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On galleries with no bad points[1]

by

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ABSTRACT

For any k we construct a simply connected compact set (art gallery) in $\mathbb{R}^3$ whose every point sees a positive fraction (in fact, more than $\frac{5}{9}$) of the gallery, but the whole gallery cannot be guarded by k guards. This disproves a conjecture of Kavraki, Latombe, Motwani, and Raghavan.

1 Introduction

Consider an art gallery (i.e., compact set) X of Lebesgue measure 1 in $\mathbb{R}^d, d \geq 2$, such that a guard placed anywhere in the gallery sees a region of Lebesgue measure at least ε. Kavraki, Latombe, Motwani, and Raghavan [KLMR] conjectured that if X has at most h holes then it can be guarded by at most $f_d(h, \varepsilon)$ guards, for some function f_d polynomial in h and $\frac{1}{\varepsilon}$.

Kalai and Matoušek [KM] proved a weaker form of the planar version of the conjecture (their function f_2 was not polynomial in h). Following some ideas of Kalai and Matoušek, the author [Va] proved the planar version of the conjecture with $f_2(h, \varepsilon) = (2 + o(1))\frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \log_2(h+2)$. Kalai and Matoušek [KM] gave an example showing that $f_2(h, \varepsilon_0) = \Omega(\log h)$ for some fixed $\varepsilon_0 > 0$. Considering $\lfloor \frac{\varepsilon_0}{\varepsilon} \rfloor$ shifted copies of the set in their example, one can get an example showing the lower bound $f_2(h, \varepsilon) = \Omega(\frac{1}{\varepsilon} \log h)$ for $\varepsilon \in (0, \varepsilon_0)$, which almost matches the above upper bound.

In this paper we disprove the conjecture of Kavraki *et al.* for $d \geq 3$. For any $k \geq 1$, we construct a simply connected gallery in $\mathbb{R}^3$, in which each point sees more than $\frac{5}{9}$ of the gallery, but the whole gallery cannot be guarded by k guards. A motivation for the problem coming from the robot path planning can be found in [KLMR].

Let X be a set in $\mathbb{R}^d, d \geq 2$. We say that a point $x \in X$ *sees* a point $y \in X$ in X if the line segment xy is fully contained in X. For $x \in X$, let $V(x) = \{y \in X \colon x \text{ sees } y\}$. We say that a subset $G \subseteq X$ *guards* X if each point of X sees at least one point of G. We say that a point $x \in X$ is *ε-good*, if $\lambda_d(V(x)) \geq \varepsilon \lambda_d(X)$, where $\varepsilon > 0$ and λ_d denotes Lebesgue measure in $\mathbb{R}^d$. We say that the set X is *ε-good*, if each point of X is ε-good.

Here is the main result:

Theorem 1 *For any $k \geq 1$, there is a $\frac{5}{9}$-good simply connected compact set in $\mathbb{R}^3$ which cannot be guarded by k points (guards).*

In the proof of Theorem 1 we show that the constant $\frac{5}{9}$ can be still slightly improved. A generalization of our construction gives, for any $d \geq 3$ and any $k \geq 1$, a simply connected compact $\left(1 - \left(\frac{d-1}{d}\right)^{d-1}\right)$-good gallery in $\mathbb{R}^d$ which cannot be guarded by k guards. On the other hand, for polyhedral sets there is a non-trivial upper bound in any dimension (we call a set in $\mathbb{R}^d$ *polyhedral* if it is a union of finitely many convex d-dimensional polytopes).

Theorem 2 *For $d \geq 2$, any polyhedral $\frac{d}{d+1}$-good gallery in $\mathbb{R}^d$ can be guarded by one guard (in other words, it is star-shaped).*

We remark that the set in Theorem 1 may be required to be polyhedral. There is a significant gap between the lower bound $1 - \left(\frac{d-1}{d}\right)^{d-1} \approx 1 - \frac{1}{e}$ in the d-dimensional version of Theorem 1 and the upper bound $\frac{d}{d+1}$ in Theorem 2. In particular, we don't have an answer to the following question.

