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Temporal Logic and Semidirect Products: An Effective Characterization of the Until Hierarchy

by

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ABSTRACT (from [TW96])

We reveal an intimate connection between semidirect products of finite semigroups and substitution of formulas in linear temporal logic. We use this connection to obtain an algebraic characterization of the 'until' hierarchy of linear temporal logic; the k-th level of that hierarchy is comprised of all temporal properties that are expressible by a formula of nesting depth k in the 'until' operator. Applying deep results from finite semigroup theory we are able to prove that each level of the 'until' hierarchy is decidable.

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1 Preface

The purpose of this technical report is to provide proofs for the results announced in [TW96]. Motivation of the problems considered and a discussion of these results can be found there.

Sections 2 and 3 introduce the notation we use, explain the conventions we adopt, and recall previous results we draw on later in this paper. In Section 4 we prove the theorem that connects substitution for past temporal logic formulas with semidirect products of pseudovarieties of semigroups. In Section 5, we provide an algebraic characterization of since and until depth, which is then shown to yield decidability of the levels of the since and until hierarchy in Section 6. Section 7 extends the results to ω-words.

Theorems 6 and 11 have changed slightly compared to the conference paper [TW96].

2 Algebraic Theory of Regular Languages

We assume familiarity with the algebraic theory of regular languages and finite semigroup theory as, for instance, presented in [Eil76, Pin86, Alm95], and adopt the notation from [Alm95] unless otherwise stated.

Given an alphabet A, the set of all nonempty strings over A is denoted by A^+, and the set of all ω-words over A is denoted by A^ω. A subset of A^+ is called a language over A; a subset of A^ω is called an ω-language over A.

The length of a word x is denoted by $|x|$. Its characters are denoted by $x(0)$, $x(1)$, $x(2)$, ... That means, for instance, that $u(|u|-1)$ is the last character of a string u. Given a string u and positions $i, j < |u|$ such that $i \leq j$, the substring $u_i \dots u_{j-1}$ is denoted by $u[i, j]$.

A language $L \subseteq A^+$ is said to be *recognized* by a semigroup S if there exists a semigroup homomorphism $h\colon A^+ \to S$ and a subset $P \subseteq S$ such that $L = Ph^{-1}$, i. e., if L is the inverse image of some subset of S under h. The syntactic semigroup of a language L is denoted by $S(L)$.

Given a class $\mathcal{C}$ of finite semigroups, we denote by $L(\mathcal{C})$ the class of all (regular) languages that are recognized by members of $\mathcal{C}$. The boolean closure of a class $\mathcal{C}$ of languages is denoted by $\mathrm{BC}(\mathcal{C})$.

2.1 Finite semigroups and pseudovarieties

$B(1,2)^1$ denotes the semigroup $\{1,a,b\}$ whose multiplication is given by $xa = a$, $xb = b$, and $x1 = x$. The subsemigroups $\{a,b\}$ and $\{1,a\}$ of $B(1,2)^1$ are denoted by $B(1,2)$ and Sl_2, respectively.

Given a class $\mathcal{C}$ of finite semigroups, $\mathbf{V}(\mathcal{C})$ denotes the smallest pseudovariety of semigroups containing $\mathcal{C}$. We write $\mathbf{V}*\mathbf{W}$ for the semidirect product of pseudovarieties $\mathbf{V}$ and $\mathbf{W}$, and $\mathbf{V}^n$ for the iterated semidirect product $\mathbf{V}*\cdots*\mathbf{V}$ with n factors. The smallest pseudovariety of semigroups closed under semidirect products that contains $\mathcal{C}$ is denoted by $\mathbf{V}^*(\mathcal{C})$. The class of all semigroups whose reverse belongs to $\mathcal{C}$ is denoted by C^ρ.

For $n \geq 1$, $\mathbf{D}_n$ is the pseudovariety of semigroups defined by the equation

$$yx_1 \dots x_n = x_1 \dots x_n \ , \tag{1}$$

and

$$\mathbf{D} = \bigcup_{n>0} \mathbf{D}_n \ . \tag{2}$$

$\mathbf{R}$ is the pseudovariety of all $\mathcal{R}$-trivial finite semigroups, $\mathbf{SL}$ is the pseudovariety of all finite semilattices, and $\mathbf{MD}_1$ is the pseudovariety of semigroups generated by all semigroups of the form S^1 with $S \in \mathbf{D}_1$.

Equivalent descriptions of $\mathbf{D}_1$, $\mathbf{D}$, $\mathbf{R}$, and $\mathbf{MD}_1$ are given below, see, e. g., [Alm95].

$$\begin{aligned}
\mathbf{D}_1 &= \mathbf{V}(\{B(1,2)\}) \\
\mathbf{D} &= \mathbf{V}^*(\{B(1,2)\}) \\
\mathbf{SL} &= \mathbf{V}(\{Sl_2\}) \\
\mathbf{R} &= \mathbf{V}^*(\{Sl_2\}) \\
\mathbf{MD}_1 &= \mathbf{V}(\{B(1,2)^1\})
\end{aligned}$$

$\mathbf{MD}_1$ can also be defined by the following two identities:

$$xyx = yx \ , \tag{3}$$

$$x^2 = x \ . \tag{4}$$

2.2 Switching rules

We sum up P. Stiffler's switching rules, see [Sti73].[1]

[1] Stiffler's notation is different from the one we use: he denotes $B(1,2)$, Sl_2, and $B(1,2)^1 1$ by U_1, U_2, and U_3, respectively. Notice also that in [Eil76], U_n denotes $B(1,n)^1$ (for

Theorem 1 (Stiffler) *1.* $\mathbf{D} = \mathbf{V}^*(\{B(1,2)\})$, *in particular,* $\mathbf{D}{*}\mathbf{D} \subseteq \mathbf{D}$.

2. $\mathbf{R} = \mathbf{V}^*(\{Sl_2\}) = \bigcup_{n\geq 1} \mathbf{SL}^n$, *in particular,* $\mathbf{R}{*}\mathbf{R} \subseteq \mathbf{R}$.

3. $\mathbf{LR} = \mathbf{R}{*}\mathbf{D} = \mathbf{V}^*(\{B(1,2), Sl_2\})$, *in particular,* $\mathbf{R}{*}\mathbf{D}{*}\mathbf{R} \subseteq \mathbf{R}{*}\mathbf{D}$.

4. For every monoidal pseudovariety $\mathbf{V}$ *of semigroups,* $\mathbf{D}{*}\mathbf{V} \subseteq \mathbf{V}{*}\mathbf{D}$.

The following theorem is Corollary 2.4(c) from [Sti73].

Theorem 2 (Stiffler) *For every* $l > 0$,

$$B(1,2)^{1^l} \circ Sl_2 \prec Sl_2^{4l} \circ Sl_2 \circ B(1,2)^{1^{2l}}; \tag{5}$$

here, every semigroup stands for its right regular representation, e. g., the semigroup $B(1,2)^{1^l}$ *stands for the transformation semigroup usually denoted by* $(B(1,2)^{1^l}, B(1,2)^{1^l})$.

2.3 Semidirect/wreath products and sequential functions

We review the connection between wreath products and sequential functions just in the form we need it.

A function $\sigma: A^+ \to B^+$ is called a sequential function if it is realized by a transducer. Recall that a transducer is a tuple

$$(Q, A, B, q_I, \delta, \lambda) \tag{6}$$

where Q is a finite set of states, $q_I \in Q$ is the initial state, A and B are alphabets, $\delta: Q \times A \to Q$ is a transition function, and $\lambda: Q \times A \to B^+$ is an output function. The transition function δ is inductively extended to a function $Q \times A^+ \to Q$ by setting $\delta(q, wa) = \delta(\delta(q,w), a)$. The function $\sigma: A^+ \to B^+$ realized by the transducer is inductively defined by

$$\begin{aligned} a\sigma &= \lambda(q_I, a) && \text{for } a \in A \text{, and} \\ wa\sigma &= w\sigma(\lambda(\delta(q_I, w), a)) && \text{for } a \in A,\ w \in A^+. \end{aligned}$$

A transducer as in (6) is called *synchronous* if $\lambda(q,a) \in B$ for all $q \in Q$, $a \in A$. Accordingly, a sequential function is called synchronous if it is realized by a synchronous transducer.

$n \geq 0$).

The transformation semigroup of a transducer as in (6) is (Q,T) where T is the subsemigroup of Q^Q generated by $q \mapsto \delta(q,a)$ for $a \in A$.

We will use the following well-known fact about wreath products and regular languages, see, e. g., [Pin86].

Proposition 1 *Let $\sigma\colon A^+ \to B^+$ be a sequential function given by a transducer whose transformation semigroup is (Q,T), and let $L \subseteq B^+$ be a language recognized by a semigroup S. Then $L\sigma^{-1}$ is recognized by the semigroup of $(S^1,S)\circ(Q,T)$.*

We will also use the wreath product principle, introduced in [Str78]:

Proposition 2 *Let $L \subseteq A^+$ be a language recognized by the semigroup of a wreath product $(P,S)\circ(Q,T)$ of transformation semigroups via a homomorphism $h\colon A^+ \to S^Q * T$. Denote the projection of h onto the second component by h_2, and let $B = T^1 \times A$. Define $\sigma\colon A^+ \to B^+$ by*

$$a_0 \ldots a_{n-1}\sigma = (1,a_0)(a_0h_2,a_1)(a_0a_1h_2,a_2)\ldots(a_0\ldots a_{n-2}h_2,a_{n-1}) \ . \tag{7}$$

This function is a synchronous sequential function, which can be realized by a transducer whose transformation semigroup is T's right regular representation.

