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k-Weak Orders: Recognition and a Tolerance Result

by

Ann N. Trenk[1]
Department of Mathematics
Wellesley College
Wellesley, MA 02181 USA

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Abstract

In this paper we introduce a family of ordered sets we call *k-weak orders* which generalize weak orders, semi-orders, and bipartite orders. For each k, we give a polynomial-time recognition algorithm for k-weak orders and a partial characterization. In addition, we prove that among 1-weak orders, the classes of bounded bitolerance orders and totally bounded bitolerance orders are equal. This enables us to recognize the class of totally bounded bitolerance orders for 1-weak orders.

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1 Introduction

The ordered sets in this paper will be irreflexive with "$\prec$" denoting the relation, unless otherwise specified. If x and y are incomparable elements we write $x \sim y$. We denote by $\underline{r} + \underline{s}$ the ordered set consisting of two disjoint chains, one with r elements, the other with s elements. We write $x_1 \prec x_2 \prec \cdots \prec x_r || y_1 \prec y_2 \prec \cdots \prec y_s$ to denote the $\underline{r} + \underline{s}$ whose chains are labeled $x_1 \prec x_2 \prec \cdots \prec x_r$, and $y_1 \prec y_2 \prec \cdots \prec y_s$.

An ordered set is called a *weak order* (or a *preorder*) if and only if it has no $\underline{2} + \underline{1}$ as an induced suborder. It is easy to see that $P = (X, \prec)$ is a weak order if and only if it admits a real-valued function $u : X \rightarrow \mathbf{R}$ so that $x \prec y$ in P iff $u(x) < u(y)$ [3]. Such a function is called an *ordinal utility function* [6]. It places an ordering on the elements of P which is similar to a linear ordering except that ties are allowed. The ordering determined by the function u is useful in that it provides a ranking of elements, and it is therefore desirable to extend this notion to a wider class of ordered sets.

In this paper we generalize weak orders to k-weak orders, where $k \geq 0$ is an integer. When $k = 0$, the definition reverts to that of a weak order. (When $k < 0$ the definition is that of a linear order, but we will confine our attention to $k \geq 0$.)

Definition 1 *Given an ordered set $P = (X, \prec)$, an integer-valued function $lev : X \rightarrow \mathbf{Z}$ is a k-*leveling function *of P if it satisfies the following for all $x, y \in X$.*

- **Rule A:** *If $x \prec y$ then $lev(x) < lev(y)$.*
- **Rule B:** *If $x \sim y$ then $|lev(x) - lev(y)| \leq k$.*

Definition 2 *An ordered set $P = (X, \prec)$ is k-*weak *if there exists a k-leveling function of P.*

Given a k-weak order with k-leveling function lev, we say that element v is on *level i* (or is a *level i element*) when $lev(v) = i$.

Rule A ensures that if y is preferred to x then y gets a higher level (ranking) than x does. Note that this rule is independent of k. If x and y are incomparable, then Rule B ensures that the levels they receive are not too far apart, i.e., at most k units. To force a k-leveling function to be as close as possible to an ordinal utility function, we use the smallest value of k which admits a k-leveling. We will call this value of k the *weakness* of P.

The following are easy consequences of Definitions 1 and 2 which we state without proof.

Proposition 3 *If $P = (X, \prec)$ is a k-weak order with k-leveling function lev, then*

1. *The set $\{v : lev(v) = i\}$ of level i elements is an antichain.*
2. *If $lev(y) - lev(x) \geq k + 1$ then $x \prec y$.*
3. *Any induced suborder of P is k-weak.*

Each k-leveling lev of order $P = (X, \prec)$ is a member of an equivalence class $[lev(x)] = \{lev(x) + c : c \in \mathbf{Z}\}$ of k-leveling functions of P. Thus in searching for a k-leveling function of P, we may pick a representative of each equivalence class by choosing a base element $b \in X$ and fixing $lev(b) = 1$. This is done in Step 2 of Algorithm k-Weak Leveling.

The rest of the paper is organized as follows. In §2 we give an algorithm for recognizing k-weak orders and prove the algorithm runs in time $O(n^4 k)$ for an n-element ordered set. In §3 we explore the relationship between the weakness of P and the largest $\underline{r} + \underline{s}$ induced in P. In §4, we characterize the class of totally bounded bitolerance orders for 1-weak orders, which was our original motivation for studying k-weak orders.

