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Stable Families of Coalitions and Normal Hypergraphs[1]

by

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Abstract

The core of a game is defined as the set of outcomes acceptable for *all* coalitions. This is probably the simplest and most natural concept of cooperative game theory. However, the core can be empty because there are too many coalitions. Yet, some players may not like or know each other, so they cannot form a coalition. Let $\mathcal{K}$ be a fixed family of coalitions. The $\mathcal{K}$-core is defined as the set of outcomes acceptable for all the coalitions from $\mathcal{K}$. The family $\mathcal{K}$ is called *stable* if the $\mathcal{K}$-core is not empty for any normal form game (or equivalently, for any game in generalized characteristic function form).

Normal hypergraphs can be characterized by several equivalent properties, e.g. they are *dual* to the *clique hypergraphs* of *perfect graphs*. We prove that a family $\mathcal{K}$ of coalitions is stable iff $\mathcal{K}$ as a hypergraph is normal.

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1 Introduction

The simplest model of a cooperative game without side payments is the characteristic function (CF) form of it. A criterion of non-emptyness of the core for such games was given by Bondareva (1962, 1963; in Russian) and then independently by Shapley (1965, 1967). It is a rather simple criterion based on Farkas (1901) lemma on duality in linear programming. According to this criterion, a CF has to satisfy a system of linear inequalities, one for any balanced subfamily of coalitions. A family of coalitions $\mathcal{K}$ is called v-stable if the $\mathcal{K}$-core is not empty for any superadditive CF. The Bondareva-Shapley criterion easily generalizes for $\mathcal{K}$-cores. This generalization results immediately in the following criterion given by Gurvich and Vasin (1977, 1978b; in Russian), and then independently by Kaneko and Wooders (1982). A family $\mathcal{K}$ is v-stable iff $\mathcal{K}$ is partitionable, i.e. if any balanced weighting of $\mathcal{K}$ is a positive linear combination of partitions. Examples and interpretations of such families were consider by Gurvich and Vasin (1977, 1978b), Kaneko and Wooders (1982), Greenberg and Weber (1986), Demange(1990), Aharoni, Hajnal and Milner (1994), Kuipers (1994).

A more general model of cooperative games, able to take into account side payments, is the generalized characteristic function (GCF) form. A sufficient condition of non-emptyness of the core for these games were given by Scarf (1967). This result is based on an important combinatorial lemma related to a fixed point theorem, see Danilov and Sotskov (1992). However, Scarf's conditions are also stated in terms of balanced families, and formally they are very close to Bondareva-Shapley's inequalities. A family $\mathcal{K}$ is called V-stable if the $\mathcal{K}$-core is not empty for any compact superadditive GCF. Scarf's theorem easily generalizes for $\mathcal{K}$-cores. This generalization immediately results in the following criterion given by Gurvich and Vasin (1978a; in Russian). Family $\mathcal{K}$ is V-stable iff the extended family $\mathcal{K} \cup [I]$ is partitionable, where $[I]$ denotes the family of all the "singletons", i.e. coalitions of cardinality 1.

Thus the classes of v- and V-stable families of coalitions in fact coincide, provided all the singletons are included. Such an assumption is natural for game theory. Indeed, the main reasons to consider not all the coalitions but only a fixed family was that not all the coalitions can be organized. However, nobody can prevent a single player to organize the coalition which consists only of himself. On the other hand, from a combinatorial point of view, these two conditions: $\mathcal{K}$ is partitionable and $\mathcal{K} \cup [I]$ is partitionable differ a lot. The second one is much stronger and, in a sense, is "more natural". Namely, Boros and Gurvich (1994) proved that $\mathcal{K} \cup [I]$ is partitionable iff the hypergraph $\mathcal{K}$ is normal. The same result was obtained independently by Kuipers (1994).

Normal hypergraphs were introduced by Berge (1970). Using normal hypergraphs Lovász (1972b) proved the so called weak perfect graph conjecture of Berge (1961).

A family $\mathcal{K}$ is g-stable if $\mathcal{K}$-core is not empty for any game in normal form. It was also proved by Gurvich and Vasin (1978a) that V-stability is equivalent to g-stability. Here we provide a shorter proof based on a result by Moulin and Peleg (1982) characterizing effectivity functions generated by game forms (as monotone and superadditive). We collect here all the results mentioned above, and reproduce the proofs (some of them in a simplified form) given by Gurvich and Vasin (1977, 1978a; in Russian).

Let us remark also that the criterion of stability for effectivity functions, given by Keiding (1985), connects the notion of g-stability to kernels in perfect graphs, and in particular to a conjecture by Berge and Duchet (1983) which was proved by Boros and Gurvich (1994).

2 Balanced weightings and partitionable families of coalitions

Let us consider a family $\mathcal{K}$ of coalitions and a non-negative integer valued weight function on it, $w : \mathcal{K} \to \mathbb{Z}_+$. The function w is called a *balanced weighting* of $\mathcal{K}$ with multiplicity m if the equation $\sum_{K \mid i \in K \in \mathcal{K}} w(K) = m$ holds for every player $i \in I$, or in other words, if each player participates in the same number of coalitions (taking weights into account). We shall call a family $\mathcal{K}$ *balanced* if it has a strictly positive balanced weighting $w : \mathcal{K} \to \mathbb{Z}_+$. For an arbitrary non-negative weighting w, the subfamily $\mathcal{K}_w = \{K \in \mathcal{K} | w(K) > 0\}$ is called the *support* of w. A balanced weighting is called *minimal* if its support does not strictly contain a support of some other balanced weighting. A balanced weighting w of multiplicity $m = 1$ is called a *partition weighting* (or simply a *partition*), because in this case the support $\mathcal{K}_w$ of w is a partition of I. A balanced weighting w is called a *sum of partitions* if there are partitions $w^i : \mathcal{K} \to \{0, 1\}$ and non-negative integers $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., l$, such that $w = \sum_{i=1}^{l} \alpha_i w^i$.

Proposition 1 (Shapley (1965)) *The following properties of a family $\mathcal{K}$ are equivalent.*

(a) *Any balanced weighting of $\mathcal{K}$ is a sum of partitions.*

(b) *Any balanced weighting of $\mathcal{K}$ contains a partition.*

(c) *Any* minimal *balanced weighting of $\mathcal{K}$ is a partition.*

We shall call a family $\mathcal{K}$ *partitionable* if $\mathcal{K}$ satisfies the equivalent conditions (a),(b),(c) above. Later we shall extend the list (a),(b),(c) by further equivalent properties, in terms of perfect graphs and normal hypergraphs (see Appendix).

We recall here 9 examples for partitionable and non-partitionable families, from Gurvich and Vasin (1977, 1978b).

Example 1 $I = \{1, 2\}$; $\mathcal{K} = 2^I = \{(1), (2), (1, 2)\}$. *If there are only 2 players then the family of* all *possible coalitions is partitionable.*

Example 2 $I = \{1, 2, 3\}; \mathcal{K} = 2^I - \{(1, 3)\} = \{(1), (2), (3), (1, 2), (2, 3), (1, 2, 3)\}$. *In case of 3 player, the family of all coalitions but one doubleton is partitionable.*

Example 3 $I = \{1, 2, 3\}$; $\mathcal{K} = C_3 = \{(1, 2), (2, 3), (3, 1)\}$. *A 3-cycle is not partitionable. This is the simplest balanced family which does not contain a partition.*

Example 4 *$I = \{1,2,3,4\}; \mathcal{K} = C_4 = \{(1,2),(2,3),(3,4),(4,1)\}$. A 4-cycle is partitionable. It is easily seen that the above 4-cycle is the sum of 2 partitions:* $\{(1,2),(3,4)\}$ *and* $\{(2,3),(4,1)\}$.