The language L is a boolean combination of languages of the following two types:

(i) languages of the form th_2^{-1} for some $t \in T$;

(ii) languages of the form $V\sigma^{-1}$ where $V \subseteq B^+$ is recognized by S.

2.4 Decidability criteria

We say a pseudovariety $\mathbf{V}$ of semigroups is *effectively locally finite* if it is locally finite and $m \mapsto F_m\mathbf{V}$ is computable.

The following proposition can be found, for instance, in [Str85], as Proposition 6.2, and in [Alm95], as Theorem 10.2.1.

Proposition 3 *If $\mathbf{V}$ and $\mathbf{W}$ are effectively locally finite pseudovarieties of semigroups, then $\mathbf{V}*\mathbf{W}$ is effectively locally finite.*

The following proposition is Theorem 5.7 from [Str85].

Proposition 4 (Straubing) *If* $\mathbf{V}$ *is a variety of semigroups, then* $S \in \mathbf{V}*\mathbf{D}$ *if and only if* $S \in \mathbf{V}*\mathbf{D}_{|S|}$.

The next theorem is an obvious consequence of results in [TW85] on graph congruences.

Theorem 3 (Thérien,Weiss) *If* $\mathbf{V}$ *and* $\mathbf{W}$ *are pseudovarieties such that*

- $\mathbf{LV} = \mathbf{V}*\mathbf{D}$,
- $\mathbf{V}$ *is decidable, and*
- $\mathbf{W}$ *is effectively locally finite,*

then $\mathbf{V}*\mathbf{W}$ *is decidable.*

The decision procedure we have in mind is the obvious modification of the procedure described under (ii) in Example 2.5 of [TW85]. Its correctness follows from Theorem 2.8 in [TW85].

2.5 Regular ω-languages

We use A. Arnold's notion of the syntactic congruence of a regular ω-language, see [Arn85].

Let $L \subseteq A^\omega$. The *syntactic congruence* of L, denoted by $\equiv_L$, is the binary relation on A^+ that relates u and v if and only if

$$(\forall x, y \in A^*)\,(x(uy)^\omega \in L \leftrightarrow x(vy)^\omega \in L) \tag{8}$$

and

$$(\forall x, y \in A^*)\,(\forall z \in A^+)\,(xuyz^\omega \in L \leftrightarrow xvyz^\omega \in L) \tag{9}$$

The *syntactic semigroup* of L is the quotient of A^+ modulo the syntactic congruence of L; it is denoted by $S(L)$. If L is regular, then $S(L)$ is finite, and given a representation of L (e. g., by a Büchi automaton) an isomorphic copy of $S(L)$ can be computed.

Given an arbitrary congruence $\equiv$ on A^+, we define a similarity relation $\sim$ on A^ω as follows: $\alpha \sim \beta$ iff there exist decompositions $\alpha = u_0u_1u_2\dots$ and $\beta = v_0v_1v_2\dots$ such that $u_i \equiv v_i$ for $i \geq 0$.

For a given language $L \subseteq A^\omega$, we denote by $\sim_L$ the similarity relation induced by $\equiv_L$. The following crucial property of the syntactic congruence of a regular ω-language goes back to [Arn85] but was only explicitly stated in [Péc88].

Lemma 1 (Arnold, Pécuchet) *Let $L \subseteq A^\omega$ be regular and $\sim_L$ the similarity relation induced by $\equiv_L$. If $\alpha \sim_L \beta$, then $\alpha \in L$ iff $\beta \in L$.*

3 Temporal Logic

3.1 Future Temporal Logic

A (future) *temporal logic (TL) formula* over an alphabet A is built from the elements of A and the logical constants TRUE and FALSE using boolean connectives and the usual temporal logic operators: next ($\circ$), eventually ($\Diamond$), and until (U), the two former of which are unary operators and the latter of which is a binary operator written in infix notation.

Given a word $x \in A^+ \cup A^\omega$ and a position $i < |x|$, one defines what it means for a TL formula ϕ to be true in x at i, in symbols $(x, i) \models \phi$. In this paper, we adopt the following rules:

- $(x, i) \models a$ if $x(i) = a$, for every $a \in A$,
- $(x, i) \models \neg\phi$ if $(x, i) \not\models \phi$,
- $(x, i) \models \phi_1 \vee \phi_2$ if $(x, i) \models \phi_1$ or $(x, i) \models \phi_2$,
- $(x, i) \models \phi_1 \wedge \phi_2$ if $(x, i) \models \phi_1$ and $(x, i) \models \phi_2$,
- $(x, i) \models \circ\phi$ if $i + 1 < |x|$ and $(x, i + 1) \models \phi$,
- $(x, i) \models \Diamond\phi$ if there exists j such that $i \leq j < |x|$, and $(x, j) \models \phi$,
- $(x, i) \models \phi \mathsf{U} \psi$ if there exists j such that $i \leq j < |x|$, $(x, i') \models \phi$ for every i' with $i \leq i' < j$, and $(x, j) \models \psi$.

Given a TL formula ϕ, we write $x \models \phi$ and say that x is a model of ϕ if $(x, 0) \models \phi$. We write $L(\phi)$ and $L^\omega(\phi)$ for the set of all finite models and the set of all infinite models of ϕ, i. e., $L(\phi) = \{u \in A^+ \mid u \models \phi\}$ and $L^\omega(\phi) = \{\alpha \in A^\omega \mid \alpha \models \phi\}$.

In some of the commonly accepted definitions of the semantic of the eventually operator the value of j is restricted to positions strictly larger than i. For the until operator, a similar restriction of i' to the values strictly between i and j has been considered. The until depth of a language or ω-language, however, does not depend on which of the four possible combinations is actually chosen. (For a formal definition of until depth, see below.)

True and False denote the disjunction over all atomic formulas and $a \vee \neg a$ for some $a \in A$, respectively.

3.2 Past Temporal Logic

Past temporal logic (PTL) formulas are built just as TL formulas but use a dual set of operators: instead of $\circ$, $\diamond$, and U, PTL formulas use previously ($\ominus$), sometime in the past ($\diamondminus$), and since (S). The semantics of PTL formulas is defined dually to the semantics of TL formulas: in the above definition for TL, every temporal operator is replaced by its dual and every condition that involves the ordering is replaced by the symmetric condition (and $|x|$ by -1). For instance, $(t, i) \models \ominus\phi$ if $i - 1 > -1$ (i. e., $i - 1 \geq 0$)and $(t, i - 1) \models \phi$.

The *dual* of a PTL formula is the TL formula that is obtained from ϕ by replacing every past temporal operator by its future counterpart. Accordingly, the dual of a TL formula is the PTL formula that is obtained by replacing every future temporal operator by its past counterpart. The dual of a formula ϕ is denoted by $\bar{\phi}$. Obviously, $\bar{\bar{\phi}} = \phi$.

Given a PTL formula ϕ over an alphabet A and $u \in A^+$, we write $w \models \phi$ and say that w is a model of ϕ if $(w, |w| - 1) \models \phi$.

Abusing notation, we use $L(\phi)$ to denote the set of models of both TL and PTL formulas although there are formulas that are TL and PTL formulas at the same time and their respective sets of models are different; consider, e. g., the atomic formula a. It will, however, always be clear from the context whether we view a given formula as a TL or a PTL formula.

For the set of duals of a set of formulas Φ we will write $\overline{\Phi}$.

Given a class Φ of TL or PTL formulas, we write $L(\Phi)$ for $\{L(\phi) \mid \phi \in \Phi\}$. Accordingly, we write $L^\omega(\Phi)$ for $\{L^\omega(\phi) \mid \phi \in \Phi\}$.