2 Recognition of k-Weak Orders

2.1 Preliminaries

Lemma 4 *Let $P_1 = (X_1, \prec_1), P_2 = (X_2, \prec_2), \ldots, P_m = (X_n, \prec_m)$ be k-weak ordered sets with $X_i \cap X_j = \emptyset$ for all $i \neq j$. Define $P = (X, \prec)$ to be the ordered set with $X = X_1 \cup X_2 \cup \cdots \cup X_m$ and comparabilities as follows: for any $u \in X_i$ and $v \in X_j$ and $i < j$, we have $u \prec v$, and if $u, v \in X_i$ then $u \prec v$ iff $u \prec_i v$. Then the order P is k-weak.*

Proof. Let $lev_i : X_i \to \mathbf{Z}$ be a k-leveling function of P_i for $i = 1, 2, \ldots, m$. If necessary, add an appropriate constant c_i to each value of $lev_i(x)$ so that the range of the function lev_i is $[h_i] = \{1, 2, 3, \ldots, h_i\}$. Define $lev : X \to [h_1 + h_2 + \cdots + h_m]$ by $lev(u) = lev_1(u)$ if $u \in X_1$, and $lev(u) = h_1 + h_2 + \cdots + h_{j-1} + lev_j(u)$ for $u \in X_j, j > 1$. It is easy to verify that lev gives a k-leveling of P. $\square$

Definition 5 *An ordered set $P = (X, \prec)$ is* inseparable *if it is impossible to partition the set $X = X_1 \cup X_2$ nontrivially so that $u \prec v$ for all $u \in X_1, v \in X_2$.*

The following simple algorithm partitions an ordered set into inseparable induced suborders.

Algorithm Stackem

Input: An ordered set $Q = (X, \prec)$.

Output: A partition $X = X_1 \cup X_2 \cup \cdots \cup X_m$ for which (i) $x_i \prec x_j$ whenever $i < j$ and $x_i \in X_i$ and $x_j \in X_j$, and (ii) the induced suborders $Q_i = (X_i, \prec)$ are inseparable.

Algorithm: If Q is an antichain, then clearly Q is inseparable. Otherwise, pick a minimal non-isolated element x_1 and initialize $X_1 := \{x_1\}$. This is stage 0. At stage i, search each element v which was added to X_1 in stage $i-1$, and add to X_1 all elements in X/X_1 which are incomparable to v. Stop when X_1 stabilizes, i.e., at the first stage j when no new elements are added to X_1 at stage j.

If $X_1 = X$, then stop: Q is inseparable.

Otherwise, $X = X_1 \cup (X/X_1)$ partitions X nontrivially and $Q_1 = (X_1, \prec)$ is inseparable. Repeat the algorithm on X/X_1 to obtain $X/X_1 = X_2 \cup (X_1/X_2)$ and thus $X = X_1 \cup X_2 \cup (X_1/X_2)$ where $Q_1 = (X_1, \prec)$ and $Q_2 = (X_2, \prec)$ are inseparable. Continue the process until every part of the partition is inseparable.

Proposition 6 *Algorithm Stackem produces the stated output and runs in $O(|X|^2)$ time.*

Proof. First we consider the running time. The algorithm requires a minimal non-isolate to initialize each X_i, and these can be found in $O(|X|^2)$ time once a topological ordering of the elements of Q is constructed. Additionally, the neighborhood of each element $v \in X$ is examined exactly once, requiring linear time for each v, thus the algorithm runs in time $O(|X|^2)$.

Clearly Algorithm Stackem produces a partition $X = X_1 \cup X_2 \cup \cdots \cup X_m$. We need to show that (i) the induced suborders $Q_i = (X_i, \prec)$ are inseparable, and (ii) if $i < j$ and $x_i \in X_i$ and $x_j \in X_j$, then $x_i \prec x_j$.

To show (i): Suppose Q_i were separable, and let $X_i = W_1 \cup W_2$ be a separation with $u \prec v$ whenever $u \in W_1$ and $v \in W_2$. Since X_i is initialized with a minimal element x, we know that element is in W_1. But then the elements of W_2 would never be added to X_i because none of them are incomparable to anything in W_1, a contradiction.

