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Dual Graphs on Surfaces[1]

by

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Abstract

Consider an embedding of a graph G in a surface S (*map*). Assume that the difference splits into connected components (*countries*), each one homeomorphic to an open disk. (It follows from this assumption that graph G must be connected). Introduce a graph G^* dual to G realizing the neighbor relations among countries. The graphs G and G^* have the same set of edges. More precisely, there is a natural one-to-one correspondence between their edge-sets. An arbitrary pair of graphs with common set of edges is called a *plan*. Every map induces a plan. A plan is called **geographic** if it is induced by a map. In terms of Eulerian graphs we obtain criteria for a plan to be geographic. We also give an algorithm of reconstruction a map from a geographic plan. A case when this map is unique is singled out. Partially, these results were announced by Gurvich and Shabat in 1989.

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1. **Introduction.** Consider an embedding of a graph G in a surface S (*map*). Assume that the difference splits into connected components (*countries*), each one homeomorphic to an open disk. (It follows from this assumption that graph G must be connected). Introduce a graph G^* dual to G realizing the neighbor relations among countries. The graphs G and G^* have the same set of edges. More precisely, there is a natural one-to-one correspondence between their edge-sets. An arbitrary pair of graphs with common set of edges is called a *plan*. Every map induces a plan. A plan is called **geographic** if it is induced by a map.

   In terms of Eulerian graphs we obtain criteria for a plan to be geographic, and we also give an algorithm of reconstruction a map from a geographic plan. The case when this map is unique is singled out. Some of these results were announced in [1].

2. **Graphs.** Finite undirected graphs are considered. Loops and nmultiple edges are allowed. We need the following concepts; for more details see for instance [2; ch.1, §1; ch.2, §§$1-3$, and ch.7, §1].

   A cyclic sequence of alternating vertices and edges in which any adjacent edge and vertex are incident is called a **closed route**. A closed route in which all the edges are different is called a **cycle**. A cycle is called **simple** if all its vertices are different.

   The **degree**, or **valence**, of a vertex is defined as as the number of edges incident to it, where loops are considered with multiplicity 2. The degree of vertex v in graph G is denoted by $deg(G, v)$.

   A cycle containing all the vertices of a graph, and a graph itself in which such a cycle exists, are called **Eulerian**. Clearly, an Eulerian graph must be connected. Criteria for a connected graph to be Eulerian are given by the following Lemma (see, for instance, [2, Theorem 7.1]).

   Lemma 1. For a connected graph three properties are equivalent:
   a) The graph is Eulerian.
   b) There exist a collection of simple cycles such that their edge-sets represent a partition of the edge-set of the graph.
   c) All vertices have even degrees.

3. **Surfaces.** The topological classification of surfaces (i.e. two-dimensional,compact, smooth manifolds without boundary) is well known; see, for instance, [2, ch.3]. They are divided into
   orientable: (S_p; $p = 0, 1, 2, \cdots$), and nonorientable (C_q; $q = 1, 2, \cdots$).

   For example, S_0 is a sphere, S_1 is a torus, C_1 is a projective plane, C_2 is a Klein bottle.

   The "sphere with p handles" serves as a standard model for S_p, and the "sphere with q holes pasted by Möbius strips" for C_q.

4. **Maps.** Let S be a surface and $G =< A^0, A^1 >$ be a graph where $A^0 = a^0_1, a^0_2, ...$ is the set of vertices, $A^1 = a^1_1, a^1_2, ...$ is the set of edges.

Then let ϕ be an embedding of Gin S such that the edges do not have intersections on the surface apart from their common vertices of the graph, and also they have no self-intersections apart from the vertices of loops. Let us cut surface S along the edges of graph G; in other words, partition the difference $S - \phi(G)$ into connected components (countries). Remind that every country must be homeomorphic to an open disk and therefore graph G must be connected. Denote the set of countries by $A^2 = a_1^2, a_2^2, \ldots$.

A triple $M =< S, G, \phi >$ satisfying the conditions stated above will be called a **map**. Two maps $M' =< S', G', \phi' >$ and $M'' =< S'', G'', \phi" >$ are considered **isomorphic** if there exists a homeomorphism $g : S' \to S''$ carrying $\phi'(G')$ into $\phi''(G'')$.

5. **Dual graphs.** There is an obvious incidence relation between the countries and edges. The graph of this relation will be called dual to G on surface S and denoted by $G^* =< A^2, A^1 >$. Thus two countries are neighbor if and only if they have a common edge (but it is not enough to have a common vertex). Duality is an involution, i.e. $G^{**} = G$. Dual graph is also connected and it induces a dual map on the same surface. See§31 for more details concerning dual graphs, maps and plans.

6. **The Euler characteristic.** It is well known that for every map on a given surface S the number $\#A_0 - \#A_1 + \#A_2$ takes the same value, called the Euler characteristic of the surface and denoted by $\chi(S)$. For the surfaces considered in §2 one has

$$\chi(S_p) = 2 - 2p,\ p = 0, 1, 2, \ldots;\ \ \chi(C_q) = 2 - q,\ q = 1, 2, \ldots. \tag{1}$$

Therefore always $\chi \leq 2$, and $\chi = 2$ only in case of a sphere; then only two surfaces can have a given Euler characteristic χ less than 2 : these are C_q with $q = 2 - \chi$ and S_p with $p = 1 - \frac{\chi}{2}$, where the second one exists only for even χ.

7. **Plans.** Let A_0, A^1 and A^2 be three pairwisely disjoint sets, whose elements will be callyed 0-, 1- and 2-cells respectively, and let $G_{01} =< A^0, A^1 >$ and $G_{21} =< A^2, A^1 >$ be two graphs with the same set of edges A^1. The pair $P =< G_{01}, G_{21} >$ will be called a **plan**.

8. **Geographic plans.** Every map M induces in the obvious way a plan $P = P(M)$ in which the 0-, 1- and 2-cells are respectively the vertices, edges and countries of the map, while $G_{01} = G$ and $G_{21} = G^*$ are the dual graphs neighborhood for vertices and countries respectively.

A plan will be called **geographic** if it is induced by a map.

Note that the Euler characteristic of the corresponding surface (but not the map itself) is unambiguously determined by the geographic plan due to formula

$$\chi = \#A_0 - \#A_1 + \#A_2.$$

The aim of this note is to characterize the geographic plans.

9. Loops. In any plan, every 1-cell is incident to one or two 0-cells and 2-cells. Let 1-cell a^1 be incident to unique 0-cell a^0 (2-cell a^2). Then a^1 is called a **loop incident to** a^0 **in** G_{01} (a **loop incident to** a^2 **in** G_{21}). In case of a geographic map or plan a_0 is called a **loop with vertex** a^0 (an **interior edge of country** a^2).