Problem 1 *Is there an $\varepsilon < 1$ such that for any $d \geq 2$ any simply connected ε-good (polyhedral) gallery in $\mathbb{R}^d$ can be guarded by at most $k = k(d)$ guards?*

We can solve Problem 1 if other Borel measures than the uniform (Lebesgue) one are considered. This is shown in Section 3, where we also prove Theorem 2. Section 2 is devoted to the proof of Theorem 1. Throughout the paper, $B(x, r)$ denotes the r-neighborhood of x.

2 The construction

In this section we prove Theorem 1. Let S be a regular tetrahedron of edge length 1 with vertices A, B, C, D, let U be the center of S, and let V be the center of the triangle $T = ABC$. Our construction is based on the following observation:

Observation 1 *For every $\varepsilon < \frac{5}{9}$, there are two real numbers $\eta > 0$ and $\alpha > 0$ such that the following holds. Let T' be an equilateral triangle contained in $T \cap B(V, \eta)$, with sides parallel to those of T. Let V' be the center of T', and let E be a point inside S such that the angle between the line $V'E$ and the plane containing T is bigger than $\frac{\pi}{2} - \alpha$. Then the set $S \setminus S'$, where $S' = \operatorname{conv}(T' \cup \{E\})$, is ε-good.*

Observation 1 follows from the fact that each of the three planes containing the points U, V, D and cutting the triangle $T = ABC$ in a line parallel to some side of T cuts the tetrahedron S into two parts of measure $\frac{4}{9}\lambda_3(S)$ and $\frac{5}{9}\lambda_3(S)$, respectively. Observation 1 yields the following lemma.

Lemma 1 *For every two positive real numbers $\eta, \delta \in (0, 0.01)$, for every point $W \in (B(V, \eta) \cap T)$, for every $t > 0$, and for every point $P \in S \setminus B(W, 2t)$, there exist a tetrahedron $S' = S'(\eta, \delta, W, t, P) \subset S$ and a real number $t' = t'(\eta, \delta, W, t, P) > 0$ such that*

(i) $S' \subset B(W, 2t) \setminus B(W, 2t')$,

(ii) $S \setminus S'$ is simply connected and $(\frac{5}{9} - g_1(\eta, \delta))$-good, where $g_1(\eta, \delta) \to 0$ as $\eta + \delta \to 0$,

(iii) every point of $S \setminus (B(W, 2t) \setminus B(W, 2t'))$ is $(1 - g_2(\delta))$-good in $S \setminus S'$, where $g_2(\delta) \to 0$ as $\delta \to 0$,

(iv) W sees no point of $B(P, \delta^2 \cdot \operatorname{dist}(P, W))$ in $S \setminus S'$.

(The functions g_1 and g_2 are independent of W, t, P.)

Proof. Let η, δ, W, t, P satisfy the conditions of Lemma 1. Consider a plane ρ containing the points W, P and perpendicular to the plane τ containing T (ρ is uniquely determined unless the line WP is perpendicular to τ). Choose the Cartesian coordinate system in ρ so that the line $\rho \cap \tau$ coincides with the x-axis, W coincides with the origin, and both the two coordinates of P are non-negative. Denote $W' = (\delta t, 0)$, $E = (-\delta t, t)$, and return to $\mathbb{R}^3$. Let $T' \subset T$ be the equilateral triangle with center W' and with sides of length $10\delta^2 t$ parallel to those of T. Then $S' = \operatorname{conv}(T' \cup \{E\})$ satisfies the lemma: (i) and (iii) hold with $t' = \delta^2 t$, (ii) follows from Observation 1, and (iv) follows from the construction. □

Let $k \geq 1$. Repeated applications of Lemma 1 give the following consequence.

Lemma 2 *For every two positive real numbers $\eta, \delta \in (0, 0.01)$, for every point $W \in (B(V, \eta) \cap T)$, for every $t > 0$, and for every k points $P_1, \ldots, P_k \in S \setminus B(W, 2t)$, there exists a set (union of k tetrahedra) $S'' = S''(\eta, \delta, W, t, \{P_1, \ldots, P_k\}) \subset S$ such that*

(i) $S'' \subset B(W, 2t) \setminus \{W\}$,

(ii) $S \setminus S''$ is simply connected and $(\frac{5}{9} - h(\eta, \delta))$-good, where $h(\eta, \delta) \to 0$ as $\eta + \delta \to 0$,

(iii) W sees no point of $\bigcup_{i=1}^{k} B(P_i, \delta^2 \cdot \operatorname{dist}(P_i, W))$ in $S \setminus S''$.