To show (ii): Suppose $i < j$ and there exist $x_i \in X_i$ and $x_j \in X_j$, with $x_i \not\prec x_j$. By construction, $y \not\sim x_j$ for all $y \in X_i$, otherwise, x_j would have been added to X_i. Thus we may partition $X_i = W_1 \cup W_2$ where $W_1 = \{y \in X_i : y \prec x_j\}$ and $W_2 = \{y \in X_i : x_j \prec y\}$. Note that by transitivity, $u \prec v$ for all $u \in W_1$ and $v \in W_2$, so all minimal elements in X_i are in W_1.

W_1 is not empty, because the algorithm initializes X_i with a minimal element, and W_2 is not empty because $x_i \in W_2$. But now $X_i = W_1 \cup W_2$ separates the inseparable $Q_i = (X_i, \prec)$, a contradiction. □

We remark on two special cases. If $Q = (X, \prec)$ itself is inseparable, the output partition is $X = X_1$. If Q is a weak order (i.e., 0-weak) with ordinal utility function u, then each resulting Q_i is an antichain (in fact, the Q_i correspond to distinct values of $u(x)$).

2.2 The Main Algorithm

We next present an algorithm which determines if an ordered set is k-weak, and in the affirmative case, constructs a k-leveling function. We discuss correctness and complexity afterwards.

Algorithm k-Weak Leveling

Input: An ordered set $Q = (Y, \prec)$.

Output: A k-weak leveling function $lev : Y \to \mathbf{Z}$ of Q, or the statement that Q is not k-weak.

The algorithm:

Step 1: Use Algorithm Stackem to partition Q into inseparable suborders Q_i. Do Steps 2–6 (below) on each of the Q_i. If any of them fail to be k-weak, STOP; Q is not k-weak. If all of them are k-weak, use the construction in the proof of Lemma 4 to obtain a k-leveling of Q.

To simplify the notation, we let $P = (X, \prec) = Q_i$.

Step 2: [Initialization Step] Choose a particular element $b \in X$ to be the base of the k-leveling, and assign $lev(b) = 1$. Let $U = X - \{b\}$.

Form a $\{0,1\}$-matrix M whose rows and columns are indexed by U. Initialize:

$$M_{xy} = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Step 3: [Assign Ranges] Assign the range $R(x) = [\ell(x), u(x)]$ to each $x \in U$ as follows:

- If $b \prec x$ set $R(x) = [2, \infty)$.
- If $x \prec b$ set $R(x) = (-\infty, 0]$.
- If $b \sim x$ set $R(x) = [1-k, 1+k]$.

Step 4: [Narrowing the Ranges]
Narrowing Steps (NS): Pick two distinct elements $x, y \in U$ with $M_{xy} = 0$. Label them so that either $x \prec y$ or $x \sim y$.

1. If $x \prec y$ and $\ell(y) \leq \ell(x)$, increase $\ell(y)$ to $\ell(x) + 1$.
2. If $x \prec y$ and $u(x) \geq u(y)$, decrease $u(x)$ to $u(y) - 1$.
3. If $x \sim y$ and $u(y) \geq u(x) + k + 1$, decrease $u(y)$ to $u(x) + k$.
4. If $x \sim y$ and $u(x) \geq u(y) + k + 1$, decrease $u(x)$ to $u(y) + k$.
5. If $x \sim y$ and $\ell(y) \leq \ell(x) - k - 1$, increase $\ell(y)$ to $\ell(x) - k$.
6. If $x \sim y$ and $\ell(x) \leq \ell(y) - k - 1$, increase $\ell(x)$ to $\ell(y) - k$.

If $\ell(x) > u(x)$ or $\ell(y) > u(y)$, STOP. P is not k-weak.

If $R(x)$ is bounded and was narrowed in this pass of the narrowing steps, set $M_{xz} = M_{zx} = 0$ for all z other than x and y. Likewise, if $R(y)$ is bounded and was narrowed in this pass of the narrowing steps, set $M_{yz} = M_{zy} = 0$ for all z other than x and y.

In any event, set $M_{xy} = M_{yx} = 1$.

If all entries of M are 1's, continue to Step 5. Otherwise, begin Step 4 again.

Step 5: Set $lev(x) = \ell(x)$ for each $x \in U$.
(End of Algorithm k-Weak Leveling)

2.3 Correctness and Complexity of Algorithm k-Weak Leveling

We establish the correctness of Algorithm k-Weak leveling using three propositions. The first verifies that the assignment made in Step 5 is well-defined, the second ensures that the algorithm terminates, and the third that if Step 5 is reached, then the resulting function lev is indeed a k-leveling function. After this we consider the complexity of the algorithm.