10. Bimatrices of plans. Let $P =< G_{01}, G_{21} >$ be a plan and let $B_{01} : A^1 \times A^0 \to 0, 1, 2$ and $B_{21} : A^1 \times A^2 \to 0, 1, 2$ be the incidence matrices of the graphs G_{01} and G_{21} respectively. These matrices have the same set of lines A^1 corresponding to 1-cells; the sets of columns A^2 and A^2correspond to 0- and 2-cells respectively. Elements take only three values 0, 1 and 2, where the last one corresponds to loops.

Sum of elements in each line is equal to 2 for both matrices.

The pair $B =< B_{01}, B_{21} >$ will be called the **bimatrix** of the plan.

11. Examples. 11.1. Let $S = S_0$ be a sphere and graph $G = G_{01}$ consists of two vertices and one edge. Then $G^* = G_{21}$ is a loop, and $B = (11|2)$. Let vise-versa G be a loop; then G^* consists of one edge with two vertices, and $B = (2|11)$. These two graphs are dual.

11.2. Let S be a sphere and graph G consists of the pair of edges with common pair of vertices. Then G is selfdual, that is $G = G_{01} = G^* = G_{21}$, and $B = \begin{pmatrix} 11 & | & 11 \\ 11 & | & 11 \end{pmatrix}$.

11.3. Selfdual bimatrix $B = (11|11)$ is not associated with a geographic plan, because it can not be induced by a map, because $\chi = \#A_0 - \#A_1 + \#A_2 = 2 - 1 + 2 = 3 > 2$; see also §§16, 17.

11.4. Let S be a sphere again and graph G consists of two edges with one common vertex. Then G^* consists of two loops with the common vertex, and $B = \begin{pmatrix} 110 & | & 2 \\ 101 & | & 2 \end{pmatrix}$.

11.5. Let S be again a sphere and G consists of two vertices, one edge joining them and one loop incident to one of them. Then G^* is isomorphic to G but the loop in G^* is identified with the edge in G and vise-versa; thus G is not selfdual, and $B = \begin{pmatrix} 11 & | & 20 \\ 20 & | & 11 \end{pmatrix}$.

11.6. Selfdual bimatrix $B = \begin{pmatrix} 11 & | & 11 \\ 02 & | & 20 \end{pmatrix}$ is not associated with a geographic plan because it can not be induced by a map. The reason will be explained in §§16, 17.

11.7. Let S be again a sphere and G consists of two vertices, one edge joining them and two loops incident to each. Then G^* consists of two adjacent edges and one loop incident to their common vertex, and $B = \begin{pmatrix} 20 & | & 110 \\ 11 & | & 020 \\ 02 & | & 011 \end{pmatrix}$.

11.8. Let $S = C_1$ be a projective plane and G is a basic loop. Then G is selfdual, that is G^* is also the loop, and $B = (2|2)$.

11.9. Let $S = S_1$ be a torus or $S = C_2$ be a Klein bottle and in both cases G consists of one vertex and two basic loops (which are not homotopic to each other and to a point). Then in both cases there is only one country, G is selfdual, and $B = \begin{pmatrix} 2 & | & 2 \\ 2 & | & 2 \end{pmatrix}$.

The last example demonstrated that the same plan can be induced by different maps and even on different surfaces but with the same Euler characteristic). Different maps on the same surface also can induce the same plan; see §§20.3 – 20.6.

12. The incidence matrix of 0- and 2-cells of a plan *defined* as a matrix product $B_{02} = B^*_{01} \times B_{21}$, where $B_{01} : A^1 \times A^0 \to 0, 1, 2$ and $B_{21} : A^1 \times A^2 \to 0, 1, 2$ are the incidence matrices of the graphs G_{01} and G_{21} respectively, and $*$ is the transposing. In other words B_{02} is a mapping equal to the scalar product of two columns corresponding to 0-cell $a^0 \in A^0$ and 2-cell $a^2 \in A^2$.

13. Circuits of countries and vertices of a map. By definition, any country is homeomorphic to an open disk. Let us fix a country a^2 in a map M and go around it along its boundary. As a result a certain cycle C_{a^2} (maybe not simple) will be selected in the graph G_{01}. This cycle contains all edges and vertices adjacent with country a^2. The same vertex a^0 may occur in it several $(b(a^0, a^2))$ times, and the same edge may occur only once or twice; and the latter possibility occurs only for interior edges of country a^2.

It is well known that an arbitrary sufficiently small neighborhood of an arbitrary point of a surface is homeomorphic to an open disk. Let us fix a vertex a^0 on a map M and go around it along the boundary of such a neighborhood. As a result a certain cycle C_{a^0} (maybe not simple) will be selected in the graph G_{21}. This cycle contains all the edges and vertices adjacent with vertex a^0. The same country a^2 may occur in it several $(b(a^2, a^0))$ times, and the same edge may occur only once or twice; and the latter possibility occurs only for loops with vertex a^0.

14. The respective degrees of vertices and countries. Lemma 2. For any map, vertex $a^0 \in A^0$ and country $a^2 \in A^2$ one has

$$2b(a^0, a^2) = 2b(a^2, a^0) = B_{02}(a^0, a^2) \tag{2}$$

Sketch of the proof. All three numbers are equal to the same sum $\sum_{a^1 \in A^1} B_{01}(a^0, a^1) \times B_{21}(a^2, a^1)$.

Consequence. For any geographic plan all the elements of matrix B_{02} are even.

15. Graphs of 0- and 1-cells of a plan. Let $P = < G_{01}, G_{21} >$ be a plan. Let us fix an arbitrary 0-cell $a^0 \in A^0$ and consider the set of all 1- and 2-cells incident to it. The subgraph of G_{21} formed by them will be denoted by G'_{a^0}. In this subgraph let us double each 1-cell a^1 corresponding to a loop (incident to a^0) in G_{01}, that is replace a^1 by a pair of multiple (parallel) edges $a^{1\prime}$, $a^{1\prime\prime}$ with the same pair of vertices. If a^1 is a loop in G_{21} also then let us double this loop. The obtained graph will be denoted by G_{a^0}.

Graphs G_{a^2} and G'_{a^2} are defined analogously for every 2-cell $a^2 \in A^2$ of the plan P. It is enough to interchange the indices 0 and 2 in the above definition.