(The function h is independent of $W, t, \{P_1, \ldots, P_k\}$.)

Proof. Given $\eta, \delta, W, t, \{P_1, \ldots, P_k\}$, an application of Lemma 1 with parameters η, δ, W, t, $P = P_1$ gives a set $S'(\eta, \delta, W, t, P_1) = S'_1$ and a number $t'(\eta, \delta, W, t, P_1) = t'_1 > 0$. Thereafter, an application of Lemma 1 with parameters $\eta, \delta, W, t = t'_1, P = P_2$ gives a set $S'(\eta, \delta, W, t'_1, P_2) = S'_2$ and a number $t'(\eta, \delta, W, t'_1, P_2) = t'_2 > 0$. We apply Lemma 1 k times (in the ith application we use parameters $\eta, \delta, W, t = t'_{i-1}, P = P_i$, and obtain a set S'_i and a number $t'_i > 0$). It is easily checked that the set $S'' = \bigcup_{i=1}^{k} S'_i$ satisfies Lemma 2. □

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. First we show a slightly weaker result. For any positive integer k and for any real number $\omega > 0$, we find a $(\frac{5}{9}-\omega)$-good simply connected gallery in $\mathbb{R}^3$ which cannot be guarded by k guards.

Choose $\eta, \delta \in (0, 0.01)$ such that $h(\eta,\delta) < \frac{\omega}{2}$, where h is the function from condition (ii) in Lemma 2. Choose arbitrarily finitely many points $P_1, \ldots, P_n$ in S such that $S \subset \bigcup_{i=1}^{n} B(P_i, \frac{\delta^2\eta}{2\sqrt{k}})$, and denote $I = \{1, \ldots, n\}$. A standard area argument shows that for any $J \in \binom{I}{k}$, there is a point $W(J) \in T \cap B(V, \eta)$ such that $\mathrm{dist}(P_i, W(J)) > \frac{\eta}{2\sqrt{k}}$ for every $i \in J$ (as usual, $\binom{I}{k}$ denotes the set of all k-element subsets of I). Moreover, the points $W(J), J \in \binom{I}{k}$, can be taken pairwise distinct.

For any $J \in \binom{I}{k}$ and $t \in (0, \frac{\eta}{4\sqrt{k}})$, there is a set $S''(\eta, \delta, W(J), t, \{P_i : i \in J\})$ satisfying Lemma 2 (in particular, $S \setminus S''(\eta, \delta, W(J), t, \{P_i : i \in J\})$ is $(\frac{5}{9} - \frac{\omega}{2})$-good since $h(\eta,\delta) < \frac{\omega}{2}$). It is easy to see that the compact set

$$X = \overline{\bigcap_{J\in\binom{I}{k}} S \setminus S''(\eta, \delta, W(J), t, \{P_i : i \in J\})}$$

is simply connected and $(\frac{5}{9}-\omega)$-good for a sufficiently small $t > 0$. We claim that X cannot be guarded by k guards. Suppose to a contrary that X is guarded by a set G of k guards. Certainly, $G \subset \bigcup_{i\in J} B(P_i, \frac{\delta^2\eta}{2\sqrt{k}})$ for some $J \in \binom{I}{k}$. Then, by condition (iii) in Lemma 2, the point $W(J) \in X$ sees no point of

$$\bigcup_{i\in J} B(P_i, \delta^2 \mathrm{dist}(P_i, W(J))) \supset \bigcup_{i\in J} B(P_i, \frac{\delta^2\eta}{2\sqrt{k}}) \supset G$$

in X, a contradiction.

To get a $\frac{5}{9}$-good set from the set X, it suffices to remove a proper (quite small) region from X in the neighborhood of the center of each of the triangles ABD, ACD, and BCD. E.g., in the neighborhood of the center V_0 of the triangle $T_0 = ABD$, we can remove the tetrahedron $\mathrm{conv}(T' \cup \{U_0\})$, where $T' \subset T_0$ is the equilateral triangle with center V_0 and with sides of length $\frac{1}{10}$ parallel to those of T_0, and U_0 is the point on the segment UV_0 at distance $\frac{1}{100}$ from V_0. Similarly, we can remove the other two regions. In this way we even obtain a $(\frac{5}{9}+c)$-good simply connected gallery which cannot be guarded by k guards ($c > 0$ is a constant). $\square$