Proposition 7 *If Algorithm k-Weak Leveling reaches Step 5, all ranges are bounded. Thus the assignment $lev(x) = \ell(x)$ made in Step 5 is well-defined.*

Proof. Partition $X = I \cup J \cup L$ where v's membership in I, J or L is based on the range $R(v)$ at the beginning of Step 5 as follows:

- $x \in I$ if $R(x)$ has the form $(-\infty, j]$ for some $j \in \mathbf{Z}$,
- $y \in J$ if $R(y)$ is bounded,
- $z \in L$ if $R(z)$ has the form $[\ell, \infty)$ for some $\ell \in \mathbf{Z}$.

These are the only possible forms for ranges because ranges initially have one of these forms (Step 3) and in Step 4 ranges are narrowed only by increasing the lower bound or decreasing the upper bound. At the beginning of Step 5, the Narrowing Steps have been applied to all pairs of elements of X, thus $x \prec y \prec z$ for all $x \in I$, $y \in J$, $z \in L$ (or further narrowing would have occurred). We know $J \neq \emptyset$ since $b \in J$. If $I = L = \emptyset$, we are done. Otherwise, either $X = (I \cup J) \cup L$ or $X = I \cup (J \cup L)$ separates P, a contradiction. $\Box$

Proposition 8 *Either Algorithm k-Weak Leveling terminates in Step 4 when $R(x) = \emptyset$ for some $x \in U$, or otherwise after a finite number of operations,*

1. *each $R(x)$ becomes bounded and*

2. the algorithm proceeds to Step 5.

Proof. Algorithm k-Weak Leveling terminates in Step 4 if and only if $R(x) = \emptyset$ for some $x \in U$. Thus we assume $R(x) \neq \emptyset$ for all $x \in U$.

By Proposition 7, if the algorithm proceeds to Step 5, then all ranges are bounded. Therefore, it remains to consider the case in which for all $x \in U$, we have $R(x) \neq \emptyset$, and yet the algorithm does not reach Step 5. We show this leads to a contradiction.

Every time all pairs of elements in U are considered, some range must narrow to a bounded set, otherwise the matrix M would fill with 1's and the algorithm would proceed to Step 5. Once $R(x)$ becomes bounded, whenever it is further narrowed, $|R(x)|$ decreases by at least 1. Thus every time all pairs of elements of U are considered, either the quantity $G_1 = \#\{x \in U : R(x) \text{ is unbounded}\}$ or the quantity

$$G_2 = \sum_{x \in U,\ R(x)\ bnd} |R(x)|$$

decreases by at least one. By our assumption, $|R(x)| \geq 1$ for all $x \in U$. Since G_1 never increases and is bounded below by 0, and G_2 is bounded below by $|U|$, eventually $G_1 = 0$. After this point, G_2 decreases by at least 1 every time all pairs of elements in U are considered, and eventually $G_2 < |U|$, a contradiction. □

Proposition 9 *If the algorithm proceeds to Step 5, then the resulting function lev gives a k-leveling of P.*

Proof. Pick any $a, b \in U$. By Proposition 7, $R(a)$ and $R(b)$ are bounded at the beginning of Step 5, thus we may write $R(a) = [\ell(a), u(a)]$ and $R(b) = [\ell(b), u(b)]$. We will show that assigning $lev(a) = \ell(a)$ and $lev(b) = \ell(b)$ satisfies Rules A and B.

Without loss of generality, assume $a \prec b$ or $a \sim b$. Since the algorithm has reached Step 5, we know M is the matrix of all 1's, and in particular, both a's row and column and b's row and column are all 1's. Thus the values of $R(a)$ and $R(b)$ have not changed since the six narrowing steps were last applied to the pair $x = a$ and $y = b$.

If $a \prec b$ then by Narrowing Step 1, $\ell(b) \geq \ell(a) + 1$, hence $lev(b) > lev(a)$, which satisfies Rule A. (Rule B is satisfied vacuously.)

If $a \sim b$, then by Narrowing Steps 5 and 6, $\ell(a) \geq \ell(b) - k$ and $\ell(b) \geq \ell(a) - k$. Thus

$$\begin{gathered} lev(a) - k \leq \ell(b) = lev(b) \leq \ell(a) + k = lev(a) + k \\ -k \leq lev(b) - lev(a) \leq k \\ |lev(b) - lev(a)| \leq k \end{gathered}$$

satisfying Rule B. (Rule A is satisfied vacuously.) □

Theorem 10 *Algorithm k-Weak Leveling correctly determines whether P is k-weak, and in the affirmative case produces a k-leveling of P.*

Proof. In Step 1, if any of the Q_i fail to be k-weak, then Q is not k-weak by the third part of Proposition 3. Thus we may focus on Steps 2–6.

