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Stable Effectivity Functions and Perfect Graphs [1]

by

E. Boros
RUTCOR
Rutgers University, P.O.Box 5062
New Brunswick, NJ 08903
(boros@rutcor.rutgers.edu)

V. Gurvich[2]
DIMACS & RUTCOR
Rutgers University, P.O.Box 5062
New Brunswick, NJ 08903
(gurvich@rutcor.rutgers.edu)

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Abstract

An effectivity function $\mathcal{E}$ is called *stable* if the core $C(\mathcal{E}, u)$ is not empty for any payoff u. The problem to characterize stable effectivity functions seems, in general, very difficult. In this paper we apply a graph theoretic approach to this problem. Using a graph based model we obtain some necessary and some sufficient conditions for stability in terms of perfect graphs. We also demonstrate that the conjecture by Berge and Duchet (1983) related to perfect graphs and kernels, in fact, is a special case of the considered problem of stability of effectivity functions.

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[2]On leave from the International Institute of Earthquake Prediction Theory and Mathematical Geophysics, Moscow, Russia

1 Basic notions

Let us recall first some of the basic definitions, notations as well as some well-known results of game theory, see e.g. Moulin (1983), Peleg (1984a).

1.1 Effectivity functions

Let I be the set of *players* and A the set of *outcomes*. Subsets $K \subseteq I$ of the players will be called coalitions, and subsets $B \subseteq A$ of the outcomes will be called *blocks*. An *effectivity function* (EFF) is defined as a mapping

$$\mathcal{E} : 2^I \times 2^A \to \{0, 1\}.$$

In other words, an EFF is a Boolean function on the set of variables $I \cup A$. We can treat it as well as a subset of $2^I \times 2^A$, i.e a set of pairs (K, B) at which $\mathcal{E}(K, B) = 1$. The notations $\mathcal{E}(K, B) = 1$ ($\mathcal{E}(K, B) = 0$) is read "*coalition K is effective* (*is not effective*) *for block B*" and means that coalition K is able to guarantee that an outcome from B will be chosen. We shall also consider effectivity functions given by an explicit list $\mathcal{E} = \{(K_j, B_j) | j \in J\}$ of all pairs of coalitions and blocks for which $\mathcal{E}$ is effective, i.e. in this case $\mathcal{E}(K, B) = 1$ if and only if $(K, B) = (K_j, B_j)$ for some $j \in J$.

An EFF is called *K-monotone* if $\mathcal{E}(K, B) = 1$ and $K \subseteq K'$ implies $\mathcal{E}(K', B) = 1$. Similarly, $\mathcal{E}$ is called *B-monotone* if $\mathcal{E}(K, B) = 1$ and $B \subseteq B'$ implies $\mathcal{E}(K, B') = 1$. Finally, $\mathcal{E}$ is called *monotone* if it is both K- and B-monotone, i.e. if

$$\Big(\mathcal{E}(K, B) = 1, \quad K \subseteq K', \, B \subseteq B'\Big) \implies \mathcal{E}(K', B') = 1.$$

An EFF is called *superadditive* if $\mathcal{E}(K_1, B_1) = 1$, $\mathcal{E}(K_2, B_2) = 1$ and $K_1 \cap K_2 = \emptyset$ imply that $\mathcal{E}(K_1 \cup K_2, B_1 \cap B_2) = 1$. Analogously, it is called *subadditive* if $\mathcal{E}(K_1, B_1) = 1$, $\mathcal{E}(K_2, B_2) = 1$ and $B_1 \cap B_2 = \emptyset$ imply that $\mathcal{E}(K_1 \cap K_2, B_1 \cup B_2) = 1$. Finally, $\mathcal{E}$ is called *convex* if $\mathcal{E}(K_1, B_1) = 1$ and $\mathcal{E}(K_2, B_2) = 1$ imply either $\mathcal{E}(K_1 \cup K_2, B_1 \cap B_2) = 1$ or $\mathcal{E}(K_1 \cap K_2, B_1 \cup B_2) = 1$, or both.

An EFF $\mathcal{E}$ is called *additive*, if it is a threshold Boolean function. Additive effectivity functions are obviously monotone and convex.

A pair of arguments (K, B) will be called *boundary* (or *corner*) if one (or two) equations hold out of the following four: $K = \emptyset$, $K = I$, $B = \emptyset$, and $B = A$. *Boundary conditions* are defined by the following formulae:

$$\begin{array}{lll} \mathcal{E}(I, B) & = 1 & \forall B \neq \emptyset, \\ \mathcal{E}(K, A) & = 1 & \forall K \neq \emptyset, \\ \mathcal{E}(\emptyset, B) & = 0 & \forall B \subseteq A, \\ \mathcal{E}(K, \emptyset) & = 0 & \forall K \subseteq I. \end{array} \tag{1.1}$$

The *dual* of an EFF $\mathcal{E}$ is defined by $\mathcal{E}^d(K, B) = \mathcal{E}(B, K)$ (see Peleg (1984)).

In other words, *duality* d is introduced by substituting A, a, B in place of I, i, K respectively, and vice versa. Obviously, duality d is an involution, i.e. $(\mathcal{E}^d)^d = \mathcal{E}$. Futhermore, the following relations hold

$$\begin{aligned}
&\mathcal{E}_1 \leq \mathcal{E}_2 \text{ iff } \mathcal{E}_1^d \leq \mathcal{E}_2^d;\\
&\mathcal{E} \text{ is K-monotone iff } \mathcal{E}^d \text{ is B-monotone;}\\
&\mathcal{E} \text{ is monotone iff } \mathcal{E}^d \text{ is monotone;}\\
&\mathcal{E} \text{ is superadditive iff } \mathcal{E}^d \text{ is subadditive;}\\
&\mathcal{E} \text{ is convex iff } \mathcal{E}^d \text{ is convex.}
\end{aligned}$$

Another involutive mapping $*$ was introduced by Gurvich (1973,1975) as follows:

$$\mathcal{E}^*(K, B) = \mathcal{E}(I \setminus K, A \setminus B)\ \forall K \subseteq I, B \subseteq A.$$

An EFF $\mathcal{E}$ is called **-selfdual* if $\mathcal{E}^* = \mathcal{E}$, i.e. if $\mathcal{E}(K, B) = \mathcal{E}(I \setminus K, A \setminus B)$ for all $K \subseteq I,\ B \subseteq A$.

Both definitions of duality play an important role in game theory, see e.g. Moulin (1983), Peleg (1984 a,b), Gurvich (1973,1975,1990,1992,1995).

1.2 Game forms and their effectivity functions.

Let X_i denotes a finite set of strategies of the player $i \in I$, and let $X = \times_{i \in I} X_i$. Let $g : X \to A$ be a mapping, which assign an outcome $a \in A$ to every *strategy-vector* $(x_i \mid i \in I)$. This mapping g is not supposed to be injective, in other words, the same outcome can be assigned to different strategy-vectors. The mapping g (or sometimes the quadruple $< I, A, X, g >$) is called *game form*. Thus, a game form is "something like a game in normal form but without payoff which are not yet specified". To any game form g we can associate an EFF $\mathcal{E}_g$, defined by the following formula:

$$\mathcal{E}_g(K, B) = 1 \iff (\exists x_K \mid g(x_K, x_{I \setminus K}) \in B\ \forall x_{I \setminus K}))$$

for all $K \subseteq I,\ B \subseteq A$, where $x_K = \{x_i | i \in K\}$ and $x_{I \setminus K} = \{x_i | i \notin K\}$ are the strategy-vectors of the coalitions K and $I \setminus K$, respectively. In other words, we set $\mathcal{E}_g(K, B) = 1$ if and only if the members of the coalition K can select strategies $x_i \in X_i,\ i \in K$ in such a way that for any selection of strategies of all other players of $I \setminus K$, the resulting outcome will belong to the block B.

Let us call a mapping $u : I \times A \to \mathbb{R}$ a *utility function*. The real value $u(i, a)$ will be interpreted as the profit of player $i \in I$ in case outcome $a \in A$ is realized. A pair $< g, u >$ defines a game in *normal form*.

Remark 1 *Note that at this point players and outcomes are no more symmetric (dual). The preferences of a player (e.g. voter) on the set of outcomes (e.g. candidates) do matter, but not vice versa.*

1.3 Playable and playing-minor effectivity functions

An EFF is called *playable* if it is realized by a game form, i.e. if there exists a game form g such that $\mathcal{E} = \mathcal{E}_g$.

Theorem 1 (Moulin and Peleg (1982)) *An EFF is playable iff it is monotone, superadditive and satisfies the boundary conditions (1.1).*

An EFF $\mathcal{E}$ will be called a *playing-minor* if it is majorized by a playble EFF, i.e. if there exists a game form g such that $\mathcal{E} \leq \mathcal{E}_g$. The following theorem provides a necessary and sufficient condition for an effectivity function to be a playing-minor.

Theorem 2 (Gurvich and Vasin (1978)) *An effectivity function $\mathcal{E} = \{(K_j, B_j)|j \in J\}$ is a playing-minor iff the following condition holds:*

(*) *For every nonempty subset $J' \subseteq J$ for which the corresponding coalitions are pairwise disjoint (i.e. $K_{j_1} \cap K_{j_2} = \emptyset$ for every $j_1, j_2 \in J'$, $j_1 \neq j_2$), the corresponding blocks have a nonempty intersection (i.e. $\cap_{j \in J'} B_j \neq \emptyset$).*

Proof. For completeness we provide here a short proof based on Theorem 1.

Let us suppose first that $\mathcal{E}$ is a playing-minor, i.e. $\mathcal{E} \leq \mathcal{E}_g$ for some game form g, and let us assume that condition (*) does not hold. In other words, for some $J' \subseteq J$ we have $\mathcal{E}(K_j, B_j) = 1$ for all $j \in J'$, $K_{j_1} \cap K_{j_2} = \emptyset$, for all $j_1, j_2 \in J'$, $j_1 \neq j_2$, and $\bigcap_{j \in J'} B_j = \emptyset$.

Then $\mathcal{E}_g(K_j, B_j) = 1 \forall j \in J'$ is implied by $\mathcal{E} \leq \mathcal{E}_g$. The effectivity function $\mathcal{E}_g$ is monotone, superadditive and satisfies the boundary conditions, according to Theorem 1. Thus, $\mathcal{E}_g(\bigcup_{j \in J'} K_j, \bigcap_{j \in J'} B_j) = 1$ is implied by the superadditivity of $\mathcal{E}_g$. However, $\bigcap_{j \in J'} B_j = \emptyset$ in contradiction to the fourth boundary condition (1.1). This contradiction proves that condition (*) must hold.

