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On Linear Ordering of Strongly Extensional Finitely-Branching Graphs and Non-Well-Founded Sets [1]

by

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1 Introduction

Definability of a linear order in finite structures of some given class by logical means is an important issue of finite model theory and descriptive complexity theory. For example, first-order logic extended by least fixed point operator ($\mathrm{FO}+\mathrm{LFP}$) describes exactly PTIME-computability over finite *linear ordered* structures [10, 12, 23, 11] (cf. also [15, 7] where recursive finite functions and the successor operation are considered instead of $\mathrm{FO}+\mathrm{LFP}$ and the linear order, respectively). If the structures considered are not linearly ordered in advance, but some order may be uniformly defined in $\mathrm{FO}+\mathrm{LFP}$ then the same result holds for this class of structures, too.

We show in this paper that this is the case for the class of strongly extensional ($\mathcal{SE}$) *finite* graphs (i.e. arbitrary finite graphs considered up to bisimulation equivalence relation). The vertices of such graphs serve as a faithful representation of hereditarily-finite non-well-founded sets (with finite transitive closure) which constitute a universe HFA. Actually, we define in $\mathrm{FO}+\mathrm{IFP}$ (which is a version of $\mathrm{FO}+\mathrm{LFP}$) a linear ordering on arbitrary $\mathcal{SE}$ *finitely branching* graphs. These graphs may be infinite and so represent a more general class of hereditarily-finite non-well-founded sets HFA^{∞} (possibly with infinite transitive closure). Non-well-founded sets are also called *hypersets* (cf. [1, 3]).

Our interest to these questions arose from a work on characterizing the complexity classes of computable set-theoretic operations by some bounded set theory (BST) languages [16]–[22]. So, definability in the language Δ_{D} considered in [19, 20] over HFA-sets (cf. also the Appendix below) was characterized in terms of definability in $\mathrm{FO}+\mathrm{LFP}$ over finite graphs of the abovementioned kind.

Now, having definable linear ordering, we answer affirmatively the question in op. cit. on coincidence of Δ_{D}-definability with PTIME-computability over HFA.

We propose two different definitions of linear ordering, one of which is an application of the method of A.Dawar, S. Lindell and S. Weinstein to the case of $\mathcal{SE}$ finite graphs, and give a comparison of both approaches in terms of coherence of the orders on different graphs.

Also note, that these definability results may be considered in the framework of a set-theoretic approach to "nested" and "circular" semi-structured Data Bases or to *Web-like* Data Bases (WDB), by using the Δ_{D}-language regarded as a *query language*. (Cf. [18] and also more abstract and mathematically oriented paper [19] for "non-circular" and "circular" cases, respectively. A paper on the connection to the World-Wide Web and WDB using the main result of this work is in preparation.)

The following Section 2 is devoted to relating the notion of a graph to hypersets. Then, in Section 3, we define in $\mathrm{FO}+\mathrm{IFP}$ a linear ordering on strongly extensional finite and finitely branching graphs. Also, the main result on describing PTIME over these graphs and over HFA is presented there. The last Section 4 compares our approach with that of A. Dawar, S. Lindell and S. Weinstein [4]. There is also an Appendix where set theory BSTA and its language Δ_{D} for the universe HFA, as well as their expressive power, are described.

2 Graphs and sets

Remember, that *hereditarily-finite sets* are defined as finite sets whose elements are finite sets, etc., until the empty set or an *urelement* (i.e. an atomic object which is not a set) will be obtained. These sets constitute the universe HF. Analogously, it is defined the universe HC of *hereditarily-countable sets*. For simplicity, we will assume that there are no urelements involved in HF- and HC-sets, i.e. that all considered sets are *pure*.[1] It is implicit also that the universes HF $\subset$ HC are *well-founded*, i.e. that there are no infinite chains of sets satisfying

$$\ldots \in v_{i+1} \in v_i \in \ldots \in v_1 \in v_0. \tag{1}$$

We will represent HF- and HC-sets (and also sets of more general nature) by vertices of graphs g. Formally, a *graph* is defined here as arbitrary map of the form $g : |g| \to \mathcal{P}(|g|)$ with $|g| = \mathbf{dom}(g)$ being a set of *vertices*, and $\mathcal{P}(|g|)$ denoting the set of all subsets of the set $|g|$. For any $u, v \in |g|$ we write $u \in^g v$ or $g \models u \in v$ instead of $u \in gv$ and say that "u is an element of v in the sense of g". Of course, $\in^g$ used here is *not* a real membership relation between sets. This is just a binary relation between vertices u and v which says that there exists an edge between u and v (in a fixed direction you choose, say $u \in^g v$ iff the edge $u \leftarrow v$ is in g).

Depending on the cardinality of sets gv or $|g|$, there are three important classes of graphs: *countably branching*, $\mathcal{CBG}$, *finitely branching*, $\mathcal{FBG}$, and just *finite* graphs, $\mathcal{FG}$. Also, g is called *well-founded* graph, $\mathcal{WG}$, if it has no infinite chains of vertices like (1). For finite graphs this means that there are no (directed) *cycles*, i.e. these are *finite acyclic graphs*, $\mathcal{FAG}$, or $\mathcal{FWG}$.

Let us define *denotational semantics* $[\![v]\!]_g$ (called also *decoration* [1] or *collapsing* in the case of $\mathcal{WG}$ [2, 16, 18]) for the vertices v of any graph g. According to the set-theoretic symbolism like $\in^g$ we let $[\![v]\!]_g$ be a *set of sets of sets, etc.* defined recursively by the identity

$$[\![v]\!]_g = \{[\![u]\!]_g | u \in^g v\} \quad \text{i.e.} \quad \forall z (z \in [\![v]\!]_g \Leftrightarrow \exists u \in^g v (z = [\![u]\!]_g)) \tag{2}$$

For example, if g contains exactly five edges $u \in^g v$, $u \in^g v'$, $u \in^g w$, $v \in^g w$ and $v' \in^g w$ then $[\![u]\!]_g = \emptyset$, $[\![v]\!]_g = [\![v']\!]_g = \{\emptyset\}$, and $[\![w]\!]_g = \{\{\emptyset\}, \emptyset\}$. If g is any well-founded graph, then there is no problem with "calculating" $[\![v]\!]_g$ so that if g is countably/finitely branching then $[\![v]\!]_g$ is an HC-, respectively, HF-set.

However, in non-well-founded case, such as in the graph $\circlearrowleft$, there is a problem. We need sets like $\Omega = \{\Omega\}$. This leads to rather unusual *non-well-founded set theory* whose universe may contain sets serving as their own members, etc. [1, 3]. We confine ourselves to its subuniverse HCA of hereditarily-countable (or even hereditarily-finite) sets. Its main property, besides countability[2] of each set $x \in$ HCA, consists in the following "external" *anti-foundation axiom* (which is just a *finality* property in an appropriate category $\mathcal{CBG}$):

---

[1] The case of urelements or, more generally, of sets of *labelled* elements is important for application to databases; cf. the corresponding approach in [18]. Another, proof-theoretic application of urelements to some kind of bounded arithmetic is presented, e.g., in [20]

[2] More precisely, cardinality of x is countable *or* finite.

For any $\mathcal{CBG}$ g there exists a unique denotational semantics map $[\![\text{-}]\!]_g : |g| \to \mathrm{HCA}$ *satisfying the above identity (2) with set-theoretic denotations like* $\{\ldots\}$ *and the membership relation* $\in$ *understood in the sense of* $\langle \mathrm{HCA}, \in^{\mathrm{HCA}} \rangle$.