3.3 Fragments of TL and PTL

The *until depth* of a TL formula is its nesting depth in terms of the until operator. When ϕ is a formula, $\mathrm{ud}(\phi)$ denotes its until depth. We define inductively:

$$\begin{aligned} \mathrm{ud}(a) &= 0, \\ \mathrm{ud}(\neg\phi) &= \mathrm{ud}(\phi) \text{ , for } a \in A, \end{aligned}$$

$$\begin{aligned}
\mathrm{ud}(\phi_1 \vee \phi_2) &= \max\{\mathrm{ud}(\phi_1), \mathrm{ud}(\phi_2)\}\ , \\
\mathrm{ud}(\phi_1 \wedge \phi_2) &= \max\{\mathrm{ud}(\phi_1), \mathrm{ud}(\phi_2)\}\ , \\
\mathrm{ud}(\bigcirc\phi) &= \mathrm{ud}(\phi)\ , \\
\mathrm{ud}(\Diamond\phi) &= \mathrm{ud}(\phi)\ , \\
\mathrm{ud}(\phi_1 \mathsf{U}\, \phi_2) &= \max\{\mathrm{ud}(\phi_1), \mathrm{ud}(\phi_2)\} + 1\ .
\end{aligned}$$

The class of all TL formulas of until depth at most k is denoted by $\mathtt{Ut}_k$. The *residual depth* of a TL formula ϕ, denoted by $\mathrm{rd}(\phi)$, is defined to be the depth of the formula in terms of the TL operators other than U:

$$\begin{aligned}
\mathrm{rd}(a) &= 0, \\
\mathrm{rd}(\neg\phi) &= \mathrm{rd}(\phi)\ , \text{for } a \in A, \\
\mathrm{rd}(\phi_1 \vee \phi_2) &= \max\{\mathrm{rd}(\phi_1), \mathrm{rd}(\phi_2)\}\ , \\
\mathrm{rd}(\phi_1 \wedge \phi_2) &= \max\{\mathrm{rd}(\phi_1), \mathrm{rd}(\phi_2)\}\ , \\
\mathrm{rd}(\bigcirc\phi) &= \mathrm{rd}(\phi) + 1\ , \\
\mathrm{rd}(\Diamond\phi) &= \mathrm{rd}(\phi) + 1\ , \\
\mathrm{rd}(\phi_1 \mathsf{U}\, \phi_2) &= \max\{\mathrm{rd}(\phi_1), \mathrm{rd}(\phi_2)\}\ .
\end{aligned}$$

The *since depth* of a PTL formula is its nesting depth in terms of the since operator; we denote the class of all PTL formulas of since depth at most k by $\mathtt{Sn}_k$. We have, for instance, $\overline{\mathtt{Sn}_k} = \mathtt{Ut}_k$. This implies $\overline{L(\mathtt{Ut}_k)} = L(\mathtt{Sn}_k)$. Here, $\overline{L(\mathtt{Ut}_k)}$ stands for the class of all languages that contains the reverse of each language from $L(\mathtt{Ut}_k)$.

We denote the class of atomic PTL formulas by $\mathtt{At}$. The class of all PTL formulas that are disjunctions of atomic formulas is referred to by $\mathtt{Dj}$; by convention, the logical constant False is a disjunction of atomic formulas and stands for the empty disjunction. The class of PTL formulas in which neither S nor $\diamondminus$ occur is denoted by $\mathtt{Pv}$. The class of all formulas in $\mathtt{Sn}_1$ in which neither $\diamondminus$ nor $\ominus$ occur is denoted by $\mathtt{Sn}$. The class of all PTL formulas in which neither S nor $\ominus$ occur is denoted by $\mathtt{St}$.

We will use the following theorem, which is the dual of Theorem 4.2 from [CPP93].

Theorem 4 (Cohen, Perrin, Pin)

$$L(\mathbf{LR}) = L(\mathtt{Sn}_0)\ . \tag{10}$$

We will also look at a refinement of the until hierarchy: $\mathtt{Ut}_k^l$ is defined to be the set of all TL formulas of until depth at most k and residual depth at most l.

3.4 Substitution

If ϕ is a PTL formula over an alphabet B and $(\psi_b)_{b\in B}$ a family of PTL formulas over A, then $\phi[b \mapsto \psi_b]$ stands for the PTL formula over A obtained from ϕ by replacing every occurrence of a letter $b \in B$ by ψ_b.

Formally, substitution is defined inductively as follows.

$$\begin{aligned}
b'[b \mapsto \psi_b] &= \psi_{b'} && (11)\\
\neg\phi[b \mapsto \psi_b] &= \neg(\phi[b \mapsto \psi_b]) && (12)\\
\phi \vee \phi'[b \mapsto \psi_b] &= (\phi[b \mapsto \psi_b]) \vee (\phi'[b \mapsto \psi_b]) && (13)\\
\phi \wedge \phi'[b \mapsto \psi_b] &= (\phi[b \mapsto \psi_b]) \wedge (\phi'[b \mapsto \psi_b]) && (14)\\
\ominus\phi[b \mapsto \psi_b] &= \ominus(\phi[b \mapsto \psi_b]) && (15)\\
\diamondminus\phi[b \mapsto \psi_b] &= \diamondminus(\phi[b \mapsto \psi_b]) && (16)\\
\phi \mathsf{S} \phi'[b \mapsto \psi_b] &= (\phi[b \mapsto \psi_b]) \mathsf{S} (\phi'[b \mapsto \psi_b]) && (17)
\end{aligned}$$

We first note that the substitution process is associative.

Remark 1 (associativity) *Let ϕ be a PTL formula over an alphabet C, $\{\psi_c\}_{c\in C}$ a family of PTL formulas over an alphabet B, and $\{\chi_b\}_{b\in B}$ a family of PTL formulas over an alphabet A. Then*

$$\phi[c \mapsto \psi_c[b \mapsto \chi_b]] = \phi[c \mapsto \psi_c][b \mapsto \chi_b] \ . \qquad (18)$$

It is also true that substituents in a substitution process can be replaced by equivalent formulas without changing the set of models of the resulting formula:

Remark 2 (equivalent substituents) *Let ϕ be a TL formula over B, and $\{\psi_b\}_{b\in B}$ and $\{\psi'_b\}_{b\in B}$ families of TL formulas over an alphabet A such that ϕ_b and ϕ'_b are equivalent for every $b \in B$. Then $\phi[b \mapsto \psi_b]$ and $\phi[b \mapsto \psi_b]$ are equivalent.*

Observe that if ϕ and ϕ' are equivalent TL formulas over A, then $\phi[b \mapsto \psi_b]$ and $\phi'[b \mapsto \psi_b]$ need not be equivalent: consider the formulas $\phi = a \wedge b$, $\phi' = \textsc{False}$, $\psi_a = c$, and $\psi_b = c$.

Given classes Φ and Ψ of PTL formulas, $\Phi[\Psi]$ stands for the class of PTL formulas that can be written in the form $\phi[b \mapsto \psi_b]$ with $\phi \in \Phi$ and $\psi_b \in \Psi$ for each $b \in B$.

Remark 3 *For every $k \geq 0$,*

$$\begin{aligned}
\mathtt{Sn}_{k+1} &= \mathtt{Sn}_0[\mathtt{Sn}[\mathtt{Sn}_k]] \\
&= \mathtt{Sn}_0[\mathtt{Sn}][\mathtt{Sn}_k] \\
&= \mathtt{Sn}_0 \underbrace{[\mathtt{Sn}][\mathtt{Sn}_0] \dots [\mathtt{Sn}][\mathtt{Sn}_0]}_{k+1 \ \textit{times}} .
\end{aligned} \tag{19}$$

We will also write $\mathtt{Sn}_0([\mathtt{Sn}][\mathtt{Sn}_0])^{k+1}$ for the term (19).

3.5 The EF game for TL

The following definition of the equivalence relation $\equiv_k^l$ is a combinatorial description of game equivalence as defined in [EW96].

Let $x, y \in A^+ \cup A^\omega$, $i < |x|$, and $j < |y|$.

We define $(x, i) \equiv_k^l (y, j)$ to hold if and only if the following conditions and their symmetric counterparts (where the roles of (x, i) and (y, j) are interchanged) are met.

next condition (required if $l > 0$) If $i + 1 < |x|$, then $j + 1 < |y|$ and $(x, i+1) \equiv_k^{l-1} (y, j+1)$.

eventually condition (required if $l > 0$) For every i' with $i \leq i' < |x|$, there exists j' with $j \leq j' < |y|$ such that $(x, i') \equiv_k^{l-1} (y, j')$.

until condition (required if $k > 0$) For every i' with $i \leq i' < |x|$, there exists j' with $j \leq j' < |y|$ such that:

- **(UC1)** if $i' = i$, then $j' = j$;
- **(UC2)** $(x, i') \equiv_{k-1}^l (y, j')$;
- **(UC3)** for every j'' with $j \leq j'' < j'$, there exists i'' with $i \leq i'' < i$ such that $(x, i'') \equiv_{k-1}^l (y, j'')$.

We write $x \equiv_k^l y$ for $(x, 0) \equiv_k^l (y, 0)$.

The following theorem is an adapted version of Theorem 2.1 from [EW96].

Theorem 5 (Etessami, Wilke) *Let x and y be words and $k, l \geq 0$. Then $x \equiv_k^l y$ if and only if x and y are models of the same $\mathtt{Ut}_k^l$-formulas.*

3.6 Switching rules for the previously operator

Remark 4 *For PTL formulas ϕ and ψ,*

$$\begin{aligned} L(\ominus(\phi \mathsf{S} \psi)) &= L(\ominus\phi \mathsf{S} \ominus\psi) \ , \\ L(\ominus\diamondsuit\phi) &= L(\diamondsuit\ominus\phi) \ , \\ L(\ominus(\phi \wedge \psi)) &= L(\ominus\phi \wedge \ominus\psi) \ , \\ L(\ominus(\phi \vee \psi)) &= L(\ominus\phi \vee \ominus\psi) \ , \\ L(\ominus\neg\phi) &= L(\neg\ominus\phi \wedge \ominus\textsc{True}) \ . \end{aligned}$$

4 Substitution and Semidirect Products

4.1 Substitution and sequential functions

Remark 5 *Let $\sigma: A^+ \to B^+$ be a synchronous sequential function, $u \in A^+$, and $v = u\sigma$. Then $|u| = |v|$, and $u[0,i]\sigma = v[0,i]$ for every $i \leq |u|$.*

Remark 6 *For PTL formulas ϕ and ϕ' over an alphabet A, the following equations hold.*

$$L(\neg\phi) = A^+ \setminus L(\phi) \tag{20}$$
$$L(\phi \vee \phi') = L(\phi) \cup L(\phi') \tag{21}$$
$$L(\ominus\phi) = L(\phi)B \tag{22}$$
$$L(\diamondsuit\phi) = L(\phi)B^* \tag{23}$$

Lemma 2 *Let $\sigma: A^+ \to B^+$ be a synchronous sequential function and $L, L' \subseteq B^+$. Then the following equations hold.*

$$(B^+ \setminus L)\sigma^{-1} = A^+ \setminus L\sigma^{-1} \tag{24}$$
$$(L \cup L')\sigma^{-1} = L\sigma^{-1} \cup L'\sigma^{-1} \tag{25}$$
$$LB\sigma^{-1} = L\sigma^{-1}A \tag{26}$$
$$LB^*\sigma^{-1} = L\sigma^{-1}A^* \tag{27}$$

Proof. The first two equations, (24) and (25), are trivial, and do not rely on the assumption that σ is a synchronous sequential function. They hold for every function $\sigma: A^+ \to B^+$.