For the converse direction let us suppose that an EFF $\mathcal{E}$ satisfies condition (*). We shall show that $\mathcal{E}$ is a playing-minor. Indeed, let us define another effectivity function $\mathcal{E}'$ as follows: $\mathcal{E}'(K, B) = 1$ if there exists a nonempty index-subset $J' \subseteq J$ such that the corresponding coalitions are pairwise disjoint (i.e. $K_{j_1} \cap K_{j_2} = \emptyset$ for every $j_1, j_2 \in J'$, $j_1 \neq j_2$), $\bigcup_{j \in J'} K_j \subseteq K$, and $\bigcap_{j \in J'} B_j \subseteq B$, or $\mathcal{E}'(K, B) = 1$ if $K = I$ and $B \neq \emptyset$, or if $B = A$ and $K \neq \emptyset$. It is not difficult to check that the obtained EFF $\mathcal{E}'$ is playable, because it satisfies the conditions of Theorem 1. Since we have $\mathcal{E}'(K_j, B_j) = \mathcal{E}(K_j, B_j) = 1$ for all $j \in J$, $\mathcal{E}' \geq \mathcal{E}$ follows, and thus $\mathcal{E}$ is a playing-minor. □

Remark 2 *The dual claim also holds. If the EFF $\mathcal{E}^d$ is playable then there exists an EFF $\mathcal{E}'$ which is monotone,* subadditive, *satisfies the boundary conditions and $\mathcal{E} \leq \mathcal{E}'$.*

The special case $|J'| = 2$ will be important. Suppose $\mathcal{E}$ satisfies the condition

$$\mathcal{E}(K_1, B_1) = 1, \mathcal{E}(K_2, B_2) = 1, K_1 \cap K_2 = \emptyset, B_1 \cap B_2 = \emptyset \tag{1.2}$$

for some coalitions $K_1, K_2 \subseteq I$ and blocks $B_1, B_2 \subseteq A$. Then neither $\mathcal{E}$ nor $\mathcal{E}^d$ can be a playing-minor.

1.4 Core and $\mathcal{K}$-core.

Let us denote by $PR(K, a, u)$ the set of all those outcomes that coalition K strictly and unanimously prefers to outcome a with respect to the utility function u, i.e.

$$PR(K, a, u) = \{a' \in A \,|\, u(i, a') > u(i, a)\ \forall i \in K\}.$$

Note that $a \notin PR(K, a, u)$.

Given an EFF $\mathcal{E}$ and a utility function u, we say that the coalition K *rejects* an outcome a if $\mathcal{E}(K, PR(K, a, u)) = 1$, i.e. if K is able to guarantee an outcome, better than a for all of its participants. The *core* $C(\mathcal{E}, u)$ is defined as the set of not rejected outcomes, i.e.

$$C(\mathcal{E}, u) = \{a \in A \,|\, \mathcal{E}(K, PR(K, a, u)) = 0\ \forall K \subseteq I\}.$$

Given $\mathcal{E}$, u and a family of coalitions $\mathcal{K}$, the $\mathcal{K}$-*core* $C(\mathcal{E}, u, \mathcal{K})$ is defined as the set of outcomes not rejected by the coalitions from $\mathcal{K}$.

$$C(\mathcal{E}, u, \mathcal{K}) = \{a \in A \mid \mathcal{E}(K, PR(K, a, u)) = 0\ \forall K \in \mathcal{K}\}.$$

1.5 Stable families of coalitions

A family of coalitions $\mathcal{K}$ is called *g-stable* if the $\mathcal{K}$-core is not empty for any game in normal form, or in other words, if $C(\mathcal{E}_g, u, \mathcal{K}) \neq \emptyset$ for any utility function u and for any game form g. Here we give necessary and sufficient conditions for g-stability based on the notion of *balanced weighting.*

Let us consider a family $\mathcal{K}$ of coalitions and a non-negative integer valued weight function on it, $w : \mathcal{K} \to \mathbb{Z}_+$. The function w is called a *balanced weighting* of $\mathcal{K}$ with multiplicity σ if for every player $i \in I$ the equation $\sum_{K \mid i \in K \in \mathcal{K}} w(K) = \sigma$ holds. In other words, if each player participate the same number of coalitions (taking weights into account). We shall call a family $\mathcal{K}$ *balanced* if it has a strictly positive balanced weighting $w : \mathcal{K} \to \mathbb{Z}_+$.

For an arbitrary non-negative weighting w, the subfamily $\mathcal{K}_w = \{K \in \mathcal{K} | w(K) > 0\}$ is called the *support* of w. A balanced weighting is called *minimal* if its support does not contain properly the support of some other balanced weighting. A balanced weighting w of multiplicity $\sigma = 1$ is called a *partition weighting* (or simply a *partition*), because in this case the support $\mathcal{K}_w$ of w is a partition of I. A balanced weighting w is called a *sum of partitions* if there are partitions $w^i : \mathcal{K} \to \{0, 1\}$ and non-negative integers $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., l$, such that $w = \sum_{i=1}^{l} \alpha_i w^i$.

Lemma 1 (Shapley (1965)) *The following properties of a family $\mathcal{K}$ are equivalent:*

(a) *Any balanced weighting of $\mathcal{K}$ is a sum of partitions.*

(b) *Any balanced weighting of $\mathcal{K}$ contains a partition.*

(c) *Any* minimal *balanced weighting of $\mathcal{K}$ is a partition.*

We shall call a family $\mathcal{K}$ *partitionable* if $\mathcal{K}$ satisfies (a),(b),(c) above.

Let us denote by $[I]$ the family of all single-player coalitions (singletons) of I, i.e. $[I] = \{\{i\}|i \in I\}$. The following theorem characterizes the g-stability of a family of coalitions.

Theorem 3 (Gurvich and Vasin (1978)) *A family of coalitions $\mathcal{K}$ is g-stable iff $\mathcal{K} \cup [I]$ is partitionable.*

The proof of this Theorem is rather long and was published only in Russian (with author's abstract in English). A modified proof was given by Boros, Gurvich and Vasin (1995).

2 Stable effectivity functions

An effectivity function $\mathcal{E}$ is called *stable* if its core $C(\mathcal{E}, u)$ is not empty for any utility function u, and $\mathcal{E}$ is called $\mathcal{K}$*-stable* if its $\mathcal{K}$-core $C(\mathcal{E}, u, \mathcal{K})$ is not empty for any u. Evidently, stability is negatively monotone, i.e. if $\mathcal{E}_1$ is stable ($\mathcal{K}$-stable) and $\mathcal{E}_1 \geq \mathcal{E}_2$ then $\mathcal{E}_2$ is stable ($\mathcal{K}$-stable). The problem *to characterize stable EFFs* seems in general intractable. But for some special cases it can be solved.

2.1 Sufficient conditions for stability

In this subsection let us restrict ourself to monotone EFFs satisfying the boundary conditions. The following results are known.

Theorem 4 (Moulin and Peleg (1982)) *Additive EFFs are stable.*

Theorem 5 (Peleg (1984)) *Convex EFFs are stable.*

An EFF is called *balanced* if for any balanced family $\{K_j|j \in J\}$ of coalitions of I, and for any partition $\{B_i|i \in I\}$ of the set of outcomes A the equation $\mathcal{E}(K_j, A \setminus \bigcup_{i \in K_j} B_i) = 0$ must hold for at least one $j \in J$.

Theorem 6 (Danilov and Sotskov (1987, 1991)) *Balanced EFFs are stable.*

Remark 3 *We do not need the assumption of K-monotonicity in the above theorem.*

The family of EFFs simultaneously sub- and superadditive seems very close to the family of stable EFFs. However, there are EFFs subadditive and superadditive but not stable. An example was given by Peleg (1984b) (see also Gurvich (1990)).

Theorem 7 (Abdou (1981)) *A *-selfdual effectivity function is stable iff it is subadditive and superadditive.*

Moulin (1983) and Danilov and Sotskov (1987) conjectured that any stable EFF is majorized by a *-selfdual and stable one. This conjecture together with Theorem 7 would provide a general characterization of stable EFFs. However, this conjecture was disproved by a counterexample given by Gurvich (1990) (see also Danilov and Sotskov (1991)). The same example also disproves an analogous conjecture by Danilov and Sotskov (1987) saying that any stable EFF is majorized by a convex one.

2.2 Necessary conditions for stability

Let us consider the notion of an (m,l)-cycle introduced by Danilov and Sotskov (1987). Let us say that an EFF $\mathcal{E}$ is (m,l)-cyclic if m and l are positive integers satisfying $1 \le l \le m$, $J = \{1,...,m\}$, and there exist families of coalitions $\{K_j | j \in J\}$, and blocks $\{B_j | j \in J\}$ such that $K_j \subseteq I$, $B_j \subseteq A$ for all $j \in J$, the sets $\{K_j | j \in J\}$ are pairwisely disjoint, the sets $\{B_j | j \in J\}$ form a cover of A, and for any $j \in J$ the coalition $K_j \cup K_{j+1} \cup ... \cup K_{j+l-1}$ is effective for the block $A \setminus (B_j \cup B_{j+1} \cup ... \cup B_{j+l-1})$ (where indices $j+t$ are taken mod m.)

Remark 4 *In case of K-monotone (resp. B-monotone) EFFs we can assume without loss of generality that the family $\{K_j | j \in J\}$ (resp. $\{B_j | j \in J\}$) is a partition of I (of A).*

Below we consider some important special cases. If the EFF $\mathcal{E}$ satisfies the condition (1.2), then it is $(2,1)$-cyclic, and consequently neither $\mathcal{E}$ nor its dual $\mathcal{E}^d$ are stable. Indeed, the condition

$$\mathcal{E}(K_1, B_1') = 1, \mathcal{E}(K_2, B_2') = 1, K_1 \cap K_2 = \emptyset, B_1' \cap B_2' = \emptyset \text{ for some } K_1, K_2, B_1', B_2'$$

will coincide with the definition of a $(2,1)$-cycle after the substitution $B_1 = A \setminus B_1'$, $B_2 = A \setminus B_2'$. Let us now consider a utility function u such that K_1 strictly prefers B_1' to B_2', K_2 strictly prefers B_2' to B_1', and all the players prefer $B_1' \cup B_2'$ to $A \setminus (B_1' \cup B_2')$. Obviously the core $C(\mathcal{E}, u)$ is empty.

