DIMACS Technical Report 96-20
July 1996

Asymptotics of the total chromatic number for simple graphs

by

P. Mark Kayll[1]
Department of Mathematical Sciences
The University of Montana
Missoula, MT 59812-1032
kayll@charlo.math.umt.edu

[1]Author was an unofficial DIMACS visitor during the preliminary stages of preparing this paper.

DIMACS is a cooperative project of Rutgers University, Princeton University, AT&T Bell Laboratories and Bellcore.

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

For simple graphs, the ratio of the total chromatic number to its linear analogue, the fractional total chromatic number, tends to one as the latter parameter grows large. Two short proofs of this fact are presented. One is based on a theorem of Kahn (*J. Combin. Theory Ser. A* **73** (1996), 1–59) concerning the asymptotics of the list-chromatic index, the other on recent progress of Molloy and Reed in the direction of the "Behzad-Vizing Conjecture". The main result has the flavour of several recent others establishing asymptotic agreement of certain integral (hyper)graph parameters with their fractional analogues.

1 Introduction

The purpose of this paper is to point out and prove an asymptotic connection between the total chromatic number χ_t and its fractional analogue χ_t^* for (simple) graphs with large maximum degree. Recent and far-reaching progress [12, 20] in different areas of colouring theory leads to two independent, satisfyingly simple, proofs of our main result, which is in the spirit of several others (*e.g.* [12]–[16]) connecting various integer-valued (hyper)graph parameters with their fractional counterparts. An important aspect of our contribution is the placement of χ_t into the class of graph parameters exhibiting "good asymptotic behaviour", in the sense elucidated, *e.g.*, in [11].

Postponing definitions, we state our main result:

(1.1) Theorem *For graphs G,*

$$\chi_t(G) \sim \chi_t^*(G) \quad as \quad \chi_t^*(G) \to \infty. \tag{1}$$

That is, for each $\varepsilon > 0$ there exists $D = D(\varepsilon)$ such that every graph G with $\chi_t^*(G) > D$ satisfies

$$(1+\varepsilon)^{-1} < \frac{\chi_t(G)}{\chi_t^*(G)} < 1+\varepsilon. \tag{2}$$

(Note, in particular, that the convergence in (1) is "uniform" in G, *i.e.*, does not depend on the choice of G as long as $\chi_t^*(G)$ is large.) Two proofs of Theorem 1.1 are presented in §3.

Notation, terminology, simple bounds

We shall assume the reader is familiar with standard graph theory terminology, as can be found, *e.g.*, in [3] or [18]. Here we pause to fix notation and introduce a few ideas which may not be quite as familiar. In this paper, *graph* means simple graph, and *colouring* means proper colouring (in whichever context — vertex-, edge-, or alternate type of colouring — this term arises).

For a graph G, we use $V(G)$, $E(G)$, $\Delta(G)$, $\chi(G)$ and $\chi'(G)$, respectively, for its vertex set, edge set, maximum degree, ordinary (vertex) chromatic number, and chromatic index (edge chromatic number), but usually the argument '(G)' is omitted when the graph is clear from context. Of the many well-known relationships between the last three parameters, those needed for our purposes are the obvious

$$\chi \leq \Delta + 1 \tag{3}$$

and

$$\chi' \geq \Delta. \tag{4}$$

Behzad [2] and Vizing [25] introduced independently a notion that is a "hybrid" of vertex- and edge-colouring. A *total colouring* of a graph G is a map $\sigma : E \cup V \to \Gamma$ — where Γ is a set of "colours" — such that $\sigma(A) \neq \sigma(B)$ whenever A, B are adjacent vertices, distinct non-disjoint edges, or a vertex-edge pair with the edge incident on the vertex. The *total chromatic number*, $\chi_t(G)$, of G is the least size of a Γ admitting such a colouring. Of course,

$$\chi_t \geq \Delta + 1, \tag{5}$$

since any total colouring exhausts $\Delta + 1$ different colours at a vertex of degree Δ. Both Vizing and Behzad conjectured that (5) is close to the "truth":