In Step 2 we choose a particular element b to be the base of our k-leveling and assign $lev(b) = 1$. As noted earlier, choosing a base simply picks one representative of each equivalence class of k-levelings.

Once $lev(b) = 1$ is fixed, the range sets are initialized (Step 3) and narrowed (Step 4). The initialization in Step 3 is accomplished by applying Rules A and B to the pair (b, x) for each $x \in U$. The narrowing steps proceed by applying Rule A (in Narrowing Steps 1,2) or Rule B (in Narrowing Steps 3–6) to the pair (x, y). Because the initialization and narrowing of the ranges is accomplished only by using Rules A and B, the range sets satisfy the following invariant:

$(*)$ If lev is a k-leveling of P with $lev(b) = 1$ then $lev(x) \in R(x)$ for all $x \in U$.

If at any time in Step 4, $R(x) = \emptyset$ for some $x \in U$, then by $(*)$ there is no possible value to assign to $lev(x)$, thus P is not k-weak. This proves the assertion made just after Narrowing Step 6.

Proposition 8 implies that the algorithm either terminates in Step 4 (with the conclusion that P is not k-weak) or reaches Step 5 after a finite number of operations. In the latter case, Proposition 9 ensures that the function lev assigned in Step 5 is indeed a k-leveling, and thus that P is k-weak. □

The next lemma is needed to prove the complexity bound in Theorem 12.

Lemma 11 *Let Q be an ordered set and $P = (X, \prec)$ an inseparable suborder of Q with $|X| = n$. If Steps 2 – 5 of Algorithm k-Weak Leveling are applied to P then at all times, for all $x \in U$, either $R(x)$ is unbounded or (i) $R(x) \subseteq [1-k, 1+k]$, or (ii) $R(x) \subseteq [2, 1+(n+1)k]$ or (iii) $R(x) \subseteq [1-(n+1)k, 0]$.*

Proof. Partition $U = A \cup B \cup C$ where

- $x \in A$ if $b \prec x$ (x is above the base element b),
- $y \in B$ if $y \prec b$ (y is below the base element b),
- $z \in C$ if $b \sim z$ (z is incomparable to the base element b).

At the end of Step 3 of Algorithm k-Weak Leveling, $R(z) = [1-k, 1+k]$ for all $z \in C$, and $R(v)$ is unbounded for all $v \in A \cup B$. Since ranges only narrow in Step 4, the elements in C automatically satisfy the desired conditions (type (i)). We consider $a \in A$, the case $b \in B$ is similar and is omitted.

As Step 4 progresses, some (or all) of the ranges $R(a)$ become bounded for $a \in A$ by Proposition 8. Let $A = \{a_1, a_2, \ldots, a_r\}$ where $i < j$ means $R(a_i)$ becomes bounded before $R(a_j)$. We will show, by induction on t, that when $R(a_t)$ first becomes a bounded set, $R(a_t) \subseteq [2, 1+(t+1)k]$.

First we handle the base case. Let w be the element of U for which applying the Narrowing Steps (hereafter referred to as NS) to the pair (w, a_1) results in $R(a_1)$'s first becoming bounded. Note that only NS 2, 3 and 4 can result $R(a_1)$ becoming bounded, thus $a_1 \sim w$ (using NS 3 or 4) or $a_1 \prec w$ (using NS 2). If $w \in B$, then $w \prec b \prec a_1$, a contradiction. If $w \in A$ then $u(w) = \infty$, so the conclusions of NS 2,3, and 4 would not result in a bounded $R(a_1)$. Thus $w \in C$. If $a_1 \prec w$ then $b \prec a_1 \prec w$, contradicting $w \in C$. Thus $w \in C$ and $R(w) \subseteq [1-k, 1+k]$ and $a_1 \sim w$. Now after applying by NS 3 and 4, to the pair (w, a_1) the result is $R(a_1) \subseteq [2, 1+2k]$, as desired.