Example 5 *$I = \{1,2,...,n\}; \mathcal{K} = C_n = \{(1,2),(2,3),...,(n-1,n),(n,1)\}, n \geq 3$. An even cycle is partitionable (clearly, it is the sum of 2 partitions), but an odd cycle is not.*

This is the first reminiscence with perfect graphs.

Example 6 *$I = \{1,2,...,n\}$, $\mathcal{K} = S_n^{n-1}$, $n \geq 2$. The family S_n of all the $(n-1)$-subsets of I is partitionable if $n = 2$ and it is not partitionable if $n > 2$.*

Example 7 *$I = \{1,2,...,n\}$, $\mathcal{K} = \{(1),(2),...,(n)\} \cup \{K \subseteq I | 1 \in K\}$, In other words, $\mathcal{K}$ contains all the singletons and all the coalitions including player 1. Clearly, $\mathcal{K}$ is partitionable.*

One can interpret player 1 in the above example as a spy, whose duty is to participate in any coalition consisting of more than one player.

Example 8 *$I = \{1,2,...,n\}$, $\mathcal{K} = \{K \subseteq I | i', i'' \in K,\ i' < j < i'' \Rightarrow j \in K\}$. In other words, I is a finite completely ordered set, and $\mathcal{K}$ is the family of all the intervals of I. Clearly, the family $\mathcal{K}$ is partitionable.*

We can interpret the above example such that players are owners of the shops situated along a railway line in a remote region. Then, it is natural to suppose that only a few close neighbors can organize a coalition (and effectively coordinate the prices, for example). In this case the customers living within the corresponding interval cannot resist. But if an intermediate player fails to participate then the whole coalition fails.

This example was considered also by Greenberg and Weber (1986). It was generalized by Aharoni, Hajnal and Milner (1994) for the case of an infinite ordered set.

The next example is another generalization. It was considered also by Demange (1990) and by Kuipers(1994).

Example 9 *Let $T =< I, E >$ be a finite tree where I is the set of vertices (identified with the players) and E is the set of edges. The family $\mathcal{K}$, of all the connected subgraphs (i.e. subtrees) of T, is partitionable.*

This example generalizes Examples 7 and 8, in which the corresponding tree T is a star and a simple path, respectively, i.e. $E(T) = \{(1,2),(1,3),...,(1,n)\}$ and $E(T) = \{(1,2),(2,3),...,(n-1,n)\}$, respectively, where $I = \{1,2,...,n\}$.

An interpretation, similar to that of Example 8, can be given in terms of a tree structured road (or railway) system. Another interpretation was given by Gurvich and Menshikov (1989). Let us fix an arbitrary vertex $r \in T$. It will be called the root and the corresponding player will be called the "big boss" The obtained rooted tree will be treated as a hierarchical

scheme (of a company). It is supposed that if two players $i', i'' \in I$ participate in a coalition then the players along the path connecting the corresponding vertices also have to be included in this coalition. The boss of the minimal rank common for the players i' and i'' and also all the intermediate bosses belong to this path. (The path is unique because T is a tree.) Thus "the spy" from Example 7 can be substituted by "the boss".

3 Characteristic functions

A *characteristic function* (CF) is defined as a mapping $v : 2^I \to \mathbb{R}$. The real values $v(K)$, $K \subseteq I$, can be interpreted as anticipated profits of the coalitions.

A CF v is called *superadditive* if

$$v(K' \cup K'') \geq v(K') + v(K'') \quad \forall K', K'' \subseteq I, \quad K' \cap K'' = \emptyset.$$

For any $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $K \subseteq I$ let $x_K = \sum_{i \in K} x_i$.

The set of *outcomes (or possible allocations)* is defined as $A = \{x | x_I \leq v(I)\} \subseteq \mathbb{R}^I$, i.e. A consists of all the vectors which the coalition of all the players is able to obtain.

An outcome x is called *acceptable* for a coalition $K \subseteq I$ if $x_K \geq v(K)$, i.e. if this coalition cannot expect more than given by this outcome.

The *core* $C(v)$ is the set of outcomes acceptable for all the coalitions $K \subseteq I$. In other words, the core is defined by the following system of linear inequalities

$$C(v) = \{x \in \mathbb{R}^I | x_I \leq v(I) \text{ and } x_K \geq v(K) \, \forall K \subseteq I\}.$$

Given an arbitrary set of coalitions $\mathcal{K} \subseteq 2^I$, the $\mathcal{K}$-core $C(v, \mathcal{K})$ is defined as the set of outcomes acceptable for all the coalitions $K \in \mathcal{K}$, i.e.

$$C(v, \mathcal{K}) = \{x \in \mathbb{R}^I \mid x_I \leq v(I) \text{ and } x_K \geq v(K) \, \forall K \in \mathcal{K}\}.$$

According to the above definitions, $C(v, \mathcal{K}) = C(\mathcal{K})$ for $\mathcal{K} = 2^I$.

A set $\mathcal{K}$ is called *v-stable* if $C(\mathcal{K}, v) \neq \emptyset$ for any superadditive CF v.

Theorem 1 (Bondareva (1962,1963), Shapley(1965, 1967)) *The core $C(v)$ is not empty iff any (minimal) balanced weighting $w : 2^I \to \mathbb{Z}+$ of all the coalitions satisfies the inequality $\sum_{K \subseteq I} w(K)v(K) \leq v(I)$.*

Proof follows directly from the theory of duality applied to the system of linear inequalities defining the core, see Farkas (1902), and also Fan Ku (1956).

Theorem 2 (Gurvich and Vasin (1977)) *The core $C(v, \mathcal{K})$ is not empty iff any (minimal) balanced weighting $w : \mathcal{K} \to \mathbb{Z}_+$ of $\mathcal{K}$ satisfies the inequality $\sum_{K \in \mathcal{K}} w(K)v(K) \leq v(I)$.*

This theorem is an easy generalization of the previous one. The proof follows exactly the same lines. Let us consider 3- and 4-cycles for instance

Example 10 *$I = \{1,2,3\}$, $\mathcal{K} = C_3 = \{(1,2),(2,3),(3,1)\}$. The $\mathcal{K}$-core is given by*

$$\begin{array}{cccc} x_1 & +x_2 & & \geq & v(1,2) \\ x_1 & & +x_3 & \geq & v(1,3) \\ & +x_2 & +x_3 & \geq & v(2,3) \\ x_1 & +x_2 & +x_3 & \leq & v(1,2,3) \end{array}$$

This system yields the inequality

$$v(1,2) + v(1,3) + v(2,3) \leq 2v(1,2,3)$$

which is not a consequence of superadditivity. For example, it is violated by the superadditive (and simple) CF v given by

$$v(1) = v(2) = v(3) = 0, \quad v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = 1$$

Example 11 *$I = \{1,2,3,4\}$, $\mathcal{K} = C_4 = \{(1,2),(2,3),(3,4),(4,1)\}$. The $\mathcal{K}$-core is given by*

$$\begin{array}{cccc} x_1 & +x_2 & & \geq & v(1,2) \\ & +x_2 & +x_3 & \geq & v(2,3) \\ & & +x_3 & +x_4 & \geq & v(3,4) \\ x_1 & & & +x_4 & \geq & v(1,4) \\ x_1 & +x_2 & +x_3 & +x_4 & \leq & v(1,2,3,4) \end{array}$$

Obviously, this system is consistent iff

$$v(1,2) + v(3,4) \leq v(1,2,3,4), \quad \textit{and} \quad v(2,3) + v(1,4) \leq v(1,2,3,4)$$

which is true for any superadditive CF v.