(1.2) Conjecture ([2, 25]) *Every graph G satisfies $\chi_t(G) \leq \Delta(G) + 2$.*

Keeping in mind the similarity between (5) and (4), the reader should compare Conjecture 1.2 with Vizing's Theorem [24] (which for graphs, states that $\chi' \leq \Delta+1$). Unlike the latter result, this conjecture has seen only limited progress towards a proof (see *e.g.* [6]–[10], [20, 23]). Nevertheless, Conjecture 1.2 has inspired many interesting results, including, in addition to those already cited, some fractional variants [17, 21]. Some of these (Theorems 2.1, 2.4) are needed as lemmas for our proofs in §3.

Another graph colouring parameter introduced by Vizing [26] (and independently by Erdős, Rubin and Taylor [5]) is the *list-chromatic index*, χ'_ℓ. For a graph G, $\chi'_\ell(G)$ is the least t such that whenever $\mathcal{S}(A)$ is a set of size t for each $A \in E$, there exists an edge-colouring σ of G with $\sigma(A) \in \mathcal{S}(A)$ for each $A \in E$ (such a colouring is an *$\mathcal{S}$-list-colouring*). Considering the case when all the "lists" $\mathcal{S}(A)$ are identical shows that we always have $\chi'_\ell \geq \chi'$, and combining this observation with (4) gives

$$\chi'_\ell \geq \Delta. \tag{6}$$

(The route to (6) through (4) is clearly not the most efficient, but it has the advantage of illuminating an elementary connection between χ'_ℓ and χ'.) One natural reason for studying list-colourings is that an ordinary edge-colouring problem in which some colours have already been assigned is a list-colouring problem. Our interest in χ'_ℓ is motivated by its connection with χ_t noted in Lemma 2.2 below. See [1] for a survey of (fairly) recent progress in the theory of list-colouring and [14, 15] for some exciting, still more recent developments on, resp., edge- and list-colouring (and *many* more references than we give here!).

Fractional matters

We turn now to the definition of χ^*_t, the fractional analogue of χ_t; for this, it is natural first to formulate an integer programming problem defining χ_t (see [4] or [22] for omitted LP/IP terminology). For a graph G and $S \subseteq V$, let $\delta(S)$ denote the set of edges of G with exactly one end in S. Following [17], we define a *total stable set* of G to be any subset of $E \cup V$ of the form $M \cup S$, where $M \subseteq E$ is a matching, $S \subseteq V$ is a stable set, and $\delta(S) \cap M = \varnothing$. We use $\mathcal{T} = \mathcal{T}(G)$ for the family of total stable sets of G.

Notice that a total colouring is simply an $f : \mathcal{T} \to \{0, 1\}$ satisfying

$$\sum_{A \in T \in \mathcal{T}} f(T) = 1 \quad \text{for each } A \in E \cup V, \tag{7}$$

and χ_t is the optimal value of an IP:

$$\chi_t(G) = \min \left\{ \sum_{T \in \mathcal{T}} f(T) : f \text{ is a total colouring of } G \right\}. \tag{8}$$

This leads to the fractional versions of (7), (8). An $f : \mathcal{T} \to \mathbf{R}^+$ satisfying (7) is a *fractional total colouring*, and, writing $f(G)$ for $\sum_{T \in \mathcal{T}} f(T)$, we have the *fractional total chromatic number*

$$\chi_t^*(G) = \min \{ f(G) : f \text{ is a fractional total colouring of } G \}.$$

It is easy to see that each fractional total colouring f satisfies $f(G) \geq \Delta(G) + 1$, which gives a lower bound for χ_t^*:

$$\chi_t^* \geq \Delta + 1. \tag{9}$$

The upper bound

$$\chi_t^* \leq \chi_t \tag{10}$$

follows since χ_t^* is the optimal value of the linear relaxation of (8).

2 Lemmas

Both of our proofs of Theorem 1.1 depend on connections between Δ and χ_t^*, χ_t. The (Δ, χ_t^*)-connection is provided by (9) and the following result — a fractional version of Conjecture 1.2 — due to Kilakos and Reed.

