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A Circular Graph - Counterexample to the Duchet Kernel Conjecture

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Abstract

We construct a directed graph G such that a) G is strongly connected, b) G has the circular symmetry, c) G is not a directed odd cycle, d) G has no kernel but e) after removing any edge from G the obtained graph has a kernel.

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Let $G =< V, A >$ be a directed graph (digraph). A kernel is a subset K of V which is

(i) independent, i.e. there are no arcs between its elements, and

(ii) dominating, i.e. for every vertex v outside K there is an arc from a vertex of K to v.

A digraph is called kernel-less if it has no kernel.

Von Neumann proved that every acyclic digraph has a unique kernel. Later Richardson proved that in any kernel-less digraph there is an odd directed cycle. A very short proof of the last fact is given by Berge and Duchet in their survey [1] on kernels in digraphs. Thus, odd directed cycles are the simplest kernel-less digraphs. Let us note that removing an edge from such a cycle we get an acyclic digraph which has a (unique) kernel.

In the late 70-s [1] Pierre Duchet [2] conjectured that there are no other connected digraphs with this property. In other words, for every connected kernel-less digraph which is not an odd directed cycle, there exists an edge which can be removed and the obtained digraph is still kernel-less. Clearly, this would strengthen the Richardson theorem. However, here we give a counterexample.

Let $n > 16$ and let $G(n)$ be a digraph whose vertices are $V = [n] = \{1, ..., n\}$ and whose arcs are $A = \{[i, i + j(\text{mod } n)] \mid i = 1, ..., n;\ j = 1, 7, 8\}$.

Lemma 1 *$G(n)$ has a kernel iff $n = 0(mod\ 3)$ or $n = 0(mod\ 29)$.*

Proof: Let $K = \{k(1), ..., k(m)\}$ be a kernel in $G(n)$, and let $d(i)$ be the circular distance between the i-th and $(i + 1)$-th vertices, that is

$$k(i + 1) = k(i) + d(i)(\text{mod } n);\quad i = 1, ..., m;\quad k(m + 1) = k(1).$$

Let us note that $d(i)$ can take the values 2, 3, 4, 5, 6 and 9 only. Indeed, if $d(i) = 1$, or 7, or 8 then K could not be independent, and if $d(i) > 9$ then K could not be dominating. Moreover, the following assertions also result from the fact that K is independent and dominating:

if $d(i) = 3$ then $d(i + 1) = 2$ and $d(i + 2) = 4$, or $d(i + 1) = d(i + 2) = 3$, or $d(i + 1) = 6$;
if $d(i) = 4$ then $d(i + 1) = 2$ and $d(i + 2) = 3$, or $d(i + 1) = 5$;
if $d(i) = 5$ then $d(i + 1) = 4$ and $d(i + 2) = 2$;
if $d(i) = 6$ then $d(i + 1) = 3$ and $d(i + 2) = 2$;
if $d(i) = 9$ then $d(i + 1) = d(i + 2) = d(i + 3) = 2$,
where all $i + j$ are taken mod $m; j = 1, 2$. Thus, all the kernels are given by the periodical circular sequences of distances with periods:

$[3]$, or $[9, 2, 2, 2]$, or $[6, 3, 2, 4, 5, 4, 2, 3]$.

In other words, the only possible kernels are

$\{3, ..., 3t\}$, $\ n = 3t$, $\ t = 6, 7, ...$, or

[1] In the first version of this paper we numbered the vertices $0, 1, ..., n - 1$ and attributed the paper Duchet (1980) to the early 80-s. In the present version we renumbered the vertices by $1, ..., n$ and respectively reattributed the paper to the late 70-s.

$\{9, 11, 13, 15, ..., 15t + 9, 15t + 11, 15t + 13, 15t + 15\}$, $n = 15(t + 1); t = 1, 2, ...$, or

$\{6, 9, 11, 15, 20, 24, 26, 29, ..., 29t + 6, 29t + 9, 29t + 11, 29t + 15, 29t + 20, 29t + 24, 29t + 26, 29t + 29\}; n = 29(t + 1), t = 0, 1, ...$

This proves the lemma. □

In particular, digraph $G(43)$ is kernel-less. But removing any edge from $G(43)$ we create a kernel, in contradiction to Duchet's conjecture. Because of circularity, it is sufficient to check only 3 edges from 129: one of length 1, one of length 7, and one of length 8. And indeed, in all three cases kernels do appear. Respectively,

$$K_1 = \{6, 9, 11, 15, 20, 24, 25, 29, 34, 38, 40, 43\},$$
$$K_7 = \{7, 9, 11, 13, 22, 24, 26, 28, 37, 39, 41, 43\},$$
$$K_8 = \{3, 5, 8, 14, 17, 19, 23, 28, 32, 34, 37, 43\}.$$

Let us note that circular distances between the neibours

$$6, 3, 2, 4, 5, 4, 1, 4, 5, 4, 2, 3;$$
$$7, 2, 2, 2, 9, 2, 2, 2, 9, 2, 2, 2;$$
$$3, 2, 3, 6, 3, 2, 4, 5, 4, 2, 3, 6.$$

are related to periods $[9, 2, 2, 2]$ and $[6, 3, 2, 4, 5, 4, 2, 3]$.

The conjecture may be true if we restrict ourself by circular digraphs with only two edge-lengths involved. (In our example above there are three: $1, 7$, and 8.) At least, we considered the case of two lengths: 1 and i, where i is an integer (positive or negative). And, though we could not prove the conjecture, still we checked its validity for $n < 10,000,000$.

The family of circular graphs can be used as a test not only in this case; see also [3]. We could compare circular graphs with a bore-well which is very narrow but it can be made deep enough to meet some interesting layers of graphs with some prescribed ("circular") properties.

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References

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