at each step, any vertex knowing I can share it with *one* of its neighbors.

Let $V_t,\ t = 0, 1, 2, \dots$ denote the set of vertices which have I at the beginning of step t, $V_0 = \{v_0\}$. Clearly $|V_{t+1}| \leq 2|V_t|$, and so it takes at least $\gamma = \lceil \log_2 n \rceil$ steps before I is sent to all the vertices in G. It is not hard to show that the edges used by the optimal broadcast algorithm forms a tree. The broadcast problem is to find the tree and the minimum broadcast time.

Finding the optimal broadcast tree in general graphs is NP-hard. But efficient approximation algorithm exists for the single source broadcast problem which sends a piece of information to every nodes in the network. See Ravi [10]. Various works have been done on more practical broadcast models, see [1] [2] [5]

We consider the problem that for a random graph $G_{n,p}$ what is the threshold for p such that **whp** [1] a piece of information can be broadcasted in the graph in exactly $\lceil \log_2 n \rceil$ many steps.

The best upperbound to date is due to Frieze, Molloy [8] who show that, if $p \geq 18 \ln n/n$ then **whp** there exists a rooted broadcating tree on every node. A trivial lower bound for the problem can be calculated using the fact that each node needs to have at least $\lceil \log_2 n \rceil$ neighbors with probability $1 - o(n^{-1})$, which is $\gamma \ln n/n$ (γ is a constant which we will specify later).

In this paper, we show that the trivial lower bound is also sufficient for the existence of a rooted optimal broadcast tree on each node of a random graph $G_{n,p}$ by using a different scheme to construct the optimal broadcast tree.

2 The Lower Bound

We consider the following necessary condition for the existence of a rooted optimal broadcast tree on each node in the graph :

To ensure that there exists an optimal broadcast tree rooted at any $v \in V$, the degree of v is at least $\lceil \log_2 n \rceil$.

The probability for this condition to be held is stated in the following Lemma,

[1] An event ϵ_n is said to occur whp (with high probability) if $\mathbf{Pr}[\epsilon_n] = 1 - o(1)$as $n \to \infty$.

Lemma 1 *Given a random graph $G_{n,p}$, the probability threshold that each vertex $v \in V$ have degree at least $\lceil \log_2 n \rceil$* **whp** *is $p = \gamma \ln n/n$, where $\gamma = 3.864\cdots$ is the solution of equation*

$$1 - \gamma + \log_2 \gamma + \log_2 e + \log_2 \ln 2 \;=\; 0. \tag{1}$$

Proof We first show that it is sufficient. Let E_v be the event that vertex v has less than $\lceil \log_2 n \rceil$ neighbors. Then

$$\begin{aligned}
& \mathbf{Pr}[\text{ there exists a vertex with degree} < \lceil \log_2 n \rceil] \\
= \;& \mathbf{Pr}[\bigvee_{v \in V} E_v] \\
\le \;& \sum_{v \in V} \mathbf{Pr}[E_v] \\
\le \;& 2n \binom{n}{\lceil \log_2 n \rceil - 1} p^{\lceil \log_2 n \rceil - 1} (1-p)^{n - \lceil \log_2 n \rceil} \\
\le \;& 2n \left(\frac{ne}{\lceil \log_2 n \rceil} \right)^{\lceil \log_2 n \rceil - 1} \left(\frac{\gamma \ln n}{n} \right)^{\lceil \log_2 n \rceil - 1} e^{-\gamma \ln n} \\
\le \;& 2n^{1-\gamma} (\gamma e \log_2 e)^{\lceil \log_2 n \rceil - 1} \\
= \;& 2^{2-\gamma} (2^{1-\gamma} \gamma e \ln 2)^{\lceil \log_2 n \rceil - 1}
\end{aligned}$$

The probability tends to 0 as $n \to \infty$ if $p = (\gamma + \epsilon) \ln n/n$, where ϵ is any fixed constant.