In particular, there exists a unique Ω in HCA. If g ranges only over $\mathcal{FG}$, respectively, over $\mathcal{FBG}$ then $[\![v]\!]_g$ will range over the corresponding subuniverses $\mathrm{HFA} \subset \mathrm{HFA}^\infty$ of HCA. [3] HFA consists of hereditarily-finite sets $x \in \mathrm{HCA}$ whose *transitive closure*

$$\mathsf{TC}(x) \rightleftharpoons \{z | z \in z_1 \in \ldots \in z_n \in x \quad \text{for some} \quad z_1, \ldots, z_n \in \mathrm{HCA}\}$$

is finite. HFA^∞ consists also of hereditarily-finite sets. However, their transitive closure may be infinite (just countable). Also note, that HFA is a countable universe, whereas HFA^∞ is of continual cardinality [6, 13]. Evidently, $x \in y \in \mathrm{HFA}$ implies $x \in \mathrm{HFA}$ for all $x, y \in \mathrm{HCA}$ and the same for HFA^∞ in place of HFA. Also sets from the subuniverses $\mathrm{HF} \subseteq \mathrm{HFA}$ and $\mathrm{HC} \subseteq \mathrm{HCA}$ correspond to vertices of finite and, respectively, countable well-founded graphs. It can be proved in the line of [1]

Theorem 1 *There exists the unique, up to isomorphism, universe* HCA *consisting only of finite or countable sets and satisfying anti-foundation axiom (and analogously also for* HFA^∞ *and* HFA*).*

□

Consider a binary relation $\approx^{g,g'} \subseteq |g| \times |g'|$ (which is an equivalence relation $\approx^g$ in the case $g = g'$)

$$u \approx^{g,g'} u' \rightleftharpoons [\![u]\!]_g = [\![u']\!]_{g'}.$$

It can be redefined in terms of the graphs g, g' only, without any mentioning (the semantics $[\![\text{-}]\!]$ in) a universe of sets like HCA, HFA or HFA^∞.

A set of pairs $R \subseteq |g| \times |g'|$ is called *bisimulation relation* between g and g' if it satisfies the implication

$$uRu' \Rightarrow \forall v \in^g u \exists v' \in^{g'} u'(vRv') \& \forall v' \in^{g'} u' \exists v \in^g u(vRv'). \tag{3}$$

Evidently, the class of bisimulation relations is closed under unions, compositions and taking the inverse relations.

Proposition 1 (Essentially due to [1].) *The largest bisimulation relation between g and g' coincides with $\approx^{g,g'}$. Moreover, the implication $\Rightarrow$ in (3) may be replaced by equivalence $\Leftrightarrow$ for the case of $\approx^{g,g'}$.*

Note that in the *finitely branching* case the relation $\approx^{g,g'}$ may be obtained as intersection

$$\approx^{g,g'} = \bigcap_{i=0}^{\infty} \approx_i^{g,g'} \tag{4}$$

[3]Other authors are using denotations HF_0 for HF, $\mathrm{HF}_{1/2}$ for HFA and HF_1 for HFA^∞.

where i ranges over natural numbers[4] and

$$u \approx_0^{g,g'} u' \rightleftharpoons \textbf{true} \quad \text{and}$$
$$u \approx_{i+1}^{g,g'} u' \rightleftharpoons \forall v \,\epsilon^g\, u \exists v' \,\epsilon^{g'}\, u'(v \approx_i^{g,g'} v') \& \forall v' \,\epsilon^{g'}\, u' \exists v \,\epsilon^g\, u(v \approx_i^{g,g'} v')$$

so that $\approx_0^{g,g'} \supseteq \approx_1^{g,g'} \supseteq \ldots \supseteq \approx^{g,g'}$. For the finite graphs this sequence evidently stabilizes on some *finite* step i: $\approx_i^{g,g'} = \approx^{g,g'}$, so that $\approx^{g,g'}$ is polytime *computable* wrt cardinalities of $|g|, |g'|$. Moreover, it follows that $\not\approx^{g,g'}$ and therefore $\approx^{g,g'}$ are definable in FO + LFP.

Remember, that FO+LFP [14, 10, 11, 23] is an extension of the language of first-order logic by the predicates definable by the least fixed point operator allowing recursive definitions of the form $P(\bar{x}) \Leftrightarrow \varphi(\bar{x}, P)$ with the predicate variable P having only positive occurrences in FO-formula φ. The least fixed point P may be obtained as a limit of the sequence $P_0(\bar{x}) \rightleftharpoons \textbf{false}$, $P_{n+1}(\bar{x}) \rightleftharpoons \varphi(\bar{x}, P_n)$ (which should be generalized to the case of any ordinal n for interpreting FO + LFP in an infinite model). Alternatively, we will get so called IFP construct (cf. e.g. [8]) by taking $P_{n+1}(\bar{x}) \rightleftharpoons P_n(\bar{x}) \vee \varphi(\bar{x}, P_n)$ for *arbitrary* φ, non-necessary positive in P.

For the finitely branching graphs with infinite number of vertices (4) gives only *semidecidability* of $\not\approx^{g,g'}$ (wrt oracles g, g' [5]).

A graph g is called *extensional*, i.e. $g \in \mathcal{EG}$, if the ordinary *extensionality axiom*

$$\forall u' \in u \exists v' \in v(u' = v') \& \forall v' \in v \exists u' \in u(u' = v') \Rightarrow u = v$$

holds in g for $\in$ interpreted as ϵ^g. *Strongly extensional graphs*, $\mathcal{SEG}$, satisfy the axiom

$$u \approx^g v \Rightarrow u = v.$$

For well-founded graphs these two versions of extensionality axiom are equivalent (by using ϵ-induction). Note, that HCA, HFA$^\infty$, HFA, HC and HF are strongly extensional wrt the corresponding membership relation $\in$.

According to the well-known Ackerman's bijection between HF and the natural numbers [2], it was defined in [16, 18] corresponding (lexicographical) linear ordering on HF as the unique relation $< \; = \; <_{\mathrm{HF}}$ satisfying the following equivalence

$$x < y \Leftrightarrow \exists u \in y \setminus x \forall w \in x \cup y(u < w \Rightarrow w \in x \cap y) \quad \text{or} \tag{5}$$
$$x < y \Leftrightarrow \exists u \in y \setminus x \forall w \in x \dot{-} y(u \not< w).$$

Here $x \dot{-} y \rightleftharpoons (x \cup y) \setminus (x \cap y)$. The same formula defines a linear ordering $<^g$ on the vertices of any extensional finitely branching[6] well-founded graph ($\mathcal{EFBWG}$ [7]) g with $\in$ interpreted

[4]In general case i should range in this equation over *ordinals*, instead of natural numbers. Then we have to write, e.g. $\approx_\omega^{g,g'} \rightleftharpoons \bigcap_{i \in \omega} \approx_i^{g,g'}$, and in finitely branching case we have $\approx^{g,g'} = \approx_\omega^{g,g'}$.

[5]Note, that a graph may be represented by a *computable* function $g : |g| \to \mathcal{P}(|g|)$. Let you imagine, e.g., that a vertex v is an address (URL) of a page of the World-Wide Web and invoking gv results in a set of "clickable" addresses on that page. Cf. also the corresponding brief note in the Introduction.

[6]Finite branching is an essential requirement to get *linear* order $<$.

[7]Note, that any $\mathcal{EFBWG}$ may be isomorphically embedded onto some transitive subset of HF.

as ϵ^g. Of course, $u \in y \setminus x$ is understood as abbreviation of $u \in y \& u \notin x$ and analogously for $w \in x \cap y$ so that the above definition may be considered as a first-order formula in the signature $\{\epsilon\}$. Let us call this ordering $<^g$ on g (and $<^{\mathrm{HF}}$ on HF) *canonical* one.

The above equivalence gives $<$ as a unique fixed point of the operator defined by its right-hand-side. It is not difficult to rewrite this as a first-order formula which is *positive* wrt $<$ (cf. (14) in Section 3) so that $<$ proves to be uniformly definable in FO + LFP over the class $\mathcal{EFBWG}$.