Let us assign a CF $v_{\mathcal{K}}$ to a family of coalitions $\mathcal{K}$. For this, let us first add to $\mathcal{K}$ all the unions of pairwise disjoint coalitions from $\mathcal{K}$ and let us denote the obtained expanded family by $\mathcal{K}'$. Then let us define $v_{\mathcal{K}}$ by

$$v_{\mathcal{K}}(K) = \max_{K' :\, K \supseteq K' \in \mathcal{K}'} |K'|.$$

Lemma 1 *The CF $v_{\mathcal{K}}$ is superadditive, takes nonnegative integer values and satisfies the following relations:*

$$v_{\mathcal{K}}(K) = |K| \;\; \forall K \in \mathcal{K}', \quad \textit{and} \;\; v_{\mathcal{K}}(K) < |K| \;\; \forall K \notin \mathcal{K}'.$$

Furthermore, if the family $\mathcal{K}$ is not partitionable, then $C(v_{\mathcal{K}}, \mathcal{K}) = \emptyset$.

Proof. The first four claims are straightforward. Let us consider the last one, and let us assume that $\mathcal{K}$ is a non-partitionable family. Then there exists a minimal balanced weighting $w : \mathcal{K} \to \mathbb{Z}_+$ which is not a partition. Let us consider the support $\mathcal{K}_w$ and the corresponding CF $v_{\mathcal{K}_w}$. Obviously, $C(v_{\mathcal{K}_w}, \mathcal{K}_w) \supseteq C(v_{\mathcal{K}}, \mathcal{K})$. Let us prove that $C(v_{\mathcal{K}_w}, \mathcal{K}_w) = \emptyset$. Indeed, on the one hand $v_{\mathcal{K}_w}(I) < |I|$ because $I \notin \mathcal{K}'_w$, and on the other hand $\sum_{K \in \mathcal{K}_w} w(K) v_{\mathcal{K}_w}(K) = \sum_{K \in \mathcal{K}_w} w(K)|K| = |I|$ because the weighting w is balanced. Thus the inequality $v_{\mathcal{K}_w}(I) < \sum_{K \in \mathcal{K}_w} w(K) v_{\mathcal{K}_w}(K)$ is implied, and hence $C(v_{\mathcal{K}_w}, \mathcal{K}_w) = \emptyset$ follows by Theorem 2. □

Theorem 3 (Gurvich and Vasin (1977), Kaneko and Wooders (1982)) *A family $\mathcal{K}$ is v-stable iff $\mathcal{K}$ is partitionable.*

Proof. The statement follows directly from Theorem 2 and the previous lemma.

($\Leftarrow$): If $w : \mathcal{K} \to \mathbb{Z}_+$ is a partition, then the inequality of Theorem 2 follows from the superadditivity of the CF v.

($\Rightarrow$): Let $w : \mathcal{K} \to \mathbb{Z}_+$ be a minimal balanced weighting of $\mathcal{K}$ which is not a partition. Let us consider the support $\mathcal{K}_w = \{K \in \mathcal{K} | w(K) \neq 0)\}$ and the CF $v_{\mathcal{K}_w}$. According to the previous lemma, this CF is superadditive and the core $C(v_{\mathcal{K}_w}, \mathcal{K}_w)$ is empty. Thus $\mathcal{K}_w$ is not v-stable, and consequently $\mathcal{K}$ is not v-stable, either. □

4 Generalized characteristic functions

A generalized characteristic function (GCF) is defined as a mapping $V : 2^I \to 2^{\mathbb{R}^I}$. The subsets $V(K) \subseteq \mathbb{R}^I$ are interpreted as the expected sets of possible allocations for the coalitions $K \subseteq I$, provided side payments are not allowed. The following 3 properties of GCFs will be always assumed.

P1 $V(K) \neq \emptyset$ for any $K \subseteq I$.

P2 $V(K) = V^0(K) \times \mathbb{R}^{I \setminus K}$ for any $K \subseteq I$, where $V^0(K) \subseteq \mathbb{R}^K$, i.e. a coalition $K \subseteq I$ is not able to influence the allocations of the complementary coalition $I \setminus K$.

P3 $V(K)$ is a comprehensive subset of $\mathbb{R}^I$ for any $K \subseteq I$, i.e. $x \in V(K)$ and $x'_i \leq x_i$ for all $i \in K$ implies $x' \in V(K)$.

A GCF is called *compact* if it has the following 3 properties.

P4 $V(K)$ is a closed subset of $\mathbb{R}^I$ for any $K \subseteq I$.

P5 For all players $i \in I$ there exists $v_i \in \mathbb{R}$ such that $V(\{i\}) = \{x \in \mathbb{R}^I \mid x_i \leq v_i\}$, i.e. each player's profit is bounded from above.

P6 The set $V^0(K) \cap \{x \in \mathbb{R}^K \mid x_i \geq v_i \ \forall i \in K\}$ in $\mathbb{R}^K$ is a compact.

Let us denote by $V_{PO}(K)$ the set of Pareto optimal vectors of $V^0(K)$, i.e.

$$V_{PO}(K) = \{x \in V^0(K) |\ \nexists y \in V^0(K) : y \neq x \text{ and } y_i \geq x_i \forall i \in K\}.$$

A GCF V is called *finitely generated* if $V_{PO}(K)$ is a finite not empty set for any $K \subseteq I$. Obviously, any finitely generated GCF is compact.

A GCF is called *superadditive* if

P7 $V(K' \cup K'') \supseteq V(K') \cap V(K'')$ for all coalitions $K', K'' \subseteq I$ with $K' \cap K'' = \emptyset$.

It is called *monotone* if

P8 $V^0(K') \subseteq PR_{K'}V^0(K'')$ for all coalitions $K' \subseteq K'' \subseteq I$, where $PR_{K'}V^0(K'')$ denotes the projection of $V^0(K'')$ on $\mathbb{R}^{K'}$.

Proposition 2 *Any superadditive GCF is monotone, i.e. P7 implies P8.*

Proof. Let us fix a superadditive GCF V, a pair of coalitions K' and K'' such that $K' \subseteq K''$, and an arbitrary vector $x \in V^0(K')$. We have to prove P8, that is $x' \in PR_{K'}V^0(K'')$. Let us fix an arbitrary vector $x \in V^0(K'' \setminus K')$. Coalitions K' and $K'' \setminus K'$ are disjoint and their union is K''. Thus, according to P7, there exists a vector $x'' \in V^0(K'')$ such that $x''_i \geq x'_i$ for any $i \in K'$. □

The set of outcomes is defined as $A = V(I) \subseteq \mathbf{R}^I$, i.e. A contains all those allocations which the coalition of all the players is able to guarantee. An outcome x is called *acceptable* for a coalition $K \subseteq I$ if this coalition cannot guarantee more for all the participants, i.e. if there is no $x' \in V(K)$ such that $x'_i > x_i$ for all the players $i \in K$. The *core* $C(V)$ is defined as the set of outcomes acceptable for all the coalitions $K \subseteq I$. Given an arbitrary family of coalitions $\mathcal{K} \subseteq 2^I$, the $\mathcal{K}$-core $C(V, \mathcal{K})$ is defined as the set of outcomes acceptable for all the coalitions $K \in \mathcal{K}$. According to these definitions, $C(V, \mathcal{K}) = C(V)$ for $\mathcal{K} = 2^I$. A family $\mathcal{K}$ will be called *V-stable* if $C(V, \mathcal{K}) \neq \emptyset$ for any compact, superadditive GCF V.