For the proof of (26), let u be a string over A and $n = |u|$. If $u \in LB\sigma^{-1}$, then $u\sigma \in LB$, say $u\sigma = vb$ with $v \in L$ and $b \in B$. By Remark 5, this implies $u[0, n-1]\sigma = v$, hence $u[0, n-1] \in L\sigma^{-1}$ and thus $u \in L\sigma^{-1}A$. On the other hand, if $u \in L\sigma^{-1}A$, say $u = u'a$ with $u' \in L\sigma^{-1}$ and $a \in A$, then, by Remark 5, $u\sigma = u'\sigma b$ for some $b \in B$, thus $u \in LB\sigma^{-1}$.

For the proof of (27), let u be a string over A and $n = |u|$, just as in the previous paragraph. If $u \in LB^*\sigma^{-1}$, then $u\sigma \in LB^*$, say $u\sigma = v'v''$ for some $v' \in L$ and $v'' \in B^*$. By Remark 5, $u[0, |v'|]\sigma = v'$, hence $u \in L\sigma^{-1}A^*$. On the other hand, if $u \in L\sigma^{-1}A^*$, say $u = u'u''$ with $u'\sigma \in L$ and $u'' \in B^*$, then $u\sigma[0, |u'|] \in L$ by Remark 5, and thus $u \in LB^*\sigma^{-1}$. □

Lemma 3 *Let $\sigma: A^+ \to B^+$ be a synchronous sequential function and $\{\psi_b\}$, $b \in B$, a B-indexed family of PTL formulas over A such that $L(b)\sigma^{-1} = L(\psi_b)$ for every $b \in B$. Then, for every PTL formula ϕ over B,*

$$L(\phi)\sigma^{-1} = L(\phi[b \mapsto \psi_b]) \ . \tag{28}$$

Notice that $L(b)$ is nothing else than B^*b.

Proof. The proof is by induction on the structure of ϕ. The base case is trivial. In the induction step, we have to consider negation, disjunction, the previously operator, the sometime in the past operator, and the since operator. The proofs for the first four cases all follow the same pattern. They use the respective equations from Remark 6, Lemma 2, the definition of the substitution operation, and the induction hypothesis in the same way. We give only a proof for the sometime in the past operator:

$$\begin{array}{rcll}
L(\diamondminus\phi)\sigma^{-1} & = & (L(\phi)B^*)\sigma^{-1} & \text{by (23)} \\
& = & L(\phi)\sigma^{-1}A^* & \text{by (27)} \\
& = & L(\phi[b \mapsto \psi_b])A^* & \text{by induction hypothesis} \\
& = & L(\diamondminus(\phi[b \mapsto \psi_b])) & \text{by (23)} \\
& = & L(\diamondminus\phi[b \mapsto \psi_b]) & \text{by definition of substitution} \ .
\end{array}$$

If ϕ is of the form $\phi_1 \mathsf{S} \phi_2$, we establish the claim as follows. Assume u is an arbitrary string over A. Let $v = u\sigma$. Then, by Remark 5, $|v| = |u|$. The string u belongs to $L(\phi_1 \mathsf{S} \phi_2)\sigma^{-1}$ if and only if there exists $i \leq |v|$ such that

- $v[0, i] \models \phi_2$, and
- $v[0, j] \models \phi_1$ for every j with $i < j < |v|$.

By induction hypothesis, we know that $v[0,i] \models \phi_2$ if and only if $u[0,i] \models \phi_2[b \mapsto \psi_b]$. (Remember that, by Remark 5, $u[0,i]\sigma = v[0,i]$.) Similarly, $v[0,j] \models \phi_1$ if and only if $u[0,j] \models \phi_1[b \mapsto \psi_b]$ for every $j < |u|$. Therefore, the above assertion is equivalent to the existence of $i \leq |u|$ such that

- $u[0,i] \models \phi_2[b \mapsto \psi_b]$, and
- $u[0,j] \models \phi_1[b \mapsto \psi_b]$ for every j with $i < j$.

According to the semantic of PTL formulas, the latter is true if and only if $u \models (\phi_1[b \mapsto \psi_b])\ \mathsf{S}\ (\phi_2[b \mapsto \psi_b])$. However, $(\psi_1[b \mapsto \psi_b])\ \mathsf{S}\ (\psi_2[b \mapsto \psi_b]) = (\phi_1\ \mathsf{S}\ \phi_2)[b \mapsto \psi_b]$ by (17). □

The lemma actually holds under weaker assumptions; we only need that σ is *length-preserving,* i.e., that σ satisfies $|u\sigma| = |u|$ for every $u \in A^+$, and that it *respects prefixes* in the following sense: $uv\sigma \in u\sigma B^+$ for all $u, v \in B^+$.

4.2 Substitution and semidirect products of pseudovarieties

Proposition 5 *Let* $\mathbf{V}$, $\mathbf{W}$ *be pseudovarieties of semigroups and* Φ, Ψ *classes of PTL formulas such that* $L(\mathbf{V}) = L(\Phi)$ *and* $L(\mathbf{W}) = L(\Psi)$. *If* $\Phi[\mathtt{Dj}] \subseteq \Phi$, *then*

$$L(\Phi[\Psi]) \subseteq L(\mathbf{V}{*}\mathbf{W}) \ . \tag{29}$$

Proof. Let $\phi \in \Phi$ be a formula over an alphabet B and $\{\psi_b\}_{b \in B}$ a family of formulas from Ψ. We want to show that $L(\phi[b \mapsto \psi_b])$ is recognized by an element of $\mathbf{V}{*}\mathbf{W}$.

Since $\mathbf{W}$ is assumed to be a pseudovariety, we can find a single semigroup T and an onto homomorphism $h\colon A^+ \to T$ such that every language $L(\psi_b)$ is recognized by h. For every $t \in T$, there exists a formula ψ_t such that $th^{-1} = L(\psi_t)$. That is, for every $b \in B$, there exists a disjunction χ_b of atomic formulas over T (viewed as an alphabet) such that $L(\psi_b) = L(\chi_b[t \mapsto \psi_t])$. Thus

$$\begin{aligned} L(\phi[b \mapsto \psi_b)]) &= L(\phi[b \mapsto \chi_b[t \mapsto \psi_t]]) && \text{by Remark 2} \\ &= L(\phi[b \mapsto \chi_b][t \mapsto \psi_t]) && \text{by Remark 1} \ . \end{aligned}$$

Write ϕ' for $\phi[b \mapsto \chi_b]$. This formulas is in Φ, for $\Phi[\mathtt{Dj}] \subseteq \Phi$. It is enough to show that $L(\phi'[t \mapsto \psi_t])$ is recognized by an element of $\mathbf{V}{*}\mathbf{W}$.

Note first that since $\phi' \in \Phi$, there is an element $S \in \mathbf{V}$ that recognizes $L(\phi')$.

Consider the synchronous sequential function $\sigma: A^+ \to T^+$ given by the transducer $(T^1, A, T, 1, \delta, \lambda)$ where $\delta(t, a) = \lambda(t, a) = t(ah)$ for all $t \in T^1$, $a \in A$. The transformation semigroup of this transducer is isomorphic to (T^1, T). We have $u\sigma \in T^*(uh)$ for every $u \in A^+$, i. e., $L(t)\sigma^{-1} = th^{-1}$ for every $t \in T$. Since $th^{-1} = L(\psi_t)$, this implies $L(t)\sigma^{-1} = L(\psi_t)$ for every $t \in T$. Thus, by Lemma 3, $L(\phi'[t \mapsto \psi_t]) = L(\phi')\sigma^{-1}$. On the other hand, $L(\phi')\sigma^{-1}$ is recognized by the semigroup of $(S^1, S) \circ (T^1, T)$ according to Lemma 1. Hence, $L(\phi'[t \mapsto \psi_t])$ is recognized by $S^{S^1} * T$, which is an element of $\mathbf{V}{*}\mathbf{W}$. □

Proposition 6 *Let* $\mathbf{V}$, $\mathbf{W}$ *be pseudovarieties of semigroups and* Φ, Ψ *classes of PTL formulas such that* $L(\mathbf{V}) = L(\Phi)$ *and* $L(\mathbf{W}) = L(\Psi)$.