For arbitrary graph g (which may be non-well-founded) and its vertices let us define the following operations and relations

$$
\begin{aligned}
&[\![x]\!]_g^0 \rightleftharpoons \emptyset,\\
&[\![x]\!]_g^{i+1} \rightleftharpoons \{[\![y]\!]_g^i | y \,\epsilon^g\, x\},\\
&x <_k^g y \rightleftharpoons [\![x]\!]_g^k <^{\mathrm{HF}} [\![y]\!]_g^k.
\end{aligned}
\tag{6}
$$

Here all $[\![x]\!]_g^n$ are HF-sets for *finite* n (independently of the degree of graph branching!). [8]

Proposition 2

$$x \approx_k^g y \Leftrightarrow [\![x]\!]_g^k = [\![y]\!]_g^k \quad \textit{(even for } k \textit{ any ordinal; cf. the footnote 8).} \tag{7}$$

Proof It suffices to note that

$$
\begin{aligned}
&x \approx_0^g y \Leftrightarrow \mathbf{true} \quad \text{and}\\
&[\![x]\!]_g^{k+1} = [\![y]\!]_g^{k+1} \Leftrightarrow \forall u \,\epsilon\, x \exists v \,\epsilon\, y([\![u]\!]_g^k = [\![v]\!]_g^k) \& \forall v \,\epsilon\, y \exists u \,\epsilon\, x([\![u]\!]_g^k = [\![v]\!]_g^k).
\end{aligned}
$$

□

The following statements are immediate consequences of this Proposition and of the above definitions

$$
\begin{aligned}
&x \approx_{k+1}^g y \Rightarrow x \approx_k^g y, &(8)\\
&x \approx^g y \Leftrightarrow \forall i(x \approx_i^g y) \quad (i \text{ must range over ordinals if } g \notin \mathcal{FBG}),\\
&\qquad \Leftrightarrow \forall i([\![x]\!]_g^i = [\![y]\!]_g^i),\\
&[\![([\![x]\!]_g^k)]\!]_{\mathrm{HF}}^n = [\![x]\!]_g^{\min(k,n)},\\
&x <_0^g y \Leftrightarrow \mathbf{false},\\
&<_k^g \text{ is a strict linear preorder (slp-o) with corresponding equivalence relation}\\
&x \approx_k^g y \Leftrightarrow x \not<_k^g y \& y \not<_k^g x,\\
&x <_{k+1} y \Leftrightarrow \exists u \,\epsilon\, y[\forall v \,\epsilon\, x(u \not\approx_k v)\ \& &(9)\\
&\qquad \forall w \,\epsilon\, x \cup y(u <_k w \rightarrow \exists p \,\epsilon\, x \exists q \,\epsilon\, y(p \approx_k w \ \&\ q \approx_k w))].
\end{aligned}
$$

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[8] However this will not be the case if we extend the definition of $[\![x]\!]_g^{\alpha+1}$ for arbitrary non-limit ordinals as above and take for limit ordinals $[\![x]\!]_g^\delta \rightleftharpoons \langle [\![x]\!]_g^0, [\![x]\!]_g^1, \ldots, [\![x]\!]_g^\alpha, \ldots \rangle_{\alpha<\delta}$.
HF-set of the form $[\![x]\!]^i$ plays the role of i-th approximation to $[\![x]\!]$. This notion and some its version were considered by M.Boffa and, respectively, by M.Mislove, L.Moss and F.Oles (for protosets); cf. [13].

Define *rank* of any vertex of a graph g by taking

$$\mathrm{rk}(x) = 0 \quad \text{iff} \quad x \textbf{ is } \emptyset,$$
$$\mathrm{rk}(x) = \sup\{\mathrm{rk}(u) + 1 | u \in x\} \quad \text{and}$$
$$\mathrm{rk}(x) = \infty \quad \text{iff there exists an infinite chain} \quad \ldots \in x_{i+1} \in x_i \in \ldots \in x_1 \in x.$$

where x **is** $\emptyset \rightleftharpoons \neg\exists v \in x.\mathbf{true}$, i.e. there are no edges $x \to \ldots$. In general, $\mathrm{rk}(x)$ may be any, possibly infinite, ordinal or ∞. For finitely branching (and also for finite) graphs, $\mathrm{rk}\, x$ may be a natural number (i.e. a finite ordinal) or ∞. The inequality $\mathrm{rk}(x) \neq \infty$ characterizes the vertices x lying in the *well-founded part* of g.

Proposition 3 *For arbitrary graphs g*
(a) $[\![x]\!]^i_g = [\![x]\!]^{i+1}_g$ *iff* $\mathrm{rk}(x) \leq i$ *and*
(b) $[\![x]\!]^i_g \leq^{\mathrm{HF}} [\![x]\!]^{i+1}_g$.

Proof (a&b): The base case $i = 0$ is trivial both for (a) and (b). For the induction step it suffices to show (for (a) and (b), respectively) that

$$[\![x]\!]^{i+1} = [\![x]\!]^{i+2} \Leftrightarrow \forall u \in x([\![u]\!]^i = [\![u]\!]^{i+1}) \quad \text{and} \tag{10}$$
$$[\![x]\!]^{i+1} < [\![x]\!]^{i+2} \Leftrightarrow \exists u \in x([\![u]\!]^i < [\![u]\!]^{i+1}), \tag{11}$$

assuming that (a&b) holds for i. The converse implication for (10) and the direct implication for (11) follow by the definition of $[\![x]\!]^k$ and by induction hypothesis. For the rest, let us suppose that $[\![u]\!]^i = [\![u]\!]^{i+1}$ fails for some $u \in x$. Then, by induction hypothesis ($[\![u]\!]^i \leq [\![u]\!]^{i+1}$), we should have $[\![u]\!]^i < [\![u]\!]^{i+1}$. Take $u \in x$ with the largest such $[\![u]\!]^{i+1}$ (in the sense of $<^{\mathrm{HF}}$). This is possible because $[\![u]\!]^{i+1} \in [\![x]\!]^{i+2}$, for $x \in u$, and $[\![x]\!]^{i+2} \in \mathrm{HF}$. Then

$$[\![u]\!]^{i+1} \in [\![x]\!]^{i+2} \setminus [\![x]\!]^{i+1}$$

because, otherwise, we would have $[\![u]\!]^{i+1} \in [\![x]\!]^{i+1}$, $[\![u]\!]^i < [\![u]\!]^{i+1} = [\![v]\!]^i$ for some $v \in x$ and, by (8), $[\![u]\!]^{i+1} \neq [\![v]\!]^{i+1}$ and, therefore, $[\![u]\!]^{i+1} = [\![v]\!]^i < [\![v]\!]^{i+1}$, by induction hypothesis ($[\![v]\!]^i \leq [\![v]\!]^{i+1}$), what contradicts to the choice of u. Moreover, $[\![u]\!]^{i+1}$ witnesses that $[\![x]\!]^{i+1} < [\![x]\!]^{i+2}$ (cf. (5) and (9)):

$$\forall w \in x([\![u]\!]^{i+1} < [\![w]\!]^i \vee [\![u]\!]^{i+1} < [\![w]\!]^{i+1} \Rightarrow [\![w]\!]^i = [\![w]\!]^{i+1} \in [\![x]\!]^{i+1} \cap [\![x]\!]^{i+2})$$

by using induction hypothesis ($[\![w]\!]^i \leq [\![w]\!]^{i+1}$) and by the choice of u.

$\square$

3 Linear ordering of strongly extensional finitely branching graphs

We will consider in this section mainly finitely branching graphs, up to bisimulation. (The latter means that we actually deal with strongly extensional graphs.) However, some considerations are valid for arbitrary graphs.

Our goal is to define by reasonable means some global strict linear preordering on the vertices of arbitrary $\mathcal{FBG}$ g, up to bisimulation, and therefore linear ordering on the universes HFA and HFA$^\infty$.