Theorem 4 (Scarf (1967)) *The core $C(V)$ is not empty if V is compact and for any (minimal) balanced weighting $w : 2^I \to \mathbb{Z}_+$ of all the coalitions the following inclusion holds:*

$$\bigcap_{K \subseteq I, w(K) \neq 0} V(K) \subseteq V(I). \tag{4.1}$$

Remark 1 *Scarf's Theorem provides only sufficient conditions for non-emptiness of the core while Theorem 1 provides necessary and sufficient conditions.*

Remark 2 *If w is a partition weighting then P7 (superadditivity) implies (4.1).*

Theorem 5 (Gurvich and Vasin (1978a)) *The $\mathcal{K}$-core $C(V, \mathcal{K})$ is not empty if the GCF V is compact and for any (minimal) balanced weighting $w : \mathcal{K} \cup [I] \to \mathbb{Z}_+$ of the extended family $\mathcal{K} \cup [I]$ the inclusion (4.1) holds.*

Proof. The proof is based on Theorem 4. Let us introduce a new GCF V' obtained from V by a "discrimination" of the coalitions which does not belong to the family $\mathcal{K}$, i.e.by defining

$$V'(K) = V(K) \text{ if } K \in \mathcal{K} \text{ and } V'(K) = \bigcap_{i \in K} V(\{i\}) \text{ if } K \notin \mathcal{K}.$$

According to this formula, $V'(\{i\}) = V(\{i\})$ for any $i \in I$.

If the conditions of the theorem hold for a GCF V then the conditions of Theorem 4 hold for the GCF V', because for any balanced weighting $w : 2^I \to \mathbb{Z}_+$ we can substitute any coalition $K \notin \mathcal{K}$ by the set of the singletons $\{\{i\}|i \in K\}$ and obtain a balanced weighting $w' : \mathcal{K} \cup [I] \to \mathbb{Z}_+$. Thus, $C(V') \neq \emptyset$ by Theorem 4. On the other hand, $C(V') \subseteq C(V, \mathcal{K})$, because the GCFs V and V' enable the coalitions of $\mathcal{K}$ with "the same power". □

Theorem 4 is a special case of the last theorem corresponding to $\mathcal{K} = 2^I$.

To any CF v we can assign a GCF V_v defined by

$$V_v^0(K) = \{x \in \mathbf{R}^K | x_K = \sum_{i \in K} x_i \leq v(K)\}$$

for all coalitions $K \subseteq I$.

Lemma 2 *For any CF v the corresponding GCF V_v is compact and monotone, and it is superadditive iff CF v is superadditive. Furthermore, $C(V_v, \mathcal{K}) = C(v, \mathcal{K})$ for any $\mathcal{K}$.*

Proof. The monotonicity follows from the equality $PR_{K'}V^0(K'') = \mathbf{R}^{K'}$ for any $K', K'' \subseteq I$ for which $K' \subseteq K''$. The compactness is evident.

To prove superadditivity, let us choose coalitions $K', K'' \subseteq I$ such that $K' \cap K'' = \emptyset$. Then $(V(K') \cap V(K'') \subseteq V(K' \cup K'')$ iff $\sum_{i \in K'} x_i \leq v(K')$ and $\sum_{i \in K''} x_i \leq v(K'')$, which implies that $\sum_{i \in K' \cup K''} x_i \leq v(K' \cup K'')$ iff $v(K') + v(K'') \leq v(K' \cup K'')$.

Finally, the last claim follows by the definition of $\mathcal{K}$-cores. □

To any GCF V let us assign the *truncated* GCF V' defined by

$$V'^0(K) = \{x \in V^0(K) | \exists y \in V^0(K) \text{ for which } y_i \geq x_i,\ y_i \geq v_i \forall i \in I\}$$

for all coalitions $K \subseteq I$. Equivalently,

$$V'_{PO}(K) = V_{PO}(K) \cap \{x \in \mathbb{R}^K | x_i \geq v_i \forall i \in K\}.$$

In other words, $V'_{PO}(K)$ consists of all the vectors of $V_{PO}(K)$ acceptable for every player $i \in K$. Obviously, the truncated GCF V' is compact if V is compact. Let us note also that every allocation $x \in V'(K) \setminus V(K)$ is not acceptable for at least one player $i \in K$.

Example 12 $\mathcal{K}' = \{(1,2),(1,3),(2,3,4)\}$ *is partitionable, because $\mathcal{K}'$ does not contain a partition at all, while $\mathcal{K}' \cup [I]$ is not partitionable. Indeed, $\mathcal{K} = \{(1,2),(1,3),(2,3,4),(4)\}$ is a minimal balanced subfamily of $\mathcal{K}' \cup [I]$, and obviously $\mathcal{K}$ is not a partition.*

Let us consider the CF $v_{\mathcal{K}}$ and the GCF $V = V_{v_{\mathcal{K}}}$. Then

$$\begin{array}{lcll} V_{PO}(1,2) & = & \{(x_1,x_2) & |x_1+x_2=2\} \\ V_{PO}(1,3) & = & \{(x_1,x_3) & |x_1+x_3=2\} \\ V_{PO}(2,3,4) & = & \{(x_2,x_3,x_4) & |x_2+x_3+x_4=3\} \\ V_{PO}(1,2,3,4) & = & \{(x_1,x_2,x_3,x_4) & |x_1+x_2+x_3+x_4=3\} \end{array}$$

etc., while

$$\begin{array}{lcl} V'_{PO}(1,2) & = & \{(x_1,x_2)|x_1+x_2=2,\ x_1\geq 0, x_2\geq 0\} \\ V'_{PO}(1,3) & = & \{(x_1,x_3)|x_1+x_3=2,\ x_1\geq 0, x_3\geq 0\} \\ V'_{PO}(2,3,4) & = & \{(x_2,x_3,x_4)|x_2+x_3+x_4=3,\ x_2\geq 0, x_3\geq 0, x_4\geq 1\} \\ V'_{PO}(1,2,3,4) & = & \{(x_1,x_2,x_3,x_4)| \begin{array}{l} x_1+x_2+x_3+x_4=3, \\ x_1\geq 0, x_2\geq 0, x_3\geq 0, x_4\geq 1 \end{array} \}, \end{array}$$

etc.

Finally, to a GCF V let us assign another GCF V'' defined by $V''_{PO}(K) = INT(n, V'_{PO}(K))$, where $INT(n,X)$ is the set of all the outcomes of V the coordinates of which are all multiple of $1/n$. It is easy to check that V'' is finitely generated.

Theorem 6 (Gurvich and Vasin (1978)) *A family of coalitions $\mathcal{K}$ is V-stable iff the extended family $\mathcal{K} \cup [I]$ is partitionable.*

Here we shall prove a stronger claim.

Theorem 7 . *If $\mathcal{K} \cup [I]$ is partitionable then the $\mathcal{K}$-core is not empty for any compact superadditive GCF. Furthermore, if $\mathcal{K} \cup [I]$ is not partitionable then there exists a superadditive finitely generated GCF such that its $\mathcal{K}$-core is empty.*

Proof. For the first claim, let us apply Theorem 5. If $w : \mathcal{K} \cup [I] \to \mathbb{Z}_+$ is a partition then (4.1) is implied by the superadditivity (P7).