3 Related results

In this section we prove Theorem 2 and some related results for other Borel measures than the uniform (Lebesgue) one. A Borel measure μ on $\mathbb{R}^d$ is called *locally finite* if $\mu(K)$ is finite for any compact measurable set $K \subset \mathbb{R}^d$. Let X be a set in $\mathbb{R}^d$, $d \geq 2$, and let μ be a Borel measure on $\mathbb{R}^d$. We say that a point $x \in X$ is *(μ,ε)-good*, if $\mu(V(x)) \geq \varepsilon\mu(X)$. We say that the set X is *(μ,ε)-good*, if each point of X is (μ,ε)-good. An analogue of Theorem 2 for locally finite Borel measures gives almost a tight result.

Theorem 3 *Let $\omega > 0$. Then*

(i) for any locally finite Borel measure μ in $\mathbb{R}^d, d \geq 2$, any polyhedral $(\mu, \frac{d}{d+1}+\omega)$-good set of positive μ-measure in $\mathbb{R}^d$ can be guarded by one guard,

(ii) for any $d \geq 3$, there is a locally finite Borel measure μ_d in $\mathbb{R}^d$ such that for any $k \geq 1$ there is a polyhedral $(\mu_d, \frac{d-1}{d}-\omega)$-good simply connected gallery of positive μ_d-measure which cannot be guarded by k guards.

Thus, an analogue of the question in Problem 1 for locally finite Borel measures has a negative answer. We remark that in the plane the number of guards needed in a (μ,ε)-good gallery with h holes is bounded by $(2+o(1))\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\log_2(h+2)$ for any locally finite Borel measure μ, like for the uniform measure. This is because the proof in [Va] for the uniform (Lebesgue) measure doesn't use any specific properties of the uniform measure, and therefore can be applied for any locally finite Borel measure.

We now prove Theorems 2 and 3.

Proof of Theorems 2 and 3(i). Let $d \geq 2$, $\omega > 0$, and let μ be a locally finite Borel measure in $\mathbb{R}^d$. Let X be a $(\mu, \frac{d}{d+1}+\omega)$-good (or $\frac{d}{d+1}$-good, if $\mu = \lambda_d$) polyhedral gallery in $\mathbb{R}^d$ with $\mu(X) > 0$. Let P be a centerpoint[1] of X with respect to μ. We claim that P sees every point in X. Suppose to a contrary that P doesn't see a point $Q \in X$. Let $Q' \in X$ be the point of X on the segment PQ closest to P with the property that it doesn't see P. Then there is a point Q'' of X in a small neighborhood of Q' which sees a part of X contained in a closed halfspace not containing P. From the centerpoint property of P, this implies that Q'' isn't $(\mu, \frac{d}{d+1}+\omega)$-good (or $\frac{d}{d+1}$-good, if $\mu = \lambda_d$), a contradiction. □

Proof of Theorem 3(ii). We outline the construction for $d = 3$. The construction is analogous for the other values of d. For $d = 3$, the gallery is similar to the gallery from the proof of Theorem 1. We again consider a regular tetrahedron $S = ABCD$ with edge length 1. Consider the Borel measure μ_3 which is the Lebesgue measure on the $\frac{1}{10}$-neighborhood of each of the vertices A, B, C, and the zero measure on the rest of $\mathbb{R}^3$. The proof now goes literally as the proof of Theorem 1 in Section 2 (including Observation 1 and Lemmas 1 and 2) with the last paragraph omitted and with the exceptions that every occurance of $\frac{5}{9}$ is replaced by $\frac{2}{3}$, and that the measure μ_3 is considered instead of the Lebesgue measure (thus, the notion ε-good is always replaced by (μ_3,ε)-good). □

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[1] P is a centerpoint of a compact set $X \subset \mathbb{R}^d$ with respect to a locally finite Borel measure μ if every halfspace in $\mathbb{R}^d$ containing P contains a subset of X of measure $\geq \frac{1}{d+1}\mu(X)$. The existence of a centerpoint for any compact set $X \subset \mathbb{R}^d$ with respect to a locally finite Borel measure follows easily from the existence of a centerpoint for any finite set in $\mathbb{R}^d$ with respect to Lebesgue measure (see [Ed]).