A simple approach consists in defining for arbitrary vertices $x, y \in |g|$

$$x \lhd_k y \;\rightleftharpoons\; \exists i \leq k(x <_i y \& \forall j \leq i(x \approx_i y)) \quad \text{and} \tag{12}$$

$$\begin{aligned} x \lhd y \;&\rightleftharpoons\; \exists i(x <_i y \& \forall j \leq i(x \approx_i y))] \\ &\Leftrightarrow\; \exists k(x \lhd_k y) \end{aligned} \tag{13}$$

Evidently, $\lhd_k^g$ and $\lhd^g$ are slp-o with the corresponding equivalence relations $\approx_k^g$ and $\approx_\omega^g$ ($\approx_\omega^g = \approx^g$ for $g \in \mathcal{FBG}$). We actually need to have a *uniform* definition of $\lhd^g$ (or any other appropriate) slp-o in terms of FO+LFP (or IFP) over $g \in \mathcal{FBG}$. Note, that this can be done by imitating an appropriate segment of the natural numbers (for $g \in \mathcal{FG}$; cf. the variables i, j in the definition of $\lhd$) with the help of *stage comparison theorem* of Y.N. Moschovakis [14]. It was this way how the main result of this paper (on definability of a global linear order in any $\mathcal{SEFG}$ and Theorem 3) was first obtained. However, in this paper we present more elegant solution in Theorem 2 by defining differently some global slp-o $\prec^g$ and showing in Theorem 2 that $\prec^g$ is definable in FO+IFP (and even in FO+LFP, if g ranges over $\mathcal{FG}$) and, in Proposition 8, that $\prec = \lhd$.

To define $\prec$ we could try to use (only with a partial success) the following modification of the formula (5), now with the *positive* occurrence of $<$, so that both the least and the largest fixed points of the corresponding operator over binary relations may be considered. First, let us return to $<^{\mathrm{HF}}$ in this framework.

Proposition 4 *The canonical strict linear order* $<^{\mathrm{HF}}$ *(axiomatized by (5)) and its non-strict version* $\leq^{\mathrm{HF}}$ *on the well-founded universe* HF *are also the unique fixed points of:*

$$x < y \Leftrightarrow \exists u \in y \setminus x \forall w \in x \dot{-} y(w < u \vee w = u), \tag{14}$$

$$x \leq y \Leftrightarrow \forall u \in x \setminus y \exists w \in x \dot{-} y(u \leq w \& u \neq w) \quad \textit{and} \tag{15}$$

$$x \leq y \Leftrightarrow \forall u \in x \setminus y \exists w \in y \setminus x(u \leq w \& u \neq w).$$

□

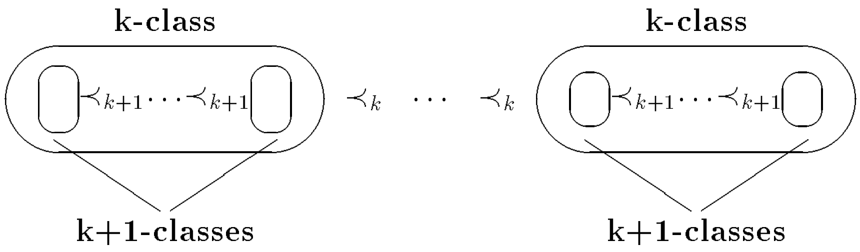
Note the following general lemma which is actually used here to switch between (14) and (15). Let $F : \mathcal{P}(X) \to \mathcal{P}(X)$ be any monotone operator with $\mathcal{P}(X)$ denoting the powerset of X. Define its *adjoint* operator $F^* : \mathcal{P}(X) \to \mathcal{P}(X)$ by $F^*(Q) \rightleftharpoons \overline{F(\bar{Q})}$ where $\bar{P} \rightleftharpoons X \setminus P$ for any $P \in \mathcal{P}(X)$. (In our case $X = |g|^2$ and $P, Q \in \mathcal{P}(|g|^2)$ are binary relations such as $<$.)

Lemma 1 *$F^{**} = F$, and $P = F(P)$ iff $\bar{P} = F^*(\bar{P})$ so that, moreover, P is the least (resp., largest) fixed point of F iff is $\bar{P}$ is the largest (resp., least) fixed point of F^*.*

□

Proposition 5 *For the non-well-founded cases of* HFA *and* HFA$^\infty$ *any fixed point of (14) restricted to* HF *coincides with $<^{\mathrm{HF}}$. However, the largest fixed point $<^a$ is not a linear order because we have $\Omega <^a \alpha$ and $\alpha <^a \Omega$ with $\Omega = \{\Omega\}$ and $\alpha = \{\alpha, \emptyset\}$ two different sets in* HFA*. The least fixed point $<_a$ is also not a linear order because α and Ω are incomparable wrt $<_a$. (Note, that $\emptyset <_a \Omega$ and $\{\alpha\} <_a \alpha$, so that both α and Ω are in* $\mathbf{dom}(<_a)$.*)*

We see that, if used straightforwardly, this approach does not work to define a linear order on *all* vertices of arbitrary (strongly) extensional finite branching graphs which may contain cycles/infinite paths. However, we may define the required strict linear preorder $\prec^g$ on any, even non-extensional $\mathcal{FBG}$ g, up to bisimulation $\approx^g$, by ordering step-by-step, for $k = 0, 1, \ldots$, the equivalence classes, called *k-classes*, corresponding to the equivalence relations $\approx^g_k$ according to the following picture. (Actually, the following definition of $\prec^g_k$, $k = 0, 1, 2, \ldots$, works for *arbitrary* graphs.)



First note, that there exists only one 0-class, just the set of all vertices $|g|$, and only finite number of k-classes for each k. So, we let $x \prec_0 y \rightleftharpoons \mathbf{false}$ for all $x, y \in |g|$. Let us suppose that we have defined a strict linear order on the k-classes and therefore corresponding strict linear preorder $\prec_k$ on $|g|$. Then we can order $k+1$-classes by a strict linear order $\prec_{k+1}$ inside any k-class *lexicographically* wrt previously defined order $\prec_k$. I.e. for any $x \approx_k y$ (inside one k-class) define recurrently linear *pre*order relation on the vertices (instead of classes)

$$
\begin{aligned}
x \prec_{k+1} y \rightleftharpoons & \exists u \in y[\forall v \in x(u \not\approx_k v) \; \& \\
& \forall w \in x \cup y(u \prec_k w \to \exists p \in x \exists q \in y(p \approx_k w \;\&\; q \approx_k w))].
\end{aligned}
$$

In the case of $x \not\approx_k y$, i.e. of different k-classes, take $x \prec_{k+1} y \rightleftharpoons x \prec_k y$.

It is evident that $x \prec_{k+1} y$ does not depend on the choice of representatives x and y inside their $k+1$ classes so that this is indeed a strict linear order on these classes: $x \approx_k y \Leftrightarrow x \not\prec_k y \& y \not\prec_k x$. We also have $\approx_0 \supseteq \approx_1 \supseteq \ldots$ and $\prec_0 \subseteq \prec_1 \subseteq \ldots$. In the case of finite g this process stabilizes: k-classes $=$ $k+1$-classes and $\prec_k = \prec_{k+1}$ for some k. This results in some strict linear preorder $\prec^g$ on $|g|$ corresponding to the largest bisimulation $\approx^g$. In the general case of arbitrary graph g take

$$\prec^g_\omega \rightleftharpoons \bigcup_{k=0}^{\infty} \prec^g_k \quad \text{i.e.} \quad x \prec^g_\omega y \rightleftharpoons \exists k(x \prec^g_k y).$$

It is easy to show that $\prec^g_\omega$ is slp-o such that $x \approx^g_\omega y \Leftrightarrow x \not\prec^g_\omega y \& y \not\prec^g_\omega x$. In the case of $\mathcal{FBG}$ we have

$$\approx^g = \approx^g_\omega \quad \text{and define}$$
$$\prec^g \rightleftharpoons \prec^g_\omega .$$

We can resume the above iterative definition for *arbitrary* $x, y \in |g|$ (with the superscript g omitted) as

$$\begin{aligned} & x \prec_{k+1} y \rightleftharpoons x \prec_k y \vee \\ & \quad \{x \approx_k y \ \& \ \exists u \in y[\forall v \in x(u \not\approx_k v) \ \& \\ & \quad \forall w \in x \cup y(u \prec_k w \to \exists p \in x \exists q \in y(p \approx_k w \ \& \ q \approx_k w)]\} \end{aligned}$$

where expressions like $y \approx_k y$ may be replaced by $x \not\prec_k y \& y \not\prec_k x$. This means that we have proved simultaneously