Lemma 3. For any $a^0 \in A^0$, $a^2 \in A^2$ the chain of equalities (2) can be prolonged in the following way

$$B_{02}(a^0, a^2) = deg(G_{a^0}, a^2) = deg(G_{a^2}, a^0) \tag{3}$$

Each 1-cell $a^1 \in A^1$ occurs exactly twice in the set of graphs G_{a^0}, $a^0 \in A^0$ as well as in the set G_{a^2}, $a^2 \in A^2$, but some 1-cells can occur only once in the sets G'_{a^0}, $a^0 \in A^0$, G'_{a^2}, $a^2 \in A^2$.

Proof follows from the definitions.

16. Properties of plans. A plan $P =< G_{01}, G_{21} >$ will be called:

a) **even** if all the elements of its matrix B_{02} are even;
b) **connected** if graph G_{01} (or equivalently G_{21}) is connected;
c) **locally connected** if the graphs G_{a^0}, G_{a^2}(or equivalently (G'_{a^0}, G'_{a^2}) are connected for any $a^0 \in A^0$, $a^2 \in A^2$.
d) **Eulerian** if the graphs G_{a^0}, G_{a^2} (not G'_{a^0}, G'_{a^2}) are Eulerian for any $a^0 \in A^0$, $a^2 \in A^2$.
Evidently d) is equivalent to a) & c), according to Lemma 1.

17. Criterion for plans to be geographic. Theorem 1. A plan is geographic if and only if it is even, connected and locally connected.
Proof of $\Rightarrow$. Let a plan be geographic, that is induced by a map. Then this plan is connected and graphs G_{a^0}, G_{a^2} are Eulerian for any $a^0 \in A^0$, $a^2 \in A^2$, according to §§13 – 15. Thus, considered plan is also Eulerian, that is even and locally connected, according to Lemmas 1-3.

Proof of $\Leftarrow$is more complicated and will be given later in §23.

Remark. A connected and locally connected plan can be induced by different maps on the same surface or even on different surfaces; see example in §11.9. Still the Euler characteristic is unambiguously determined by the plan, according to equality $\chi = \#A_0 - \#A_1 + \#A_2$. Thus in any case there exist not more than two such surfaces, there is only one if $\chi = 2$ or $\chi = 2i + 1 < 2$, and there is none if $\chi > 2$.

In particular, inequality $\#A_0 - \#A_1 + \#A_2 \leq 2$ holds for any plan which is even connected and locally connected, because otherwise there exists no surface for a corresponding map.

18. The plane representation of maps. The classical combinatorial approach to surfaces will be considered here briefly; see more details in [3], chapter 3. Fix few polygons on a plane. Each one can have any number of edges including even 2 and 1. The sum of all these numbers is supposed to be even, and the set of all the edges is supposed to be divided into pairs. Fix an arbitrary direction for each edge and denote the edges directed clockwise by a, b, $c, \cdots$ and counterclockwise by $\overline{a}$, $\overline{b}$, $\overline{c}, \cdots$ Two edges in any pair are supposed to be denoted by the same letter. Thus in the set of all the polygon each letter occurs twice. It is supposed also that this property does not hold for the subsets, that is for any subset of the set of all the polygons there exist a letter which occurs once. A surface will be obtained if one glues all the pairs of edges denoted by the same letter in accordance with their directions. The last assumption provides the obtained surface to be connected, but not a disjunctive sum of few different surfaces. Note also that except for the surface one obtaines a map on it. For this map the number of countries is equal to the number of polygons; the number of edges is equal to the number of different letters; the number of vertices is more difficult to compute, but still it is unambiguously determined by the identification of edges in polygons.

The following operations preserve both the surface and the map.

a) A cyclic shift of letters in a polygon. For example,

$$(a\ \overline{a}\ c\ d\ \overline{c}\ b\ e\ f\ \overline{f}\ \overline{e}) \qquad\qquad (d\ \overline{c}\ b\ e\ f\ \overline{f}\ \overline{e}\ a\ \overline{a}\ c).$$

b) Reorientation of an edge: $a \to \overline{a}$, $overlinea \to a$
c) Reorientation of a polygon. That is a combination of two operations: the redirection of all the edges of the polygon according to b), and replacement of the cyclic order of letters in the polygon by the inverse one. For example, $(a\ c\ \overline{c}\ b\ d\ \overline{e})$ $(e\ \overline{d}\ \overline{b}\ c\ \overline{c}\ \overline{a})$.

A map and the corresponding surface are called **orientable** if there exist orientations of all the polygons such that each edge occurs in both directions. In other words, it is possible to apply few times operation c) above in such a way that any letter a will occur twice only in combination $(a,\ \overline{a})$, but not $(a,\ a)$ or $(\overline{a},\ \overline{a})$.

One can find out in [3] more operations which preserve only the surface given by a map but not the map itself. These operations enable us to obtain the classification given in §3. The surfaces S_p and C_q can be represented by the following normal forms.

$$S_p = (a_1\ b_1\ \overline{a}_1\ \overline{b}_1\ a_2\ b_2\ \overline{a}_2\ \overline{b}_2\ \cdots\ a_p\ b_p\ \overline{a}_p\ \overline{b}_p),\ S_0 = (a\ \overline{a});$$

$$C_q = (c_1\ c_1\ c_2\ c_2\ \cdots\ c_q\ c_q);\ p, q \in \{1, 2, \cdots\}. \tag{4}$$

Note that each map contains only one polygon and only one vertex. For this reason the Euler characteristics are given by formula (1).
Surfaces S_p are orientable and C_q are not.

19. The maps inducing a given plan. Fix a plan $P = < G_{01}, G_{21} >$ which is even connected and locally connected. Consider all the graphs of 2-cells G_{a^2}, $a^2 \in A^2$

. Remind that each 0-cell $a^0 \in A^0$ occurs in these graphs at least once but maybe more, and each 1-cell $a^1 \in A^1$ occurs exactly twice; see §§13 – 15. Fix directions of all the edges (1-cells) in the graphs G_{a^2}, $a^2 \in A^2$ arbitrarily, but one condition: the directions of doubled edges must be the same. Remind that all the graphs G_{a^2}, $a^2 \in A^2$ are Eulerian and chose an arbitrary Eulerian cycle in each one. These cycles induce polygons and these polygons induce a map, according to§18. Theorem 1 seems proved, however there is one important "but". The sets of countries and edges of the map are the sets A^2 and A^1 respectively, but the set of vertices can differ from A^0. Of course, different vertices of A^0 can not be glued in the map, but the same vertex can occur few times. The maps with identified vertices and countries (mm-maps) will be considered later; see§§25 – 28. But Theorem 1 claim the existence of a standard map, and it is not still proved. Thus our aim is to chose Eulerian cycles so that to minimize the number of vertices and make it equal to $\#A_0$, or equivalently, to minimize Euler characteristic and make it equal to $\#A_0 - \#A_1 + \#A_2$.