Lemma 2 *For a monotone EFF $\mathcal{E}$ the following 3 properties are equivalent:*

- *$\mathcal{E}$ is (m,l)-cyclic,*
- *$\mathcal{E}^d$ is $(m, m-l)$-cyclic,*
- *$\mathcal{E}^*$ is $(m, m-l)$-cyclic.*

Proof. Proof follows directly from the definitions of d-dual, *-dual and cyclic EFFs. □

Proposition 1 (Danilov and Sotskov (1987)) *An (m,l)-cyclic EFF is not stable.*

Lemma 3 *An EFF $\mathcal{E}$ is not a playing-minor iff it is $(m,1)$-cyclic for some m.*

Proof. We can show the statement using Theorem 2 and the de Morgan formula $\bigcup_{j\in J} B_j = A \setminus \bigcap_{j \in J}(A \setminus B_j)$. In particular, $\bigcup_{j\in J} B_j = A$ iff $\bigcap_{j\in J}(A \setminus B_j) = \emptyset$.

Let us suppose at first that the EFF $\mathcal{E} = \{K_j, B_j | j \in J\}$ is $(m,1)$-cyclic, i.e. there is an integer m and there are two families $\{K_j | j \in J'\}$, and $\{B_j | j \in J'\}$ for some $J' = \{1,...,m\} \subseteq J$ such that $K_j \subseteq I$, $B_j \subseteq A$ for all $j \in J'$, the sets $\{K_j | j \in J'\}$ are pairwise disjoint, the sets $\{B_j | j \in J'\}$ form a cover of A, and $\mathcal{E}(K_j, A \setminus B_j) = 1$ for any $j \in J'$. These imply $\bigcap_{j\in J'}(A \setminus B_j) = A \setminus \bigcup_{j \in J'} B_j = \emptyset$, from which it follows that $\mathcal{E}$ is not a playing-minor, according to Theorem 2.

Conversly, let us suppose that $\mathcal{E}$ is not a playing-minor. According to condition (*) of Theorem 2, this means that there exist families of coalitions and blocks $\{K_j | j \in J'\}$, and $\{B_j | j \in J'\}$ for some $J' = \{1, ..., m\} \subseteq J$ such that $\mathcal{E}(K_j, B_j) = 1$ for all $j \in J'$, the coalitions $\{K_j | j \in J'\}$ are pairwise disjoint, and the blocks $\{B_j | j \in J'\}$ have no outcome in common, $\bigcap_{j \in J'} B_j = \emptyset$. Then $\bigcap_{j \in J'} B_j = A \setminus \bigcup_{j \in J'}(A \setminus B_j) = \emptyset$, and consequently, $\bigcup_{j \in J'}(A \setminus B_j) = A$, i.e. the blocks $\{(A \setminus B_j); j \in J'\}$ form a cover of A. Thus $\mathcal{E}$ is $(m, 1)$-cyclic for $m = |J'|$. $\square$

Corollary 1 *Playing-minority is a necessary condition for stability. Moreover, if $\mathcal{E}$ or $\mathcal{E}^d$ is not a playing-minor then $\mathcal{E}$ and $\mathcal{E}^d$ are both not stable.*

Proof. . According to Lemmas 2 and 3, $\mathcal{E}$ (resp. $\mathcal{E}^d$) is not a playing minor iff it is $(m, 1)$-cyclic (resp. $(m, m-1)$-cyclic) for some m. Thus, according to Proposition 1, if $\mathcal{E}$ or $\mathcal{E}^d$ is not a playing-minor then $\mathcal{E}$ and $\mathcal{E}^d$ are both not stable. $\square$

Proposition 2 (Gurvich (1995)) *A monotone *-selfdual EFF is superadditive (subadditive) iff it is not (3,1)-cyclic (not (3,2)-cyclic).*

2.3 Keding's criterion of stability

The following theorem by Keiding (1985) provides necessary and sufficient conditions for stability, which conditions however are rather difficult to check..

Let us consider an EFF $\mathcal{E} = \{K_j, B_j | j \in J\}$, a utility function u, and let us denote by $R_j \subseteq \{a \in A | u(i, a) < u(i, b) \; \forall b \in B_j \; \forall i \in K_j\}$ a set of the outcomes (not necessary all of them) which are strictly worse than any one from B_j for all the players from K_j. Evidently, $B_j \cap R_j = \emptyset$ for any $j \in J$. By the definition, all the outcomes in R_j are rejected by the coalition K_j, and thus cannot belong to the core $C(\mathcal{E}, u)$.

Let us consider now the extended list $\{(K_j, B_j, R_j) | \; j \in J\}$, such that K_j is effective for B_j and rejects R_j for all $j \in J$. The core $C(\mathcal{E}, u)$ is empty if $\bigcup_{j \in J} R_j = A$, or in other words, if all the outcomes are rejected (by some coalitions). In this case such a list will be called a *rejecting table* of the pair $(\mathcal{E}, u)$ and denoted by $RT(\mathcal{E}, u)$.

Let us suppose now that we are given only $\mathcal{E}$ but not u, and let us consider an arbitrary list $\{K_j, B_j, R_j | \; j \in J\}$. Such a list will be called a *rejecting table of* $\mathcal{E}$ and will be denoted by $RT(\mathcal{E})$, if $\mathcal{E}(K_j, B_j) = 1$ for all $j \in J$, and $\bigcup_{j \in J} R_j = A$. The rejecting table $RT(\mathcal{E})$ will be called *utilizable* if there exists a utility function u such that $RT(\mathcal{E}) = RT(\mathcal{E}, u)$. Evidently, $\mathcal{E}$ is not stable iff there exists a utilizable rejecting table $RT(\mathcal{E})$. Let $\{(K_j, B_j, R_j) | \; j \in J\}$ be an arbitrary $RT(\mathcal{E})$. Let us consider a set of outcomes $A_c = \{a_{j_1}, a_{j_2}, ..., a_{j_r}\} \subseteq A$, $r \geq 2$. This set will be called a *cycle in* $RT(\mathcal{E})$ if there exists a subset $J_c = \{j_1, j_2, ..., j_r\} \subseteq J$ such that

$$a_{j_1} \in B_{j_1} \cap R_{j_2}, a_{j_2} \in B_{j_2} \cap R_{j_3}, ..., a_{j_r} \in B_{j_r} \cap R_{j_1}.$$

Such a cycle will be called a *common-player cycle* if all the corresponding coalitions have a player in common, i.e. if $\bigcap_{t=1}^{r} K_{j_t} \neq \emptyset$.

Lemma 4 *A $RT(\mathcal{E})$ is utilizable iff it contains no common-player cycle.*

Proof. If there exists a common-player cycle then $RT(\mathcal{E})$ cannot be utilizable, because the preferences of any common player $i \in \bigcap_{t=1}^{r} K_{j_t}$ would be cyclic. Conversely, if there are no common-player cycles in $RT(\mathcal{E})$ then this RT induces for any player $i \in I$ an acyclic partial preference. Clearly this preference can be extended to a linear order. Let us chose a utility function $u(i,*)$ realizing the same linear order. Obviously $RT(\mathcal{E})$ is realized by the obtained utility vector $u = (u(i,*)|i \in I)$. □

Theorem 8 (Keiding (1985)) *An EFF $\mathcal{E}$ is stable iff for any rejecting table $RT(\mathcal{E})$ there exists a common-player cycle.*

Proof. Proof follows directly from the previous lemma. □

3 "Hypergraphic" approach to effectivity functions

In this section we use standard notations and definitions from the theory of graphs and hypergraphs. See Appendix for these definitions and for the proofs of basic properties.

3.1 Co-occurrence and intersection graphs

Let us consider a hypergraph $\mathcal{H}$, its dual hypergraph $\mathcal{H}^d$ (defined by the transposed incidence matrix), and its co-occurrence graph $G_c(\mathcal{H}^d) = G(\mathcal{H})$. Clearly, for any hypergraph $\mathcal{H}$ the intersection graph of $\mathcal{H}$ and the co-occurrence graph $\mathcal{H}^d$ coincide (see Lemma 6), i.e.

$$G_i(\mathcal{H}) = G_c(\mathcal{H}^d) = G(\mathcal{H}).$$

Let us remark that multiple edges are permitted for both hypergraphs $\mathcal{H}$ and $\mathcal{H}^d$, and are forbidden for the graph $G(\mathcal{H})$.

According to the definition of the co-occurrence graph $G_c(\mathcal{H}^d)$, we can treat an edge of $\mathcal{H}^d$ (or, which is the same, a vertex of $\mathcal{H}$) as a clique of the graph $G(\mathcal{H})$. The set of all the edges of $\mathcal{H}^d$ in this way can be treated as a collection of cliques of $G(\mathcal{H})$, which form an edge-cover, i.e. each edge of the graph belongs to a clique from this collection.

Let us fix a vertex j of $G(\mathcal{H})$ and let us consider the subcollection $\mathcal{H}_j$ of all the edges of $\mathcal{H}^d$, i.e. cliques of $G(\mathcal{H})$, which contain j. Then let us consider the family of these subcollections for all the vertices of $G(\mathcal{H})$. This family is exactly the hypergraph $\mathcal{H}$. Vice versa, given an arbitrary graph G and an arbitrary family $\mathcal{H}^d$ of its cliques covering all the edges of the graph, then we have $G = G_c(\mathcal{H}^d) = G(\mathcal{H})$, where each edge of $\mathcal{H}$ is a collection of cliques containing a fixed vertex of G.

The hypergraph $\mathcal{H}$ will be called *clique-maximal* if it contains all the maximal cliques of its co-occurrence graph $G_c(\mathcal{H})$ and *strictly clique-maximal* if it is clique-maximal and contains no other (not maximal) cliques. Hypergraph $\mathcal{H}$ is clique-maximal iff the dual hypergraph $\mathcal{H}^d$ satisfies the Helly property (see Lemma 7).

3.2 A graphical representation of EFFs

An EFF $\mathcal{E}$ can be specified by explicitly listing all the pairs (K, B) such that the coalition K is effective for the block B, i.e. $\mathcal{E} = \{K_j, B_j | j \in J\}$. In other words, $\mathcal{E}$ is specifyed by two hypergraphs $\mathcal{K} = \{K_j | j \in J\}$ (a family of coalitions) and $\mathcal{B} = \{B_j | j \in J\}$ (a family of blocks) the edges of which are numbered by a common set of indeces J. We shall use the notation $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$, as well. The dual hypergraphs $\mathcal{K}^d$, $\mathcal{B}^d$, and the graphs $G(\mathcal{K}) = G_i(\mathcal{K}) = G_c(\mathcal{K}^d)$ and $G(\mathcal{B}) = G_i(\mathcal{B}) = G_c(\mathcal{B}^d)$ will play an important role. Note that the above hypergraphs can have multiple edges, as before, but both graphs are simple.