For the second claim, let us suppose that $\mathcal{K} \cup [I]$ is not partitionable, and let us consider a minimal balanced weighting $w : \mathcal{K} \cup [I] \to \mathbb{Z}_+$ which is not a partition. Let us consider its support $\mathcal{K}_w = \{K \in \mathcal{K} \cup [I] | w(K) \neq 0\}$, the CF $v_{\mathcal{K}_w}$ and the GCF $V_{v_{\mathcal{K}_w}}$. According to Lemma 1, $v_{\mathcal{K}_w}$ is superadditive and $C(v_{\mathcal{K}_w}, \mathcal{K}_w) = \emptyset$. Thus, $V_{v_{\mathcal{K}_w}}$ is superadditive, compact, and $C(V_{v_{\mathcal{K}_w}}, \mathcal{K}_w) = \emptyset$, by Lemma 2.

Unfortunately, we cannot just finish our proof yet, since $\mathcal{K}_w$ can contain singletons which are not in $\mathcal{K}$, and in principle, it could happen that the $\mathcal{K}_w$-core is empty while the $(\mathcal{K}_w \setminus [I])$-core is not.

Let us consider the truncated GCF $V'_{v_{\mathcal{K}_w}}$. It is not difficult to check that it is monotone, superadditive, compact, and by the definition, any Pareto optimal allocation is acceptable for all the singletons from $\mathcal{K}$. Thus $C(V'_{v_{\mathcal{K}_w}}, \mathcal{K}_w \setminus [I]) = \emptyset$. This implies that $C(V'_{v_{\mathcal{K}_w}}, \mathcal{K}) = \emptyset$, since $\mathcal{K}_w \setminus [I] \subseteq \mathcal{K}$. Thus the family $\mathcal{K}$ is not V-stable.

Since the GCF $V'_{v_{\mathcal{K}_w}}$ is not finitely generated, let us use instead the GCF $V''_{v_{\mathcal{K}_w}}$. Clearly, it is superadditive and by the definition, it is finitely generated.

Let us prove that $C(V''_{v_{\mathcal{K}_w}}, \mathcal{K}_w \setminus [I]) = \emptyset$.

Let us recall that we denote by $n = |I|$ the number of players, and by m the multiplicity of the balanced family $\mathcal{K}_w$. Furthermore, for $x = (x_1, ..., x_n) \in \mathbf{R}^n$, let $x_K = \sum_{i \in K} x_i$ for all $K \subseteq I$, and in particular, let $x_I = \sum_{i \in I} x_i$. Finally, let $A = \{x = (x_1, ..., x_n) \in \mathbb{R}^n | x_I \leq v_{\mathcal{K}_w}(I)\}$ denote the set of outcomes. Let us introduce further $\delta_i = 1 - x_i$ for $i \in I$, and let $\delta_K = |K| - x_K$ for $K \subseteq I$. In particular, $\delta_I = n - x_I$. Let us note that

$$v_{\mathcal{K}_w}(I) \leq n - 1, \;\; \delta_I = n - x_I \geq 1, \;\; \sum_{K \in \mathcal{K}_w} \delta_K \geq m, \;\; \sum_{K \in \mathcal{K}_w} |K| = nm.$$

We claim that for any allocation $x \in A$ there exists a coalition $K \in \mathcal{K}_w$ such that $\delta_K \geq |K|/n$. Indeed, if $\delta_K < |K|/n$ for all $K \in \mathcal{K}_w$ then we get a contradiction by

$$m \leq \sum_{K \in \mathcal{K}_w} \delta_K < (1/n) \sum_{K \in \mathcal{K}_w} |K| = m$$

Let us now prove that the allocation x (above) is not acceptable for the coalition K (above) with respect to $V''_{v_{\mathcal{K}_w}}$. Let us increment x_i by a strictly positive value to the nearest multiple of $1/n$ for any $i \in K$. (If x_i itself is a multiple of $1/n$ then we still add $1/n$.) The obtained vector belongs to $V''_{v_{\mathcal{K}_w}}$ and strictly and unanimously preferred to x by K. □

5 Game forms and effectivity functions

As before, let I denote the set of players and let A denote the set of outcomes. Subsets $K \subseteq I$ and $B \subseteq A$ denote, respectively, coalitions of players and blocks of outcomes.

An *effectivity functions* (EFF) is defined as a mapping

$$\mathcal{E} : 2^I \times 2^A \to \{0, 1\}.$$

In other words, an EFF is a Boolean function on the set of variables $I \cup A$. We can as well treat it as a subset of $2^I \times 2^A$, i.e. as the set of pairs (K, B) for which $\mathcal{E}(K, B) = 1$.

The notation $\mathcal{E}(K, B) = 1$ (resp. $\mathcal{E}(K, B) = 0$) is read as "coalition K is effective (resp. is not effective) for block B" and means that K is (resp. is not) able to guarantee that an outcome from B will be realized.

An EFF is called K-monotone if $\mathcal{E}(K, B) = 1$ and $K \subseteq K'$ implies $\mathcal{E}(K', B) = 1$. Analogously, it is called B-monotone if $\mathcal{E}(K, B) = 1$ and $B \subseteq B'$ implies $\mathcal{E}(K, B') = 1$. Finally, an EFF $\mathcal{E}$ is called *monotone* if it is both K- and B-monotone, i.e. if

$$\mathcal{E}(K, B) = 1, \; K \subseteq K' \; B \subseteq B' \; \Rightarrow \; \mathcal{E}(K', B') = 1.$$

An EFF is called *superadditive* if

$$(\mathcal{E}(K_1, B_1) = 1, \; \mathcal{E}(K_2, B_2) = 1, \; K_1 \cap K_2 = \emptyset) \Rightarrow (\mathcal{E}(K_1 \cup K_2, B_1 \cap B_2) = 1.$$

The so called *boundary conditions* are defined as

$$\begin{array}{l}\mathcal{E}(I,B)=1 \text{ for any } B \text{ except for } B=\emptyset,\\ \mathcal{E}(K,A)=1 \text{ for any } K \text{ except for } K=\emptyset,\\ \mathcal{E}(\emptyset,B)=0 \text{ for any } B, \text{ in particular } \mathcal{E}(\emptyset,A)=0,\\ \mathcal{E}(K,\emptyset)=0 \text{ for any } K, \text{ in particular } \mathcal{E}(I,\emptyset)=0.\end{array}$$

Let X_i denote a finite set of strategies of player $i \in I$, and let $X = \times_{i\in I} X_i$. Let $g : X \to A$ be a mapping, which assign an outcome $a \in A$ to any strategy-vector $(x_i \,|\, i \in I)$. This mapping g is not supposed to be injective, i.e. the same outcome can be assigned to different strategy-vectors.

The mapping g (or sometimes the quadruple $< I, A, X, g >$) is called a *game form*. Thus, a game form is "something like a game in normal form but without payoff", which is not yet specified. To any game form g we can assign an EFF $\mathcal{E}_g$ defined by

$$\mathcal{E}_g(K,B)=1 \iff (\exists x_K \mid g(x_K, x_{I\setminus K}) \in B \; \forall x_{I\setminus K}))$$

for all $K \subseteq I$, $B \subseteq A$, where $x_K = \{x_i | i \in K\}$ and $x_{I\setminus K} = \{x_i | i \notin K\}$ are the strategie-vectors of the coalitions K and $I \setminus K$, respectively. The right hand part means that coalition K is able to guarantee that the resulting outcome belong to block B independently on the actions of the complementary coalition $I \setminus K$.

An effectivity function $\mathcal{E}$ is called *playable* if it is realized by a game form, i.e. if there exists a game form g such that $\mathcal{E} = \mathcal{E}_g$.

Theorem 8 (Moulin-Peleg (1982)) *An EFF is playable iff it is monotone, superadditive and satisfies the boundary conditions.*

6 Cores of effectivity functions

Let us call *utility function* a mapping $u : I \times A \to \mathbf{R}$. The real number $u(i, a)$ is interpreted as the profit of player $i \in I$ in case outcome $a \in A$ is realized.