20. Examples. Considered plans will be given by their bimatrices .

20.1. Let $B = (2|2)$, then $\#A_0 = \#A_1 = \#A_2 = \chi = 1$ Graph G_{a^2} of the unique 2-cell $a^2 \in A^2$ consists of the unique 0-cell $a^0 \in A_0$ and the unique 1-cell $a = a^1 \in A^1$, which is a loop. This loop must be doubled in G_{a^2} ; see §15. There are two Eulerian cycles in G_{a^2} :$(a\ a)$ and $(a\ \overline{a})$,
The first one $(a\ a)$generates a map on projective plane C_1. It is easy to check that this map has only one vertex, and thus really induces the considered plan given by bimatrix $B = (2|2)$.
The second cycle $(a\ \overline{a})$ generates a map on sphere S_0 with two vertices, which must be identified because there is only one vertex in the plan. (Thus, it is not really a map but a mm-map; see§§25 – 28.)

20.2 Let $B = \begin{pmatrix} 2 & | & 2 \\ 2 & | & 2 \end{pmatrix}$, then $\#A_0 = \#A_2 = 1$, $\#A_1 = 2$,$\chi = 0$. Now graph G_{a^2} contains not one but two doubled loops a and b. There exist five different Eulerian cycles $(a\ b\ \overline{a}\ \overline{b})$, $(a\ a\ b\ b)$, $(a\ b\ \overline{b}\ \overline{a})$, $(a\ a\ b\ \overline{b})$, $(a\ b\ a\ b)$, The first two cycles generate the normal form maps on torus S_1 and on Klein bottle C_2 respectively. The third one generates a map on sphere S_0 with three vertices, which must be identified because there is only one vertex in the plan. The last two cycles generate two different (not isomorphic) maps on projective plane C_1; each map has two vertices which also must be identified.

20.3 Let $B = \begin{pmatrix} 2 & | & 2 \\ 2 & | & 2 \\ 2 & | & 2 \end{pmatrix}$ then $\#A_0 = \#A_2 = 1$,$\#A_1 = 3$, $\chi = -1$. Graph G_{a^2} contains three doubled loops a, b and c. Eulerian cycles $(a\ a\ b\ b\ c\ c)$, $(a\ b\ c\ c\ b\ a)$ generate two different maps on the same surface C_3. Cycle $(a\ b\ c\ a\ b\ c)$ generates a maps on projective plane C_1 with three vertices which must be identified.

20.4 Let $B = \begin{pmatrix} 2 & | & 2 \\ 2 & | & 2 \\ 2 & | & 2 \\ 2 & | & 2 \end{pmatrix}$ then $\#A_0 = \#A_2 = 1$,$\#A_1 = 4$, $\chi = -2$. Graph G_{a^2} contains four doubled loops a, b, c and d. Eulerian cycles $(a\ b\ \overline{a}\ \overline{b}\ c\ d\ \overline{c}\ \overline{d})$ $(a\ b\ c\ d\ \overline{a}\ \overline{b}\ \overline{c}\ \overline{d})$ and $(a\ a\ b\ b\ c\ c\ d\ d)$, $(a\ a\ b\ b\ c\ d\ d\ c)$ generate two pairs of different maps on two different surfaces: S_2 and on C_4 respectively. Each map has only one vertex.

20.5. Let us generalize the four previous examples and consider $\#A_0 = \#A_2 = 1$, $\#A_1 = r$, $\chi = 2 - r$. If r is even then this plan is induced by different maps on two different surfaces S_p and C_q where $p = \frac{r}{2}$ and $q = r$. Between these maps there are the normal forms of these two surfaces respectively given by two cycles

$$(a_1^1\ a_2^1\ \overline{a}_1^1\ \overline{a}_2^1\ a_3^1\ a_4^1\ \overline{a}_3^1\ \overline{a}_4^1\ \cdots\ a_{r-1}^1\ a_r^1\ \overline{a}_{r-1}^1\ \overline{a}_r^1),\quad (a_1^1\ a_1^1\ a_2^1\ a_2^1\ \cdots\ a_r^1\ a_r^1)$$

If r is odd then maps on S_p no longer exist. The same plan can be induced by different maps on any surface. For C_3, C_4 and S_2 it was demonstrated by examples 20.3 and 20.4. The following example will demonstrate this fact for S_0.

20.6 Let $A_1 = \{a,\ b,\ c,\ d,\ e,\ f\}$ $\#A_1 = 6$, $\#A_0 = 7$, $\#A_2 = 1$; then $\chi = 7 + 1 - 6 = 2$, graph $G_{01} = G_{a^2}$ must be a tree on a sphere. It is not difficult to check that two Eulerian cycles

$$(a\ \overline{a}\ c\ d\ \overline{d}\ \overline{c}\ b\ \overline{b}\ e\ f\ \overline{f}\ \overline{e})\ and\ (a\ \overline{a}\ b\ \overline{b}\ c\ d\ \overline{d}\ \overline{c}\ e\ f\ \overline{f}\ \overline{e})$$

from graph G_{a^2}generate two maps which are not isomorphic, but the induced trees, and consequently plans, are equal. The above example is due to the following simple topological remark: isomorphism of two trees not always can be prolonged for the whole spheres containing these trees.

20.7. Let $B = \begin{pmatrix} 11 & | & 2 \\ 11 & | & 2 \\ 11 & | & 2 \end{pmatrix}$, then $\#A_0 = 2$, $\#A_2 = 1$, $\#A_1 = 3$, $\chi = 0$. Graph G_{a^2} contains two vertices a_2^0, a_1^0 and three doubled edges a, b and c connecting them. Let us direct a and $c$from a_1^0 to a_2^0, and b from a_2^0 to a_1^0. Check that $(a\ b\ c\ \overline{a}\ \overline{b}\ \overline{c})$ and $(a\ b\ c\ b\ a\ \overline{c})$ are Eulerian cycles, which generate two different maps on a torus and on a Klein bottle respectively.

20.8. Let $B = \begin{pmatrix} 20 & | & 2 \\ 11 & | & 2 \\ 11 & | & 2 \end{pmatrix}$, then $\#A_0 = 2$, $\#A_2 = 1$, $\#A_1 = 3$, $\chi = 0$. Graph G_{a^2} contains two vertices a_1^0, a_2^0, one doubled loop a incident to a_1^0 and two doubled edges b and c . Let us direct them oppositely, b from a_1^0 to a_2^0 and c from a_2^0 to a_1^0. Check that $(a\ b\ c\ \overline{a}\ \overline{c}\ \overline{b})$ and $(a\ a\ b\ c\ b\ c)$ are Eulerian cycles, which generate two different maps on a torus and Klein bottle respectively.