To obtain the dual EFF $\mathcal{E}^d$, we have to exchange the role of players and outcomes, i.e. to replace $(\mathcal{K}, K, I, i)$ respectively by $(\mathcal{B}, B, A, a)$, and vice versa.

Let us note, for example, that

$$E(G(\mathcal{K})) \cup E(G(\mathcal{B})) = E(J) = \{(j_1, j_2) | j_1, j_2 \in J,\ j_1 \neq j_2\} \tag{3.3}$$

is a necessary condition for stability. Indeed, there exists an edge $(j_1, j_2) \in E(J)$ (i.e. $j_1, j_2 \in J$) which neither belongs to $G(\mathcal{K})$ nor to $G(\mathcal{B})$ iff there are coalitions $K_1, K_2 \in \mathcal{K}$ and blocks $B_1, B_2 \in \mathcal{B}$ satisfying (1.2). Thus, $\mathcal{E}$ is not stable because it is (2,1)-cyclic.

3.3 Criterion of playing-minority

We can easily translate the necessary and sufficient condition (*) of Theorem 2 to the language of graphs and hypergraphs.

Theorem 9 . *An EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ is a playing-minor iff it satisfies the condition*

(**) *any clique of the graph $G(\mathcal{B})$ which has no edges in common with $G(\mathcal{K})$ must be covered by a hyperedge of $\mathcal{B}^d$.*

In particular, if the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ have no common edges at all then they must be complementary and the hypergraph $\mathcal{B}^d$ must be clique-maximal, otherwise $\mathcal{E}$ cannot be a playing-minor.

Proof. Let us show that conditions (*) and (**) are equivalent. For this end, let us recall that a coalition $K_j \in \mathcal{K}$ (resp. block $B_j \in \mathcal{B}$) is formed by the collection of cliques of the graph $G(\mathcal{K})$ (resp. $G(\mathcal{B})$), belonging as edges to the hypergraph $\mathcal{K}^d$ (resp. $\mathcal{B}^d$) and containing the given vertex $j \in J$. Let us fix an arbitrary subset $J' \subseteq J$. Obviously, the coalitions $\{K_j | j \in J'\}$ are pairwise disjoint iff (see (*)) the graph $G(\mathcal{K})$ has no edge within J'. Let us now recall that any edge of $E(J)$ must belong either to $G(\mathcal{K})$ or to $G(\mathcal{B})$, otherwise $\mathcal{E}$ is (2,1)-cyclic and therefore can not be a playing-minor. Thus J' is a clique of $G(\mathcal{B})$ (see (**)) Furthermore, the blocks $\{B_j | j \in J'\}$ have an outcome in common iff (see (*)) the clique J' is a subset of a hyperedge of $\mathcal{B}^d$ (see (**)). Thus (*) implies (**) and vice versa. $\square$

3.4 The "hypergraphic" version of Keiding's Theorem

Let us formulate the graph theoretic version of Theorem 8.

Let us fix an EFF $\mathcal{E} = \{K_j, B_j | j \in J\}$. A directed graph D obtained from a simple graph G by orientation of some of its edges (in one way each) will be called a *suborientation.* Given a graph G and a suborientation D of it, a set of vertices $J' \subseteq J$ is said to be *rejected* if there exists a vertex $j \notin J'$ such that any edge (j, j'), $j' \in J'$, between j and J' is directed in D from j' towards j. A directed cycle in D is called a *clique-cycle* if its vertices induce a clique of G. A suborientarion D of the graph $G(\mathcal{K})$ is called $\mathcal{B}^d$*-rejecting* if any clique in $G(\mathcal{K})$, which is an edge of $\mathcal{B}^d$, is rejected. A suborientation D of $G(\mathcal{K})$ is called $\mathcal{K}^d$*-clique-acyclic* if no directed cycle induces a subset of an edge of $\mathcal{K}^d$. Given an EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$, we shall call a suborientation D of $G(\mathcal{K})$ *clique-acyclic rejecting suborientation* (CARS) if it is $\mathcal{K}^d$-clique acyclic and $\mathcal{B}^d$-rejecting simultaneously.

Keding's theorem can now be reformulated as follows.

Theorem 10 *An EFF $\mathcal{E} = \{K_j, B_j | j \in J\}$ is stable iff the corresponding graph $G(\mathcal{K})$ has no CARS.*

Proof. Let us apply Keiding's theorem to $\mathcal{E}$, and let us assume that $\mathcal{E}$ is not stable. Then there exists a rejecting table (RT) without common-player cycles. Let us construct from this RT a CARS D of the graph $G(\mathcal{K})$ as follows. For every line of RT of the form (K_j is effective for B_j and rejects R_j) let us consider all the cliques of the graph $G(\mathcal{B}) = G_c(\mathcal{B}^d)$ from $\mathcal{B}^d$ corresponding to the outcomes in R_j. None of these cliques contain the vertex j. Let us orient all the edges of the graph $G(\mathcal{K}^d)$, between these cliques and vertex j, towards j. Let us note first of all that we indeed obtain a *suborientation* of $G(\mathcal{K})$. In principle, there could be a bidirected edge in $G(\mathcal{K})$ but then there is a 2-cycle in the RT, which is evidently a common-player cycle. Let us note that all the cliques from $\mathcal{B}^d$ are rejected in D, because the table is rejecting.

Conversely, let us suppose that there exists a CARS, i.e. a suborientation D of $G(\mathcal{K})$ which is simultaneously $\mathcal{K}^d$-clique-acyclic and $\mathcal{B}^d$-rejecting. Let us construct for this suborientation a RT without common-player cycles. For every vertex j in $G(\mathcal{K}^d)$ let us consider the family (maybe empty) of all the cliques from $\mathcal{B}^d$ rejected by this vertex. Let R_j be the corresponding set of outcomes. For every j let us form a line of the form (K_j is effective for B_j and rejects R_j). We claim that we obtain a RT. Indeed, all the outcomes are rejected because the suborientation D was $\mathcal{B}^d$-rejecting, and there are no common-player cycles because the suborientation D was $\mathcal{K}^d$-clique acyclic. □

Remark 5 *There is an obvious one-to-one correspondence between clique-cycles in D and common-player cycles in RT. Thus, there exists a $\mathcal{K}^d$-clique acyclique and $\mathcal{B}^d$-rejecting suborientation (i.e. a CARS) iff there exists a rejecting table for $\mathcal{E}$ with no common-player cycles, i.e. iff $\mathcal{E}$ is not stable. Let us note however that in general there may be much fewer directed cycles in the suborientation than cycles in RT.*

3.5 Stability preserving operations

Keiding's theorem implies that stability is monotone with respect the following three partial (pre)-orders.

A hypergraph $\mathcal{H}$ can contain equal (multiple) edges, or edges containing one another. Let us remove all the strictly included edges, and for any set of multiple edges let us remove all but one copy. We shall call the obtained hypergraph *WIME* (Without Included and Multiple Edges) and denote it by $\mathcal{H}_0$. Clearly, any WIME hypergraph is strictly clique-maximal.

Let us introduce a partial pre-order $\preceq$ for the hypergraps with a common set of vertices. We shall say that $\mathcal{H} \preceq \mathcal{H}'$ if for each edge $H \in \mathcal{H}$ there exists an edge $H' \in \mathcal{H}'$ such that $H \subseteq H'$. Clearly, $\mathcal{H} \simeq \mathcal{H}'$ (i.e. $\mathcal{H} \preceq \mathcal{H}'$ and $\mathcal{H}' \preceq \mathcal{H}$) iff the corresponding WIME hypergraphs $\mathcal{H}_0$ and $\mathcal{H}'_0$ coincide. In particular, $\mathcal{H} \simeq \mathcal{H}_0$ for any hypergraph $\mathcal{H}$.

We shall demonstrate that the finer the edges of a hypergraph are the less stable is the corresponding EFF. For an arbitrary EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ we shall introduce the corresponding WIME EFF $\mathcal{E}_0 =< \mathcal{K}_0, \mathcal{B}_0 >$. This WIME operation removes "passive and equivalent" players from $\mathcal{K}$ and "passive and equivalent" outcomes from $\mathcal{B}$.

Let us introduce a pre-order for EFFs with common players and outcomes. We shall wright $\mathcal{E} \preceq \mathcal{E}'$ iff $\mathcal{K}^d \preceq \mathcal{K}'^d$ and $\mathcal{B}^d \preceq \mathcal{B}'^d$. As before we have $\mathcal{E} \simeq \mathcal{E}_0$.

Proposition 3 *If the EFF $\mathcal{E}$ is stable and $\mathcal{E} \preceq \mathcal{E}'$ then $\mathcal{E}'$ is stable, too.*

Proof. Evidently, if a suborientation of the graph $G(\mathcal{K})$ is $\mathcal{K}'^d$-clique-acyclic then it is $\mathcal{K}^d$-clique-acyclic too. Analogously, if it is $\mathcal{B}'^d$-rejecting then it is $\mathcal{B}^d$-rejecting too. Therefore any CARS of $G(\mathcal{K}'^d)$ is as well a CARS of $G(\mathcal{K}^d)$. Hence, if $\mathcal{E}$ is stable then $\mathcal{E}'$ must be stable too, according to Theorem 10. $\square$

In particular, an EFF $\mathcal{E}$ is stable iff the WIME EFF $\mathcal{E}_0$ is stable. Or in other words, passive and equivalent players and outcomes cannot influence stability.

We shall demonstrate now that "induced subEFFs" are more stable than the inducing EFF. Given an EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >= \{K_j, B_j | j \in J\}$ and an arbitrary subset $J' \subseteq J$, let us consider in the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$, their subgraphs induced by J', and also the subhypergraphs obtained from $\mathcal{K}^d$ and $\mathcal{B}^d$, by the restriction of their edges on the subset J'. We shall call the obtained EFF $\mathcal{E}'$ the *subEFF* of $\mathcal{E}$ induced by the subset $J' \subseteq J$ and denote it by $\mathcal{E}_{J' \subseteq J} =< \mathcal{K}, \mathcal{B},\ J' \subseteq J >= \{K_j, B_j,\ J' \subseteq J\}$.

