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On Perfect Matchings and Hamiltonian Cycles in Sums of Random Trees

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ABSTRACT

We prove that the sum of two random trees almost surely possesses a perfect matching and the sum of five random trees almost surely possesses a Hamiltonian cycle.

1 Introduction

In this paper we prove that an appropriately defined sum of two random trees almost surely possesses a perfect matching. Secondly, we show that the sum of five random trees almost surely possesses a Hamiltonian cycle.
For an integer n, we use $[n]$ to denote the set $\{1, \ldots, n\}$. A *random tree* on the set $[n]$ is a tree on this set chosen uniformly at random from the family of all trees on the set $[n]$.

Definition 1 *(Sums) Let k be a positive integer. For trees $T_1, \ldots, T_k$, all of them on the set $[n]$, we define their sum $\mathcal{ST}(T_1, \ldots, T_k)$ as the graph on the vertex set $[n]$ and edge set being the union of edge sets of the trees $T_1, \ldots, T_k$, where the parallel edges coalesce.*

Let f be a mapping from $[n] \to [n]$. Let $D(f)$ be its associated functional digraph i.e. the graph with vertex set $[n]$ and edges $(i, f(i))$, $i \in [n]$. For a set $f_1, \ldots, f_k$ of such mappings we define their sum $\mathcal{SM}(f_1, \ldots, f_k)$ as the union of the digraphs $D(f_i)$, $1 \leq i \leq k$.
Let k be a positive integer. Consider k random trees $T_1, \ldots, T_k$ on $[n]$. We use the notation $\mathbf{ST}_n(k)$ for $\mathcal{ST}(T_1, \ldots, T_k)$.
A *random mapping* $f : [n] \to [n]$ is a mapping from the set $[n]$ to itself chosen uniformly at random from the family of all mappings $[n] \to [n]$. Similarly, as in the case for trees, we use $\mathbf{SM}_n(k)$ to denote the sum of k random mappings.
$\mathbf{SM}_n(k)$ is a well studied model of random graph. Frieze [4] showed that $\mathbf{SM}_n(2)$ almost surely has a perfect matching (see also Shamir and Upfal [11] who showed that $\mathbf{SM}_n(6)$ almost surely has a perfect matching). Cooper and Frieze [2] have shown that $\mathbf{SM}_n(4)$ almost surely has a Hamilton cycle but the problem of whether or not $\mathbf{SM}_n(3)$ almost surely has a Hamilton cycle is one of the most important open questions in the theory of Random graphs.
There is also a well known bipartite mapping model $\mathbf{SM}_{n,n}(k)$. Walkup [12] had earlier shown that $\mathbf{SM}_{n,n}(2)$ almost surely has a perfect matching.
$\mathbf{ST}_n(k)$ is less well studied. Schmutz [9] computed the expected number of perfect matchings in $\mathbf{ST}_n(2)$ and showed that asymptotically it is $(4/e)^n$. He also studied a bipartite model $\mathbf{ST}_{n,n}(k)$ where the trees involved are random subtrees of the complete bipartite graph $K_{n,n}$ and showed that $\mathbf{ST}_{n,n}(2)$ almost surely has a perfect matching.
In Section 2, we prove the following theorem

Theorem 2

$$\text{(a)} \qquad \lim_{\substack{n\to\infty \\ n \text{ even}}} \mathrm{Prob}(\mathbf{ST}_n(1) \text{ has a perfect matching }) = 0 .$$

$$\text{(b)} \qquad \lim_{\substack{n\to\infty \\ n \text{ even}}} \mathrm{Prob}(\mathbf{ST}_n(2) \text{ has a perfect matching }) = 1 .$$

Using the proof methodology of Frieze and Luczak [5] who showed that $\mathbf{SM}_n(5)$ almost surely has a Hamilton cycle, we prove a result on the existence of Hamiltonian cycles in $\mathbf{ST}_n(5)$ in Section 3.