20.9. Let $B = \begin{pmatrix} 20 & | & 2 \\ 20 & | & 2 \\ 11 & | & 2 \end{pmatrix}$, and again $\#A_0 = 2$, $\#A_2 = 1$, $\#A_1 = 3$, $\chi = 0$. Graph

G_{a^2} contains two vertices a_1^0, a_2^0 , two doubled loops a and b incident to a_1^0 and one doubled edge $c$directed from a_1^0 to a_2^0. Check that $(a\ b\ \overline{a}\ \overline{b}\ c\ \overline{c})$ and $(a\ a\ b\ b\ c\ \overline{c})$ are Eulerian cycles, which generate two different maps on a torus and Klein bottle respectively.

21. A geographic plan which can be induced by maps on a Klein bottle but not on a torus. Let us consider geographic plans with three 1-cells and zero Euler characteristic, that is $\#A_0 - \#A_1 + \#A_2 = 0$, $\#A_1 = 3$. There are four such plans $B^1 = \begin{pmatrix} 11 & | & 2 \\ 11 & | & 2 \\ 11 & | & 2 \end{pmatrix}$,

$B^2 = \begin{pmatrix} 20 & | & 2 \\ 11 & | & 2 \\ 11 & | & 2 \end{pmatrix}$, $B^3 = \begin{pmatrix} 20 & | & 2 \\ 20 & | & 2 \\ 11 & | & 2 \end{pmatrix}$, $B^4 = \begin{pmatrix} 20 & | & 2 \\ 02 & | & 2 \\ 11 & | & 2 \end{pmatrix}$

(and also of course four dual plans, which can be obtained from $B^1 - B^4$ by permutation of matrices B_{01} and B_{21}). The first three plans can be induced by maps on toruses; see examples 20.7-20.9 and also [4]. Let us check that the fourth plan can not. Really, there is only one country. Its graph G_{a^2} consists of two vertices, two doubled loops a and b incident to each one, and one doubled edge c joining both. (This graph looks like "spectacles".) There is only one Eulerian cycle $(a\ \overline{a}\ c\ b\ \overline{b}\ \overline{c})$ which induce a map on **orientable** surface; see §18. But this surface is a sphere with two identified vertices, not a torus.
At the same time there exist Eulerian cycle in G_{a^2} which induces nondegenerate maps on a Klein bottle, for example $(a,\ a,\ c,\ b,\ b,\ \overline{c})$.
Thus Theorem 1 will not hold if we restrict ourself by orientable surfaces only and the following four problems appear: in the set of all the geographic plans to find the subsets of plans which can be induced by maps on: a) orientable surfaces, b) nonorientable surfaces, c) both, d) some. The last problem is solved by Theorem 1, and all four problems are solved by this theorem provided χ is odd or $\chi \geq 2$, because then not more than one surface can exist.

22. A proof of Lemma 1 was proposed in 1736 by Euler in the his-torically first publication on graph theory dealing with the problem of circuits of the K nigsberg bridges.
a) $\Rightarrow$ b). Let us go along an Eulerian cycle. It is divided by any vertex into a sum of cycles with pairwisely disjoint edge-sets. Apply the same approach to each of these cycles and so on. Thus a given Eulerian cycle can be split into edge-disjoint simple cycles. Note that they must have common vertices because the graph is connected.
b) $\Rightarrow$ a). Chose any two cycles with a common vertex. They can be united into one cycle (not simple). It can be even done by at least two different ways. This remark is of no use now but it will be a matter of principle when we shall prove Theorem 1 in §23. Now we can use any of two ways. Then chose another two cycles with a common vertex and so on. We shall finish with only one cycle left, which must be Eulerian.
b) $\Rightarrow$ c). The degree of a vertex in a simple cycle is equal to 2.

c) $\Rightarrow$ b). Let us go along the edges of the graph. There can not be dead ends because all the degrees are even. Fix the first time we have come twice to the same vertex. Then our route contains a cycle which must be simple. Subtract the edges of this cycle from the graph. In the obtained graph the degrees of all the vertices are even again. Repeat the whole procedure from the beginning and so on.

23. The rest of the proof of Theorem 1. Implication $\Rightarrow$ was proved in §17. A proof of $\Leftarrow$ was started in §19. Arbitrary Eulerian cycles were chosen in graphs G_{a^2}, $a^2 \in A^2$. They generate a map according to §18. But this map possibly has duplicated vertices and induces the given plan only if these vertices are identified; see examples in §20.
In other words, the map induces a partition of the Eulerian graphs G_{a^0}, $a^0 \in A^0$ into simple cycles with pairwisely disjoint sets of edges. If there is only one cycle (Eulerian) for any $a^0 \in A^0$ then O.K., but if there are few cycles for at least one vertex then the obtained map is degenerate. However it can be reconstructed.
Now an algorithm is suggested which will diminish one by one the total number of cycles in graphs G_{a^0}, $a^0 \in A^0$ until all these cycles became Eulerian, while the cycles in graphs G_{a^2}, $a^2 \in A^2$ are changed but still remain Eulerian. This algorithm is analogous to Eulerian one; see §22, b) $\Rightarrow$ a).
Suppose all the cycles are already Eulerian then the present map is nondegenerate and O.K. Suppose not. Then there exist $a^0 \in A^0$, $a^2 \in A^2$ such that graph G_{a^0} contains two cycles with common 2-cell a^2. Then the Eulerian cycle of graph G_{a^2} can be split into two cycles with common 0-cell a^0 . Thus we have two "eights" respectively in the graphs G_{a^0} and G_{a^2} and with a^2 and a^0 in the middle.
For each "eight" there exist three different circuits; one of them ("bad") consists of two separate cycles, and two others ("nice") consist of one cycle each. Really, start the circuit of an "eight" in the middle of it; go along the first cycle in any one of two possible directions; return again to the middle; then go along the second cycle; now two directions are no longer equivalent; thus two different "nice" cycles can be obtained.
So we have two "eights". The associated two triplets of circuits are in correspondence. In each triplet there are two "nice" circuits and only one "bad". Thus we can chose two corresponding circuits, one from each triplet. In other words we can unite two cycles in graph G_{a^0} by two different ways, one of them spoils the Eulerian cycle of graph G_{a^2} but the other one does not, that is the cycle changes but still remains Eulerian.
Then chose another vertex $a^{0\prime} \in A^0$ and two cycles with a common vertex in $G_{a^{0\prime}}$ and so on, until all the cycles in graphs G_{a^0}, $a^0 \in A^0$ and all the corresponding cycles in graphs G_{a^2}, $a^2 \in A^2$ be Eulerian.
Consider the simplest example. Let plan P is given by bimatrix $B = (2|2)$, then $\#A_0 = \#A_1 = \#A_2 = \chi = 1$; see §20.1. Both graphs G_{a^0} and G_{a^2} are "simple eights", that is each one consists of doubled loop a. The corresponding two triplets are $\{(a\ \overline{a}),\ (a\ a),\ ((a),\ (a))\}$ and $\{((a),\ (a)),\ (a\ a),\ (a\ \overline{a})\}$ respectively. Thus Eulerian cycle $(a\ \overline{a})$ in graph G_{a^2} does not fit because it generates a map on a sphere with two identified vertices instead of one. But the next Eulerian cycle $(a\ a)$ generates the

normal form map for a projective plane. This map has only one vertex and realizes plan P.