Theorem 2 *The strict linear preorder $\prec^g$ (corresponding to the bisimulation equivalence relation $\approx^g$) on any $\mathcal{FBG}$ g defined above is inflationary fixed point of the operator*

$$\begin{aligned} & \varphi(R, x, y) \rightleftharpoons (\neg R(x, y) \& \neg R(y, x)) \& \exists u \in y[\forall v \in x(R(u, v) \vee R(v, u)) \ \& \\ & \quad \forall w \in x \cup y(R(u, w) \to \\ & \quad \exists p \in x \exists q \in y(\neg R(p, w) \ \& \ \neg R(w, p) \ \& \ \neg R(q, w) \ \& \ \neg R(w, q))]. \end{aligned}$$

□

It is clear that for any $\mathcal{SEFBG}$ g (i.e. such $\mathcal{FBG}$ that $\approx^g$ coincides with the equality on g) the relation $\prec^g$ is a strict linear order on $|g|$. In particular, we have defined strict linear orders $\prec^{\mathrm{HFA}^\infty}$ and $\prec^{\mathrm{HFA}}$ ($= \prec^{\mathrm{HFA}^\infty}|_{\mathrm{HFA}}$).

It is well known [8] that the expressive power of $\mathrm{FO} + \mathrm{IFP}$ over *finite* structures is the same as $\mathrm{FO} + \mathrm{LFP}$. Thus we get:

Corollary 1 *The strict linear preorder $\prec^g$ over the class of all finite graphs (or strict linear order $\prec^g$ over the class of all strongly extensional finite graphs) is uniformly definable in* $\mathrm{FO} + \mathrm{LFP}$.

□

By definability of PTIME in terms of $\mathrm{FO} + \mathrm{LFP}$ + a linear order (cf. the Introduction), this implies positive answers to the open questions from [19, 20] (cf. also Appendix for the corresponding material from [19]):

Theorem 3 *(a) The class of global predicates definable in* FO + LFP *over strongly extensional finite graphs ($\mathcal{SEFG}$) and therefore the class of graph transformers $\mathcal{SEFG} \to \mathcal{FG}$ definable in* FO + LFP *coincide with those PTIME-computable.*

(b) It follows that the class of Δ_{D}-definable operations over HFA *coincides with the class of all* PTIME*-computable operations with respect to the graph encoding of sets via* $[\![\text{-}]\!]$ (cf. also Appendix and Theorem 5 there).

□

It is easy to prove (by using induction on k) the following *coherency* property of $\prec_k$, $\approx_k$, $\prec$ and $\approx$:

Proposition 6 *For any two $\mathcal{FBG}$ $(g; x, y)$ and $(g'; x', y')$, with two distinguished vertices in each,*

$$x \approx^{g,g'} x' \;\&\; y \approx^{g,g'} y' \Rightarrow (x \approx^g y \Leftrightarrow x' \approx^{g'} y')\&(x \prec^g y \Leftrightarrow x' \prec^{g'} y')$$

and the same for $\approx_k$ and $\prec_k$.

Corollary 2 *The global slp-o $\prec$ over the class $\mathcal{FG}$ or countable $\mathcal{FBG}$ (or, equivalently, global slo $\prec$ over the class $\mathcal{SEFG}$ or countable $\mathcal{SEFBG}$) induces a unique strict linear order on the entire universe* HFA *or, respectively,* HFA^∞ *via decoration operation as follows:*

$$[\![x]\!]_g \prec^{\mathrm{HFA}} [\![y]\!]_g \rightleftharpoons \exists(g; x, y) \in \mathcal{FG}(x \prec^g y),$$
$$[\![x]\!]_g \prec^{\mathrm{HFA}^\infty} [\![y]\!]_g \rightleftharpoons \exists(g; x, y) \in \mathcal{FBG}(x \prec^g y).$$

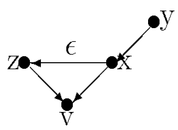
Moreover, this order coincides with $\prec^{\mathrm{HFA}}$ or, respectively, $\prec^{\mathrm{HFA}^\infty}$ defined previously.

□

Since HF is a subuniverse of HFA we have now two linear orders for HF, defined in different ways: $<^{\mathrm{HF}}$ and $\prec|_{\mathrm{HF}}$.

Proposition 7 *The orders $<$ and $\prec$ are different on the universe* HF.

Proof It suffices to demonstrate an acyclic extensional graph $g = \langle |g|, \epsilon^g \rangle$ and two its vertices x and y, such that $x < y$, but $y \prec x$:



We have $v \in x \in y$, $v \in z$, $z \in x$ and therefore $x < y$, because for the canonical order the following property holds: $t \in s \Rightarrow t < s$. Further, we have $\{v\} \prec_1 \{x, y, z\}$ for 1-classes. But the next step gives $v \prec_2 z \prec_2 y \prec_2 x$ and therefore $y \prec x$.

$\square$

Let us also present explicitly the slp-o $\prec_1$ and $\prec_2$:

$$\begin{aligned}
&x \prec_1 y \Leftrightarrow x \textbf{ is } \emptyset \,\&\, y \textbf{ is not } \emptyset\\
&x \textbf{ is } \emptyset \,\&\, y \textbf{ is } \emptyset \Rightarrow x \approx_k y \quad \text{for all } k\\
&x \textbf{ is not } \emptyset \,\&\, y \textbf{ is not } \emptyset \Rightarrow\\
&\quad \{x \prec_2 y \Leftrightarrow \exists u \in y[u \textbf{ is not } \emptyset \,\&\, \forall v \in x(v \textbf{ is } \emptyset)] \vee\\
&\quad \exists u \in y[u \textbf{ is } \emptyset \,\&\, \forall v \in x(v \textbf{ is not } \emptyset) \;\&\; (\exists p \in x \exists q \in y(p \textbf{ is not } \emptyset \;\&\; q \textbf{ is not } \emptyset)]\}.
\end{aligned}$$

Note, that the preorders $\prec_k$ and $\prec$ may be characterized also as lexicographical wrt HF-approximations $[\![x]\!]^n$ (cf. (6), (7), (12) and (13) above) as follows.

Proposition 8 *$x \prec_k y \Leftrightarrow x \lhd_k y \rightleftharpoons \exists i \le k(x <_i y \& \forall j \le i(x \approx_i y))$ and therefore $\prec = \lhd$ in any $\mathcal{FBG}$.*

Proof Let us define for each n slp-o $x \lhd'_{n+1} y$ as

$$\begin{aligned}
x \lhd'_{n+1} y \rightleftharpoons \exists u \in y[\forall v \in x(u \not\approx_n v) \;\&\\
\forall w \in x \cup y(u \lhd_n w \to \exists p \in x \exists q \in y(p \approx_n w \;\&\; q \approx_n w))].
\end{aligned} \tag{16}$$

Note, that both $x \lhd_{n+1} y$ and $x \lhd'_{n+1} y$ generate the same equivalence relation $\approx_{n+1}$. Evidently, $x \lhd_n y \Rightarrow x \lhd_{n+1} y$ for all $x, y \in |g|$ and $n = 0, 1, \ldots$. Therefore, to prove Proposition we must show that for all $x \approx_n y$

$$x \lhd_{n+1} y \Leftrightarrow x \lhd'_{n+1} y \quad \text{i.e. that} \quad x <_{n+1} y \Leftrightarrow x \lhd'_{n+1} y.$$

Let, for the contrary, $x \lhd'_{n+1} y$, but $y <_{n+1} x$. According to (9), we have equivalence (for *all* x, y)

$$\begin{aligned}
y <_{n+1} x \Leftrightarrow \exists v \in x[\forall u \in y(v \not\approx_n u) \;\&\\
\forall w \in x \cup y(v <_n w \to \exists p \in x \exists q \in y(p \approx_n w \;\&\; q \approx_n w))].
\end{aligned} \tag{17}$$

Then we have, by (16) and (17), some $u \in y$ and $v \in x$ such that $u \not\approx_n v$, $\forall v \in x(u \not\approx_n v)$, $\forall u \in y(v \not\approx_n u)$, $\forall w \in x \cup y(v <_n w \to \exists p \in x \exists q \in y(p \approx_n w \;\&\; q \approx_n w))$, and $\forall w \in x \cup y(u \lhd_n w \to \exists p \in x \exists q \in y(p \approx_n w \;\&\; q \approx_n w))$.