On the other hand it is also necessary. Let X_u be the indicator of random variable E_u, and $X = \sum_{u \in V} X_u$ be the random variable of number of vertices which has fewer than $\lceil \log_2 n \rceil$ neighbors. It is not hard to show that

$$\mathbf{E}[X^2] = \mathbf{E}[X](1 + (1 + o(1))\mathbf{E}[X])$$

and

$$\mathbf{E}[X] \to \infty \quad \text{when} \quad p = (\gamma - \epsilon) \ln n/n.$$

Therefor, since

$$\mathbf{Pr}[X > 0] \ge \frac{\mathbf{E}^2[X]}{\mathbf{E}[X^2]},$$

$\mathbf{Pr}[X > 0] \to 1$ as $n \to \infty$. □

3 Hall's Theorem Revisited

The analysis for the threshold of perfect bipartite matching in a random bipartite graph $G_{n,n,p}$ relies heavily on the sufficient and necessary condition of Hall's Theorem. We will extend Hall's condition (2) to the existence of perfect b-matching and near perfect matching in a bipartite graph, as a tool for the analysis of our algorithm.

Theorem 1 (Hall's Theorem) *Let $G(V_1, V_2; E)$ be a bipartite graph, such that $|V_1| = |V_2| = n$. Then G a perfect matching of size n if and only if*

$$\forall U \subseteq V_1, \qquad |N(U)| \geq |U|, \tag{2}$$

where $N(U) \subseteq V_2$ are the neighbors of U.

Let $G(V_1, V_2; E)$ be a bipartite graph such that $|V_1| = n$ and $|V_2| \geq kn$. A k-matching[2] of G is a set of pairs $\{(u; M_u) | u \in U \subseteq V_1, M_u \subset V_2, |M_u| = k\}$ such that for $u \neq u'$, $M_u \cap M_{u'} = \emptyset$. The k-matching is called maximum if $|U| = |V_1| = n$.

Hall's theorem can be extended in the following two ways.

Theorem 2 *Let $G(V_1, V_2; E)$ be a bipartite graph such that $|V_1| = n$ and $|V_2| \geq kn$. Then G has a maximum k-matching if and only if the following holds,*

$$\forall U \subseteq V_1, \qquad |N(U)| \geq k|U|. \tag{3}$$

Proof The Theorem can be proved in the same way by duplicating each vertex in U to k copies. We omit the complete proof here. □

Theorem 3 *Let $G(V_1, V_2; E)$ be a bipartite graph such that $|V_1| = n$ and $|V_2| \geq n$. Then g has a matching of size $n - 1$ if and only if*

$$\forall U \subseteq V_1, |N(U)| \geq |U| - 1.$$

This is stated as an exercise in Bondy and Murty [4].

4 The Upper Bound

The best upperbound to date is due to Frieze and Molloy [8]. They show that when $p > 18 \ln n/n$, **whp** there exists a rooted broadcast tree in $G_{n,p}$ on every node of the graph such that the broadcast time is $\lceil \log_2 n \rceil$. Even their result is asymptotically tight, it still shows a wide gap between the lower bound we derived in the previous section.

In this section we show that the lower bound is also sufficient for the existence of an optimal broadcast tree. Thus provides the exact threshold for the problem.

[2]This definition should be distinguished from the usual definition of a b-matching.

4.1 The Algorithm

For the simplicity of presenting and analyzing the algorithm, we assume that $|V| = 2^\nu$, where $\nu = \log_2 n$ is an integer. The general case can easily be proved with a small modification of our proof.

In [8] Frieze and Molloy show a tight upperbound up to a constant factor, but their method does not seem possible to find the exact threshold due to the perfect matching bottleneck. We overcome the problem by introducing a different way of evolving the broadcast tree and construct a perfect 1-matching using an almost perfect 1-matching followed by an iterative fixing step.

We consider a different evolution of the broadcast tree. A *rooted* broadcast tree with root v_0 of degree ν, $T_\nu(v_0)$, is defined as follow:

v_0 is connected to ν neighbors $v_{1,1}, v_{1,2}, \ldots, v_{1,\nu}$ of v_0, such that under certain permutation, $v_{1,i}$ $(1 \leq i \leq \nu)$ is the root of a $T_{i-1}(v_{1,i})$ tree.