24. An algorithm for the construction of a map realizing a given geographic plan is obtained. This algorithm reduces to the construction of two associated sets of Eulerian cycles in two given collections of Eulerian graphs $\{G_{a^0}, a^0 \in A^0\}$ and $\{G_{a^2},\ a^2 \in A^2\}$. One can use the standard method of successive union of cycles with a common vertex applied by Euler in 1736. But only one from two possible ways of such a union should be chosen on each step.

25. Multiplanet and multicolored maps (mm-maps). It will be convenient to expand the class of maps so that any even plan could be induced. A union of maps on few different surfaces (planets) will be called a multiplanet map. Then let us color all the vertices $a^0 \in A^0$ and all the countries $a^2 \in A^2$ of such a map with two different sets of colors Co^0 and Co^2, where $Co^0 \cap Co^2 = \emptyset$. We assume that all the vertices and all the countries of the same color are identified.
A coloring will be called:
proper if any two vertices (countries), incident to the same edge, that is adjacent, are colored with different colors;
perfect if any two vertices (countries), incident to the same country (vertex), that is neighbor, are colored with different colors;
motley if any two vertices (countries) are colored with different colors. A motley coloring is perfect, and a perfect coloring is proper.
A mm-map is referred to as a map if it is singleplanet and motley colored.

26. The corresponding operations with plans and their bimatrices. Let us consider an arbitrary plan $P =< G_{01}, G_{21} >$, where $G_{01} =< A^0, A^1 >$ and $G_{21} =< A^2, A^1 >$. Then color all the 0-cells $a^0 \in A^0$ and all the 2-cells $a^2 \in A^2$ with two different sets of colors Co^0 and Co^2, where $Co^0 \cap Co^2 = \emptyset$. Then identify all the 0-cells and all the 2-cells of the same color and denote the obtained plan by $P' =< G'_{01}, G'_{21} >$.
Now let us consider bimatrices $B =< B_{01}, B_{21} >$ and $B' =< B'_{01}, B'_{21} >$ corresponding to the plans P and P'. Note that B' can be obtained from B by the following simple operations. For each color $c \in Co^0$, (resp. $c \in Co^2$) replace all the columns in matrix B_{01} (resp. B_{21}), corresponding to the 0-cells (2-cells) colored with c , by one column equal to the sum of all these columns. Let us consider the examples

$$B = \begin{pmatrix} 101 & | & 200 \\ 020 & | & 020 \\ 002 & | & 101 \end{pmatrix} \to B' = \begin{pmatrix} 11 & | & 20 \\ 20 & | & 02 \\ 02 & | & 11 \end{pmatrix};\ (11|11) \to (2|11) \Rightarrow (2|2).$$

Note that the first plan is locally connected but not connected, while the second one is connected but not locally connected; the third plan is not even, while all the others are even. Remind also that the sum of elements in each line is equal to 2 for any plan.

Lemma 4. If plan P is even then P' is also even.
Proof. Let b'_{01}, b''_{01} be two columns from B_{01} and $b_{01} = b'_{01} + b''_{01}$. Then $< b_{01}, b_{21} >=< b'_{01} + b''_{01}, b_{21} >=< b'_{01}, b_{21} > + < b''_{01}, b_{21} >$ for any column b_{21} from B_{21}, and the sum of two even numbers is even.
Inverse implication does not hold; see the second example above. Still there is an important special case when it holds.
Lemma 5. Any plan P' is generated by the unique locally connected and properly colored plan P . Plans P and P' can be even only simultaneously.
Sketch of the proof. Let us chose in P' all the 0-cells from G_{01} (2-cells from G_{21}) which are not connected, split them into connected components, replace each component by a separate cell, and color this cells by the same color. The obtained plan P is locally connected and properly colored. To prove the second statement note that two columns of B'_{01} (resp. B'_{21}) are associated with adjacent 0-cells (2-cells) if and only if they do not intersect, that is their scalar product is equal to 0. Thus elements of matrices B and B' take the same values and there may be only more elements equal to 0 in B. The analogous statement holds for matrices B_{02} and B'_{02}.
See the first example above; note that P' is connected while P is not. For this reason we can not restrict ourself by consideration of connected plans only.

27. **Plans induced by mm-maps.** Let us fix a mm-map M. Exchange provisionally its coloring by the motley one. Denote the obtained multiplanet map by M'. Each planet of M' is a map, thus it induces a plan. Take the union of these plans for all the planets of M' and denote the obtained plan by $P' =< G'_{01}, G'_{21} >$. Identify in the graphs G'_{01} and G'_{21} respectively 0- and 2-cells, corresponding to vertices and countries of M' colored with the same color in M. Denote the obtained plan by $P = P(M)$.

28. **Criterion for plans to be mm-geographic. Theorem 2**. For any mm-map the corresponding plan is even, and any even plan is induced by a properly colored mm-map. Moreover any locally connected even plan is induced by a motley colored mm-map, but it can not be induced by a map which is colored perfectly but not motley.
Sketch of the proof. Let us fix an even plan P'; then construct the locally connected properly colored plan P , according to Lemma 5. (If P' is locally connected itself then $P = P'$.) Replace the coloring of P by the motley one. Each connected component of the obtained plan is induced by a map according to Theorem 1. Take the union of these maps realized by separate planets, and identify the colors backwards according to the coloring of P . The obtained mm-map induces P'.
All the operations applied above are unambiguous except for one: few maps and mm-maps can induce the same even connected and locally connected plan; see the proof of Theorem 1 in §23. Still, all the maps are motley colored by the definition, and there is no perfectly colored one between mm-maps considered in §23, because any Eulerian graph is connected and therefore coloring of P is either motley or not perfect.