Remark 3 *We could substitute preferences instead of utility functions as well.*

Remark 4 *Due to utilities, players and outcomes are no more symmetric (dual). Indeed, the preferences of players on the set of outcomes do matter, but not vice versa.*

Let us denote by $PR(K, a, u)$ the set of all those outcomes that coalition K strictly and unanimously prefers to a with respect to the utility function u, i.e.

$$PR(K,a,u) = \{a' \in A \,|\, u(i,a') > u(i,a) \;\; \forall i \in K\}.$$

Note that $a \notin PR(K, a, u)$.

Given an EFF $\mathcal{E}$ and a utility function u, we say that the coalition K *rejects* an outcome a if $\mathcal{E}(K, PR(K, a, u)) = 1$, i.e. if K is able to guarantee an outcome, better for all of its participants than a. The *core* $C(\mathcal{E}, u)$ is defined as the set of not rejected outcomes, i.e.

$$C(\mathcal{E}, u) = \{a \in A \,|\, \mathcal{E}(K, PR(K, a, u)) = 0 \;\; \forall K \subseteq I\}.$$

Given $\mathcal{E}$, u and a family of coalitions $\mathcal{K}$, the $\mathcal{K}$-*core* $C(\mathcal{E}, u, \mathcal{K})$ is defined as the set of outcomes not rejected by the coalitions from $\mathcal{K}$, i.e.

$$C(\mathcal{E}, u, \mathcal{K}) = \{a \in A \mid \mathcal{E}(K, PR(K, a, u)) = 0 \;\; \forall K \in \mathcal{K}\}.$$

A *game in normal form* is defined as a pair $< g, u >$. If it is given then the corresponding cores $C(\mathcal{E}_g, u)$ and $\mathcal{K}$-core $C(\mathcal{E}_g, u, \mathcal{K})$ are determined, too.

A family of coalitions $\mathcal{K}$ is called *g-stable* if $\mathcal{K}$-core is not empty for any game in normal form, or in other words, if $C(\mathcal{E}_g, u, \mathcal{K}) \neq \emptyset$ for any utility function u and for any game form g.

7 V-stability is equivalent to g-stability

In this section we allow the set of outcomes A and the sets of strategies $X_i, i \in I$ to be infinite, but the set of players I is still supposed to be finite. Given an EFF $\mathcal{E}$ and a utility function $u : I \times A \to \mathbb{R}$, let us define a CF $v_{\mathcal{E},u}$ and a GCF $V_{\mathcal{E},u}$ by

$$v_{\mathcal{E},u}(K) = sup_{B \subseteq A, \mathcal{E}(K,B)=1} \; inf_{a \in B} \sum_{i \in K} u(i, a),$$

and

$$V_{\mathcal{E},u}(K) = \bigcup_{B \subseteq A, \mathcal{E}(K,B)=1} \; \bigcap_{a \in B} \bigcap_{i \in K} U(i, a),$$

where $U(i, a) = \{x \in \mathbb{R}^I | x_i \leq u(i, a)\}$. In other words, the CF $v_{\mathcal{E},u}$ and the GCF $V_{\mathcal{E},u}$ for any $K \subseteq I$ are, respectively, the sum and the sets of allocations which coalition K is able to guarantee in the game $< I, A, \mathcal{E}, u >$, given in effectivity function form.

Remark 5 *The core and the $\mathcal{K}$-core of such a game is defined by the corresponding GCF, i.e. $C(\mathcal{E}, u, \mathcal{K}) = C(V_{\mathcal{E},u}, \mathcal{K})$. The core and $\mathcal{K}$-core of a normal form game $< g, u >$ can be defined equivalently via the corresponding EFF, i.e. $C(g, u, \mathcal{K}) = C(\mathcal{E}_g, u, \mathcal{K})$. Thus, going from (g, u) through $(\mathcal{E}_g, u)$ to $V_{\mathcal{E}_g,u}$ we can lose some information about the game, but not about its core. However, if we substitute GCF V by CF v then this is no more true.*

Remark 6 *If case of an infinite EFF $\mathcal{E}$ we shall always restrict ourself by a utility function u such that the CF $v_{\mathcal{E},u}$ and the GCF $V_{\mathcal{E},u}$ are well defined, i.e. finite and non-empty, respectively.*

.

The following claims are straightforward.

Lemma 3 *Let $\mathcal{E}$ be an arbitrary EFF and u be a utility function. Then the following hold.*

(c1) *If $\mathcal{E}$ is superadditive then both $v_{\mathcal{E},u}$ and $V_{\mathcal{E},u}$ are superadditive.*

(c2) *If $\mathcal{E}$ is monotone then the GCF $V_{\mathcal{E},u}$ is monotone.*

(c3) *If $\mathcal{E}$ is finite then the GCF $V_{\mathcal{E},u}$ is well defined and finitely generated.*

(c4) *The GCF $V_{\mathcal{E},u}$ has the properties P1,P2,P3.*

Theorem 9 (von Neuman and Morgenstern (1953), §157.2,3) *Every superadditive CF v is realizable by a finite game $< g, u >$ in normal form, i.e. $v = v_{\mathcal{E},u}$ for some playable EFF $\mathcal{E} = \mathcal{E}_g$ and utility function u.*

In fact a much stronger claim holds: For any superadditive v there is a game such that any coalition $K \subseteq I$ is able to guarantee not only the sum $v(K)$, but any allocation of this sum between its participants, as well. Of course, this game (or more exactly the strategy set X) is not finite. Moreover, the inverse claims to (c2), (c3), and (c4) hold too.

Theorem 10 (Gurvich and Vasin (1978a)) *Every superadditive GCF V is realizable by a (continuous) game $< g, u >$ in normal form, i.e. $V = V_{\mathcal{E},u}$ for some playable EFF $\mathcal{E} = \mathcal{E}_g$ (with a continuous set of outcomes) and utility u.*

Theorem 11 *Every finitely generated superadditive GCF V is realizable by a finite game $< g, u >$ in normal form, i.e. $V = V_{\mathcal{E},u}$ for some finite playable EFF $\mathcal{E} = \mathcal{E}_g$ and utility u.*

We provide here simpler proofs based on the theorem by Moulin and Peleg (1982).

Proof of Theorem 10. Given a GCF V, let us define an EFF $\mathcal{E}$ and a utility u such that $V_{\mathcal{E},u} = V$. The set of players I is already given. Let $A = V(I)$ and let $\mathcal{E}$ be defined by

$$\mathcal{E}(K, B) = 1 \quad \text{iff} \quad B \supseteq \{x \in V(I) | \exists x^0 \in V^0(K) : x_i = x_i^0 \, \forall i \in K\}$$

or in other words, a coalition K is able to force any allocation from $V^0(K)$, but it cannot at all influence the allocations for players not in K.

Let u be defined by $u_i(x) = x_i$ for all $x \in A$. It is not difficult to check that $\mathcal{E}$ is monotone if V is monotone, $\mathcal{E}$ is superadditive if V is superadditive, and $\mathcal{E}$ always satisfies the boundary conditions, with maybe the exception of $\mathcal{E}(K, \emptyset) \neq 0$ for some K. Moreover, if V is monotone then $\mathcal{E}(K, \emptyset) = 0$ for all $K \subseteq I$. Let us recall also that, according to Proposition 2, any superadditive V is monotone. Thus, according to Theorem 8, V is realizable by a (continuous) playable EFF and a utility function. □

Proof of Theorem 11. Given a finitely generated GCF V, let us define an EFF $\mathcal{E}$ and a utility function u such that $V_{\mathcal{E},u} = V$. The set of players I is already given.