Now assume that when $R(a_j)$ is first narrowed to a bounded set that $R(a_j) \subseteq [2, 1+(j+1)k]$ for $j = 1, 2, \ldots, s-1$. Consider a_s and let (a_s, w) be the pair chosen when $R(a_s)$ is first narrowed to a bounded set. As before, $a_s \sim w$ or $a_s \prec w$, thus $w \notin B$. Again, if $w = a_p$ for some $p > s$, then $u(a_p) = \infty$ and applying NS 2, 3 and 4 to the pair a_s, a_p will not result in a bounded $R(a_p)$.

Thus either $w \in C$ or $w = a_i$ for some $i < s$. In the first case, the argument given in the base case shows $R(a_s) \subseteq [2, 1+2k]$. Now consider

$w = a_i$ for some $i < s$. By the induction hypothesis and the fact that ranges only narrow as the algorithm continues, $R(a_i) \subseteq [2, 1 + (i+1)k]$. If $a_s \sim a_i$, $R(a_s)$ is narrowed so that $R(a_s) \subseteq [2, 1 + (i+1)k + k] \subseteq [2, 1 + (s+1)k]$. If $a_s \prec a_i$, $R(a_s)$ is narrowed so that $R(a_s) \subseteq [2, 1+(i+1)k-1] \subset [2, 1+(s+1)k]$. □

Theorem 12 *With input $Q = (Y, \prec)$, Algorithm k-Weak Leveling runs in $O(|Y|^4 k)$ time for $k \geq 1$, and $O(|Y|^2)$ time for $k = 0$.*

Proof. Algorithm Stackem runs in $O(|Y|^2)$ time by Proposition 6. The remaining steps are applied to each of the $Q_i = (X_i, \prec)$. Let $|X_i| = n_i$. Clearly Steps 2, 3, and 5 run in time $O({n_i}^2)$ for each i, and thus in time $O(|Y|^2)$ all together. So we focus attention on Step 4.

First we consider the case $k \geq 1$. We compute an upper bound on the number of times some range set is narrowed to a bounded set. Each non-base element x of X_i can have its range narrowed to a bounded set at most $(n_i + 1)k$ times by Lemma 11 and the fact that $|R(x)|$ decreases by at least 1 each time. Since there are $n_i - 1$ non-base elements in X_i, there are most $(n_i - 1)(n_i + 1)k \leq n_i^2 k$ narrowings to bounded sets overall.

Furthermore, after all $\binom{n_i-1}{2} < n_i^2$ pairs of non-base elements in X_i are examined, a narrowing to a bounded set must occur, otherwise M is filled with 1's and the algorithm proceeds to Step 5. Thus Step 4 is completed in $O(n_i^4 k)$ time for each Q_i, and in $O(|Y|^4 k)$ time overall.

Now consider the case $k = 0$. By the remark at the end of Section 2.1, each Q_i is an antichain. Thus at the beginning of Step 4, each range set is $R(x) = \{1\}$, and no narrowings take place. After all $\binom{n_i-1}{2} < n_i^2$ pairs of non-base elements in X_i are examined, the matrix M is filled with 1's and the algorithm proceeds to Step 5. Thus when $k = 0$, Step 4 is completed in $O(n_i^2)$ time for each Q_i, and in $O(|Y|^2)$ time overall. □

3 Partial Characterizations of k-Weak Orders

Finding a forbidden suborder characterization of k-weak orders is currently an open problem. However, the orders $\underline{r} + \underline{s}$ seem to play a central role in this problem as well as in characterizing other important classes of orders. With this in mind, we make the following definition.

Definition 13 *An ordered set is* $(M,2)$-free *if it has no induced* $\underline{r}+\underline{s}$ *where* $r,s \geq 1$ *and* $r+s=M$.

More generally, an ordered set is (M,t)-free if it has no induced $\underline{r_1}+\underline{r_2}+\cdots+\underline{r_t}$ where each $r_i \geq 1$ and $r_1+r_2+\cdots+r_t=M$. However, we will only be concerned with the case $t=2$.

The (2,2)-free orders are those with no induced $\underline{1}+\underline{1}$ which are precisely the linear orders. The (3,2)-free orders are those with no induced $\underline{2}+\underline{1}$, which is the class of 0-weak (or just weak) orders. Using the Scott-Suppes Theorem [11] (or see [12]), semi-orders (also known as unit interval orders) are the class of (4,2)-free orders, that is, those with no induced $\underline{2}+\underline{2}$ and no induced $\underline{3}+\underline{1}$. Corollary 15 is a general result relating k-weak orders to $(M,2)$-free orders.