We can recursively define the tree, where T_1 is simply an edge and T_0 is empty. It can easily be shown that this is actually an optimal broadcast tree on $n = 2^\nu$ nodes. The degree sequence (outgoing edges) of the roots of these subtrees can be sorted as follow:

1 node of degree $\log_2 n$, 1 node of degree $\log_2 n - 1$, 2 nodes of degree $\log_2 n - 2, \ldots, 2^t$ nodes of degree $\log_2 n - t - 1, \ldots, \frac{n}{8}$ nodes of degree 2, $\frac{n}{4}$ nodes of degree 1. For simplicity, a node of outgoing degree k $(1 \leq k \leq \nu)$ will be called a k-node. We will use this term throughout the paper.

We will construct the broadcast tree in the descending order of the degree sequence.

The algorithm can be described in two stages:

Algorithm Rooted_Broadcast_Tree.

Let $G = (V, E)$ and v_0 be the root.

Stage 1. Quasi-Tree grow.

Find ν arbitrary neighbors for the ν-node v_0. If fails then halt.

In the order of $i = \nu - 1, \nu - 2, \ldots, 3, 2$, pick a set of $2^{\nu-i-1}$ leaves of the current tree (none of two leaves can share a common parent), find i neighbors from the remaining vertex set for each of them. If fails then halt.

Stage 2. Final matching.

Find a perfect matching of the remaining $\frac{n}{4}$ nodes to $\frac{n}{4}$ leaves of the current tree (none of two can share the same parent in the current tee) to form a complete optimal broadcast tree.

4.2 Analysis of Rooted_Broadcast_Tree

In this section we show that **whp** both stage 1 and stage 2 of Rooted_Broadcast_Tree succeed with probability $1 - o(n^{-1})$. Thus will lead to the following main Theorem,

Theorem 4 *Let $G = (V, E)$ be a random graph $G_{n,p}$, and γ be the solution of equation*

$$1 - \gamma + \log_2 \gamma + \log_2 e + \log_2 \ln 2 \quad = \quad 0.$$

Then for any positive constant ϵ,

(i) If $p = \gamma(1-\epsilon)\ln n/n$, then **whp** *$\exists\ v \in V$ such that there is no optimal broadcast tree rooted on v.*

(ii) If $p = \gamma(1+\epsilon)\ln n/n$, then **whp** *$\forall\ v \in V$, there is an optimal broadcast tree rooted on v.*

4.2.1 Analysis of Stage 1

The analysis of this stage relies on the following result of Erdös and Rényi [6].

Theorem 5 *Let $G_{n,n,p}$ be a random bipartite graph with edge probability p. If $p \geq (1+\delta)\ln n/n$ where δ is any positive constant, then* **whp** *a $G_{n,n,p}$ has a perfect matching.*

Remark 1 *The probability is the same as the probability to ensure that each node has at least one neighbor* **whp***.*

To show that with probability at least $1-o(n^{-1})$ stage 1 will succeed, we notice that stage 1 consists of a sequence of k-matchings, i.e. a ν-matching of $(1, n-1)$[3], a $(\nu-1)$-matching of $(1, n-\nu-1)$, a $(\nu-2)$-matching of $(2, n-2\nu-2)$, ..., a 3-matching of $(\frac{n}{16}, \frac{11n}{16})$ and a 2-matching of $(\frac{n}{8}, \frac{n}{2})$.

Lemma 2 *Let $G = (V, E)$ be a random graph $G_{n,p}$ where $p = \gamma(1+\epsilon)\ln n/n$ and γ as defined in (1). Then with probability at least $1 - o(n^{-1})$, finding all the k-matchings in stage 1 except for $k = 2$ and 3 will succeed.*

Proof First notice that according to the order of the sequence of the matchings, the edge slots used in i-matching is disjoint with those used in j-matchings, $i+1 \leq j \leq \nu$.