For any $K \subseteq I$ let us consider the finite set of generators $B(K) \subseteq V^0(K)$, and define the set of outcomes as their union, i.e. $A = \cup_{K \subseteq I} B(K)$. For any outcome $b \in B(K)$ let us

introduce the set of outcomes B_b which consists of all the outcomes $a \in B(K')$ such that $K \subseteq K'$ and $x_i(a) \geq x_i(b)$ for all $i \in K$. In other words, B_b consists of the outcome b and all the generators majorizing b. Now let us define an EFF $\mathcal{E}$ by

$$\mathcal{E}(K, B) = 1 \text{ iff } \exists b \in B(K) | B_b \subseteq B.$$

for any K except for $K = I$ and $K = \emptyset$. In other words, K is effective for any set of outcomes which consists of a generator from $B(K)$ and all its majorizing generators, and of course, K is effective also for any set of outcomes containing one of the above mentioned ones. Let us define now

$$\mathcal{E}(I, B) = 1 \; \forall B \neq \emptyset, \;\; \mathcal{E}(I, \emptyset) = 0, \;\; \text{and} \;\; \mathcal{E}(\emptyset, B) = 0 \; \forall B.$$

By this definition, the EFF $\mathcal{E}$ is finite, B-monotone and satisfies the boundary conditions. Clearly, $\mathcal{E}$ is also K-monotone if V is monotone, and superadditive if V is superadditive (according to Proposition 2, any superadditive V is monotone.) Let us define a utility function u by seting $u_i(x) = x_i$ for all $x \in A$. Thus, we obtain $V_{\mathcal{E},u} = V$. Therefore, according to Theorem 8, any finitely generated superadditive GCF is realizable by a finite playable EFF and a utility function. □

Remark 7 *Let us note, that Theorem 11 is a rather strong generalization of Theorem 9. However the proof given above is quite short, due to the use of the Moulin-Peleg theorem.*

8 Appendix: Perfect graphs and normal hypergraphs

Theorem 12 (Boros and Gurvich (1994)) *A family of coalitions $\mathcal{K}$ is g-stable iff the hypergraph $\mathcal{K}$ is normal.*

Here we provide the proof together with the necessary definitions and results concerning perfect graphs and normal hypergraphs.

8.1 Graphs

Let $G =< V, E >$ be finite non-oriented graph without loops and multiple edges; V is the set of vertices and E is the set of edges. The number of vertices $|V|$ and edges $|E|$ is denoted by n and m, and is called the *order* and the *size* of graph G respectively.

An *subgraph* $G' =< V', E' >$ of graph $G =< V, E >$ is defined by the subset of vertices $V' \subseteq V$ and by the subset of edges $E' \subseteq E$ which join *some* of these vertices. A subgraph is called *induced* if *all* the edges of E which join the vertices of $V' \subseteq V$ are included in E'

The *complement* of a graph $G =< V, E >$ is defined as $G^c =< V, E^c >$, i.e. the set of vertices is the same, and two vertices are adjacent in G^c iff they are not adjacent in G. Evidently the complementation is an involution, i.e. $(G^c)^c = G$.

A graph is called *complete* if any two vertices are connected by an edge. The complement to a complete graph is called *edge-free*. A complete subgraph is called a *clique*, and an

edge-free induced subgraph is called a *stable set*. Each clique (stable set) which does not strictly belong to another clique (stable set) is called *maximal*, and each clique (stable set) of maximal size is called *maximum*. The order of a maximum clique (stable set) is called *clique number* (*stability number*) and will be denoted by $\omega(G)$ (by $\alpha(G)$). The following relations always hold

$$\alpha(G^c) = \omega(G),\ \omega(G^c) = \alpha(G),\ n(G^c) = n(G),\ m(G^c) = n(n-1)/2 - m(G).$$

A mapping $c : V \to \{1, .., j\}$ is called a *coloring* of the graph (by j colors). A coloring is called *proper* if any two adjacent vertices are colored by different colors. The minimal number of colors by which a proper coloring exists is called the *chromatic number* of the graph and is denoted by $\chi(G)$. Evidently $\chi(G) \geq \omega(G)$ for any graph G.

8.2 Perfect graphs

A graph G is called *perfect* if for any induced subgraph G' of G the equation $\chi(G') = \omega(G')$ holds. There is a well-known simple criterion of perfectness.

Proposition 3 *A graph G is perfect iff in any of its induced subgraphs there exists a stable set which has a vertex (exactly one) from any maximum clique.*

The following simple generalization will be important.

Proposition 4 *A graph is perfect iff for any non-negative weighting $w : V \to \mathbf{R}_+$ of its verices there exists a stable set which has a vertex (exactly one) from any clique of maximum weight.*

If the weights of vertices take values 0 and 1 only then the above proposition coincide with the previous one. The following criterion is more difficult.

Theorem 13 (Lovász (1972 a,b)) *A graph G is perfect iff for any induced subgraph G' of G the inequality $\alpha(G') \times \omega(G') \geq n(G')$ holds.*

Corollary 1 *Graph G is perfect iff the complementary graph G^c is perfect.*

Perfect graphs were introduced by Berge (1961 a,b), where he suggested two conjectures. The last statement was called Berge's weak perfect graph conjecture (BWPGC), and since 1972 it is called Lovász perfect graph theorem. The following Berge Strong Perfect Graph Conjecture (BSPGC) is still open.

A *hole* is a simple chordless cycle of the length at least 4, and an *antihole* is the complement to a hole. Odd holes C_{2i+1} and odd antiholes C^c_{2i+1} satisfy the following equations

$$\omega(C_{2i+1}) = \alpha(C^c_{2i+1}) = 2,\ \alpha(C_{2i+1}) = \omega(C^c_{2i+1}) = i.$$

In both cases $\alpha \times \omega = 2i < 2i + 1 = n$. Thus odd holes and odd antiholes are not perfect.

By the definition, a graph cannot be perfect if it contains an induced subgraph isomorphic to an odd hole or an odd antihole. The inverse implication is BSPGC.

8.3 Hypergraphs

A *hypergraph* $\mathcal{H}$ is defined as a nonempty family of a nonempty subsets, called *edges*, of a finite set which elements are called *vertices*. Let us denote by $E(\mathcal{H})$ the set of edges and by $V(\mathcal{H})$ the set of vertices. Multiple edges are allowed for hypergraphs. A hypergraph $\mathcal{H}$ can also be given by *incidence matrix* A whose rows and columns are labelled respectively by the edges and the vertices of the hypergraph, and where $A(H, j) = 1$ iff vertex j belongs to the edge H. We assume that each row and each column of A contain ats least one non-zero. The transposed matrix A^T defines the dual hypergraph $\mathcal{H}^d$ whose edges and vertices are assigned respectively to the vertices and the edges of $\mathcal{H}$, i.e. $E(\mathcal{H}) = V(\mathcal{H}^d)$ and $V(\mathcal{H}) = E(\mathcal{H}^d)$.

Given a hypergraph $\mathcal{H}$, let us define the *co-occurrence* graph, $G_c(\mathcal{H})$, with the same set of vertices $V(\mathcal{H})$, and where two vertices are connected by an edge iff they belong to the same edge of $\mathcal{H}$. Let us define also the *intersection* graph, $G_i(\mathcal{H})$, with vertex-set is $E(\mathcal{H})$, and where two vertices are connected iff the corresponding two edges of $\mathcal{H}$ have a non-empty intersection.