(2.1) Theorem ([17]) *Every graph G satisfies $\chi_t^*(G) \leq \Delta(G) + 2$.*

The first proof bridges the (Δ, χ_t)-connection through χ'_ℓ, which, as noted above, is related to χ_t via:

(2.2) Lemma *Every graph G satisfies $\chi_t(G) \leq \chi'_\ell(G) + 2$.*

Proof. We use a simple modification of an observation from [8]. Let $\Gamma = \{1, \ldots, \chi'_\ell + 2\}$. To obtain a total colouring $\sigma : E \cup V \to \Gamma$ of G, begin with a vertex-colouring $\sigma_V : V \to \Gamma$, which certainly exists since $\chi < \chi'_\ell + 2$ (by (3), (6)). Now define lists $\mathcal{S}(A)$, for $A = \{x, y\} \in E$, by $\mathcal{S}(A) = \Gamma \setminus \{\sigma_V(x), \sigma_V(y)\}$. Since $|\mathcal{S}(A)| = \chi'_\ell$ for each $A \in E$, there exists an $\mathcal{S}$-list-colouring $\sigma_E : E \to \Gamma$, and our choice of $\mathcal{S} : E \to \binom{\Gamma}{\chi'_\ell}$ ensures that σ_E extends σ_V to the desired total colouring σ of G. ∎

To complete the bridge, we will rely on a result, due to Kahn, asserting that the bound in (6) is at least asymptotically tight:

(2.3) Theorem ([12]) *For graphs G, we have $\chi'_\ell(G) \sim \Delta(G)$ as $\Delta(G) \to \infty$.*

That is, for each $\gamma > 0$ there exists $d = d(\gamma)$ such that every graph G with $\Delta(G) > d$ satisfies $\chi'_\ell(G) < (1+\gamma)\Delta(G)$. (Note that as in Theorem 1.1, the convergence here is uniform in G.) Theorem 2.3 is actually a corollary of a far more general result of Kahn, who proved the theorem for hypergraphs of bounded edge size that satisfy a natural local sparseness condition. For our purposes, *i.e.* for the first proof of Theorem 1.1, we need only the special case stated here.

Our second proof will be more direct, exploiting a recent result of Molloy and Reed in the direction of Conjecture 1.2. The latest in a sequence (see [6, 10, 8]) of improvements to upper bounds on χ_t, their result is

(2.4) Theorem ([20]) *Every graph G satisfies $\chi_t(G) \leq \Delta(G) + C$, where C is an absolute constant.*

The best value of C obtained at this writing is enormous [19]; nevertheless, Theorem 2.4 gives substantially more than we need for proving Theorem 1.1 (see the remark following the second proof in §3).

3 Proofs

The formulation of Theorem 1.1 was inspired by a pair of results [14, 15] of Kahn. The contrast between the complexity of the proofs of these results and that of the following proofs derives from the power of Theorems 2.3, 2.4: with the foundations laid (in §§1–2), each essentially yields our main result as a corollary.

First proof of Theorem 1.1. In light of (10), it remains only to establish the right hand inequality in (2) for sufficiently large χ_t^*. Given $\varepsilon > 0$, let $\gamma = \varepsilon/2$, and choose d so large (according to Theorem 2.3) that

$$\Delta > d \quad \text{implies} \quad \chi'_\ell < (1+\gamma)\Delta. \tag{11}$$

Provided $\chi_t^* > D := \max\{d+2, 2/\gamma\}$, we then have

$$\chi_t \leq \chi'_\ell + 2 < (1+\gamma)\Delta + 2 < (1+\gamma)\chi_t^* + \gamma\chi_t^* = (1+\varepsilon)\chi_t^*$$

(justifying the inequalities, resp., by: Lemma 2.2; (11), Theorem 2.1 and $\chi_t^* > d+2$; (9) and $\chi_t^* > 2/\gamma$), which is what we want. ∎