Theorem 3

$$\lim_{n\to\infty} \text{Prob}(\mathbf{ST}_n(5) \text{ has a Hamiltonian cycle }) = 1 \ .$$

2 Perfect Matchings - Proof of Theorem 2

(a) This follows immediately from Meir and Moon's result [8] that the size of the largest matching in a random tree is almost surely asymptotic to $(1-\rho)n \approx .432\,n$ where $\rho e^\rho = 1$.
(b) The proof of the second limit in Theorem 2 consists of several lemmas. Before giving the lemmas we need some notation. Let $G = (V, E)$ be a graph. For $U \subseteq V$, let $G[U] = (U, E_U)$ be a subgraph of G induced on U, i.e., $E_U = \{e \in E\ ;\ e \subseteq U\}$. Further, let $N_G(U) = \{v \in V \setminus U\ ;$ there is $u \in U$ such that $\{u, v\} \in E\}$ denote the neighborhood of the set U and set $N(U) = N_{\mathbf{ST}_n(2)}(U)$. A subset $U \subseteq V$ is said to be *stable* if $E_U = \emptyset$.
The following lemma is due to Gallai [6] and Edmonds [3] (cf. [4]).

Lemma 4 *If a graph G does not have a perfect matching then there exists $K \subseteq V(G), |K| = k \geq 0$ such that if $H = G[V(G) \setminus K]$ then*

$$H \text{ has at least } k+1 \text{ components with an odd number of vertices } ; \tag{1}$$

$$\text{No odd component of } H, \text{ which is not an isolated vertex, is a tree } . \tag{2}$$

The set K guaranteed by Lemma 4 will be called a *bad set.*
In the following sequence of lemmas, we are going to show that almost surely $\mathbf{ST}_n(2)$ has no bad set, for n even.
Before starting with the lemmas, we recall the following two formulas: the number of forests on n vertices with k fixed roots is equal to kn^{n-k-1}, and the number of forests on n vertices with k roots (there roots can be any k of the n vertices) is equal to $\binom{n-1}{k-1}n^{n-k}$.

Lemma 5 *For sets $K, L \subseteq V_n$, let $\mathcal{A}_1(K, L)$ be the event that $N(L) \subseteq K$. For positive integers k, l define the event $\mathcal{A}_1(k, l)$ by*

there exist $K, L \subseteq V_n, K \cap L = \emptyset, |K| = k, |L| = l$ such that $\mathcal{A}_1(K, L)$ occurs.

For $0 < \epsilon < 1$ let $u(\epsilon) = [\frac{1-\epsilon}{3e^4(1+\epsilon)^{1+\epsilon}}]^{1/\epsilon}$ and suppose that $u = u(\epsilon)$ satisfies $(5e^4)^u/u^u \leq 2^{1/8e^2}$. Then setting $n_1 = \lfloor un \rfloor$ and $l_1 = \lceil (1+\epsilon)k \rceil$ and

$$\mathcal{A}_1(\epsilon) = \bigcup_{k=1}^{n_1} \bigcup_{l=l_1}^{\lfloor n/2 \rfloor} \mathcal{A}_1(k, l) \ ,$$

we have

$$\lim_{n\to\infty} \text{Prob}(\mathcal{A}_1(\epsilon)) = 0 \ .$$

Proof
To bound Prob $(\mathcal{A}_1(k,l))$ we are going to divide the ranges of k and l into the following two cases:

(A) $l \le \lfloor n/(2e^2)\rfloor$ and any k, and

(B) $l > \lfloor n/(2e^2)\rfloor$ and any k.

Fix K, L and the lowest numbered vertex $v \in K$. Now, each tree T with $N_T(L) \subseteq K$ is considered to be oriented towards v.
Case A. Let T be a tree oriented as described above. Delete edges oriented out of vertices in L. This leaves a forest F' with $l+1$ roots and n vertices. There are at most $(l+1)n^{n-l-2}$ such forests, (not all forests with $l+1$ roots and n vertices respect $N_T(L) \subseteq K$). To obtain T we construct a forest F'' with vertex set $K \cup L$ and roots K and take $T = F' \cup F''$. We can construct F'' in $k(k+l)^{l-1}$ ways. Hence,

$$\begin{aligned}
\text{Prob } (\mathcal{A}_1(k,l)) &\le \binom{n}{k}\binom{n}{l}\left(\frac{(l+1)n^{n-l-2}k(k+l)^{l-1}}{n^{n-2}}\right)^2 \qquad (3)\\
&\le \frac{(ne)^{k+l}}{k^k l^l}k^2\frac{l^{2l}(1+\frac{k}{l})^{2l}}{n^{2l}}\\
&\le \frac{n^k k^2 e^{3k}}{k^k}\left(\frac{el}{n}\right)^l .
\end{aligned}$$