Then

$$L(\mathbf{V}{*}\mathbf{W}) \subseteq BC(L(\Phi[\mathtt{Pv}][\mathtt{At} \cup \Psi] \cup \Psi)) \ . \tag{30}$$

Proof. Assume $L \subseteq A^+$ is a language recognized by an element of $\mathbf{V}{*}\mathbf{W}$. By Proposition 2, there exist transformation semigroups (P, S) and (Q, T) with $S \in \mathbf{V}$ and $T \in \mathbf{W}$ such that L is recognized by the semigroup of $(P, S) \circ (Q, T)$ via some homomorphism $h: A^+ \to S^Q * T$.

Let B, h_2, and σ be as in Proposition 2. It is enough to show that the two types of languages described in Lemma 2 can be defined by formulas from $\Phi[\mathtt{Pv}][\mathtt{At} \cup \Psi]$ or Ψ.

Clearly, every language of type (i) is definable by a formula from Ψ.

To conclude the proof, we show that every language of type (ii) is definable by a formula from $\Phi[\mathtt{Pv}][\mathtt{At} \cup \Psi]$. Let $V \subseteq B^+$ be recognized by an element S of V.

By assumption, there exists a formula $\phi \in \Phi$ such that $V = L(\phi)$. Also, for every $t \in T$, there exists a formula $\psi_t \in \Psi$ such that $th_2^{-1} = L(\psi_t)$.

We want to apply Lemma 3. For every $(t, a) \in B$, we have to construct formulas $\psi_{(t,a)}$ such that $L((t, a))\sigma^{-1} = L(\psi_{(t,a)})$. By definition of σ, the $\psi_{(t,a)}$'s can be chosen as follows:

$$\psi_{(t,a)} = \begin{cases} a \wedge \ominus\psi_t & \text{if } t \in T \setminus \{1\}, \\ a \wedge (\ominus\psi_t \vee \neg\ominus\textsc{True}) & \text{otherwise.} \end{cases}$$

(Notice that $\neg\ominus$ TRUE defines the set of all strings of length exactly 1.) We now can apply Lemma 3 and conclude $L(\phi)\sigma^{-1} = L(\phi[(t,a) \mapsto \psi_{t,a}])$, hence $V\sigma^{-1} = L(\phi[(t,a) \mapsto \psi_{t,a}])$.

Every formula $\psi_{(t,a)}$ can be written as $\chi_{(t,a)}[d \mapsto \psi'_d]$ where $\chi_{(t,a)} \in \mathtt{Pv}$ and $\psi'_d \in \mathtt{At} \cup \Psi$. We can then rewrite $\phi[(t,a) \mapsto \psi_{t,a}]$ as $\phi[(t,a) \mapsto \chi_{(t,a)}[d \mapsto \psi'_d]]$, which is identical to $\phi[(t,a) \mapsto \chi_{(t,a)}][d \mapsto \psi'_d]$, by Remark 1. The latter formula belongs to $\Phi[\mathtt{Pv}][\mathtt{At} \cup \Psi]$. □

Theorem 6 *Let* $\mathbf{V}$, $\mathbf{W}$ *be pseudovarieties of semigroups and* Φ, Ψ *classes of PTL formulas such that* $L(\mathbf{V}) = L(\Phi)$ *and* $L(\mathbf{W}) = L(\Psi)$.

If $\mathtt{At} \subseteq \Phi$, $\mathtt{At} \subseteq \Psi$, $\Phi[\mathtt{Dj}] \subseteq \Phi$, $\Phi[\mathtt{Pv}][\Psi] \subseteq \Phi[\Psi]$, *and if* Φ *is syntactically closed under negation and disjunction, then*

$$L(\mathbf{V}*\mathbf{W}) = L(\Phi[\Psi]) \ . \tag{31}$$

Proof. From Proposition 5 follows $L(\Phi[\Psi]) \subseteq L(\mathbf{V}*\mathbf{W})$. From Proposition 6 follows $L(\mathbf{V}*\mathbf{W}) \subseteq \mathrm{BC}(L(\Phi[\mathtt{Pv}][\mathtt{At} \cup \Psi] \cup \Psi))$. By assumption, $\Phi[\mathtt{Pv}][\mathtt{At} \cup \Psi] \subseteq \Phi[\Psi]$. Since we have $\mathtt{At} \subseteq \Phi$, we also know $\Psi \subseteq \Phi[\Psi]$, hence $\Phi[\mathtt{Pv}][\mathtt{At} \cup \Psi] \cup \Psi = \Phi[\Psi]$. It remains to be shown that $\mathrm{BC}(L(\Phi[\Psi])) = L(\Phi[\Psi])$; but this follows from the fact that Φ is syntactically closed under negation and disjunction. □

5 Characterization of Since and Until Depth

Lemma 4 1. *A language over* A *is recognized by* $B(1,2)^1$ *if and only if it is of the form* A^*BC^* *for some* $B, C \subseteq A$.

2. *A language over* A *is recognized by a semigroup in* $\mathbf{MD}_1$ *if and only if it is a boolean combination of languages of the form* A^*BC^* *for sets* $B, C \subseteq A$.

3. $L(\mathtt{Sn}) = L(\mathbf{MD}_1)$.

Similar facts were proven in [CPP93], see Lemma 1.3.

Proof. For the proof of part 1, let $h\colon A^+ \to B(1,2)^1$ be a homomorphism. Set $M_s = sh^{-1} \cap A$ for every $s \in B(1,2)^1$. Then $ah^{-1} = A^*M_aM_1^*$, $bh^{-1} = A^*M_bM_1^*$, and $1h^{-1} = M_1^+$. Therefore, every language recognized by $B(1,2)^1$

is already of the form A^*BC^* for $B, C \subseteq A$. On the other hand, for every possible choice of $B, C \subseteq A$, the set A^*BC^* is recognized by $B(1,2)^1$, as can be seen as follows. The claim is trivial if $B = \emptyset$. If $B \neq \emptyset$, consider the homomorphism $h: A^+ \to B(1,2)^1$ defined by

$$ah = \begin{cases} a & \text{if } a \in B, \\ 1 & \text{if } a \in B \setminus C, \\ b & \text{otherwise.} \end{cases}$$

Then $ah^{-1} = A^*BC^*$.

We now come to the proof of part 2. Since $\mathbf{MD}_1$ is generated by $B(1,2)^1$, a language is recognized by an element of $B(1,2)^1$ if and only if it is recognized by a finite direct product of $B(1,2)^1$ with itself (i.e., by a finite power of $B(1,2)^1$). This is equivalent to that the language is a boolean combination of languages recognized by $B(1,2)^1$. So the claim follows from part 1.

Part 3 is a trivial consequence of part 2 in view of the following equation, where $B = \{b_1, \ldots, b_m\}$ and $C = \{c_1, \ldots, c_n\}$ are (possibly empty) subsets of A.

$$L((c_1 \vee c_2 \vee \ldots \vee c_n)\ \mathsf{S}\ (b_1 \vee b_2 \vee \ldots \vee b_m)) = A^*BC^* \tag{32}$$

Recall also that every formula in $\texttt{Sn}$ is a boolean combination of formulas as in (32) and atomic formulas. □

Let $\mathbf{W}_0 = \mathbf{LR}$ and $\mathbf{W}_{k+1} = \mathbf{LR}{*}\mathbf{MD}_1{*}\mathbf{W}_k$.

Theorem 7 *For every $k \geq 0$,*

$$L(\texttt{Sn}_k) = L(\mathbf{W}_k)\ . \tag{33}$$

The case $k = 0$ is Theorem 4.

Proof. We proceed by induction on k. The induction base is Theorem 4. For the induction step, assume $L(\texttt{Sn}_k) = L(\mathbf{W}_k)$.

We apply Theorem 6 twice. By Lemma 4, part 3, we know $L(\texttt{Sn}) = L(\mathbf{MD}_1)$. Obviously, $\texttt{At} \subseteq \texttt{Sn}$, $\texttt{At} \subseteq \texttt{Sn}_k$, $\texttt{Sn}[\texttt{Dj}] \subseteq \texttt{Sn}$, and $\texttt{Sn}[\texttt{Pv}][\texttt{Sn}_k] \subseteq \texttt{Sn}[\texttt{Sn}_k]$. Furthermore, $\texttt{Sn}$ is closed under negation and disjunction. Thus, by Theorem 6,

$$L(\texttt{Sn}[\texttt{Sn}_k]) = L(\mathbf{MD}_1{*}\mathbf{W}_k)\ . \tag{34}$$

On the other hand, $L(\texttt{Sn}_0) = \mathbf{LR}$ by induction base. Obviously, $\texttt{At} \subseteq \texttt{Sn}_0$, $\texttt{At} \subseteq \texttt{Sn}[\texttt{Sn}_k]$, $\texttt{Sn}_0[\texttt{Dj}] \subseteq \texttt{Sn}_0$ and $\texttt{Sn}_0[\texttt{Pv}][\texttt{Sn}[\texttt{Sn}_k]] \subseteq \texttt{Sn}_0[\texttt{Sn}[\texttt{Sn}_k]]$. Also, $\texttt{Sn}_0$ is closed under disjunction and negation. Thus, again by Theorem 6,

$$L(\texttt{Sn}_0[\texttt{Sn}[\texttt{Sn}_k]]) = L(\mathbf{LR}{*}\mathbf{MD}_1{*}\mathbf{W}_k)\ . \tag{35}$$

In view of Remark 3, this yields (33) for $k+1$ instead of k. $\square$

From Theorem 7, we immediately get:

Theorem 8 *For every $k \geq 0$,*

$$L(\mathtt{Ut}_k) = L(\mathbf{W}_k^\rho) \ . \tag{36}$$

6 Decidability

6.1 An alternative description of $\mathbf{W}_k$

Lemma 5 *For $l, m > 0$,*

$$B(1,2)^{1^l} \circ \underbrace{Sl_2 \circ \ldots \circ Sl_2}_{m\ factors} \prec$$
$$Sl_2^{4l} \circ Sl_2 \circ Sl_2^{8l} \circ Sl_2 \circ Sl_2^{16l} \circ \ldots \circ Sl_2^{2^{m-1}l} \circ Sl_2 \circ B(1,2)^{1^{2^m l}}.$$

Here, every semigroup stands for its right regular representation.