Second proof. We dispense with the "epsilontics" and prove (1) directly. Combining (9), (10) and Theorem 2.4, we have

$$\Delta + 1 \leq \chi_t^* \leq \chi_t \leq \Delta + C \tag{12}$$

(where C is a constant), from which it is obvious that $\chi_t \sim \chi_t^*$ ($\sim \Delta$) as $\Delta \to \infty$, and that $\Delta \to \infty$ if and only if $\chi_t^* \to \infty$. ∎

Note that the second proof would still carry through even if the upper bound for χ_t in (12) were replaced by $\Delta + o(\Delta)$ (with $o(\Delta)$ any function such that $o(\Delta)/\Delta \to 0$ as $\Delta \to \infty$). Such bounds were obtained previously by Hind [6], Chetwynd and Häggkvist (see [10]) and Hind *et al.* [8].

Acknowledgement Thanks to Mike Molloy for making available manuscripts of the articles [8, 9] and for informative conversations about the subject matter of this paper.

References

[1] N. Alon, Restricted colorings of graphs, in "Surveys in Combinatorics, Keele, 1993," *London Math. Soc. Lecture Note Ser.* **187** (1993), 1–33.

[2] M. Behzad, *Graphs and Their Chromatic Numbers*, Dissertation, Michigan State University, East Lansing, MI, 1965.

[3] B. Bollobás, *Graph Theory*, Springer-Verlag, New York, 1979.

[4] V. Chvátal, *Linear Programming*, W.H. Freeman and Company, New York, 1983.

[5] P. Erdős, A. Rubin and H. Taylor, Choosability in graphs, *Congr. Numer.* **26** (1979), 125–157.

[6] H. Hind, An upper bound for the total chromatic number, *Graphs Combin.* **6** (1990), 153–159.

[7] H. Hind, Recent developments in total colouring, *Discrete Math.* **125** (1994), 211–218.

[8] H. Hind, M. Molloy and B. Reed, Total colouring with $\Delta + \mathrm{poly}(\log\Delta)$ colours, submitted.

[9] H. Hind, M. Molloy and B. Reed, Colouring a graph frugally, submitted.

[10] T. Jensen and B. Toft, *Graph Colouring Problems*, Wiley, New York, 1995.

[11] J. Kahn, Recent results on some not-so-recent hypergraph matching and covering problems, in "Extremal Problems for Finite Sets, Visegrád, 1991," *Bolyai Soc. Math. Studies* **3** (1994), 305–353.

[12] J. Kahn, Asymptotically good list-colorings, *J. Combin. Theory Ser. A* **73** (1996), 1–59.

[13] J. Kahn, A linear programming perspective on the Frankl-Rödl-Pippenger theorem, *Random Struct. Alg.* **8** (1996), 149–157.

[14] J. Kahn, Asymptotics of the chromatic index for multigraphs, *J. Combin. Theory Ser. B*, to appear.

[15] J. Kahn, Asymptotics of the list-chromatic index for multigraphs, submitted.

[16] J. Kahn and P.M. Kayll, Fractional v. integral covers in hypergraphs of bounded edge size, submitted.

[17] K. Kilakos and B. Reed, Fractionally colouring total graphs, *Combinatorica* **13** (1993), 435–440.

[18] L. Lovász and M.D. Plummer, *Matching Theory*, North-Holland, New York, 1986.

[19] M. Molloy, private communication, February 1996.

[20] M. Molloy and B. Reed, A bound on the total chromatic number, in preparation.

[21] J. Ryan, Fractional total colouring, *Discrete Appl. Math.* **27** (1990), 287–292.

[22] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.

[23] N. Vijayaditya, On total chromatic number of a graph, *J. London Math. Soc. (2)* **3** (1971), 405–408.

[24] V. G. Vizing, On an estimate of the chromatic class of a p-graph (in Russion), *Diskret. Analiz* **3** (1964), 25–30.

[25] V. G. Vizing, Some unsolved problems in graph theory, *Russian Math Surveys* **23** (1968), 125–141.

[26] V. G. Vizing, Coloring the vertices of a graph in prescribed colors (in Russian), *Diskret. Analiz* **29** (1976), 3–10, 101.