Proposition 4 *If an EFF $\mathcal{E}$ is stable then any induced subEFF of it $\mathcal{E}_{J' \subseteq J}$ is stable, too.*

Proof. This claim again results from Theorem 10. Let us suppose that $\mathcal{E}_{J' \subseteq J}$ is not stable, and let us consider a suborientation of the induced subgraph $(G(\mathcal{K}) | J' \subseteq J)$ which is simultaneously $(\mathcal{K}^d | J' \subseteq J)$-clique-acyclic and $(\mathcal{B}^d | J' \subseteq J)$-rejecting. Then let us orient all the edges between $J \setminus J'$ and J' towards J'; and orient all the edges of within $J \setminus J'$ arbitrarily, but acyclically. Clearly, the obtained suborientation of $G(\mathcal{K})$ is $\mathcal{K}^d$-clique-acyclic and $\mathcal{B}^d$-rejecting. Thus $\mathcal{E}$ cannot be stable, too, by Keiding's theorem. $\square$

Let us consider, for example, the set J of cardinality 2, the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ with no edges at all, and the strictly clique-maximal hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$. We obtain an EFF $\mathcal{E}_2$ which is (2,1)-cyclic and therefore it is not stable. This is the simplest, not stable EFF. Thus any EFF which induces $\mathcal{E}_2$ as a subEFF cannot be stable. That means exactly, that the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ of any stable EFF in union must contain all the pairs of J as edges.

Now we shall demonstrate that the less is the overlap of the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ the less stable is the corresponding EFF. For an arbitrary EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ let us consider the hypergraph $\mathcal{K}^d$ and the graph $G(\mathcal{K})$, remove an edge e from $G(\mathcal{K})$, denote the obtained graph by $G_e(\mathcal{K})$ and define a clique covering $\mathcal{K}_e^d$ in the following way: If a clique from $G(\mathcal{K})$ does not contain the edge e then include this clique unchanged in $\mathcal{K}_e^d$. If the clique from $G(\mathcal{K})$ consists of the only edge e then remove this clique. Any other clique C from $G(\mathcal{K})$, which contains the edge $e = (j_1, j_2)$ and some other edges, will be substituted in $\mathcal{K}_e^d$ by two cliques, $C_1 = C \setminus \{j_1\}$ and $C_2 = C \setminus \{j_2\}$. Analogously, we can define the graph $G_e(\mathcal{B})$ and the hypergraph $\mathcal{B}_e^d$. Obviously, these operations respect clique-maximality but not always respect strict clique-maximality of the hypergraphs. Let us consider the following three EFFs: $\mathcal{E} =< K, B >$, $\mathcal{E}(\mathcal{K}, e) =< \mathcal{K}_e, \mathcal{B} >$, and $\mathcal{E}(\mathcal{B}, e) =< \mathcal{K}, \mathcal{B}_e >$.

Proposition 5 *If either of $\mathcal{E}(\mathcal{K}, e)$ and $\mathcal{E}(\mathcal{B}, e)$ is stable then $\mathcal{E}$ is stable, too.*

Proof. The proof uses again Theorem 10. It is not difficult to verify that, all the $\mathcal{B}^d$-rejecting suborientations of the graph $G(\mathcal{K})$ are also $\mathcal{B}_e^d$-rejecting, and all the $\mathcal{K}^d$-clique-acyclic suborientations of the graph $G(\mathcal{K})$ are also $\mathcal{K}_e^d$-clique-acyclic. Therefore any CARS in $\mathcal{E}$ is a CARS in both EFFs $\mathcal{E}(\mathcal{K}, e)$ and $\mathcal{E}(\mathcal{B}, e)$. □

3.6 Necessary conditions for stability

We have already shown that any stable EFF must be a playing-minor and cannot be (m, l)-cyclic. Moreover, a stable EFF cannot induce unstable subEFFs. The simpliest unstable EFF, $\mathcal{E}_2$, is generated by the graph G_2 with 2 vertices and no edges. (Therefore, as we mentioned earlier, the graphs $G(\mathcal{K})$ and $G(\mathcal{B})\}$ of any stable EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ must contain in union all the pairs $\{(j_1, j_2) | j_1, j_2 \in J\}$ as edges.)

Another important example is provided by the so called odd holes and odd antiholes (see Appendix). Let us consider the graphs $G(\mathcal{K})$, $G(\mathcal{B})\}$ denoting respectively an odd hole and the complementary odd antihole, and let us suppose that both hypergraphs $\mathcal{K}^d$, $\mathcal{B}^d$ are clique-maximal. Then the corresponding EFF is not stable (see CARS in Appendix). Hence, any EFF, which induces such a subEFF, cannot be stable either.

Let us note that for all these cases both the EFFs and their duals are not stable.

Proposition 6 *If an EFF $\mathcal{E}$ or its dual $\mathcal{E}^d$ is stable then they both must be playing-minors.*

Proof. This claim follows immediately from Lemmas 2, 3 and Proposition 1.

3.7 The case of complementary graphs

Let us consider an EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ such that the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ have no edges in common. According to Theorem 9, $\mathcal{E}$ (resp. $\mathcal{E}^d$) is a playing-minor iff the hypergraph $\mathcal{B}^d$ (resp. $\mathcal{K}^d$) is clique-maximal. We shall assume that both $\mathcal{B}^d$ and $\mathcal{K}^d$ are clique-maximal, since otherwise, according to Corollary 1, $\mathcal{E}$ is not stable.

Furthermore, if there exists a pair (j_1, j_2) $j_1, j_2 \in J$ which is not an edge in either of the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$, then $\mathcal{E}$ is not stable, again. If there is no such pairs then the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ are complementary on the common vertex-set J. According to the Perfect Graph Theorem (Lovász (1972b), see Appendix) they are perfect or imperfect simultaneously.

Let us suppose that these graphs induce a complementary pair of odd hole and antyhole. Then $\mathcal{E}$ is not stable. It was conjectured by Berge (1961) that otherwise both graphs are perfect. (This is so called Strong Perfect Graph Conjecture, see Appendix).

In the case when the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ are complementary and perfect and the hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$ are clique-maximal, we shall show below that the EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ is stable.

Conjecture 1 *If the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ are complementary but not perfect then the effectivity function $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ is not stable.*

Let us note that the above conjecture is equivalent with the conjecture BDPGC2 by Berge-Duchet (1983), and would result from the Strong Perfect Graph Conjecture (see Appendix).

Theorem 11 (Boros and Gurvich (1994)) *If the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ are complementary and perfect, and the hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$ are clique-maximal then the EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ is stable.*

Proof. The EFF $\mathcal{E}$ is a playing-minor because the hypergraphs are clique-maximal. Hence $\mathcal{E}$ is majorized by some playable EFF $\mathcal{E}_g$. Let us note that the family of coalitions $\mathcal{K}$ is g-stable because the graph $G(\mathcal{K})$ is perfect (see Theorem 3 in Section 1 and Theorem 14 in Appendix). In other words, the $\mathcal{K}$-core of a normal form game $< g, u >$ is not empty for any utility function u. Thus $C(\mathcal{E}_g, u, \mathcal{K}) \neq \emptyset$ follows. Then, the $\mathcal{K}$-core $C(\mathcal{E}, u, \mathcal{K})$ is also not empty, because $\mathcal{E} \leq \mathcal{E}_g$ and the core is monotone. Let us note finally that $C(\mathcal{E}, u, \mathcal{K}) = C(\mathcal{E}, u)$ by the definition of $\mathcal{E}$ and $\mathcal{K}$. Hence $C(\mathcal{E}, u) \neq \emptyset$ follows, too. □

3.8 Perfect split and stability

We already know that the convex EFFs and the balanced EFFs are stable. Here we provide a new "hypergraphic" criterion for stability. The main result of the present paper is the following perfect split theorem.

Theorem 12 . *Given an EFF* $\mathcal{E} =< \mathcal{K}, \mathcal{B} >= \{(K_j, B_j) | j \in J\}$, *let us suppose that there exist subgraphs (not necessarily induced)* $G_k =< J, E_k >$ *of* $G(\mathcal{K})$ *and* $G_b =< J, E_b >$ *of* $G(\mathcal{B})$ *such that*

(a) G_k *and* G_b *are complementary on the common vertex-set* J,

(b) G_k *and/or* G_b *are perfect (here "and = or" because of the Perfect Graph Theorem),*

(c) *any (maximal) clique of* G_k *(resp.* G_b*) is covered by an edge of* $\mathcal{K}^d$ *(resp.* $\mathcal{B}^d$*).*

Then $\mathcal{E}$ *is stable.*

Proof. Let us introduce the hypergraphs $\mathcal{K}'^d$ and $\mathcal{B}'^d$ the edges of which are exactly all the maximal cliques of graphs G_k and G_b respectively. In other words, $\mathcal{K}'^d$ and $\mathcal{B}'^d$ are strictly clique-maximal hypergraphs with the co-occurrence graphs $G_k = G(\mathcal{K})$ and $G_b = G(\mathcal{B})$, respectively. Let us consider the EFFs $\mathcal{E}' =< \mathcal{K}', \mathcal{B}' >$, and $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$, and let us remove the common edges of the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ one by one (if any) until we obtain $\mathcal{E}'$ from $\mathcal{E}$. Since $\mathcal{E}'$ is stable, according to Theorem 11, the EFF$\mathcal{E}$ must be stable too, according to Proposition 5. □

Let us note that the case of complementary graphs can be specifyed by requiring that the graphs $G_k = G(\mathcal{K})$ and $G_b = G(\mathcal{B})$ are perfect and complementary on the common vertex-set J, and the hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$ are (strictly) clique-maximal.

3.9 Stability is not selfdual

Let us underline that for all the cases above, except for Keiding's theorem, the dual EFFs $\mathcal{E}$ and $\mathcal{E}^d$ appear to be stable or not stable simulaneously. But in general, this is not true. The simplest example was given by Peleg (1984a) (see Example 6.3.16 and Remark 6.3.17.) The corresponding EFF $\mathcal{E}$ and rejecting table are given below.

Example 1 Let $I = \{i_1, i_2, i_3, i_4\}$, and $A = \{a_1, a_2, a_3, a_4\}$.

$$\begin{array}{llll}
\{i_4\} & \text{is effective for } \{a_1, a_2, a_3\} & \text{and rejects } a_4, \\
\{i_2, i_3\} & \text{is effective for } \{a_1, a_4\} & \text{and rejects } a_2, \\
\{i_3, i_1\} & \text{is effective for } \{a_2, a_4\} & \text{and rejects } a_3, \\
\{i_1, i_2\} & \text{is effective for } \{a_3, a_4\} & \text{and rejects } a_1.
\end{array}$$

All the outcomes are rejected, hence $\mathcal{E}$ is not stable. It is not difficult to check, however, that the dual EFF $\mathcal{E}^d$ is stable.