Proof. The proof is by induction on m. The induction base, $m = 1$, is Theorem 2. In the induction step, we first apply the induction base and then the induction hypothesis:

$$\begin{aligned} & B(1,2)^{1^l} \circ \underbrace{Sl_2 \circ \ldots \circ Sl_2}_{m+1 \text{ factors}} \\ \prec \quad & Sl_2^{4l} \circ Sl_2 \circ B(1,2)^{1^{2l}} \circ \underbrace{Sl_2 \circ \ldots \circ Sl_2}_{m \text{ factors}} \\ \prec \quad & Sl_2^{4l} \circ Sl_2 \circ Sl_2^{8l} \circ Sl_2 \circ Sl_2^{16l} \circ Sl_2 \circ \ldots \circ Sl_2^{2^m l} \circ Sl_2 \circ B(1,2)^{1^{2^{m+1}l}}. \end{aligned}$$

$\square$

Theorem 9 $\mathbf{MD}_1 {*} \mathbf{R} \subseteq \mathbf{R} {*} \mathbf{MD}_1$ *.*

Proof. By Theorem 1, it is enough to show $\mathbf{MD}_1 {*} \mathbf{SL}^n \subseteq \mathbf{R} {*} \mathbf{MD}_1$ for every $n \geq 1$.

We first prove the base case $n = 1$, i.e., we show $\mathbf{MD}_1*\mathbf{SL} \subseteq \mathbf{R}*\mathbf{MD}_1$. Assume $S \in \mathbf{MD}_1*\mathbf{SL}$. Then $S \prec T * U$ for some $T \in \mathbf{MD}_1$ and some $U \in \mathbf{SL}$. The semidirect product $T * U$ divides the semigroup of the wreath product $(T^1, T) \circ (U^1, U)$. We only need to show that the semigroup of this transformation semigroup belongs to $\mathbf{MD}_1*\mathbf{R}$.

Since $T \in \mathbf{MD}_1$ and $U \in \mathbf{SL}$, we have

$$(T^1, T) \circ (U^1, U) \prec (B(1,2)^{1^l}, B(1,2)^{1^l}) \circ (Sl_2^m, Sl_2^m) \tag{37}$$

for some l and m, hence

$$(T^1, T) \circ (U^1, U) \prec (B(1,2)^{1^l}, B(1,2)^{1^l}) \circ (Sl_2, Sl_2) \circ \ldots \circ (Sl_2, Sl_2) \tag{38}$$

for some l and an appropriate number of factors (Sl_2, Sl_2), say m. From Lemma 5 follows that the transformation semigroup on the right hand side of (38) is a transformation semigroup whose semigroup belongs to $\mathbf{MD}_1*\mathbf{R}$, hence the semigroup of $(T^1, T) \circ (U^1, U)$ belongs to $\mathbf{MD}_1*\mathbf{R}$.

The induction step is established via the following chain of inclusions:

$$\begin{array}{rcll}
\mathbf{MD}_1*\mathbf{SL}^{n+1} & = & \mathbf{MD}_1*\mathbf{SL}*\mathbf{SL}^n & \\
& \subseteq & \mathbf{R}*\mathbf{MD}_1*\mathbf{SL}^n & \text{induction base} \\
& \subseteq & \mathbf{R}*\mathbf{R}*\mathbf{MD}_1 & \text{induction hypothesis} \\
& \subseteq & \mathbf{R}*\mathbf{MD}_1 & \text{by Theorem 1, part 2 .}
\end{array}$$

□

Theorem 10 *For $k \geq 0$,*

$$\mathbf{W}_k = \mathbf{R}*\mathbf{MD}_1^k*\mathbf{D}\,. \tag{39}$$

Proof. We prove the nontrivial direction (namely, $\mathbf{W}_k \subseteq \mathbf{R}*\mathbf{MD}_1^k*\mathbf{D}$) by induction on k.

The induction base is given by Theorem 1, part 3. For the induction step, assume $\mathbf{W}_k \subseteq \mathbf{R}*\mathbf{MD}_1^k*\mathbf{D}$. We have the following chain of inclusions, which uses that $\mathbf{MD}_1^{k+1}$ is a monoidal pseudovariety of semigroups:

$$\begin{array}{rcll}
\mathbf{W}_{k+1} & = & \mathbf{LR}*\mathbf{MD}_1*\mathbf{W}_k & \text{definition of } \mathbf{W}_{k+1} \\
& = & \mathbf{R}*\mathbf{D}*\mathbf{MD}_1*\mathbf{W}_k & \text{by Theorem 1, part 3} \\
& \subseteq & \mathbf{R}*\mathbf{D}*\mathbf{MD}_1*\mathbf{R}*\mathbf{MD}_1^k*\mathbf{D} & \text{induction hypothesis} \\
& \subseteq & \mathbf{R}*\mathbf{D}*\mathbf{R}*\mathbf{MD}_1*\mathbf{MD}_1^k*\mathbf{D} & \text{by Theorem 9} \\
& \subseteq & \mathbf{R}*\mathbf{D}*\mathbf{MD}_1^{k+1}*\mathbf{D} & \text{by Theorem 1, part 3} \\
& \subseteq & \mathbf{R}*\mathbf{MD}_1^{k+1}*\mathbf{D}*\mathbf{D} & \text{by Theorem 1, part 4} \\
& \subseteq & \mathbf{R}*\mathbf{MD}_1^{k+1}*\mathbf{D} & \text{by Theorem 1, part 1.}
\end{array}$$

□

6.2 A strong normal form

Theorem 11 *For every $k \geq 0$,*

$$L(\mathtt{Sn}_k) = L(\mathtt{St}[\mathtt{Sn}]^k[\mathtt{Pv}]) \ . \tag{40}$$

Proof. We first show $L((\mathbf{MD}_1 * \mathbf{D})^k) = L(\mathtt{Sn}([\mathtt{Pv}][\mathtt{Sn}])^{k-1}[\mathtt{Pv}])$ for every $k \geq 1$ by induction on k. The induction base, namely $L(\mathbf{MD}_1 * \mathbf{D}) = L(\mathtt{Sn}[\mathtt{Pv}])$, is a straightforward application of Theorem 6. In the induction step, by another application of Theorem 6 and using the induction hypothesis as well as the induction base, we get $L((\mathbf{MD}_1 * \mathbf{MD}_1) * (\mathbf{MD}_1 * \mathbf{D})^k) = L(\mathtt{Sn}([\mathtt{Pv}][\mathtt{Sn}])^k[\mathtt{Pv}])$, what we wanted to show.

Yet another application of Theorem 6 yields

$$L(\mathbf{LR} * (\mathbf{MD}_1 * \mathbf{D})^k) = L(\mathtt{Sn}_0([\mathtt{Sn}][\mathtt{Pv}])^k) \tag{41}$$

for every $k \geq 0$.

From Theorem 1, parts 1, 3, and 4, we can derive $\mathbf{LR} * (\mathbf{MD}_1 * \mathbf{D})^k \subseteq \mathbf{R} * \mathbf{MD}_1^k * \mathbf{D}$, which implies $\mathbf{LR} * (\mathbf{MD}_1 * \mathbf{D})^k = \mathbf{R} * \mathbf{MD}_1^k * \mathbf{D}$. Therefore, using Theorems 7 and 10, we can conclude $L(\mathtt{Sn}_k) = L(\mathtt{Sn}_0([\mathtt{Sn}][\mathtt{Pv}])^k)$ from (41).

The claim of the theorem finally follows from Remark 22, which says that the previously operator need only occur directly in front of atomic propositions. □

A corresponding theorem holds for (future) temporal logic; consider just the duals statement.