Putting $\mu_l = (el/n)^l$, we get $\mu_l/\mu_{l-1} < 1/2$ for $l < n/2e^2$.
Thus,

$$\sum_{l=l_1}^{\lfloor n/2e^2\rfloor}\left(\frac{el}{n}\right)^l \le 2\left(\frac{el_1}{n}\right)^{l_1}$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{n_1}\sum_{l=l_1}^{\lfloor n/2e^2\rfloor} \text{Prob } (\mathcal{A}_1(k,l)) &\le \sum_{k=1}^{\lfloor un\rfloor}\frac{n^k k^2 e^{3k}}{k^k}2\left(\frac{el_1}{n}\right)^{l_1}\\
&\le 2\sum_{k=1}^{\lfloor un\rfloor}\left(\frac{3e^4(1+\epsilon)^{1+\epsilon}k^\epsilon}{n^\epsilon}\right)^k\\
&= o(1) .
\end{aligned}$$

Case B. Let T be a tree oriented as described above. Let F' be the forest obtained by deleting edges oriented out of K and deleting vertices in L. This forms a forest with $n-l$ vertices and k roots K. There are $k(n-l)^{n-l-k-1}$ such forests and each forest can be extended in at most $k(k+l)^{l-1}n^{k-1}$ ways to form the oriented tree T. Indeed, we attach the vertices from L by constructing a forest on $K \cup L$ with roots K in at most $k(k+l)^{l-1}$ ways. The remaining $k-1$ edges oriented out of K can be chosen in at most n^{k-1} ways. Hence,

$$\begin{aligned}
\text{Prob }(\mathcal{A}_1(k,l)) &\leq \binom{n}{k}\binom{n}{l}\left[\frac{k(n-l)^{n-l-k-1}k(k+l)^{l-1}n^k}{n^{n-2}}\right]^2 \\
&\leq \frac{(ne)^{k+l}}{k^k l^l}\frac{k^4 e^{-2l+\frac{2l(l+k+1)}{n}} l^{2(l-1)} e^{2k}}{n^{2(l-1)}} \\
&\leq \frac{n^k e^{3k-l} k^4 e^{l+k+1} l^{l-2}}{k^k n^{l-2}} \\
&= e\left(\frac{nk^{4/k}e^4}{k}\right)^k \left(\frac{l}{n}\right)^{l-2} .
\end{aligned}$$

For n large enough,

$$\sum_{l=\lfloor n/2e^2\rfloor+1}^{\lfloor n/2\rfloor}\left(\frac{l}{n}\right)^{l-2} \leq n\left(\frac{1}{2}\right)^{\frac{n}{2e^2}-2} \leq \left(\frac{1}{2}\right)^{\frac{n}{4e^2}} .$$

Hence,

$$\begin{aligned}
\sum_{k=1}^{n_1}\sum_{l=\lfloor n/2e^2\rfloor+1}^{\lfloor n/2\rfloor} \text{Prob }(\mathcal{A}_1(k,l)) &\leq e\left(\frac{1}{2}\right)^{\frac{n}{4e^2}}\sum_{k=1}^{\lfloor un\rfloor}\left(\frac{nk^{4/k}e^4}{k}\right)^k \\
&\leq e\left(\frac{1}{2}\right)^{\frac{n}{4e^2}}\sum_{k=1}^{\lfloor un\rfloor}\left(\frac{5e^4 n}{k}\right)^k \\
&\leq en\left(\frac{(5e^4)^u}{2^{1/4e^2}u^u}\right)^n \\
&= o(1) .
\end{aligned}$$