29. Semiregular maps and plans. Plan $P =< G_{01}, G_{21} >$ will be called **semiregular** if it satisfies the following equivalent conditions.
a) There are no loops in graphs G_{01} and G_{21}
b) All the elements of the corresponding bimatrix are equal 0 or 1, but not 2. c) $G_{a^0} = G'_{a^0}$ $\forall a^0 \in A^0$ and $G_{a^2} = G'_{a^2}$ $\forall a^2 \in A^2$.

Map M will be called **semiregular** if it satisfies the following equivalent demands (see §§9, 13 for defininititions).
a) The corresponding plan is semiregular.
b) There are no loops and interior edges in M.
c) Circuits C_{a^0} and C_{a^2} are cycles for any $a^0 \in A^0$ and $a^2 \in A^2$.
d) Circuit C_{a^0} is a cycle but not a loop for any $a^0 \in A^0$.
e) Circuit C_{a^2} is a cycle but not a loop for any $a^2 \in A^2$.

30. Regular maps and plans. Plan $P =< G_{01}, G_{21} >$ will be called **regular** if it satisfies the following equivalent conditions.
a) P is semiregular and graphs G_{a^0} and G_{a^2} are simple cycles for any $a^0 \in A^0$ and $a^2 \in A^2$.
b) P is semiregular, graphs G_{a^0} and G_{a^2} are connected and $B_{02}(a^0, a^2) = deg(G_{a^0}, a^2) = deg(G_{a^2}, a^0)$ take only two values 0 and 2 for any $a^0 \in A^0$ and $a^2 \in A^2$.
Map M will be called **regular** if it satisfies the following equivalent conditions.
a) The corresponding plan is regular.
b) Circuits C_{a^0} and C_{a^2} are **simple** cycles for any $a^0 \in A^0$ and $a^2 \in A^2$.
c) Circuit C_{a^0} is a **simple** cycle but not a loop for any $a^0 \in A^0$.
d) Circuit C_{a^2} is a **simple** cycle but not a loop for any $a^2 \in A^2$.

Proposition 1. A map induced by a regular plan is unique (and by the definition regular).
Proof. Graphs G_{a^0} and G_{a^2} are simple cycles for any $a^0 \in A^0$ and $a^2 \in A^2$, thus circuits C_{a^0} and C_{a^2} are unique **simple** cycles.

31. Dual maps and plans. Plan P^* dual to P is defined by formulas
$P =< G_{01}, G_{21} >$, $G_{01} =< A^0, A^1 >$, $G_{21} =< A^2, A^1 >$;
$P^* =< G^*_{01}, G^*_{21} >$, $G^*_{01} =< A^{0*}, A^{1*} >$, $G^*_{21} =< A^{2*}, A^{1*} >$;
$A^{0*} = A^2$, $A^{2*} = A^0$, $G^*_{01} = G_{21}$, $G^*_{21} = G_{01}$.
Let $M =< S, G, \Phi >$ be a map. In the interior of each country select a vertex (capital), and in the interior of each edge select a vertex (custom-house). In every country a^2 connect the capital with all custom-houses by paths not intersecting the boundary of the country and each other. If an edge is interior then two different paths lead to the same custom-house. Then eliminate the custom-houses. The resulting map will be called dual of M and denoted by M^*.

Proposition 2.
a) A dual map M^* is unique and is defined on the same surface S.
b) $P((M))^* = P(M^*)$.
c) The operation * is an involution, that is $P^{**} = P$, $M^{**} = M$.
d) The operation * preserves the regularity and semiregularity of maps and plans.
e) Vertices (countries) of M^* correspond to countries (vertices) of M, and their circuits are also in correspondence.
f) Dual graphs G and G^* of maps M and M^* have the edge-sets of the same cardinality; there is a fixed one-to-one correspondence between these sets. Via this correspondence loops in G are associated with loops in G^* and conversely.
g) Pair of graphs G, G^* uniquely determines the plan of the map:

$$P(M) =< G, G^* >=< G_{01}, G_{21} >$$

Proof and more details see in [3],§4.2.

32. The problem of realizing of a given set of degrees by a map. Let us fix $\#A_1$ Then it follows from inequality $\chi \leq 2$ that $\#A_0$ and $\#A_2$ can take a finite number of integer positive values such that

$$\#A_0 + \#A_2 \leq \#A_1 + 2. \tag{5}$$

Consider two sets of integer positive numbers

$$p^0 = \{p_1^0, p_2^0, ..., p_{\#A_0}^0\}, \quad p^2 = \{p_1^2, p_2^2, ..., p_{\#A_2}^2\}. \tag{6}$$

The problem is to construct a map with sets of degrees of vertices and countries given by (6) or to prove that there is no such a map. All the degrees are unambiguously determined by the corresponding plan. Thus the problem can be radically simplified due to Theorem 1. The degrees of vertices and countries of a map are equal to the sums of elements in the columns of the corresponding bimatrix. Remind that the sum of elements in each line of each matrix must be equal to 2. Thus

$$\sum_{i=1}^{\#A_0} p_i^0 = \sum_{j=1}^{\#A_2} p_j^2 = 2\#A_1. \tag{7}$$

must hold otherwise there are no solutions. Note that there are only a finite number of solutions of (7) for any given $\#A_1$.
Proposition 3. Any two sets of degrees (6) which satisfies to (5,7) can be realized by an even plan (and consequently by a mm-map).
Proof will be constructive. The promised plan will be given by the corresponding bimatrix. We are going to chose its "bilines" one by one in such a way to eliminate all

the numbers in sets P^0 and p^2.
It will be done by two steps. At first we use bilines of the types $(2;\ 1,1)$ and $(1,1;\ 2)$ to eliminate all the odd degrees in P^0 and p^2. Then we use bilines of the type $(2;\ 2)$ to eliminate all the left even degrees. Any plan, obtained by this way, must be even because there were no bilines of the type $(1,1;\ 1,1)$ at all.
The second step is always possible, but the first one is not. Each number k_2 from p^0 (resp. p^2) enable us to convert $f(k)$ odd numbers from P^2 (resp. p^0) into even; where $f(k) = 2\lfloor \frac{k}{2} \rfloor$, that is $f(k)$ is equal to k or $k-1$ for even and odd k respectively. We can convert into even at least $2\#A_1 - \#A_0$ odd numbers from p^2 and $2\#A_1 - \#A_2$ odd numbers from p^0. Thus there are no problems if

$$2\#A_1 - \#A_0 \geq \#A_2 \textit{ and } 2\#A_1 - \#A_2 \geq \#A_0$$

Both these inequalities are equivalent to

$$\#A_0 +\ A_2 \leq 2\#A_1,\ i.e.\ \chi \leq \#A_1. \tag{8}$$

Note that $(8) \Leftarrow (5)$ if only $\#A_1 > 1$.
Remark. Proposition 3 holds even if we change (5) by a weaker inequality (8). This generalization is actual because the multiplanet maps really can really violate (5).
Thus it follows from (5,7) that (6) can be realized by an even plan and consequently by a mm-map. But it does not follow from (5,7) that (6) can be realized by a geographic (that is even, connected and locally connected plan) and consequently by a standard map. Really there are exclusions. They are given below for $\#A_1 \leq 3$.