To calculate the fail probability of the t-matching in Stage 1, where $4 \leq t \leq \nu$, we compute the fail probability of the existence of a maximum t-matching of $(2^{\nu-t-1}, n-1-2^{\nu-t-1}\nu - \sum_{i=0}^{\nu-t-1} 2^i(i+1))$ with edge probability $(1+\epsilon)\gamma \ln n/n$.

[3] We mean a t-matching of (l, m) by a maximum t-matching in a bipartite graph $G_{l,m}$.

Let v_0 be the root, we show that $\mathbf{Pr}[|N(v)| < \nu]$ dominates the other probability terms. We will consider two cases,

Case 1. $A \leq t \leq \nu$, where A is a large constant.

CLAIM 1 *There are at least as many as* $(1 - \frac{A}{2^{A-2}})n$ *nodes which haven't been matched after the* A*-matching.*

Proof of Claim. This can be shown by counting the number of nodes which have already been matched,

$$\begin{aligned} & \sum_{i=A}^{\nu} \#\text{ of nodes matched in } i\text{-matching} \\ = \ & 1 + \sum_{i=A}^{\nu-1} 2^{\nu-i-1} i \\ \leq \ & \frac{A}{2^{A-2}} n. \end{aligned}$$

□

Choose A large enough so that there are at least $(1 - \epsilon')n$ nodes remaining after the A-matching, where $0 < \epsilon' < \epsilon/2$. For a fixed positive δ, we have

$$\begin{aligned} & \mathbf{Pr}[\exists t, A \leq t \leq \nu \text{ such that there exists no maximum } t\text{-matching}] \\ \leq \ & \sum_{t=A}^{\nu} \mathbf{Pr}[\text{ there does not exist a } t\text{-matching in } G_{2^{\nu-t-1},(1-\epsilon')n,p}] \\ \leq \ & \sum_{t=A}^{\nu} (1+\delta)\mathbf{Pr}[\exists v, |N(v)| < t \text{ in } G_{2^{\nu-t-1},(1-\epsilon')n,p}] \\ \leq \ & \sum_{t=A}^{\nu} (1+\delta)\frac{n}{2^{t+1}} \binom{(1-\epsilon')n}{t-1} p^{t-1}(1-p)^{(1-\epsilon')n-t+1} \\ \leq \ & \sum_{t=A}^{\nu} (1+\delta)\frac{n}{2^{t+1}} \left(\frac{\gamma(1+\epsilon)(1-\epsilon')e\ln n}{t-1}\right)^{t-1} n^{-\gamma(1+\epsilon)(1-\epsilon')} \\ = \ & O(n^{-1-(\gamma-\log_2 e)(\epsilon-\epsilon')}), \end{aligned}$$

using the fact that

$$\frac{1}{2^{t+1}} \left(\frac{\gamma(1+\epsilon)(1-\epsilon')e\ln n}{t-1}\right)^{t-1}$$

is a monotone increasing function for $t \in [A, \nu]$ and there are at most $\log_2 n$ terms in the sum.

Case 2. $4 \leq t < A$.

It is easy to show that after the maximum 4-matching has been found, there are still $\frac{11n}{16}$ nodes left. Thus the probability of failing to find a maximum t-matching for $4 \le t \le A-1$ can be bounded as,

$$\begin{aligned}
& \sum_{t=4}^{A-1} \mathbf{Pr}[\text{ there does not exist a maximum } t\text{-matching in } G_{2^{\nu-t-1},\frac{11n}{16},p}] \\
\le\ & \sum_{t=4}^{A-1} (1+\delta)\mathbf{Pr}[\exists v, |N(v)| < t \text{ in } G_{2^{\nu-t-1},\frac{11n}{16},p}] \\
\le\ & \sum_{t=4}^{A-1} (1+\delta)\frac{n}{2^{t+1}}\binom{\frac{11n}{16}}{t-1} p^{t-1}(1-p)^{\frac{11n}{16}-t+1} \\
\le\ & \sum_{t=4}^{A-1} (1+\delta)\frac{n}{2^{t+1}}\left(\frac{11e\gamma\ln n}{16(t-1)}\right)^{t-1} e^{-\frac{11}{16}\gamma\ln n} \\
=\ & O(n^{-1.5}),
\end{aligned}$$

for some $\delta > 0$. Since A is a constant and $\gamma > 3.86$. This completes the proof of Case 2. □