□

Lemma 6 *Suppose a graph G contains a bad set $K, 1 \leq k = |K| \leq u(\epsilon)n$, and no subset of K is bad. Let $H = G[V_n - K]$ have $s \geq k+1$ odd components $C_1, C_2, \ldots, C_s$ with $n_1 = n_2 = \cdots = n_p = 1 < 3 \leq n_{p+1} \leq \cdots \leq n_s$ vertices, respectively.*
Assume that $\mathcal{A}_1(\epsilon)$ does not occur. Then there exists a partition K, P, Q, R of V_n with $p = |P|, q = |Q|$ satisfying

$$N(R) \subseteq K, N(P) \subseteq K, N(Q) \subseteq K \; ; \tag{4}$$

$$P \textit{ is a stable set } ; \tag{5}$$

$$\textit{Each vertex of } K \textit{ is adjacent to at least one member of } P \cup Q \; ; \tag{6}$$

$$1 \leq k \leq u(\epsilon)n, 0 \leq p+q < (1+\epsilon)k, p + \lfloor q/3 \rfloor \geq k \textit{ and } q = 0 \textit{ implies } p \geq k+1 \; . \tag{7}$$

Proof See [4]. □

Let $\mathcal{A}_2(\epsilon)$ be the event that there is a partition satisfying (4) - (7) described in Lemma 6. We can immediately show that for any fixed integer k_0

$$\lim_{n\to\infty} \mathrm{Prob}(\mathbf{ST}_n(2) \text{ has a bad set } K, \text{ with } 1 \le |K| \le k_0) = 0.$$

Let us take $\epsilon = 1/2k_0$ and assume that $\mathcal{A}_1(\epsilon)$ does not occur. If there is a bad set K with $1 \le |K| \le k_0$ then the conditions of the Lemma 6 are satisfied for some $k \le k_0$. But (7) implies $q < 3\epsilon k/2$ which in this case forces $q < 1$ or $q = 0$. But then $p \ge k+1$ contradicts $p < (1+\epsilon)k$.

In the proof of the following lemma we assume that $k \ge k_0$ for some suitably large k_0.

Lemma 7 *For small* ϵ

$$\lim_{n\to\infty} \mathrm{Prob}(\mathcal{A}_2(\epsilon)) = 0\ .$$

Proof

Fix K, P, Q and $v \in K$. Each tree satisfying (4) – (6) can be chosen in at most $k(n-p-q)^{n-p-q-k-1}n^{k-1}k^p(k+q)^q$ ways. We first build a forest on $V \setminus (P \cup Q)$ with roots in K ($k(n-p-q)^{n-p-q-k-1}$ ways). Then each $x \in P$ is allowed to choose in K, each $y \in Q$ is allowed to choose in $K \cup Q$ and each $z \in K \setminus \{v\}$ is allowed to choose in V_n.

Let K_i be the set of vertices in K which have a neighbour in $P \cup Q$ in the tree T_i, $i = 1, 2$. There are two possibilities:

A: $|K_1| \ge .9k$.

Of the $k^p(k+q)^q$ choices ascribed to vertices in $P \cup Q$, at most a proportion $.9^k$ will make $|K_1| \ge .9k$. Indeed, for each $x \in K$ the probability it is included in such a choice is at most

$$1 - \left(1 - \frac{1}{k}\right)^p \left(1 - \frac{1}{k+q}\right)^q \le .64,$$

for large enough k. The corresponding events for each x are clearly negatively correlated – note that we do not claim this for the choice of tree T_1, but only for the choices defined by the upper estimate. Thus,

$$\begin{aligned}\mathrm{Prob}\ (|K_1| \ge .9k) &\le \binom{k}{.9k}(.64)^k \\ &\le (.9)^k.\end{aligned}$$

B: $|K_1| < .9k$. By a similar argument,

$$\mathrm{Prob}\ (K_2 \supseteq K \setminus K_1 \mid K_1,\ |K_1| < .9k) \le (.64)^{.1k}.$$