In this example the hypergraph $\mathcal{K}^d$ is not clique-maximal. Now we give another example where both hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$ are strictly clique-maximal.

Example 2 Let $J = \{1, ..., 9\}$ and let C_9 be a cycle given by the edges $(1,2)$, $(2,3)$, ..., $(8,9)$, $(9,1)$. Let us define $G(\mathcal{K})$ as C_9 plus 3 additional edges $(2,6)$, $(3,8)$, and $(5,9)$. Let $G(\mathcal{B})$ be the complement of C_9. Thus there are 3 common edges $(2,6)$, $(3,8)$, and $(5,9)$ in the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$. It was verified by computer that the EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ is stable, i.e. there is no CARS in $G(\mathcal{K})$. Indeed, in any CARS the cycle C_9 must by cyclically directed. For example, let us direct any edge $(j, j+1)$ from j to $j+1$. Then to reject all the cliques of cardinality 4 of the graph $G(\mathcal{B})$ we have to direct the 3 common edges as $[6,2)$, $[3,8)$, $[9,5)$. Then all the cliques are rejected except for $\{2,5,8\}$ (see Figure 1). On the other

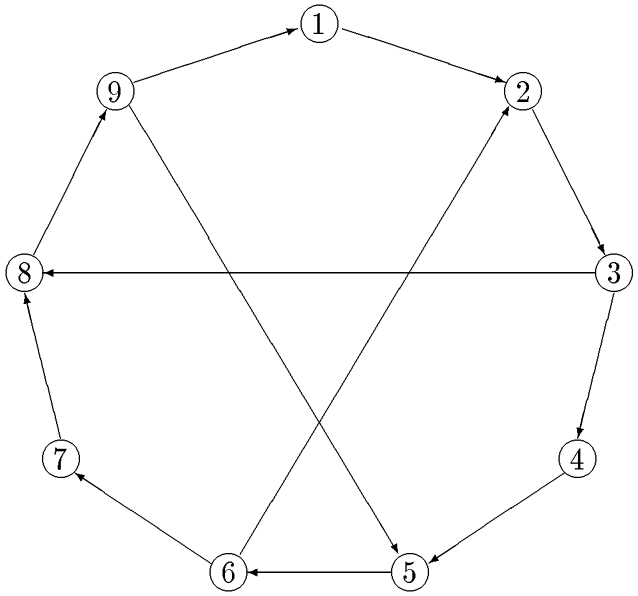

Figure 1: An example for an "almost" clique acyclic rejecting suborientation. All the stable sets, except $\{2,5,8\}$, are rejected.

hand the dual EFF $\mathcal{E}^d$ is not stable, since there exists a CARS given by the following arcs $[9,7)$, $[8,6)$, $[7,5)$, $[6,4)$, $[5,3)$, $[4,2)$, $[3,1)$, $[2,9)$, $[1,8)$, $[2,7)$, $[8,4)$, and $[5,1)$. All other edges of the graph $G(\mathcal{B})$ are not oriented. It is easy to check that all the cliques of $G(\mathcal{K})$ are rejected.

Let us note that the graphs $G(\mathcal{B})$ and $G(\mathcal{K})$ do not induce 2-colored holes neither anti-holes, however they appear as soon as we split the common edges and share them between the graphs. In other words, there exists no perfect split.

Let us note also that the set of the common edges is, in a sence, critical. Namely, if we miss one of the "central" edges then the obtained EFF is no more stable. The corresponding critical graphs $G(\mathcal{K})$ and CARS are listed below:

(a) $[1,2)$, $[2,3)$, ..., $[8,9)$, $[9,1)$, $[4,9)$, $[9,5)$, $[5,1)$, $[1,6)$, $[6,2)$, $[2,7)$.

(b) $[1,2)$, $[2,3)$, ..., $[8,9)$, $[9,1)$, $[6,2)$, $[2,7)$, $[7,3)$, $[8,4)$, $[4,9)$, $[9,5)$.

(c) $[1,2)$, $[2,3)$, ..., $[8,9)$, $[9,1)$, $[7,3)$, $[8,4)$, $[9,5)$, $[5,1)$, $[1,6)$, $[6,2)$.

In all the above cases the graph $G(\mathcal{K})$ is a 9-antihole, i.e. the complement of C_9, and both hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$ are strictly clique-maximal (see Figure 2).

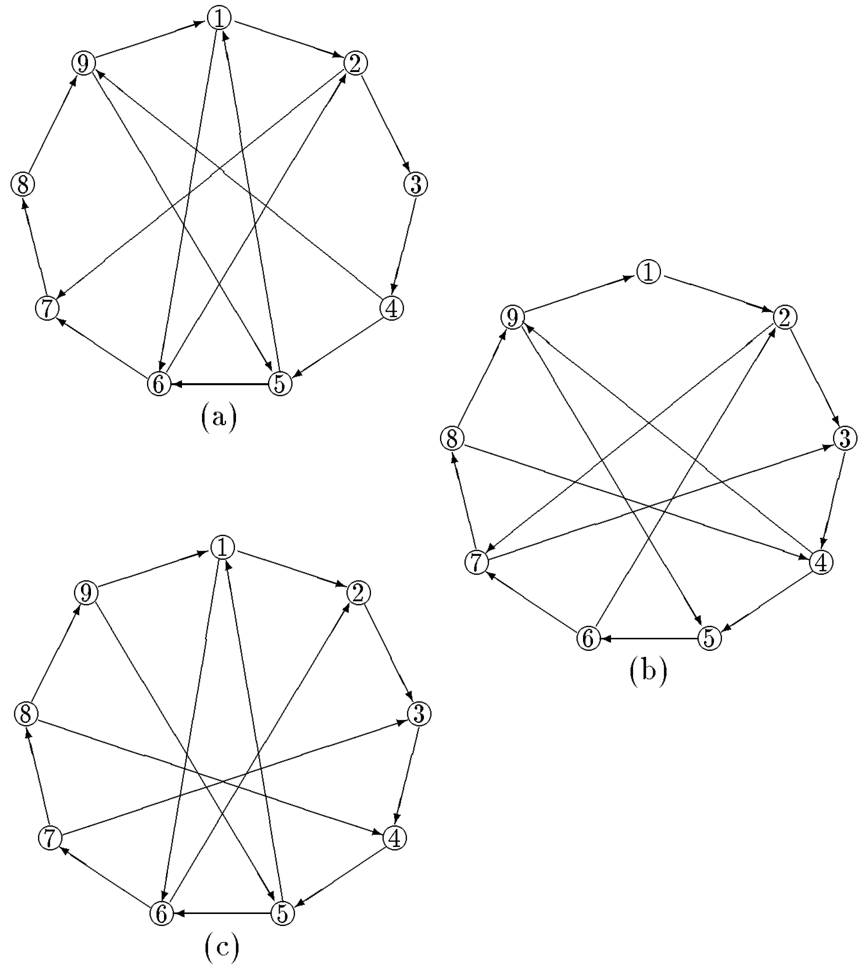

Figure 2: Examples for clique acyclic rejecting suborientation. All the stable sets are rejected.

Conjecture 2 *If the graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ are complementary and the hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$ are clique-maximal then EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ is stable iff dual $\mathcal{E}^d =< \mathcal{B}, \mathcal{K} >$ is stable.*

This conjecture would result from conjecture BDPG2 (which is not yet proved, see Appendix) and Lovász's perfect graph theorem.

3.10 The case of complete graphs

Let us suppose that both graphs $G(\mathcal{K})$ and $G(\mathcal{B})$ are complete. In this case their overlap is big, and according to Proposition 5 the EFF $\mathcal{E} =< \mathcal{K}, \mathcal{B} >$ as well as the dual $\mathcal{E}^d =< \mathcal{B}, \mathcal{K} >$ "have good chances to be stable". In particular, they are both playing-minors, according to Theorem 9. We checked by computer that this is indeed the case if $|J| < 7$. According to Proposition 3, it is sufficient to consider the critical case when the edges of the hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$ are as small as possible, i.e. the hypergraphs $\mathcal{K}^d$ and $\mathcal{B}^d$ are formed by edges of the complete graph on J, as the set of vertices.

However if $|J| = 7$ the EFF is no more stable. A CARS is given by three cycles:

$$[1,2)\ [2,3)\ \cdots\ [6,7)\ [7,1), \quad [1,3)\ [2,4)\ \cdots\ [6,1)\ [7,2) \quad \text{and} \quad [1,5)\ [2,6)\ \cdots\ [6,3)\ [7,4).$$

By this example we can try our "perfect split" sufficient condition for stability and see that it is by far not necessary. For $|J| = 4$ it still works. There exists a perfect split, G_k contains $(1,2)$, $(2,3)$, $(3,4)$, $(4,1)$; G_b contains $(1,3)$, $(2,4)$. Hence the EFF $\mathcal{E}$ is stable.

Let $|J| = 5$. Then the split of the complete graph K_5 into two 5-cycles is not perfect, because C_5 is not perfect. All other splits are not perfect because the obtained two hypergraphs cannot be both clique-maximal. Thus, there is no perfect split at all and at the same time the EFF $\mathcal{E}$ is still stable.

Analogously, if $|J| = 6$ there is no perfect split because one of the graphs have to contain a triangle, and therefore the corresponding hypergraph cannot be clique-maximal. Still, the EFF $\mathcal{E}$ is stable. Thus, for $5 \le |J| \le 6$ the perfect split criterion fails to work.

4 Appendix: Graphs and Hypergraphs.

4.1 Graphs. Preliminary definitions

Let $G =< V, E >$ be finite non-oriented graph without loops and multiple edges; V is the set of vertices and E is the set of edges. The number of vertices $|V|$ and edges $|E|$ is denoted by n and m, and is called the *order* and the *size* of graph G respectively.

An *subgraph* $G' =< V', E' >$ of graph $G =< V, E >$ is defined by the subset of vertices $V' \subseteq V$ and by the subset of edges $E' \subseteq E$ which join *some* of these vertices. A subgraph is called *induced* if *all* the edges of E which join the vertices of $V' \subseteq V$ are included in E'

The *complement* of a graph $G =< V, E >$ is defined as $G^c =< V, E^c >$, i.e. the set of vertices is the same, and two vertices are adjacent in G^c iff they are not adjacent in G. Evidently the complementation is an involution, i.e. $(G^c)^c = G$.