It follows that $v \lhd_n u$ because, otherwise, $u \lhd_n v$ would imply $\exists q \in y(v \approx_n q)$ and contradict to $\forall u \in y(v \not\approx_n u)$. Analogously, $u <_n v$. The last two formulas imply $v \lhd_{n-1} u$ and, hence, $v \not\approx_{n-1} u$, what contradicts to $x \approx_n y$.

The case when $x <_{n+1} y$, but $y \lhd'_{n+1} x$ is reduced to a contradiction in the same way. Therefore, $x \lhd'_{n+1} y \Leftrightarrow x <_{n+1} y$ holds (for $x \approx_n y$), as required.

$\square$

4 Application of DLW approach to strongly extensional graphs

The above representation of the linear order $\prec$ by lexicographical ordering of the refining equivalence classes was partially inspired by the paper [4] of A. Dawar, S. Lindell and S. Weinstein. As we will see, this 'DLW' approach itself may be used in our context of $\mathcal{SEFG}$ (and even $\mathcal{SEFBG}$), but leads to *non-coherent* linear orders in our sense, i.e. relatively to bisimulation $\approx^g$ (cf. Proposition 6). However, their orders are coherent wrt an equivalence relation $\equiv$ (cf. below) which they consider instead of $\approx$ used in this paper. (The relation $\approx$ is especially important to???e us in connections with HFA, etc.).

We begin a formal comparison with the following simple technical

Proposition 9 *Consider any binary relation $\bar{x}S\bar{y}$ between k-tuples $\bar{x}$ and $\bar{y}$ in arbitrary graph g which satisfies the following condition for all $\bar{x}, \bar{y} \in |g|$:*

$$\bar{x}S\bar{y} \Rightarrow \bar{x} \cong \bar{y} \,\&\, \bigwedge_{i=1}^{k} (\forall x_i \exists y_i (\bar{x}S\bar{y}) \,\&\, \forall y_i \exists x_i (\bar{x}S\bar{y})) \tag{18}$$

where $\bar{x} \cong \bar{y}$ means that $\bar{x}$ and $\bar{y}$ are "isomorphic", i.e. indistinguishable in g by atomic formulas in the language $\{\epsilon^g, =\}$. Then, for $k \geq 2$, $xRy \rightleftharpoons x \ldots xSy \ldots y$ is a bisimulation relation on g.

Proof Let for simplicity $k = 2$. We have to show

$$xxSyy \Rightarrow \forall u \,\epsilon\, x \exists v \,\epsilon\, y(uuSvv) \,\&\, \forall v \,\epsilon\, y \exists u \,\epsilon\, x(uuSvv).$$

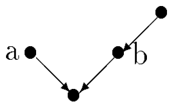
So, assume $xxSyy$ and infer the first conjunct $\forall u \,\epsilon\, x \exists v \,\epsilon\, y(uuSvv)$. Let $u \,\epsilon\, x$ and find $v \,\epsilon\, y$ such that $uuSvv$. First, (18) gives existence of such v that $uxSvy$ and $ux \cong vy$ and therefore $u \,\epsilon\, x$ gives $v \,\epsilon\, y$. Analogously, there exists v' such that $uuSvv'$ and $uu \cong vv'$ and hence $u = u$ gives $v = v'$, i.e. $uuSvv$. The second conjunct is proved symmetrically.

□

The largest S satisfying the condition (18) (for any fixed k) is an equivalence relation $\equiv^g$ on k-tuples.[9] Evidently $\equiv$ satisfies (18) even with $\Rightarrow$ replaced by $\Leftrightarrow$ and is definable in FO + LFP. Of course, the same definition of an analogous relation $\equiv^M$ makes sense not only for graphs, but also for *arbitrary* first-order structures M with finite relational signature; cf. [4, 9].

So, we have $xx \equiv^g yy \Rightarrow x \approx^g y$ (and also $xx \equiv^g_{2i} yy \Rightarrow x \approx^g_i y$). However, the inverse implication fails, as the following simple example of a graph g shows:

[9] Note, that there are two different equivalent reformulations of this relation. One is in terms of (k-) pebble games (with the quantifier alternations in (18) corresponding to moves of two players "$\forall$" and "$\exists$"), and another one is in terms of indistinguishability of the tuples $\bar{x}$ and $\bar{y}$ by the infinitary $L^k_{\infty\omega}$-formulas (or finitary L^k-formulas for the case $\mathcal{FG}$) involving only k different variables; cf. [4, 9].



$a \approx^g b$, but $aa \not\equiv^g bb$.

It follows that the $\equiv^g$-classes (respectively, $\equiv^g_{2i}$-classes) on pairs xx are finer than those defined by $\approx^g$ (respectively, by $\approx^g_i$) on the corresponding single elements x. In particular,

Corollary 3 *Strongly extensional graphs g are 2-rigid in the sense that $xx \equiv^g yy \Rightarrow x = y$ for all vertices x and y (i.e. corresponding $\equiv^g$-classes on the pairs xx are just singletons $\{xx\}$).*[10]

□

A. Dawar, S. Lindell and S. Weinstein [4] showed how to define uniformly in FO + IFP (and therefore in FO+LFP) a strict linear order on the equivalence classes of k-tuples relative to $\equiv^M$ for arbitrary finite first-order relational structures M with finite signature. The idea of the step-by-step linear ordering of refining the corresponding equivalences classes $\equiv^M_i$ is used (which works also for infinite M and each finite i).

For the classes of finite k-rigid structures this gives global FO+LFP-definable linear order on the elements of the structures. Also, for $\mathcal{SEFBG}$ g a global slo $\ll^g$ is definable by this method in FO + IFP.

Thus, their result together with Corollary 3 also gives the positive answer to the question of our interest, on defining in FO + LFP any global linear order on the vertices of arbitrary finite strongly extensional graphs.

In contrast to our approach, there are *several* 0-classes of the corresponding equivalence relation $\equiv^M_0$. If M is a graph g then there are ≤ 18 classes, or even ≤ 4 classes if g is acyclic. Each ordering $\ll_0$ of these 0-classes generates corresponding orderings $\ll_k$ of $\equiv_k$-classes and eventually an ordering $\ll_\omega$ of $\equiv_\omega$-classes, as in Section 3.

The following theorem shows that independently on the ordering $\ll_0$ of $\equiv_0$-classes the resulting global linear order $\ll^g$ over the vertices of strongly extensional *finite* graphs is not coherent relatively to bisimulation and therefore does not induce in this way a strict linear order on the entire *infinite* universe HFA, and even on HF, in contrast to $\prec^g$.

Theorem 4 *For any ordering $\ll_0$ of the 0-classes, there exist two finite strongly extensional (even acyclic) graphs with two distinguished vertices in each $(g; x, y)$ and $(g'; x', y')$ such that $[\![x]\!]_g = [\![x']\!]_{g'}$ and $[\![y]\!]_g = [\![y']\!]_{g'}$, but $x \ll^g y$ and $y' \ll^{g'} x'$.*

Proof consists in the demonstration of the required pair of graphs for every of 4!=24 possible orderings of 0-classes for acyclic graphs and is ommited.

□

Note. Such incoherent behavior of the linear orders $\ll^g$ for $\mathcal{SEFG}$ is not unexpected. The ordering of any two vertices x and y of a graph g depends here not only on vertices,

[10]We recommend to the reader to prove corresponding version of this corollary in terms of a 2-pebble game.

encoding the 'members' of 'members' of etc. of x and y. Other vertices and edges may have influence on $\ll^g$. This is a kind of "context dependency" of the ordering $x \ll^g y$.