4.2.2 Analysis of Stage 2

We only need to show that both 3-matching, 2-matching and 1-matching will suceed with probability $1 - o(n^{-1})$. Notice that when $p = (4+\epsilon)\ln n/n$, with pobability $1 - o(n^{-1})$ a perfect 2-matching exists in $G_{\frac{n}{8},\frac{n}{2},p}$, and when $p = (8+\epsilon)\ln n/n$, with probability $1 - o(n^{-1})$ there exists a perfect matching in $G_{\frac{n}{4},\frac{n}{4},p}$. By these two facts, it can be shown that when $p = (4+\epsilon)\ln n/n$, stage 2 will succeed with probability $1 - o(n^{-1})$. But this still exhibits a small gap between the lower bound.

Since a perfect matching seems indispensible to complete the broadcast tree, a different strategy need to be applied to circumvent the perfect matching bottleneck in order to show the threshold.

We break the proof into two parts. We first show that a maximum 3 and 2-matching can be constructed using one copy of the decomposed random graph with probability at least $1 - o(n^{-1})$, and an almost perfect 1-matching will be constructed with the same confidence, except that at most one node will be left unmatched. Then an iterative procedure will complete the perfect matching using the other copy of the decomposed random graph.

Let V_M ($|V_M| = v_m$) be the vertex set that has been matched and V_{UM} ($|V_{UM}| = n - v_m$) be the vertex set that hasn't been matched (Thus $v_m = \frac{5n}{16}$). Consider random graph model $G_{V,E/E(V_M),p}$ such that each edge which has at least one of its end points not belonging to V_M, appears independently with probability p. Because all the edge slots induced by V_M have been conditioned, a further exploration of these edges will certainly complicate our analysis.

We then decompose the random graph $G_{V,E/E(V_M),p}$ into two independent copies G^1 and G^2 of $G_{V,E/E(V_M),p_1}$ and $G_{V,E/E(V_M),p_2}$, where $1-p = (1-p_1)(1-p_2)$. Let $p_1 = 3(1+\epsilon)\ln n/n$, and $p_2 > (\gamma - 3)(1+\epsilon)\ln n/n$.

Part 1. Maximum 3-matching and 2-matching.

We will show that in G_1, almost surely a maximum 3-matching and a maximum 2-matching can be constructed.

Lemma 3 *Given a random bipartite graph $G_{\frac{n}{16},\frac{11n}{16},p_1}$, with probability $1-o(n^{-1})$ there exists a maximum 3-matching.*

Proof Follow from Theorem 3 and the proof of Lemma 2. □

Lemma 4 *Let B be a random bipartite graph $G_{\frac{n}{8},\frac{n}{2},p_1}$ where $p_1 = 3(1+\epsilon)\ln n/n$. Then with probability $1-o(n^{-1})$, there exists a 2-matching of size $\frac{n}{8}-1$.*

Proof Let V_1, V_2 be the two vertex sets such that $|V_1| = \frac{n}{8}$, $|V_2| = \frac{n}{2}$. It is sufficient to show with probability $1-o(n^{-1})$, $\forall U \subseteq V_1, |N(U)| \geq 2|U|-1$. By Theorem 2 and Theorem 3, we will have a 2-matching of size at least $\frac{n}{8}-1$.