6.3 Decidability of $\mathbf{W}_k$

Proposition 7 *Let $k \geq 0$, $l, m \geq 1$. The pseudovariety $\mathbf{MD}_1^k * \mathbf{D}_l$ is effectively locally finite.*

Proof. From (1) it is clear that $F_m \mathbf{D}_l$ is finite for every $m, l \geq 1$. From (3) and (4) it is clear that $\mathbf{MD}_1$ is effectively locally finite. So, by repeated application of Proposition 3, we obtain that $\mathbf{MD}_1^k * \mathbf{D}_l$ is effectively locally finite. □

Theorem 12 *Let $k \geq 0$. The problem of determining whether a given finite semigroup S belongs to $\mathbf{W}_k$ is decidable.*

Proof. In order to check whether $S \in \mathbf{W}_k$, according to Proposition 4, we only have to check for $S \in \mathbf{R}{*}\mathbf{MD}_1^k{*}\mathbf{D}_{|S|}$. This is effective by Theorem 3 in view of Theorem 1, part 3, which states $\mathbf{LR} = \mathbf{R}{*}\mathbf{D}$. □

In view of Theorems 7 and 8, the previous theorem yields:

Corollary 1 *Let $k \geq 0$. The problems of determining whether a regular language (in particular, a language represented by a TL formula) belongs to $L(\mathtt{Ut}_k)$ or $L(\mathtt{Sn}_k)$ are decidable.*

7 Extension to ω-Words

7.1 Game equivalence

Remark 7 *Let $k, l \geq 0$, $x, y \in A^\infty$, $i < |x|$, and $j < |y|$.*

1. *If $(x, i) \equiv_{k+1}^{l} (y, j)$, then $(x, i) \equiv_k^{l+1} (y, j)$.*
2. *For $u, v \in A^*$, if $(x, i) \equiv_k^l (y, j)$, then $(ux, |u| + i) \equiv_k^l (vy, |v| + j)$.*

Lemma 6 (congruence property) *Let $k, l \geq 0$.*

1. *The binary relation $\equiv_k^l$ restricted to A^+ is a congruence relation.*
2. *Let $u, v \in A^+$ and $\alpha, \beta \in A^\omega$. If $u \equiv_k^l v$ and $\alpha \equiv_k^l \beta$, then $u\alpha \equiv_k^l v\beta$.*
3. *If $u, v \in A^+$ such that $u \equiv_k^l v$, then $u^\omega \equiv_k^l v^\omega$.*

Proof. We first prove parts 1 and 2 of the lemma. Let $u, v \in A^+$, $x, y \in A^\omega$ and $i < |u|$ and $j < |v|$ such that $(u, i) \equiv_k^l (v, j)$ and $x \equiv_k^l y$. We prove by induction on $k + l$ that $(ux, i) \equiv_k^l (vy, j)$, which is sufficient for both parts of the lemma.

Induction base, $k + l = 0$. If $(u, i) \equiv_0^0 (v, j)$, then $u(i) = v(j)$. But this implies $(ux)(i) = (vy)(j)$, which, in turn, means $(ux, i) \equiv_0^0 (vy, j)$.

Induction step, $k + l > 0$. We consider the various conditions separately. Here, and in all subsequent proofs concerning the equivalence relation $\equiv_k^l$, we will encounter only symmetric situations, for which reason we don't and won't have to consider the symmetric counterparts of the conditions as stated in Subsection 3.5.

Next condition. Since we assume $(u,i) \equiv_k^l (v,j)$, we have either $|u| = i+1$ and $|v| = j+1$, or $i+1 < |u|$, $j+1 < |v|$, and $(u,i+1) \equiv_k^{l-1} (v,i+1)$. We analyze the two cases separately.

First case, $|u| = i+1$ and $|v| = j+1$. As we assume $x \equiv_k^l y$, we have $(x,0) \equiv_k^l (y,0)$, hence also $(ux,|u|) \equiv_k^{l-1} (vy,|v|)$, thus $(ux,i+1) \equiv_k^{l-1} (vy,j+1)$.

Second case, $i+1 < |u|$, $j+1 < |v|$, and $(u,i+1) \equiv_k^{l-1} (v,i+1)$. Using the induction hypothesis, we can conclude $(ux,i+1) \equiv_k^{l-1} (vy,j+1)$.

Eventually condition. We distinguish two cases.

First case, $i' < |u|$. Then, since we assume $(u,i) \equiv_k^l (v,j)$, there is a position j' such that $j \leq j' < |v|$ and $(u,i') \equiv_k^{l-1} (v,j')$. From the induction hypothesis, we can conclude $(ux,i') \equiv_k^{l-1} (vy,j')$.

Second case, $i' \geq |u|$. Write i'' for $i' - |u|$. Since we assume $x \equiv_k^l y$, there exists a position j'' in y such that $(x,i'') \equiv_k^{l-1} (y,j'')$. Thus, for $j' = |v| + j''$, we have $(ux,i') \equiv_k^{l-1} (vy,j')$ by Remark 7, part 2. (We don't need the induction hypothesis.)

Until condition. We distinguish two cases.

First case, $i' < |u|$. As we assume $(u,i) \equiv_k^l (v,j)$, there exists a position j' such that (UC1)–(U3) hold for u and v instead of x and y. By induction hypothesis, (UC1)–(UC3) also hold for ux and vy instead of x and y.

Second case, $i' \geq |u|$. Write i_1 for $i' - |u|$. Since we assume $x \equiv_k^l y$, there exists a position j_1 such that (UC1)–(UC3) hold for i_1 and j_1 instead of i' and j'. We claim that (UC1)–(UC3) hold for ux, vy instead of x and y with $j' = |v| + j_1$.

First of all, $i' \neq i$, thus (UC1) holds for ux and vy instead of x and y. Also, since $(x,i_1) \equiv_k^{l-1} (y,j_1)$, we have $(ux,i') \equiv_k^{l-1} (vy,j')$ by Remark 7, part 2. So (UC2) holds for ux and vy instead of x and y.

To see that (UC3) holds as well, let j'' be such that $j \leq j'' < j'$. We distinguish two subcases.

First subcase, $j'' \geq |v|$. Let $j_2 = j'' - |v|$ and i_2 the position in x guaranteed to exist by (UC3). We can conclude $(x,i_2) \equiv_{k-1}^l (y,j_2)$. Therefore, $(ux,i'') \equiv_{k-1}^l (vy,j'')$ with $i'' = |u| + i_2$.

Second subcase, $j'' < |v|$. From Remark 7, we know $(u,i) \equiv_{k-1}^{l+1} (v,j)$. Therefore, there exists a position i'' such that $(u,i'') \equiv_{k-1}^l (v,j'')$. By induction hypothesis, this implies $(ux,i'') \equiv_{k-1}^l (vy,j'')$.

We now turn to part 3 of the lemma. We will show by induction on $k+l$ that if $(u^m, i) \equiv_k^l (v^n, j)$ for some $m, n > 0$, then $(u^\omega, i) \equiv_k^l (v^\omega, j)$.

Induction base, $k+l=0$. The claim is trivial.

Induction step, $k+l>0$. We consider the various conditions.

Next condition. Since we assume $(u^m, i) \equiv_k^l (v^n, j)$, we have either $i+1 = m|u|$ and $j+1 = n|v|$, or $i+1 < m|u|$, $j+1 < n|v|$ and $(u^m, i+1) \equiv_k^{l-1} (v^n, i+1)$. In the second case, we obtain $(u^\omega, i+1) \equiv_k^{l-1} (v^\omega, j+1)$ immediately from the induction hypothesis. In the first case, we apply the induction hypothesis to $(u, 0)$ and $(v, 0)$ and obtain $(u^\omega, 0) \equiv_k^{l-1} (v^\omega, 0)$, which, by Remark 7, part 2, implies $(u^\omega, m|u|) \equiv_k^{l-1} (v^\omega, n|v|)$, hence $(u^\omega, i+1) \equiv_k^{l-1} (v^\omega, j+1)$.

Eventually condition. Let $i' \geq i$. Let m' be such that $m'|u| \leq i' < (m'+1)|u|$. Consider $u' = u^{m'+1}$ and $v' = v^{n+(m'-m+1)}$. From part 1 of the lemma, we can conclude $(u', i) \equiv_k^l (v', j)$. Thus there exists j' such that $(u', i') \equiv_k^{l-1} (v', j')$. Applying the induction hypothesis, we get $(u^\omega, i') \equiv_k^{l-1} (v^\omega, j')$.

Until condition. Let $i' \geq i$, and m', u' and v' be as above. From part 1 of the lemma, we can conclude $(u', i) \equiv_k^l (v', j)$. Thus there exists j' such that (UC1)–(UC3) are met with u' and v' instead of x and y. Applying induction hypothesis, we see that (UC1)–(UC3) are also met with u^ω and v^ω instead of x and y, respectively. $\square$

7.2 Reduction to strings

Lemma 7 *Let $k, l \geq 0$ and $u, v \in A^+$ be such that $u \equiv_k^l v$. For all $x, y \in A^*$ and $z \in A^+$,*

$$x(uy)^\omega \equiv_k^l x(vy)^\omega \ , \tag{42}$$

$$xuyz^\omega \equiv_k^l xvyz^\omega \ . \tag{43}$$

Proof. Relation (43) follows immediately from Lemma 6, part 1.

By the same part of that lemma, we obtain $uy \equiv_k^l vy$. Thus, by part 3 of that lemma, $(uy)^\omega \equiv_k^l (vy)^\omega$. An application of part 2 of that lemma finally yields $x(uy)^\omega \equiv_k^l x(vy)^\omega$. $\square$

Proposition 8 *Let $k \geq 0$ and $L \subseteq A^\omega$ such that $L \in L^\omega(\mathtt{Ut}_k)$. Each class of $\equiv_L$ belongs to $L(\mathtt{Ut}_k)$.*

Proof. There exists l such that L belongs to $L(\mathtt{Ut}_k^l)$. We claim that every class of the syntactic congruence of L belongs to $L(\mathtt{Ut}_k^l)$.