For the class $\mathcal{SEFBG}$ the above DLW approach also gives a global slo $\ll^g$ definable in FO+IFP. Therefore, we have $\ll^{\mathrm{HFA}}$ and $\ll^{\mathrm{HFA}^\infty}$ for HFA and HFA$^\infty$, considered as $\mathcal{SEFBG}$, as we had $\prec^{\mathrm{HFA}}$ and $\prec^{\mathrm{HFA}^\infty}$ in Section 3. It remains to understand how all these orders are related. Also, is it possible to obtain $\ll^{\mathrm{HFA}}$ and $\ll^{\mathrm{HFA}^\infty}$ as induced by *any* FO + IFP (or LFP) definable (and hence PTIME computable) global linear order on the graphs in $\mathcal{SEFG}$ via $[\![\text{-}]\!]$?

References

[1] Aczel, P. Non-Well-Founded Sets. CSLI Lecture Notes, No. 14, 1988

[2] Barwise, J.K.: Admissible Sets and Structures (Springer, Berlin, 1975).

[3] Barwise, J.K., and Moss, L., Vicious Circles: on the mathematics of circular phenomena, CSLI Lecture Notes, 1996

[4] Dawar A., Lindell S., Weinstein S., Infinitary Logic and Inductive Definability over Finite Structures, Information and Computation, v. 119, 2, pp. 160-175, 1995

[5] Fernando, T., A Primitive Recursive Set Theory and AFA: On Logical Complexity of the Largest Bisimulation, Report CS-R9213 ISSN 0169-118XCWI P.O.Box 4079, 1009 AB Amsterdam, Netherlands.

[6] Fernando, R.T.P., On Substitutional Recursion over Non-Well-Founded Sets, LICS'89, 1989, 273–282.

[7] Gurevich, Y.: 1983, Algebras of feasible functions, in: *Proc. 24th IEEE Conf. on Foundations of Computer Science*, pp. 210–214.

[8] Gurevich, Y., Shelah, S., Fixed-point extensions of first-order logic , Annals of Pure and Applied Logic 32 (1986) 265-280

[9] Hodkinson, I., Finite variable logics, Bulletin of the EATCS, 51, october 1993, pp. 111-140

[10] Immerman, N., Relational queries computable in polynomial time, in *Proccedings of 14th ACM Symposium on Theory of Computation, 1982*, pp. 147 – 152

[11] Immerman, N., Relational queries computable in polynomial time, in *Information and Control*, 68, 86–104, 1986

[12] Livchak, A.B.: 1982, Languages of polynomial queries, in: *Raschet i optimizacija teplotehnicheskih ob'ektov s pomosh'ju EVM* , Sverdlovsk, p. 41 (in Russian).

[13] M. Mislove, L. Moss, F. Oles, Non-Well-Founded Sets Modeled as Ideal Fixed Points, Information and Computation, v.93, pp. 16-54, 1991

[14] Moschovakis, Y.N.: 1974, *Elementary Induction on Abstract Structures,* Amsterdam, North-Holland.

[15] Sazonov, V.Yu.: Polynomial computability and recursivity in finite domains. Elektronische Informationsverarbeitung und Kybernetik. **16**, N7 (1980) 319–323

[16] Sazonov, V.Yu.: Bounded set theory, polynomial computability and Δ-programming. Application aspects of mathematical logic. Computing systems **122** (1987) 110–132 (In Russian) Cf. also a short English version of this paper in: Lect. Not. Comput. Sci. **278** Springer (1987) 391–397

[17] Sazonov, V.Yu.: Bounded set theory and inductive definability. Abstracts of Logic Colloquium'90. JSL **56** Nu.3 (1991) 1141–1142

[18] Sazonov, V.Yu.: Hereditarily-finite sets, data bases and polynomial-time computability. TCS **119** Elsevier (1993) 187–214

[19] Sazonov, V.Yu.: A bounded set theory with anti-foundation axiom and inductive definability. Computer Science Logic, 8th Workshop, CSL'94 Kazimierz, Poland, September 1994, Selected Papers. Lecture Notes in Computer Science **933** Springer (1995) 527–541.[11]

[20] Sazonov, V.Yu.: On Bounded Set Theory (Invited talk), in: M.L. Dalla Chiara, et al. (eds.), Logic and Scientific Methods, Volume One of the Tenth International Congress of Logic, Methodology and Philosophy of Sciences, Florence, August 1995, Kluwer Academic Publishers, Dordrecht, 1997, 85–103.

[21] Sazonov, V.Yu., Lisitsa, A.P.: Δ-languages for sets and sub-PTIME graph transformers. Database Theory – ICDT'95, 5th International Conference, Prague, Czech Republic, January 1995, Proceedings. Lecture Notes in Computer Science **893** Springer (1995) 125–138

[22] Lisitsa, A.P., Sazonov V. Yu.: Δ-languages for sets and LOGSPACE-computable graph transformers, Theoretical Computer Science, tentatively in Vol. 175, 1997, 42 pp.

[23] Vardi, M., Complexity of relational query languages, in *Proceedings of 14th Symposium on Theory of Computation, 1982*, pp. 137–146

[11]The papers [19, 20, 22] are also available at
`http://www.botik.ru/PSI/AIReC/logic/` or `ftp://ftp.botik.ru/rented/logic/papers/`.
The present paper is available at
`http://dimacs.rutgers.edu/TechnicalReports/1997.html`.

Appendix: Set theory BSTA and its language Δ_{D}

According to [19, 20], *Bounded Set Theory with Anti-foundation Axiom*, BSTA, is based on the following Δ_{D}-*language*

$$
\begin{aligned}
\Delta_{\mathsf{D}}\text{-formulas} \quad &::= \quad a \in b \parallel a = b \parallel \varphi \& \psi \parallel \varphi \vee \psi \parallel \neg\varphi \parallel \forall x \in a.\varphi \parallel \exists x \in a.\varphi \\
\Delta_{\mathsf{D}}\text{-terms} \quad &::= \quad \text{set-variables } p, q, x, y, \ldots \parallel \{a, b\} \parallel \bigcup a \parallel \mathsf{D}(t, s) \parallel \mathsf{TC}(s) \parallel \\
&\qquad \{t(x) | x \in a \& \varphi(x)\} \parallel \textsf{the-least}\; p.(p = \{x \in a | \varphi(x, p)\})
\end{aligned}
$$

where a, b, t, s are Δ_{D}-terms and φ, ψ are Δ_{D}-formulas, set-variables x, p are not free in a and all occurrences of set variable p in Δ_{D}-formula φ are only *of the form* '- $\in p$', *positive* and *not inside* of any complex subterm of φ. The condition on p guarantees that φ is monotonic under the set inclusion on this set variable and therefore (at least in the framework of the classical set theory) the least set p such that $p = \{x \in a | \varphi(x, p)\}$ must exist. In fact, we will consider also the closure of Δ_{D}-formulas under $\&, \vee, \neg$ and unbounded $\forall$ and $\exists$ with the ordinary definition of the bounded quantifiers of Δ_{D} via unbounded ones.