The probability of failure can be bounded as

$$\begin{aligned}
& \mathbf{Pr}[\ \exists U \subseteq V_1, |U| \geq 2, \text{ such that } |N(U)| \leq 2|U|-2] \\
\leq\ & \sum_{i=2}^{n/8} \binom{n/8}{i}\binom{n/2}{2i-2} p^{3i-3}(1-p_1)^{i(n/2-2i+2)} \\
\leq\ & C\binom{n/8}{2}\binom{n/2}{2} p^3(1-p_1)^{2(n/2-2)} \\
=\ & Cn\ln^3 n e^{-3\ln n} \\
=\ & \tilde{O}(n^{-2}).
\end{aligned}$$

Thus at most one vertex will be left unmatched. This completes the proof of the Lemma. □

Let this vertex be ω, now consider the other two vertices which share the same parent as ω. Let $\mathbf{E_i}$ $(i = 1, 2)$ be the events that one of the two nodes can be 2-matched to the remaining $\frac{n}{4}+2$ nodes. Since the edge slots are uncoditioned, the probability that they will both fail is bounded by,

$$\left((1-p_1)^{\frac{n}{4}+2} + \binom{\frac{n}{4}+2}{1} p_1(1-p_1)^{\frac{n}{4}+1}\right)^2 = O(n^{-1.5}).$$

Thus a perfect 2-matching can be constructed with probability at least $1-o(n^{-1})$.

Part 2. Almost perfect 1-matching and the iterative step.

We now consider completing the optimal broadcast tree by finding a perfect 1-matching with probability at least $1-o(n^{-1})$.

First, we show that an almost perfect 1-matching can be constructed with probability $1-o(n^{-1})$, i.e. at most one node will be unmatched in G^1. Notice that we have to match $\frac{n}{4}$ nodes to $\frac{n}{4}$ slots where each slot has two nodes, thus it is equivalent to think that the edge probability between the $\frac{n}{4}$ nodes and $\frac{n}{4}$ slots is $2p_1$. We have the following lemma,

Lemma 5 *Let $B(V_1, V_2)$ be a random bipartite graph $G_{\frac{n}{4},\frac{n}{4},2p_1}$ where $p_1 = 3(1+\epsilon)\ln n/n$. Then*

(i) with probability $1-o(n^{-1})$, there exists an 1-matching of size at least $\frac{n}{4}-1$,

(ii) with probability $1-o(n^{-0.5})$, there exists a perfect 1-matching.

(iii) $\mathbf{Pr}[$ *there exists exactly one isolated vertex* $|B$ *does not have a perfect 1-matching*$] = 1-o(n^{-1})$.

Proof Let $\mathcal{E}_1$ be the event that B does not have a perfect matching, $\mathcal{E}_2$ be the event that B does not have a matching of size $\frac{n}{4}-1$ and $\mathcal{E}_3$ be the event that B has exactly one isolated vertex. Follow from Theorem 3, the probability that B does not have a matching of size $\frac{n}{4}-1$ is at most,

$$\begin{aligned}
\mathbf{Pr}[\mathcal{E}_2] &\leq \mathbf{Pr}[\exists U \subseteq V_1, |N(U)| < |U|-1] \\
&\leq \sum_{i=2}^{\frac{n}{4}} \mathbf{Pr}[\exists |U| = i, |N(U)| < |U|-1] \\
&\leq \sum_{i=2}^{\frac{n}{4}} \binom{\frac{n}{4}}{i}\binom{\frac{n}{4}}{i-2} p_2^{2i-4}(1-p_2)^{i(\frac{n}{4}-i+2)} \\
&\leq (1+o(n^{-0.5}))\binom{\frac{n}{4}}{2}(1-p_2)^{i\frac{n}{4}} \\
&\leq o(n^{-1})
\end{aligned}$$

This proved (i), and (ii) can be proved in the similar way by replacing $|N(U)| < |U|-1$ with $|N(U)| < |U|$.