Combining the two cases we see that for $\delta = (.64)^{.1}$ we have

$$\begin{aligned}\mathrm{Prob}(\mathcal{A}_2(k,p,q)) &\le 2\binom{n}{k,p,q}\left[\frac{k(n-p-q)^{n-p-q-k-1}n^{k-1}k^p(k+q)^q}{n^{n-2}}\right]^2\delta^k \\ &\le 2\frac{(ne)^{k+p+q}}{k^k p^p q^q}\frac{k^{2p+2q+2}e^{2q^2/k}e^{-\frac{p+q}{n}(n-p-q-k-1)}}{n^{2p+2q}}\delta^k.\end{aligned}$$

We obtain for the probability of the event $\mathcal{A}_2(k,p,q)$, under the condition $q \geq 2$

$$\begin{aligned}\text{Prob }(\mathcal{A}_2(k,p,q)) &\leq 2\frac{e^{k-p-q}k^{2p+2q+2}}{k^k q^q p^p n^{p+q-k}} e^{2q^2/k} e^{\frac{2(1+\epsilon)(2+\epsilon)k^2+2(1+\epsilon)k}{n}} \delta^k \\ &\leq 2\left(\frac{k}{en}\right)^{p+q-k}\left(\frac{k}{p}\right)^p\left(\frac{k}{q}\right)^q k^2 e^{2q^2/k} e^{\frac{2(1+\epsilon)(2+\epsilon)k^2+2(1+\epsilon)k}{n}} \delta^k .\end{aligned}$$

We continue with the bound on Prob $(\mathcal{A}_2(k,p,q))$ using $\frac{k}{p} \leq 1+\frac{q}{p}$, $q \leq \frac{3}{2}\epsilon k$, and $p+q-k \geq 1$. Further, we use that the function x^{-x} on the interval $(0,\infty)$ has its maxima at $x = 1/e$. Thus, for every $k \geq k_0 = k_0(\epsilon)$

$$\begin{aligned}\text{Prob }(\mathcal{A}_2(k,p,q)) &\leq 2\left(\frac{k}{en}\right)^{p+q-k}\left(\frac{k}{p}\right)^p\left(\frac{k}{q}\right)^q k^2 e^{9\epsilon^2 k/2} e^{12u(\epsilon)k} e^{4u(\epsilon)} \delta^k \\ &\leq \frac{2k^3}{en} e^{4u(\epsilon)}\left(\frac{2}{3\epsilon}\right)^{3\epsilon k/2} e^{5\epsilon k+12u(\epsilon)k}\delta^k .\end{aligned}$$

Choose ϵ small enough such that $(2/3\epsilon)^{3\epsilon/2}e^{5\epsilon+12u(\epsilon)}\delta \leq \mu < 1$ for some $0 < \mu < 1$. For $k \geq k_0$ let $\mathcal{S}(k) = \{(p,q)\ ;\ k,p,q$ satisfy conditions (7) $\}$. Note that $|\mathcal{S}(k)| \leq 2k^2$ for ϵ small. We sum up over k, p, and q. Thus,

$$\begin{aligned}\text{Prob }(\mathcal{A}_2(\epsilon)) &= \sum_{k=k_0}^{\lfloor u(\epsilon)n\rfloor}\sum_{(p,q)\in\mathcal{S}} \text{Prob }(\mathcal{A}_2(k,p,q)) \\ &\leq \frac{4e^3}{n}\sum_{k=k_0}^{\lfloor u(\epsilon)n\rfloor} k^5\mu^k \\ &= o(1) .\end{aligned}$$

□

Summing up: choosing ϵ small enough and k_0 sufficiently large, so far, we have proved that there is a constant $u_0 > 0$ such that

$$\lim_{n\to\infty} \text{Prob}(\mathbf{ST}_n(2) \text{ has a bad set } K, \text{ with } 1 \leq |K| \leq u_0 n) = 0.$$

To complete the proof of Theorem 2, we need to take care about large bad sets.

Lemma 8 *Let $\mathcal{A}_3$ denote the following event:*

$\mathbf{ST}_n(2)$ *contains at least* $(\log n)^3$ *sets* $S \subseteq V_n$ *satisfying*

$$|S| \leq \log\log n\ ; \tag{8}$$

$$|E_S| \geq |S|\ . \tag{9}$$

Then $\lim_{n\to\infty}\text{Prob}(\mathcal{A}_3) = 0$.