For $\#A_1 = 1$ sets of degrees $(2;1,1)$ and $(2;2)$ are realized by geographic bimatrices $(2|11)$ and $(2|2)$ respectively; and set $(1,1;1,1)$ does not satisfy (5), because $\chi = 2+2-1 = 3 > 2$.

For $\#A_1 = 2$ sets of degrees
$(4;4),\ (4;3,1),\ (4;2,2),\ (4;2,1,1),\ (3,1;3,1),\ (2,2;2,2)$
are realized respectively by the following geographic bimatrices

$$\begin{pmatrix} 2 & | & 2 \\ 2 & | & 2 \end{pmatrix},\ \begin{pmatrix} 2 & | & 20 \\ 2 & | & 11 \end{pmatrix},\ \begin{pmatrix} 2 & | & 11 \\ 2 & | & 11 \end{pmatrix},$$

$$\begin{pmatrix} 2 & | & 110 \\ 2 & | & 101 \end{pmatrix},\ \begin{pmatrix} 20 & | & 11 \\ 11 & | & 20 \end{pmatrix},\ \begin{pmatrix} 11 & | & 11 \\ 11 & | & 11 \end{pmatrix},$$

But set $(3,1;2,2)$ is the exclusion. It can not be realized by a geographic plan, though it satisfies (5,7). All the left sets, which satisfy (7), do not satisfy (5).

For $\#A_1 = 3$ sets of degrees
$(5,1;5,1),\ (5,1;4,2),\ (5,1;3,3),\ (4,2;4,2),\ (4,2;3,3);$

$(5,1;4,1,1),\ (5,1;3,2,1),\ (4,2;3,2,1),\ (3,3;4,1,1),\ (3,3;2,2,2);$
$(6;3,1,1,1),\ (6;2,2,1,1)$
are realized respectively by the following geographic bimatrices

$$\begin{pmatrix} 20 & \mid & 11 \\ 20 & \mid & 20 \\ 11 & \mid & 20 \end{pmatrix}, \begin{pmatrix} 20 & \mid & 11 \\ 20 & \mid & 11 \\ 11 & \mid & 20 \end{pmatrix}, \begin{pmatrix} 20 & \mid & 11 \\ 20 & \mid & 20 \\ 11 & \mid & 02 \end{pmatrix}, \begin{pmatrix} 20 & \mid & 20 \\ 11 & \mid & 11 \\ 11 & \mid & 11 \end{pmatrix}, \begin{pmatrix} 20 & \mid & 11 \\ 11 & \mid & 11 \\ 11 & \mid & 11 \end{pmatrix};$$
$$\begin{pmatrix} 20 & \mid & 101 \\ 20 & \mid & 110 \\ 11 & \mid & 200 \end{pmatrix}, \begin{pmatrix} 20 & \mid & 011 \\ 20 & \mid & 110 \\ 11 & \mid & 200 \end{pmatrix}, \begin{pmatrix} 20 & \mid & 101 \\ 11 & \mid & 110 \\ 11 & \mid & 110 \end{pmatrix}, \begin{pmatrix} 20 & \mid & 110 \\ 02 & \mid & 101 \\ 11 & \mid & 200 \end{pmatrix}, \begin{pmatrix} 11 & \mid & 110 \\ 11 & \mid & 101 \\ 11 & \mid & 011 \end{pmatrix};$$
$$\begin{pmatrix} 2 & \mid & 1100 \\ 2 & \mid & 1010 \\ 2 & \mid & 1001 \end{pmatrix}, \begin{pmatrix} 2 & \mid & 1100 \\ 2 & \mid & 1010 \\ 2 & \mid & 0101 \end{pmatrix}.$$

There are also the following five exclusions:
$(3,3;3,3),\ (3,3;3,2,1),\ (4,2;4,1,1),\ (4,2;2,2,2),\ (5,1;2,2,2).$

And all the left sets are either trivial or do not satisfy (5).
Note that only one set and plan from any dual pair was considered.
Proposition 4. If plan $P =< G_{01}, G_{21} >$ is is connected and locally connected then both graphs G_{01} and G_{21} must be connected.
Proof. Suppose G_{01} is not connected. If there exists a 2-cell from G_{21} which is incident to two different components of G_{01} then plan P is not locally connected. If each 2-cell from G_{21} is incident to only one component of G_{01} then plan P is not connected.

33. Platonous maps, plans and bimatrices are defined by equalities

$$P_i^0 = p^0 \ \ \forall\, i = 1,2,...,\#A_0; \quad P_j^2 = p^2 \ \ \forall\, j = 1,2,...,\#A_2. \tag{9}$$

In other words all the degrees of vertices and of countries must be equal. Formulas (7,9) provide the following replacement of variables

$$\begin{gathered}(P^0, p^2, \chi) \ \leftrightarrow \ (\#A_0, \#A_1, \#A_2);\\ p^0 = 2\#A_0/\#A_1,\ p^2 = 2\#A_1/\#A_2,\ \chi = \#A_0 - \#A_1 + \#A_2;\\ \#A_0 = \chi/(p^0\, r),\ \#A_2 = \chi/(p^2\, r),\ \#A_1 = \chi/(2r).\\ r = \frac{1}{p^0} + \frac{1}{p^2} - \frac{1}{2} = (\#A_0 - \#A_1 + \#A_2)/(2\#A_1) = \chi/(2\#A_1).\end{gathered} \tag{10}$$

A Platonous map can exist only if numbers $\#A_0$ and $\#A_2$ are dividers of $2\#A_1$. But this is not sufficient. Consider the following example

$$(\#A_0, \#A_1, \#A_2) = (2,3,2); \quad (p^0, p^2, \chi) = (3,3,1). \tag{11}$$

Then there exists only one even plan with such degrees. This plan is given by bimatrix $B = \begin{pmatrix} 20 & | & 11 \\ 11 & | & 02 \\ 02 & | & 11 \end{pmatrix}$; connected. Thus there exists no map with given degrees $(3,3;3,3)$, in accordance with §32.

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