To show (iii), since

$$\begin{aligned}
\mathbf{Pr}[\mathcal{E}_1] &\leq \sum_{i=1}^{n/4} \mathbf{Pr}[\exists |U| = i, |N(U)| = i-1] \\
&= (1+o(n^{-2}))\mathbf{Pr}[\exists v \in V_1, |N(v)| = 0] \\
&= (1+o(n^{-2}))\sum_{j=1}^{n/4} \mathbf{Pr}[\exists \text{ exactly } j \text{ isolated nodes in } V_1] \\
&\leq (1+o(n^{-2}))(1+o(n^{-1}))\mathbf{Pr}[\exists \text{ exactly one isolated nodes in } V_1] \\
&= (1+o(n^{-1}))\mathbf{Pr}[\mathcal{E}_3],
\end{aligned}$$

therefor we have

$$\begin{aligned}\mathbf{Pr}[\mathcal{E}_3|\mathcal{E}_1] &= \frac{\mathbf{Pr}[\mathcal{E}_3 \bigwedge \mathcal{E}_1]}{\mathbf{Pr}[\mathcal{E}_1]} \\ &= \frac{\mathbf{Pr}[\mathcal{E}_3]}{\mathbf{Pr}[\mathcal{E}_1]} \\ &= (1 - o(n^{-1})).\end{aligned}$$

This completes the proof of this Lemma. □

Thus we can claim that if a perfect 1-matching is not found, then almost surely (with probability $1 - o(n^{-0.5})$. there will be exactly one isolated vertex left unmatched. Let s be the unmatched node and (t_1, t_2) be the unmatched slot, $M_3 = \{(w_i; x_i, y_i, z_i)|\ 1 \leq i \leq \frac{n}{16}\}$ be the 3-matching constructed in Part 2 and $M_2 = \{(p_i; q_i, r_i)|1 \leq i \leq \frac{n}{8}\}$ be the 2-matching constructed in Part 2. W.l.o.g., assume that $x_i = p_i, \forall i, 1 \leq i \leq \frac{n}{16}$; y_i $(1 \leq i \leq \frac{n}{16})$ and q_i $(1 \leq i \leq \frac{n}{8})$ are not used in the 1-matching.

Corollary 1 *Given that there is an unmatched vertex s after the 1-matching, then with probability $1 - o(n^{-1})$, s has a neighbor in $W \cup P$ in G^1 where $W = \{w_i|\ 1 \leq i \leq \frac{n}{16}\}$ and $P = \{p_i|\ 1 \leq i \leq \frac{n}{8}\}$.*

We need to show that s can be matched with probability at least $1 - o(n^{-0.5})$, since these two stages are independent events, the probability that they both fail is $o(n^{-1})$, which will be sufficient for the completion of Theorem 4.

We consider the following iterative procedure. Let R be the set of the remaining nodes after the 2-matching is constructed , $V_1 = W \cup P \cup \{z_i|1 \leq i \leq \frac{n}{16}\} \cup \{r_i|1 \leq i \leq \frac{n}{8}\}$ and $V_2 = \{y_i|1 \leq i \leq \frac{n}{16}\} \cup \{q_i|1 \leq i \leq \frac{n}{16}\} \cup R\backslash\{s\}$. It is easy to check the following fact:

If there exists an alternating path $s = u_0 \rightarrow v_0 \rightarrow u_1 \cdots \rightarrow v_k \rightarrow u_{k+1} \rightarrow t$ where $u_i \in V_2, v_i \in V_1, (u_i, v_i) \in E, (v_i, u_{i+1}) \in E$ and $t = t_1$ or t_2, then the size of the matching can be increased by one, thus we will get a perfect 1-matching.

To circumvent the conditioning on G^1, we use random bipartite graph G^2 during the iterative step. Recall that $p_2 = (1 + \epsilon)(\gamma - 3)\ln n/n$.

Lemma 6 *Let $B(V_1, V_2)$ be a random bipartite graph $G_{\frac{3n}{8}, \frac{3n}{8}, p_2}$, then with probability $1 - o(n^{-n^{c\epsilon}})$ where $c > 0$ is a fixed constant, the largest connected component of B, C_B has at least $(1 - o(n^{-\frac{3(\gamma-3)}{8}}))\frac{3n}{4}$ many vertices.*