We only need to show that, for all $u, v \in A^+$, if $u \equiv_k^l v$ then $u \equiv_L v$. Let $x, y \in A^*$ and $z \in A^+$.

Assume $x(uy)^\omega \in L$. From Lemma 7, we know $x(uy)^\omega \equiv_k^l x(vy)^\omega$. Thus, since $L \in L(Ut_k^l)$, $x(vy)^\omega \in L$. By symmetry, if $x(vy)^\omega \in L$ then $x(uy)^\omega \in L$. Similarly, using (43) instead of (42), one shows $xuyz^\omega \in L$ iff $xvyz^\omega \in L$. So (8) and (9) hold, which means $u \equiv_L v$. □

Lemma 8 *Let $k, l \geq 0$, $p = k + l + 1$, and $u, v \in A^*$ and $w \in A^+$ such that $uw^\omega \equiv_k^l vw^\omega$. Then $uw^p \equiv_k^l vw^p$.*

Proof. We prove that if $(uw^\omega, i) \equiv_{k'}^{l'} (vw^\omega, j)$ for $i \leq |u| + (k - k' + l - l')|w|$ and $j \leq |v| + (k - k' + l - l')|w|$, then $(uw^p, i) \equiv_{k'}^{l'} (vw^p, j)$. Write x, y, α, and β for uw^p, vw^p, uw^ω, and vw^ω.

Induction base, $k' + l' = 0$. This is trivial.

Induction step, $k' + l' > 0$. We consider the various conditions.

Next condition. Clearly, $(\alpha, i+1) \equiv_{k'}^{l'-1} (\beta, j+1)$, hence $(x, i+1) \equiv_{k'}^{l'-1} (y, j+1)$ by induction hypothesis.

Eventually condition. Let i' be such that $i \leq i' < |x|$. We distinguish two cases.

First case, $i < |u|$. Since $(\alpha, i) \equiv_{k'}^{l'} (\beta, j)$, there exists j' such that $(\alpha, i') \equiv_{k'}^{l'} (\beta, j')$. By Remark 7, part 2, we can assume $j' < |v| + (k - k' + l - l' + 1)|w|$. Thus we can apply the induction hypothesis and find $(x, i') \equiv_{k'}^{l'-1} (y, j')$.

Second case, $i' \geq |u|$. This case is simple, as we can choose $j' = |v| + i' - |u|$.

Until condition. Let i' be such that $i \leq i' < |x|$. Since $(\alpha, i) \equiv_{k'}^{l'} (\beta, j)$, there exists j' such that (UC1)–(UC3) hold for α, β, k', and l' instead of x, y, k, and l.

By Remark 7, part 2, we can find j_1 such that $j \leq j_1 < |v| + (k - k' + l - l' + 1)|w|$, $j_1 \leq j$, and $(\beta, j') \equiv_{k'-1}^{l'} (\alpha, j_1)$. We claim that (UC1)–(UC3) then hold for k', l', x, y, and j_1 instead of k, l, x, y, and j'.

(UC1) is trivially satisfied, as we know that if $i = i'$ then $j' = j$, and that $j \leq j_1 \leq j'$. By definition of j_1, we know $(\alpha, i') \equiv_{k'-1}^{l'} (\beta, j_1)$, hence, by induction hypothesis, $(x, i') \equiv_{k'-1}^{l'} (y, j_1)$. So (UC2) is satisfied.

To verify (U3), let j'' be such that $j \leq j'' < j_1$. If $j'' \geq |v|$, we simply take $i'' = |u| + j'' - |v|$. In the other case, if $j'' < |v|$, there is i'' such that $i \leq i'' < i$

and $(\alpha, i'') \equiv_{k'-1}^{l'} (\beta, j'')$. We can assume $i'' < |u| + (k - k' + l - l' + 1)|w|$. The induction hypothesis then yields $(x, i'') \equiv_{k'-1}^{l'} (y, j'')$. □

We denote by $\sim_k^l$ the similarity relation induced by $\equiv_k^l$.

Proposition 9 *Let $k, l \geq 0$, and $s, t, u, v \in A^+$ such that $st^\omega \equiv_k^{l+2} uv^\omega$. There exist $\gamma \in A^\omega$ such that $st^\omega \sim_k^l \gamma \sim_k^l uv^\omega$.*

Proof. Write α and β for st^ω and uv^ω, respectively.

We first argue that we can assume $t^\omega \equiv_k^l v^\omega$.

Since we have $st^\omega \equiv_k^{l+2} uv^\omega$, there exists j such that $(st^\omega, |s|) \equiv_k^{l+1} (uv^\omega, j)$. Consequently, there exists i' such that $(st^\omega, |s|+i') \equiv_k^l (uv^\omega, j+|u|)$. By Remark 7, part 2, we can assume $i' < |t|$ and $j < |v|$. Write v', v'', t', t'' for $v(0, i'')$, $v(i'', |v|)$, $t(0, j')$, and $t(j', |t|)$. Then $uv^\omega = uv'(v''v')^\omega$ and $st^\omega = st'(t''t')^\omega$, where $(v''v')^\omega \equiv_k^l (t''t')^\omega$. We can thus replace u by uv', v by $v''v'$, s by st', and t by $t''t'$ in order to achieve $t^\omega \equiv_k^l v^\omega$.

Let $p = k + l + 1$. Set $\gamma = s(t^p v^p)^\omega$. We claim that the requirements of the proposition are met.

We first show $\alpha \sim_k^l \gamma$. Let $u_0 = st^p$, $v_0 = st^p$, $u_i = t^{2p}$ for $i > 0$, and $v_i = v^p t^p$ for $i > 0$. Then $\alpha = u_0 u_1 u_2 \ldots$ and $\gamma = v_0 v_1 v_2 \ldots$ Trivially, $u_0 \equiv_k^l v_0$. Moreover,

$$v^p t^\omega \equiv_k^l v^p v^\omega = v^\omega \equiv_k^l t^\omega = t^p t^\omega \ ,$$

where the first equivalence follows from Lemma 6 and the second one holds by assumption. So Lemma 8 yields $v^p t^p \equiv_k^l t^p t^p$, hence $u_i \equiv_k^l v_i$ for $i > 0$, thus $\alpha \sim_k^l \gamma$.

We finally show $\gamma \sim_k^l \beta$. Let $u_0 = st^p v^p$, $v_0 = uv^p$, $u_i = t^p v^p$ for $i > 0$, and $v_i = v^{2p}$ for $i > 0$. Then $\gamma = u_0 u_1 u_2 \ldots$ and $\beta = v_0 v_1 v_2 \ldots$ As in the previous paragraph, we obtain $u_i \equiv_k^l v_i$ for $i > 0$. Similarly to what we had before,

$$st^p v^\omega \equiv_k^l st^p t^\omega = st^\omega \equiv_k^l uv^\omega \ ,$$

where the first equivalence follows from Lemma 6 and the second one holds by assumption.

From Lemma 8, we can thus conclude $uv^p \equiv_k^l st^p v^p$, which means $u_0 \equiv_k^l v_0$. □

Theorem 13 *Let $k \geq 0$ and $L \subseteq A^\omega$ regular. The following are equivalent.*

- *$L \in L^\omega(\mathtt{Ut}_k)$.*
- *Each class of $\equiv_L$ belongs to $L(\mathtt{Ut}_k)$.*

Proof. One direction follows from Proposition 8. For the other direction, assume that each class of $\equiv_L$ belongs to $L(\mathtt{Ut}_k)$. Since L is regular, $\equiv_L$ has only finitely many classes and there thus exists $l \geq 0$ such that each class of $\equiv_L$ belongs to $L(\mathtt{Ut}_k)$. We will prove that L belongs to $L(\mathtt{Ut}_k^{l+2})$.

We have to show, for $\alpha, \beta \in A^\omega$ such that $\alpha \equiv_k^{l+2} \beta$, either $\alpha, \beta \in L$ or $\alpha, \beta \notin L$. Since all classes of $\equiv_k^{l+2}$ on A^ω are regular ω-languages, it is enough to consider only ultimately periodic words, that is, we can assume that α and β are of the form st^ω and uv^ω. (Recall that every nonempty regular ω-language contains an ultimately periodic word.) By Proposition 9, there exists γ such that $\alpha \sim_k^l \gamma \sim_k^l \beta$. Since we assume that every class of $\equiv_L$ belongs to $L^\omega(\mathtt{Ut}_k)$, $\equiv_L$ is at least as coarse as $\equiv_k^l$, which implies that we also have $\alpha \sim_L \gamma \sim_L \beta$. We can now apply Lemma 1 and obtain $\alpha \in L$ iff $\gamma \in L$ iff $\beta \in L$. $\square$

In view of Corollary 1, the previous theorem yields:

Corollary 2 *Let $k \geq 0$. The problem of determining whether a regular ω-language (in particular, an ω-language defined by a TL formula) belongs to $L^\omega(\mathtt{Ut}_k)$ is decidable.*

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