Proof Fix $k \geq 2$. Let X_k be a random variable counting sets S with $|S| = k$ and $|E_S| \geq k$. Then

$$\begin{aligned} EX_k &\leq \binom{n}{k} \frac{\sum_{t=2}^{k} \sum_{\substack{i,j\geq 1\\ i+j=t}} (n-k)^t \left(\binom{k-1}{i-1} k^{k-i}\right) \left(\binom{k-1}{j-1} k^{k-j}\right)}{\left[(n-k)n^{n-(n-k)-1}\right]^2} \\ &= \binom{n}{k} \frac{k^{2k} \sum_{t=2}^{k} \binom{2(k-1)}{t-2} \left(\frac{n-k}{k}\right)^t}{[(n-k)n^{k-1}]^2} \\ &\leq e^k \left(\frac{n}{n-k}\right)^2 \left(\frac{k}{n}\right)^k \sum_{t=2}^{k} \binom{2(k-1)}{t-2} \left(\frac{n-k}{k}\right)^t . \end{aligned}$$

As we have $k \leq \log\log n$, we get

$$\begin{aligned} EX_k &\leq e^k \left(\frac{n}{n-k}\right)^2 k \binom{2k}{k} \left(\frac{n-k}{k}\right)^k \left(\frac{k}{n}\right)^k \\ &\leq (4e)^k k \left(\frac{n-k}{n}\right)^{k-2} \\ &\leq (4e)^k \log\log n . \end{aligned}$$

By the Markov inequality

$$\begin{aligned} \text{Prob } (\mathcal{A}_3) &= \text{Prob}\left(\sum_{k=3}^{\lfloor \log\log n \rfloor} X_k \geq (\log n)^3 \right) \\ &\leq \frac{2 \log\log n (4e)^{\log\log n}}{(\log n)^3} \\ &= o(1) . \end{aligned}$$

□

Now we shall make use of a known one-to-one correspondence between the family of labelled trees on n vertices with two marked vertices and the family of functional digraphs $D(f)$ of mappings $f : [n] \to [n]$. Each such digraph D consists of vertices $S(f)$ which form cycles and the remaining vertices form a set of trees $\mathcal{T}$ which are attached to the cycles. To obtain a tree T, with two appropriately marked vertices from D we shall consider vertices lying on the cycles as a permutation drawn in cyclic form. Next we write such a permutation in a line form, which in turn we treat as a directed path P. As a final step, we re-attach the trees in $\mathcal{T}$ to their vertices on P to obtain a tree with two marked vertices (these two vertices are simply the beginning and the end of P). One can easily reproduce the correspondence from trees to mappings reversing the procedure described above. We believe that the one-to-one correspondence stated above is due to Joyal. A complete description of this correspondence can be found for example in Bender and Williamson [1].
This defines a natural measure preserving mapping ϕ from the space of random mappings to the space o random trees (ϕ just "forgets" the random choice of pair of marked vertices). To finish the proof of Theorem 2 we will use ϕ to construct $\mathbf{ST}_n(2)$ in the following way: we first generate $\mathbf{SM}_n(2)$ from random functions f_1, f_2 and then apply ϕ to both of them.

Definition 9 *Let a pair of sets $K, P \subseteq V_n$ be* **matched** *if*
(ι) P is stable in $\mathbf{ST}_n(2)$;
($\iota\iota$) $N(P) = K$;
($\iota\iota\iota$) $|P| \geq |K| - \delta(n)$,
where $\delta(n) = \lceil \frac{n}{\log\log n} + (\log n)^3 \rceil$.

