DIMACS Technical Report 96-41
September 1996

A corrected version of the Duchet Kernel Conjecture[1]

by

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DIMACS is a partnership of Rutgers University, Princeton University, AT&T Research, Bellcore, and Bell Laboratories.
DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

Abstract

In 1980 Piere Duchet conjectured that odd directed cycles are the only edge minimal kernel-less connected digraphs i.e. in which after the removal of any edge a kernel appears. Although this conjecture was disproved recently by Apartsin, Ferapontova and Gurvich (1996), the following modification of Duchet's conjecture still holds: odd holes (i.e. odd non-directed chordless cycles of length 5 or more) are the only connected graphs which are not kernel-solvable but after the removal of any edge the resulting graph is kernel-solvable.

[1]The authors gratefully acknowledge the partial support of DIMACS and ONR (Grants N00014-92-J-1375 and N00014-92-J-4083).

Let $D = (V, A)$ be a directed graph (digraph). A subset $K \subseteq V$ of the vertices is called a *kernel* of D if it is (i) independent (i.e. there are no arcs between its elements), and (ii) dominating (i.e. for every vertex v outside of K there is an arc from a vertex of K to v). A digraph is called *kernel-less* if it has no kernel.

Von Neumann proved that every acyclic digraph has a unique kernel. Later Richardson proved that every kernel-less digraph has an odd directed cycle. A very short proof of the last fact is given by Berge and Duchet (1990) in their survey on kernels in digraphs. Thus, odd directed cycles are the simplest kernel-less digraphs. Let us note that removing an edge from such a cycle we get an acyclic digraph which has a (unique) kernel. Duchet (1980) conjectured that there are no other connected digraphs with this property. In other words, for every connected kernel-less digraph which is not an odd directed cycle, there exists an edge which can be removed and the obtained digraph is still kernel-less. Clearly, this would strengthen the Richardson theorem but recently Apartsin, Ferapontova and Gurvich (1996) gave a counterexample.

Here we prove that a slight modification of Duchet's (1980) conjecture holds, yet. This modification is based on the notion of kernel-solvability. A non-directed graph is called *kernel solvable* if any its orientation has a kernel whenever every clique of this orientation has a kernel. This notion was introduced by Berge and Duchet (1983) and they conjectured that perfect graphs are kernel-solvable. This conjecture was proved by Boros and Gurvich (1994), (see also Aharoni and Holzman (1995) for a shorter proof).

Theorem 1 *For every connected, non kernel-solvable graph which is not an odd hole, there exists an edge which can be removed and the obtained graph is still not kernel-solvable.*

We derive this theorem from (A) the result of Boros and Gurvich (1994), claiming that perfect graphs are kernel-solvable, from (B) a theorem of Meyniel (1984) stating that a graph is perfect if every its odd cycle has at least two chords, and from (C) the easy observation that a graph containing an odd hole or odd antihole (i.e. the complement of an odd hole) is not kernel-solvable, see e.g. Berge and Duchet (1983), Maffray (1988), Boros and Gurvich (1994).

Proof. Let G be a nondirected and non kernel-solvable graph. Then, according to (A), G is not perfect and thus, according to (B), either

(a) G has an odd hole or

(b) G has an odd cycle with only one chord.

In case (b), let us remove this chord and we get a graph G' which contains an odd hole and thus, according to (C), G' is not kernel-solvable.

In case (a), either G is an odd hole itself or it contains an odd hole and some other edges. In the latter case, let us remove one of these edges and again we obtain a graph G' which still contains the same odd hole, thus G' is not kernel-solvable by (C). At last, let us notice that an odd hole itself is not kernel solvable because a cyclically directed odd hole has no kernel but if we remove an edge from it then we get an acyclic graph which is of course kernel-solvable. □

Finally, let us remark that the above theorem parallels with an analogue result of Olaru (1972) (see also Olaru and Sachs (1984)), claiming that the odd holes are the only connected graphs which are *edge-critically imperfect*, i.e. they are not perfect but become perfect after the removal of any edge. This provides further evidence for the conjecture of Berge and Duchet (1983) stating that kernel-solvability and perfectness are equivalent properties of graphs.

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