Proof Let $C_1, C_2, \ldots, C_l$ be the connected components of B in the decreasing order of their size. Assume that $|C_1| \geq \frac{3n}{8}$, the case that $|C_1| < \frac{3n}{8}$ can easily been shown by adding smaller components until the union has size bigger than $\frac{3n}{8}$. Let $|C_2 \cup C_3 \cup \cdots \cup C_l| = t_1 + t_2$

where t_i denote the number of nodes in V_i $(i = 1, 2)$.

$$
\begin{aligned}
&\mathbf{Pr}[B \text{ has a disjoint component which has } t_1 \text{ nodes in } V_1,\ t_2 \text{ nodes in } V_2] \\
\leq\ & \binom{\frac{3n}{8}}{t_1}\binom{\frac{3n}{8}}{t_2}(1-p_2)^{t_1(\frac{3n}{8}-t_2)+t_2(\frac{3n}{8}-t_1)} \\
\leq\ & \frac{(cn)^{t_1+t_2}}{t_1^{t_1}t_2^{t_2}}e^{-\frac{3n}{8}(t_1+t_2)p_2} \\
\leq\ & \left(\frac{2cn^{1-3(\gamma-3)(1+\epsilon)/8}}{t_1+t_2}\right)^{t_1+t_2}, \\
=\ & o(n^{-\epsilon n^{1/2}}),
\end{aligned}
$$

by taking $t_1+t_2 \geq n^{1-3(\gamma-3)(1+\epsilon/2)/8}$. Therefor almost surely the largest connected component of B is of size $(1-o(n^{-\frac{3(\gamma-3)}{8}}))\frac{3n}{4}$. $\square$

Let $V(C_B)$ be the vertex set of C_B, then

Lemma 7 *With probability* $1-o(n^{-\frac{3(\gamma-3)}{4}})$, *$s$ has a neighbor in $V(C_B)\cap V_1$ and one of $\{t_1, t_2\}$ is connected to $V(C_B)\cap V_2$.*

Proof To show that s is connected to some node in T_1 given that it is the only unmatched node in the 1-matching, we consider the following two events. By Corollary 1, with probability $1-o(n^{-1})$ it has a neighbor in T_1 in G^1. Since the largest connected component C_B if of size $(1-o(n^{-\frac{3(\gamma-3)}{8}}))\frac{3n}{4}$, the probability that the neighbor does not belong to $V_1(C_B)$ is $o(n^{-\frac{3(\gamma-3)}{8}})$.

On the other hand, the probability that s does not have a neighbor in $V_1(C_B)$ in G^2 is at most,

$$
\begin{aligned}
& (1-p_2)^{(1-o(n^{-3(\gamma-3)/8}))\frac{3n}{8}} \\
\leq\ & e^{(1-o(n^{-3(\gamma-3)/8}))(-3(1+\epsilon)(\gamma-3)\ln n/8)} \\
=\ & o(n^{-3(\gamma-3)/8}).
\end{aligned}
$$

Since these two events are independent, the probability that s does not have a neighbor in $V_1(C_B)$ is $o(n^{-3(\gamma-3)/4})$.

It is similar to show that with probability $1-o(n^{-3(\gamma-3)/4})$, one of t_1, t_2 has a neighbor in $V_2(C_B)$. We omit the simple detail here. $\square$

CLAIM 2 *The probability that s can not be matched using the alternating path is* $o(n^{-3(\gamma-3)/4})$.

Proof Let $n_s \in V_1(C_B)$ be the neighbor of s and $n_t \in V_2(C_B)$ be the neighbor of t_1 or t_2. Since C_B is a connected component, there is a path ϱ from n_s to n_t. Obviously augment through path $s \to \varrho \to t_1$ or t_2 will increase the size of the matching by 1. And the probability that this will fail is $o(n^{-3(\gamma-3)/4})$. □

Thus with probability $o(n^{-3(\gamma-3)/4}) = o(n^{-0.6})$, s will fail to be matched during the iterative stage. Since the probability that a perfect 1-match exists in G^1 is at least $1-o(n^{-0.5})$, with probability $1 - o(n^{-1})$ an optimal broadcast tree could be constructed. This completes the proof of Theorem 4.

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