Lemma 4 *For any hypergraph $\mathcal{H}$ graphs $G_i(\mathcal{H})$ and $G_c(\mathcal{H}^d)$ coincide.*

Proof. This claim results directly from the definitions above. It is easy to verify that two edges of $\mathcal{H}$ have a non-empty intersection iff the corresponding two vertices of $\mathcal{H}^d$ belong to the same edge. □

Given a graph G, it is easy to obtain a hypergraph $\mathcal{H}$ such that $G = G_c(\mathcal{H})$. Let us consider an arbitrary clique covering of G, i.e. a family of cliques the union of which covers all the edges of G. Then clearly these cliques form a hypergraph $\mathcal{H}$ for which G is the co-ocurrence graph, $G = G_c(\mathcal{H})$. Moreover, all the hypergraphs with this property can be obtained in this way.

Analogously, given a graph G, we can obtain a hypergraph $\mathcal{H}$ such that $G = G_i(\mathcal{H})$. According to Lemma 4, this problem is just dual to the previous one. Let us consider an arbitrary clique covering of G again. Then the collections of those cliques, which contain a given vertex of G, form edges of a hypergraph $\mathcal{H}$ such that $G = G_i(\mathcal{H})$. Moreover, all the hypergraphs with this property can be obtained in such a way from all the clique coverings.

8.4 Normal hypergraphs

A hypergraph $\mathcal{H}$ is said to have the Helly property if any family of edges whose intersection is empty contains two disjoint edges. For example intervals of an ordered set or subtrees of a tree has this property.

A hypergraph $\mathcal{H}$ is called *clique-maximal* if it contains, as edges, all the maximal cliques of its co-ocurrence graph $G_c(\mathcal{H})$.

Lemma 5 *A hypergraph $\mathcal{H}$ has the Helly property iff its dual hypergraph $\mathcal{H}^d$ is clique-maximal.*

Proof is trivial. □

A hypergraph $\mathcal{H}$ is said to have *the Berge property* if its intersection graph $G_i(\mathcal{H})$ is perfect. It is called *normal* if it has both the Helly and Berge properties. Evidently $\mathcal{H}$ is normal iff the graph $G_c(\mathcal{H}^d)$ is perfect and its clique covering hypergraph $\mathcal{H}^d$ is clique-maximal.

Normal hypergraphs were introduced by Berge (1970), and Lovász (1972b) proved BW-PGC by means of this notion. There are many equivalent characterizations of normality, see e.g. Lovász (1972b) and Berge (1984). In the present paper we add two more characterizations: g-stability and partitionability of $\mathcal{H} \cup [I]$.

8.5 Normality and g-stability are equivalent

Theorem 14 *A hypergraph $\mathcal{K} \cup [I]$ is partitionable iff $\mathcal{K}$ is normal.*

Proof. First of all let us note that $\mathcal{K}$ has the Helly (or Berge) property iff $\mathcal{K} \cup [I]$ has the same property.

Lemma 6 *If $\mathcal{K} \cup [I]$ is partitionable then $\mathcal{K}$ has the Helly property.*

Proof. Given hypergraph $\mathcal{K} = \{K_j | j \in J\}$ let us define the maximal integer positive k such that there exists a partial hypergraph $\mathcal{K}' = \{K_j | j \in J'\}$ for some subset $J' \subseteq J$ of cardinality k, $|J'| = k$, in which any $k-1$ edges of $\mathcal{K}'$ have a vertex in common, and $\bigcap_{j \in J'} K_j = \emptyset$. Such k and $\mathcal{K}'$ are called respectively the *Nakamura number* and the *Nakamura subfamily* of $\mathcal{K}$, see Nakamura (1975). Let us define $k = 0$ if $\bigcap_{j \in J'} K_j \neq \emptyset$. Then $\mathcal{K}$ has the Helly property iff $k \leq 2$.

Let us suppose that $k > 2$ and let us consider a Nakamura subfamily $\mathcal{K}'$ of $\mathcal{K}$. Its incidence matrix contains no column with all ones, contains all k different columns with $k-1$ ones, and may contain some other columns with a smaller number of ones. Evidently, by adding some singletons to $\mathcal{K}'$ we can obtain a balanced subfamily of $\mathcal{K} \cup [I]$ which is not partitionable. □

Let us consider a hypergraph $\mathcal{K}$ together with the extended hypergraph $\mathcal{K} \cup [I]$, and their intersection graphs $G = G_i(\mathcal{K}) = G_c(\mathcal{K}^d)$ and $G' = G_i(\mathcal{K} \cup [I]) = G_c((\mathcal{K} \cup [I])^d)$. The graph G' can be obtained from G by adding "individual" vertices to all maximal cliques of G, i.e. one vertex to each of maximal clique C, incident to all the vertices of C and only to these.

Any weighting of the edges of $\mathcal{K} \cup [I]$ is a weighting of the vertices of G'. A weighting on $\mathcal{K} \cup [I]$ is balanced iff it is equiclique on G', i.e. all the maximal cliques have the same (maximum) weight. The weighting is a partition if the vertices with positive weights form a stable set of G'.

Lemma 7 *If $\mathcal{K} \cup [I]$ is partitionable then $\mathcal{K}$ has the Berge property, i.e. graph G is perfect.*

Proof. According to Lemma 6, if $\mathcal{K} \cup [I]$ is partitionable then $\mathcal{K}$ has the Helly property. Let us suppose that graph $G' = G_i(\mathcal{K} \cup [I])$ is not perfect. Then, according to Proposition 4, there is an integer non-negative weighting w' of G' such that no stable set can intersect all the cliques of maximum weight. Let us note however that w may not be an equiclique weighting. In this case let us add to all the lighter cliques the corresponding individual vertices with appropriate weights in order to make the obtained a weighting w'' equiclique. Clearly the maximum weight remains the same. The corresponding weighting of $\mathcal{K} \cup [I]$ is a balanced one, and $\mathcal{K} \cup [I]$ is partitionable, by the assumption. Thus, there is a stable set which meets all the cliques of maximum weight with respect to w''. Evidently, the same stable set meets all the cliques of maximum weight with respect to w', which yields the desired contradiction. □

Lemma 8 *If $\mathcal{K}$ has both the Helly and Berge properties, i.e if the graph $G = G_c(\mathcal{K}^d)$ is perfect and the hypergraph $\mathcal{K}^d$ is clique-maximal, then $\mathcal{K} \cup [I]$ is partitionable.*

Proof. Let us consider an arbitrary balanced weighting w of $\mathcal{K} \cup [I]$ and prove that it is partitionable. Consider the corresponding weighting of vertices of G'. Then, according to Proposition 4, there is a stable set S in G' which meets every clique of maximum weight with respect to w, because G' is perfect and $\mathcal{K}^d$ is maximal. Let us consider the subfamily of $\mathcal{K} \cup [I]$ associated with S. Evidently, it is a partition. Let us subtract unit weights of this partition from w, and obtain another balanced weighting of multiplicity one less. Repeating the above procedure, we can obtain a partition w. □

Remark 8 *A hypergraph $\mathcal{K}$ has the Helly (or Berge) property iff $\mathcal{K} \cup [I]$ has the same.*

However, this is not the case with partitionability. If $\mathcal{K} \cup [I]$ is partitionable then $\mathcal{K}$ is partitionable, too, but not vice versa. For example, $\mathcal{K} = \{(1,2),(1,3),(2,3,4)\}$ is partitionable because there are no balanced subfamilies in $\mathcal{K}$ at all, while $\mathcal{K} \cup [I]$ is not partitionable.

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