A graph is called *complete* if any two vertices are connected by an edge. The complement to a complete graph is called *edge-free*. A complete subgraph is called a *clique*, and an edge-free induced subgraph is called a *stable set*. Each clique (stable set) which does not strictly belong to another clique (stable set) is called *maximal*, and each clique (stable set) of maximal size is called *maximum*. The order of a maximum clique (stable set) is called *clique* *number* (*stability number*) and will be denoted by $\omega(G)$ (by $\alpha(G)$). The following relations always hold

$$\alpha(G^c) = \omega(G),\ \omega(G^c) = \alpha(G),\ n(G^c) = n(G),\ m(G^c) = n(n-1)/2 - m(G).$$

A mapping $c : V \to \{1, .., j\}$ is called a *coloring* of the graph (by j colors). A coloring is called *proper* if any two adjacent vertices are colored by different colors. The minimal number of colors by which a proper coloring exists is called the *chromatic number* of the graph and is denoted by $\chi(G)$. Evidently $\chi(G) \geq \omega(G)$ for any graph G.

4.2 Perfect graphs

A graph G is called *perfect* if for any induced subgraph G' of G the equation $\chi(G') = \omega(G')$ holds. There is a well-known simple criterion of perfectness.

Proposition 7 *A graph G is perfect iff in any of its induced subgraphs there exists a stable set which has a vertex (exactly one) from any maximum clique.*

The following simple generalization will be important

Proposition 8 *A graph is perfect iff for any non-negative weighting $w : V \to \mathbb{R}_+$ of its verices there exists a stable set which has a vertex (exactly one) from any clique of maximum weight.*

If the weights of vertices take values 0 and 1 only, then the above proposition coincide with the previous one. The following criterion is more difficult.

Theorem 13 (Lovász (1972 a,b)) *A graph G is perfect iff for any induced subgraph G' of G the inequality $\alpha(G') \times \omega(G') \geq n(G')$ holds.*

Corollary 2 *A graph G is perfect iff the complementary graph G^c is perfect.*

Perfect graphs were introduced by Berge (1961 a,b), where he suggested two conjectures. The last statement was called Berge's weak perfect graph conjecture (BWPGC), and since Lovász proved it in 1972 it is called the Perfect Graph Theorem. The following Berge Strong Perfect Graph Conjecture (BSPGC) is still open.

A *hole* is a simple chordless cycle of the length at least 4, and an *antihole* is the complement to a hole. Odd holes C_{2i+1} and odd antiholes C^c_{2i+1} satisfy the following equations

$$\omega(C_{2i+1}) = \alpha(C^c_{2i+1}) = 2,\ \alpha(C_{2i+1}) = \omega(C^c_{2i+1}) = i.$$

In both cases $\alpha \times \omega = 2i < 2i + 1 = n$. Thus odd holes and odd antiholes are not perfect.

By the definition, a graph cannot be perfect if it contains an induced subgraph isomorphic to an odd hole or an odd antihole. The inverse implication is known as Berge's Strong Perfect Graph Conjecture (BSPGC).

4.3 Kernels and super-orientations

A directed graph $D = (V, A)$ is called a *super-orientation* of the simple graph $G = (V, E)$, if D is obtained from G by orienting all of its edges in an arbitrary way (for some or for all of the edges $(u, v) \in E$ both arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ may be included in A).

A subset $S \subseteq V$ of the vertices of a directed graph $D = (V, A)$ is called a *kernel*, if it is a stable set and every vertex outside of S has a successor in S (i.e. if no arc $\overrightarrow{uv} \in A$ has both endpoints in S, and for every $u \notin S$ there exists a $v \in S$ such that $\overrightarrow{uv} \in A$). Let us note that if S is a kernel of a super-orientation of G, then S is a maximal stable set of G.

We shall say that a clique C of G has a kernel in a super-orientation D of G, if there is a vertex $v \in C$, which is the successor of all other vertices of C. Let us finally call a graph G *kernel solvable* if for every super-orientation D of G for which every clique has a kernel, D itself has a kernel.

Berge and Duchet (1983, 1987) studied kernels and kernel solvability (solvability), and conjectured that kernel solvability is equivalent with perfectness. More precisely, the following two conjectures were stated.

BDPGC1. Perfect graphs are kernel solvable.

BDPGC2. Kernel solvable graphs are perfect.

The BDPGC1 was proved for some special cases, including Gallai graphs by Maffray (1986), line graphs by Maffray (1988), and complements of strongly perfect graphs by Blidia, Duchet and Muffray (1988). BDPGC2 was also shown to hold for line graphs by Maffray (1988). Let us note that BDPGC2 would easily result from BSPGC.

In the paper Boros and Gurvich (1994) we proved BDPGC1 and obtained the partial answer for BDPGC2, namely we proved that it is always possible to substitute some of the vertices of a non-perfect graph by cliques so that the resulting graph is not kernel solvable.

4.4 Kernel solvability and sub-orientations

A directed graph $D = (V, A)$ is called a *sub-orientation* of the graph $G = (V, E)$, if D is obtained from G by orienting some of the edges of G (for the sake of simplicity we shall assume that no edge is bidirected in a sub-orientation).

Let us note that there is an obvious one-to-one correspondence between sub- and super-orientations of a simple graph $G = (V, E)$. In a super-orientation of G edges can be uniquely or both ways oriented, while in a sub-orientation they can be uniquely or not at all oriented. Clearly, interchanging the operations "both way orient" and "not orient at all" will establish a one-to-one correspondence between sub- and super-orientations of G. For the clarity, let us say that a super-orientation $D^+ = (V, A^+)$ and a sub-orientation $D^- = (V, A^-)$ of G are *corresponding*, if $A^- \subseteq A^+$, and denoting the set of edges of G which are uniquely (and in the same way) oriented in both directed graphs by $E^1 = \{(u, v) | \overrightarrow{uv} \in A^-\}$ we have $A^+ = A^- \cup \{\overrightarrow{uv}, \overrightarrow{vu} | (u, v) \in E \setminus E^1\}$, i.e. edges belonging to $E \setminus E^1$ are not oriented in D^-, and are both way oriented in D^+.

Let us now fix a sub-orientation $D = (V, A)$ of the graph $G = (V, E)$. We shall say that a vertex v *rejects* (in D) a stable set $S \subseteq V$ if $\overrightarrow{uv} \in A$ for all edges $(u, v) \in E$ for which $u \in S$. We shall call a sub-orientation D of G *rejecting* if every stable set of G is rejected (in D) by some vertex. (Let us note that a non-maximal stable set S is always rejected by any of the vertices $v \notin S$ for which $S \cup \{v\}$ is also stable.) Finally, let us call a sub-orientation of G *clique acyclic* if it does not contain a directed cycle inducing a clique of G.

Let us observe the following easy relations.

Lemma 5 *Let $D^+ = (V, A^+)$ and $D^- = (V, A^-)$ be a corresponding pair of super- and sub-orientations of a simple graph $G = (V, E)$. Then*

(i) *D^- is rejecting iff D^+ has no kernel;*

(ii) *D^- is clique acyclic iff every clique of G has a kernel in D^+.*

Proof. (i). If a vertex v rejects in D^- a maximal stable set S, then, by definition, every edge of G between S and v are oriented uniquely toward v both in D^- and D^+. Hence v has no successor from S in D^+, i.e. S cannot be a kernel of D^+. This implies that if D^- is rejecting, i.e. if every maximal stable set of G is rejected by at least one vertex in D^-, then none of the maximal stable sets are kernels of D^+.

For the converse direction let us observe that if a maximal stable set S is not a kernel in D^+, then there must exist a vertex $v \notin S$ which has no successor in S, i.e. for which all the edges between S and v are directed uniquely toward v (there are such edges, since S is a maximal stable set). Then, all these edges appear with the same orientation in D^-, too, and hence v rejects S in D^-.

(ii). Let $C = \{v_0, v_1, ..., v_s\}$ be the vertex set of a directed cycle in D^- inducing a clique of G. Let us say $\overrightarrow{v_i v_{i+1}} \in A^-$ for $i = 0, ..., s-1$ and $\overrightarrow{v_s v_0} \in A^-$. Then, since $A^- \subseteq A^+$ and no edge appearing in A^- is oriented both ways in A^+, none of the vertices of C is the successor of all the other, i.e. C has no kernel.

On the other hand, if D^- induces an acyclic subgraph on a clique C of G, then there exists a vertex $v \in C$ which has no successor in C in D^-. Such a vertex obviously is the successor of all other vertices of C in D^+. $\square$

A simple graph G will be called *core solvable* if it has no clique acyclic rejecting sub-orientation (CARS). It follows from the previous lemma that core solvability and kernel solvability are equivalent.

Example 3 Let us consider odd holes and odd anti-holes. Let $V = \{v_0, v_1, ..., v_{n-1}\}$, where n is an odd integer, let

$$E = \{(v_i, v_{i+2 \bmod n}) | i = 0, ..., n-1\},$$

and let

$$E' = \{(v_i, v_{i+s \bmod n}) | i = 0, 1, ..., n-1,\ 2 \leq s \leq n-2\}.$$

Then, $G = (V, E)$ is an odd hole, and $G' = (V, E')$ is an odd anti-hole. Let us define now

$$A = \{\overrightarrow{v_i v}_{i+2 \bmod n} \,| i = 0, ..., n-1\}.$$

Then it is easy to verify that $D = (V, A)$ is a rejecting, clique acyclic sub-orientation of both G and G' (see Figure 3).

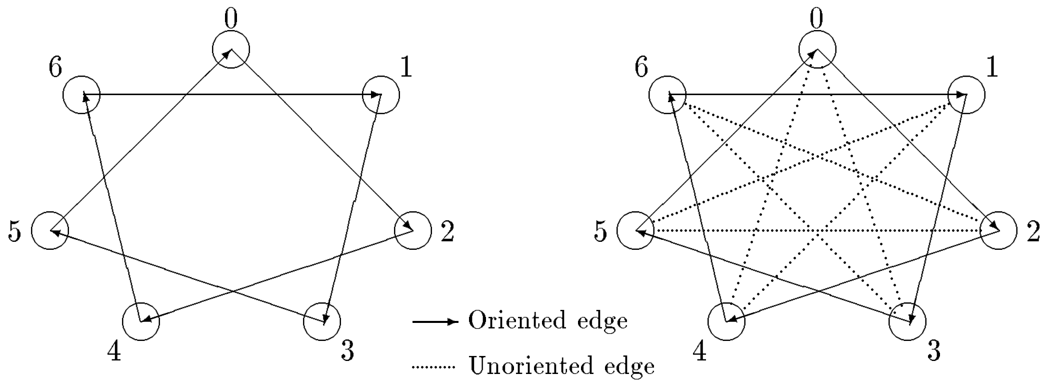

Figure 3: Clique acyclic rejecting sub-orientations of odd holes and anti-holes.