Lemma 10 *Suppose $\mathbf{ST}_n(2)$ has no bad sets of size $u_0 n$ or less but $\mathbf{ST}_n(2)$ contains a bad set $K_0, k = |K_0| > u_0 n$. Suppose K_0 does not strictly contain another bad set and $\mathcal{A}_3$ does not occur in $\mathbf{ST}_n(2)$. Let $S = S(f_1) \cup S(f_2)$. If $s = |S|$ then either $\mathbf{SM}_n(2)$ contains a matched pair K, P with*

$$|P| + \delta(n) + s \geq k \geq |K| \geq |P|$$

or

$$K_0 \text{ contains a bad set of } \mathbf{SM}_n(2).$$

Proof Arguing as in Lemma 2.7 of [4] we see that $\mathbf{ST}_n(2)$ contains a matched pair K_1, P_1 with

$$|P_1| + \delta(n) \geq k \geq |K_1| \geq |P_1|.$$

Let $P = P_1 \setminus S$. Then P is stable and $|P| \geq |P_1| - s$. Also, $N_{\mathbf{SM}_n(2)}(P) \subseteq K_1$. Now take $K = N_{\mathbf{SM}_n(2)}(P)$. Either $|K| < |P|$ and K is a bad set of $\mathbf{SM}_n(2)$ or $|K| \geq |P|$ and K, P is the required matched pair. $\square$

Both possibilities in Lemma 10 are shown almost surely not to happen in [4], completing the proof of Theorem 2. (We observe first that $s = O(\sqrt{n})$ almost surely cf Kolchin [7]. The definition of matched pair in [4] has to be amended to $\delta(n) + O(\sqrt{n})$, but this does not affect the proof there given in any significant way.)

3 Hamiltonian Cycles - Proof of Theorem 3

Frieze and Łuczak [5] proved that there is almost surely a Hamiltonian cycle in $\mathbf{SM}_n(5)$. We will use the same proof technique here, giving only a sketch as the main ideas are very similar.

We consider $\mathbf{ST}_n(5)$ to be the union of $\mathbf{ST}_n(4)$ and a random tree T_5. We observe first that Theorem 2 shows that $\mathbf{ST}_n(4)$ almost surely contains the union of two perfect matchings M_1, M_2. We can argue (see Lemma 2 of [5]) that M_1 and M_2 are an independent pair of matchings, chosen uniformly from the set of all possible perfect matchings. Furthermore, (see Lemma 3 of [5]) $M_1 \cup M_2$ is almost surely the union of at most $3 \log n$ vertex disjoint cycles – some cycles may possibly just be double edges.

We show next that $\mathbf{ST}_n(4)$ almost surely has good expansion properties. For sets $K, L \subseteq V_n$, let $\tilde{A}_1(K, L)$ be the event that $N_{\mathbf{ST}_n(4)}(L) \subseteq K$ and let

$$\mathcal{A}_4 = \bigcup_{\substack{|K| \leq 10^{-3} n \\ |L| = 2|K|}} \tilde{A}_1(K, L).$$

Lemma 11 Prob($\mathcal{A}_4$) $= o(1)$.

Proof It follows from (3) that

$$\begin{aligned}
\mathrm{Prob}(\mathcal{A}_4) &\leq \sum_{k=1}^{10^{-3}n} \binom{n}{k}\binom{n}{2k}\left(\frac{(k+1)n^{n-k-2}2k(3k)^{k-1}}{n^{n-2}}\right)^4 \\
&\leq \sum_{k=1}^{10^{-3}n} (k+1)^4\left(\frac{81e^3k}{4n}\right)^k \\
&= o(1).
\end{aligned}$$

□

The idea now is to use the extension-rotation procedure (as described in [5]). The main idea that we get from [5] is to reserve the edges of T_5 for closing paths. More precisely, at some points of our extension-rotation procedure we will have a set A, $|A| \geq 10^{-3}n$ and for each $a \in A$ there is a collection of paths with endpoints $B(a)$, $|B(a)| \geq 10^{-3}n$ and we succeed if we always find a T_5-edge of the form (a, b) where $b \in B(a)$. Almost surely we only need to attempt this at most $3\log n$ times (from Lemma 11). Let us suppose that the edges of T_5 come from a random mapping f_5 where an *adversary* has altered the edges coming out of a set S of $O(\sqrt{n})$ nodes. When given A, $\{B(a):\ a \in A\}$ we choose the lowest numbered $a \in A \setminus S$ whose f_5 value has not been examined. So, almost surely, we examine a further $O(\log n)$ a's before finding one with $f_5(a) \in B(a)$. Thus, almost surely, the number of edges examined and altered throughtout the procedure is $O(\sqrt{n})$ and we succeed in finding a Hamilton cycle.

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