For example, the following are Δ_{D}-formulas and Δ_{D}-terms: $a \subseteq b \rightleftharpoons \forall x \in a.(x \in b)$, $\mathsf{Tran}(y) \rightleftharpoons \forall u \in y(u \subseteq y)$, $\langle x, y\rangle \rightleftharpoons \{\{x\}, \{x, y\}\}$. Note, that for any set R of the universe we will have $\langle x, y\rangle \in R \Rightarrow x, y \in \bigcup\bigcup R$. Therefore, we may define in terms of Δ_{D} that R is a *bisimulation relation* relative to $\in$ as follows:

$$
\begin{aligned}
Bis_{\in}(R) \rightleftharpoons \forall x, y \in \textstyle\bigcup\bigcup R.(xRy \Rightarrow \\
(\forall x' \in x \exists y' \in y. x'Ry') \;\&\; (\forall y' \in y \exists x' \in x. x'Ry')).
\end{aligned}
$$

Non-logical axioms of BSTA (which also explain the intended *meaning* of the above Δ_{D}-constructs) are the following ones:

Extensionality: $a \subseteq a' \;\&\; a' \subseteq a \Rightarrow a = a'$;

Pair: $t \in \{a, b\} \Leftrightarrow t = a \vee t = b$;

Union: $t \in \bigcup a \Leftrightarrow \exists x \in a.t \in x$;

Δ_{D}-Image: $s \in \{t(x) | x \in a \;\&\; \varphi(x)\} \Leftrightarrow \exists x \in a.(\varphi(x) \;\&\; s = t(x))$;

Recursive Δ_{D}-Separation: If $p_0 = \textsf{the-least}\; p.(p = \{x \in a | \varphi(x, p)\})$ then

$$p_0 = \{x \in a | \varphi(x, p_0)\} \;\&\; (\{x \in a | \varphi(x, p)\} \subseteq p \Rightarrow p_0 \subseteq p);$$

Transitive closure:

$$
\begin{aligned}
&x \subseteq \mathsf{TC}(x), \;\; \mathsf{Tran}(\mathsf{TC}(x)), \\
&x \subseteq y \Rightarrow \mathsf{TC}(x) \subseteq \mathsf{TC}(y) \quad \text{and} \quad \mathsf{Tran}(y) \Rightarrow \mathsf{TC}(y) \subseteq y;
\end{aligned}
$$

Anti-Foundation Axioms:

(AFA_1) $\mathsf{D}(G, y) = \{\mathsf{D}(G, x) | \langle x, y\rangle \in G\}$ (compare with the definition of $[\![\text{-}]\!]$ in (2)),
(AFA_2 or *Strong Extensionality*) $Bis_{\in}(R)$ & $xRy \Rightarrow x = y$;

Δ_{D}-Collection: $\forall x \in a \exists y.\varphi(x, y) \Rightarrow \exists z \forall x \in a \exists y \in z.\varphi(x, y)$.

where a, b, t, s are any Δ_{D}-terms and φ is any Δ_{D}-formula. Note, that Δ_{D}-Collection is the only axiom of BSTA which itself is not a Δ_{D}-formula.

It was defined in [19], in the lines of [1] (cf. also a related approach for Primitive Recursive Set Theory [5]), a universe of sets $\mathcal{VA}$ of set theory BSTA on the base of any given universe of sets $\mathcal{V}$ satisfying a simple subtheory of BSTA, called there KPR_0 [12]. In particular, we may obtain in this way the universes $\mathcal{VA}$ = HFA or HCA of set theory BSTA from $\mathcal{V}$ = HF or HC, respectively.

Let $\mathcal{V}$ be any universe of sets of a set theory, like BSTA, or BST, or KPR_0 and ν : Codes $\rightarrow$ $\mathcal{V}$ be any surjective encoding of $\mathcal{V}$ by some kind of Codes. We say that an operation $q : \mathcal{V} \rightarrow \mathcal{V}$ over $\mathcal{V}$-sets is *computable* wrt ν in a specific (or may be generalized) sense of computability (say, PTIME-computability) if the following diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{q} & \mathcal{V} \\ {\scriptstyle\nu}\uparrow & & \uparrow{\scriptstyle\nu} \\ \text{Codes} & \xrightarrow{Q} & \text{Codes} \end{array}$$

commutes for some (appropriate) computable transformation Q between Codes. In other words, $q(\nu(c)) = \nu(Q(c))$ holds for all $c \in$ Codes. For example, for the case of PTIME-computability over Codes we denote corresponding classes of set-theoretic operations as $\mathcal{PTIME}_\nu$ to stress that this notion of PTIME-computability over $\mathcal{V}$ *depends* on an encoding ν.

In the case of HFA we choose *graph encoding* $\nu = [\![\text{-}]\!] : \mathcal{FG} \rightarrow$ HFA where (finite) graphs in $\mathcal{FG}$ serve as Codes and are supposed to have a *distinguished vertex* in each (such graphs are also called *pointed*) so that $[\![\text{-}]\!]$ is to be applied to this vertex.

The same notion of graph encoding may be formulated (generalized) for any $\mathcal{VA}$ (say, for HCA) and for the class of all graphs (codes) taken from the corresponding universe $\mathcal{V}$ (say, from HC) satisfying KPR_0. In any case, if Codes are graphs then Q in the above diagram should be considered as a (computable in a sense) graph transformer. In particular, we may consider transformers *definable* in terms of a logical language like FO + LFP.

Theorem 5 (Cf. [19, 20]) *Δ_{D}-definable (= provably-total Σ-definable in* BSTA*) operations and predicates q over a universe $\mathcal{VA}$ for* BSTA *(corresponding to some $\mathcal{V} \models \mathsf{KPR}_0$) coincide*

[12] KPR_0 is the Kripke-Platek set theory with Recursive Δ-Separation and without any Foundation Axiom. It is just BSTA *minus* operations D and TC and corresponding axioms. If $\mathcal{V} \models \mathsf{KPR}_0$ then $\mathcal{VA} \rightleftharpoons \langle \mathcal{G}, =_0, \in_0\rangle/_{=_0}$ with $\mathcal{G} \subseteq \mathcal{V}$ the class of *all* (pointed) graphs in the *internal* sense of $\mathcal{V}$; cf. the details in [19].

with those operations and predicates computable by graph transformers $Q : \mathcal{G} \to \mathcal{G}$ *or* $Q : \mathcal{G} \to \{\mathbf{true}, \mathbf{false}\}$ *(*$\mathcal{G} \subseteq \mathcal{V}$*) definable in the language* FO + LFP *wrt the graph encoding* $\nu = [\![\text{-}]\!] : \mathcal{G} \to \mathcal{VA}$.

□

Finally, let us explain how Theorems 5 and 3 (a) imply Theorem 3 (b).

In our case $\mathcal{V} = \mathrm{HF}$, $\mathcal{VA} = \mathrm{HFA}$ and $\mathcal{G} = \mathcal{FG}$. Given any transformer $Q : \mathcal{FG} \to \mathcal{FG}$, define its "inessential" variation $\tilde{Q} : \mathcal{FG} \to \mathcal{FG}$ as the composition of $g \mapsto \tilde{g} \rightleftharpoons g/_{\approx^g} : \mathcal{FG} \to \mathcal{SEFG}$ and $Q|_{\mathcal{SEFG}} : \mathcal{SEFG} \to \mathcal{FG}$. Evidently, this replacement of Q by $\tilde{Q}$ preserves commutativity of the above diagram so that both of them compute the same set-theoretic operation $q : \mathrm{HFA} \to \mathrm{HFA}$ (if any). If Q is PTIME-computable transformer then $Q|_{\mathcal{SEFG}}$ is definable in FO + LFP in terms of $\{\in, =\}$ (with $\in$ and $=$ interpreted, respectively, as ϵ^g and the ordinary identity on g), by Theorem 3 (a). Therefore $\tilde{Q}$ is also definable in FO + LFP because we may consider, instead of $\tilde{g}$, FO + LFP-definable relations

$$\approx^g \quad \text{and} \quad \in^g \rightleftharpoons x \in^g y \rightleftharpoons \exists x' \,\epsilon^g\, y (x' \approx^g x)$$

on g (so that $\approx^g$ ia a *congruence* wrt $\in^g$) and substitute them in the description of $Q|_{\mathcal{SEFG}}$ in place of $=$ and $\in$, respectively. Therefore Theorem 5 (with $\tilde{Q}$ in the role of Q) and Theorem 3 (a) entail Theorem 3 (b).

The papers [19, 20] contain also some interpretability and conservativity results for various versions of $\mathsf{BST(A)}$, KPR_0 and a *Logic of Inductive Definitions* LID based on the language FO + LFP.