Let us remark that while odd holes have a unique clique acyclic rejecting sub-orientation, odd anti-holes may have several, non-isomorphic ones. As an example let us consider the sub-orientation of $\overline{C}_7$ given in Figure 4. This sub-orientation is clique acyclic, rejecting, and non-isomorphic to the one in Figure 3.

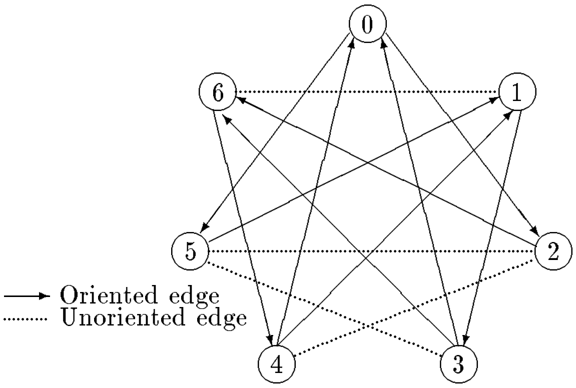

Figure 4: Another clique acyclic rejecting sub-orientation of $\overline{C}_7$.

4.5 Dual hypergraphs and their co-occurrence graphs

A *hypergraph* $\mathcal{H}$ is defined as a nonempty family of a nonempty subsets, called *edges*, of a finite set, the elements of which are called *vertices*. Let us denote by $E(\mathcal{H})$ the set of edges and by $V(\mathcal{H})$ the set of vertices. Multiple edges are allowed for hypergraphs. A hypergraph $\mathcal{H}$ can also be given by its *incidence matrix* A whose rows and columns are labelled, respectively, by the edges and the vertices of the hypergraph, and where $A(H, j) = 1$ iff vertex j belongs to the edge H. We assume that each row and each column of A contains at least one non-zero. The transposed matrix A^T defines the dual hypergraph $\mathcal{H}^d$ whose edges and vertices are assigned respectively to the vertices and the edges of $\mathcal{H}$, i.e. $E(\mathcal{H}) = V(\mathcal{H}^d)$ and $V(\mathcal{H}) = E(\mathcal{H}^d)$.

Given a hypergraph $\mathcal{H}$, let us define its *co-occurrence* graph, $G_c(\mathcal{H})$, as a graph on the same set of vertices $V(\mathcal{H})$, in which two vertices are connected by an edge iff they belong to the same hyper-edge of $\mathcal{H}$. Let us define also its *intersection* graph, $G_i(\mathcal{H})$, as a graph with vertex-set $E(\mathcal{H})$, in which two vertices are connected iff the corresponding two hyper-edges of $\mathcal{H}$ have a non-empty intersection.

Lemma 6 *For any hypergraph $\mathcal{H}$ the graphs $G_i(\mathcal{H})$ and $G_c(\mathcal{H}^d)$ coincide.*

Proof. This claim results directly from the definitions above. It is easy to verify that two edges of $\mathcal{H}$ have a non-empty intersection iff the corresponding two vertices of $\mathcal{H}^d$ belong to the same edge. □

Given a graph G, it is easy to obtain a hypergraph $\mathcal{H}$ such that $G = G_c(\mathcal{H})$. Let us consider an arbitrary clique covering of G, i.e. a family of cliques the union of which covers all the edges of G. Then clearly these cliques form a hypergraph $\mathcal{H}$ for which G is the co-ocurrence graph, $G = G_c(\mathcal{H})$. Moreover, all hypergraphs with this property can be obtained in this way.

Analogously, given a graph G, we can obtain a hypergraph $\mathcal{H}$ such that $G = G_i(\mathcal{H})$. According to Lemma 6, this problem is just dual to the previous one. Let us consider an arbitrary clique covering of G again. Then the collections of those cliques, which contain a given vertex of G, form the edges of a hypergraph $\mathcal{H}$ such that $G = G_i(\mathcal{H})$. Moreover, all hypergraphs with this property can be obtained in such a way from all the clique coverings.

4.6 Normal hypergraphs

The hypergraph $\mathcal{H}$ is said to have the *Helly property* if any family of edges whose intersection is empty contains two disjoint edges. For example intervals of an ordered set or subtrees of a tree has this property.

Lemma 7 *Hypergraph $\mathcal{H}$ has the Helly property iff its dual hypergraph $\mathcal{H}^d$ is clique-maximal, i.e. it contains, as edges, all the maximal cliques of its co-ocurrence graph $G_c(\mathcal{H}^d)$.*

Proof is trivial. □

A hypergraph $\mathcal{H}$ is said to have *the Berge property* if its intersection graph $G_i(\mathcal{H})$ is perfect. The hypergraph $\mathcal{H}$ is called *normal* if it has both Helly and Berge properties. Evidently, $\mathcal{H}$ is normal iff the graph $G_c(\mathcal{H}^d)$ is perfect and its clique covering hypergraph $\mathcal{H}^d$ is clique-maximal.

Normal hypergraphs were introduced by Berge (1970), and Lovász (1972b) proved the Perfect Graphs Theorem by means of this notion. There are many equivalent characterizations of normality, see e.g. Lovász (1972b) and Berge (1984). In the present paper we add two more characterizations: g-stability and partitionability of $\mathcal{H} \cup [I]$.

4.7 Normality and g-stability are equivalent

Theorem 14 *The hypergraph $\mathcal{K} \cup [I]$ is partitionable iff $\mathcal{K}$ is normal.*

Proof. First of all let us note that $\mathcal{K}$ has the Helly (or Berge) property iff $\mathcal{K} \cup [I]$ has the same property.

Lemma 8 *If $\mathcal{K} \cup [I]$ is partitionable then $\mathcal{K}$ has the Helly property.*

Proof. Given a hypergraph $\mathcal{K} = \{K_j | j \in J\}$ let us define the maximal integer positive k such that there exists a partial hypergraph $\mathcal{K}' = \{K_j | j \in J'\}$ for some subset $J' \subseteq J$ of cardinality k, $|J'| = k$, in which any $k-1$ edges of $\mathcal{K}'$ have a vertex in common, and $\bigcap_{j \in J'} K_j = \emptyset$. Such k and $\mathcal{K}'$ are called, respectively, the *Nakamura number* and the *Nakamura subfamily* of $\mathcal{K}$, see Nakamura (1975). Let us define $k = 0$ if $\bigcap_{j \in J'} K_j \neq \emptyset$. Then $\mathcal{K}$ has the Helly property iff $k \leq 2$. Let us suppose that $k > 2$ and consider a Nakamura subfamily. Its incidence matrix contains no column with all ones, contains all k different columns with $k-1$ ones, and may contain some other columns with a smaller number of ones. Evidently, by adding some singletons to $\mathcal{K}'$ we can obtain a balanced subfamily of $\mathcal{K} \cup [I]$ which is not partitionable. $\square$

Let us consider a hypergraph $\mathcal{K}$ together with the extended hypergraph $\mathcal{K} \cup [I]$, and their intersection graphs $G = G_i(\mathcal{K}) = G_c(\mathcal{K}^d)$ and $G' = G_i(\mathcal{K} \cup [I]) = G_c((\mathcal{K} \cup [I])^d)$. The graph G' can be obtained from G by adding "individual" vertices to all the maximal cliques of G, i.e. a vertex to any maximal clique C, incident to all the vertices of C and only to these.

Any weighting of the edges of $\mathcal{K} \cup [I]$ is a weighting of the vertices of G'. A weighting on $\mathcal{K} \cup [I]$ is balanced iff it is equiclique on G', i.e. all the maximal cliques have the same (maximum) weight. A weighting is a partition if the vertices with positive weights form a stable set of G'.

Lemma 9 *If $\mathcal{K} \cup [I]$ is partitionable then $\mathcal{K}$ has the Berge property, i.e. the graph G is perfect.*

Proof. According to Lemma 8, if $\mathcal{K} \cup [I]$ is partitionable then $\mathcal{K}$ has the Helly property. Let us suppose that the graph $G' = G_i(\mathcal{K} \cup [I])$ is not perfect. Then, according to Proposition 8, there is an integer non-negative weighting w' of G' such that no stable set can intersect all the cliques of maximum weight. Let us note however that w may not be equiclique. In this case let us increase the weight of all the lighter cliques by increasing the weight of the corresponding individual vertices in order to make the obtained weighting w'' equiclique. Clearly the maximum weight is still the same. The corresponding weighting of $\mathcal{K} \cup [I]$ is a balanced one, and $\mathcal{K} \cup [I]$ is partitionable, by the assumption. Thus, there is a stable set which meets all the cliques of maximum weight with respect to w''. Evidently, the same stable set meets all the cliques of maximum weight with respect to w', which yields the desired contradiction. $\square$

Lemma 10 *If $\mathcal{K}$ has both the Helly and Berge properties, i.e if the graph $G = G_c(\mathcal{K}^d)$ is perfect and the hypergraph $\mathcal{K}^d$ is clique-maximal, then $\mathcal{K} \cup [I]$ is partitionable.*

Proof. Let us consider an arbitrary balanced weighting w of $\mathcal{K} \cup [I]$ and prove that it is partitionable. Consider the corresponding weighting of the vertices of G'. Then, according to Proposition 8, there is a stable set S in G' which meets every clique of maximum weight with respect to w, because G' is perfect and $\mathcal{K}^d$ is maximal. Let us consider the subfamily of $\mathcal{K} \cup [I]$ associated with S. Evidently, it is a partition. Let us subtract unit weights of this partition from w, and obtain another balanced weighting of multiplicity one less. Repeating the above procedure, we can obtain a partition w. $\square$

Remark 6 *A hypergraph $\mathcal{K}$ has the Helly (or Berge) property iff $\mathcal{K} \cup [I]$ has the same. However, this is not the case with partitionability. If $\mathcal{K} \cup [I]$ is partitionable then $\mathcal{K}$ is partitionable, too, but not vice versa. For example, $\mathcal{K} = \{(1,2),(1,3),(2,3,4)\}$ is partitionable because there are no balanced subfamilies in $\mathcal{K}$ at all, while $\mathcal{K} \cup [I]$ is not partitionable.*

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