Proposition 14 *Let* $P=\underline{r}+\underline{s}$ *be the ordered set* $x_1 \prec x_2 \prec \cdots \prec x_r || y_1 \prec y_2 \prec \cdots \prec y_s$, *where* $r,s \geq 1$ *and* $r+s=2k+2$. *Then* P *is* k-*weak. Furthermore, any* k-*leveling* lev *of* P *satisfies* $lev(y_j)=lev(x_r)-k+j-1$ *and* $lev(x_i)=lev(y_s)-k+i-1$.

Proof. Let P be the ordered set above. It is easy to check that the function lev defined by $lev(x_i)=i$ for $i=1,2,\ldots,r$ and $lev(y_j)=r-k+j-1$ for $j=1,2,\ldots,s$ is a k-leveling of P. Thus it remains to justify the last sentence of the proposition.

Let lev be a k-leveling of P. First observe that by Rule A, $lev(x_1) \leq lev(x_2)-1 \leq lev(x_3)-2 \leq \cdots \leq lev(x_r)-r+1$, thus $lev(x_1)-lev(x_r) \leq 1-r$.

Now since $x_i \sim y_j$ for all i,j, by Rule B we know $lev(y_j) \in [lev(x_i)-k, lev(x_i)+k]$ for $i=1,2,\ldots,r$. Taking the intersection of these n sets, we obtain $lev(y_j) \in R(y_j)=[lev(x_r)-k, lev(x_1)+k]$. The number of integers in $R(y_j)$ is $lev(x_1)-lev(x_r)+2k+1 \leq 1-r+2k+1=2k+2-r=s$. So each $R(y_j)$ contains exactly the same s integers. However, $lev(y_1)<lev(y_2)<lev(y_3)<\cdots<lev(y_s)$ are s distinct integers, hence $lev(y_j)=lev(x_r)-k+j-1$. By symmetry, $lev(x_i)=lev(y_s)-k+i-1$. □

Corollary 15 *If* Q *is* k-*weak then* Q *is (2k+3,2)-free.*

Proof. Let $P=\underline{r}+\underline{s}$ be the order $x_1 \prec x_2 \prec \cdots \prec x_r || y_1 \prec y_2 \prec \cdots \prec y_s$, where $r,s \geq 1$ and $r+s \geq 2k+3$. By the third part of Proposition 3, it suffices to show that P is not k-weak. For a contradiction, suppose that

P were k-weak with k-leveling function lev. Consider the induced suborder $P' : x_1 \prec x_2 \prec \cdots \prec x_{r'} || y_1 \prec y_2 \prec \cdots \prec y_{s'}$, where $r', s' \geq 1$, $r' + s' = 2k+2$ and without loss of generality, $r' < r$, $s' \leq s$. Applying Proposition 14 to P' we conclude that $lev(y_1) = lev(x_{r'}) - k$.

Next consider the $\underline{r'}+\underline{s'} : x_2 \prec x_3 \prec \cdots \prec x_{r'+1} || y_1 \prec y_2 \prec \cdots \prec y_{s'}$ which is also strictly induced in P. Again applying Proposition 14 we conclude that $lev(y_1) = lev(x_{r'+1}) - k$. Thus $lev(x_{r'}) = lev(x_{r'+1})$, contradicting Rule A. $\Box$

Corollary 15 is best possible because k-weak ordered sets need not be $(2k+2, 2)$-free (see Proposition 14). For $k = 0$, the converse of Corollary 15 is true, as mentioned in the introduction. However, for $k > 0$ it is false as the following example shows.

Example 16 Fix $k > 0$ and let $P = (X \prec)$ be the ordered set with

$$X = \{x_1, x_2, \ldots, x_{k+1}, y_1, y_2, \ldots, y_{k+1}, z_1, z_2, \ldots, z_{k+2}\}$$

and whose *only* comparabilities are $x_1 \prec x_2 \prec \cdots \prec x_{k+1}$, $y_1 \prec y_2 \prec \cdots \prec y_{k+1}$, $z_1 \prec z_2 \prec \cdots \prec z_{k+2}$, $z_1 \prec x_{k+1}$ and $y_1 \prec z_{k+2}$.

It is easy to check that P is $(2k+3, 2)$-free. However, P is not k-weak, because if it were, applying Proposition 14 to the three $\underline{k+1}+\underline{k+1}$'s (i) induced by $x_1, x_2, \ldots, x_{k+1}$ and $z_2, z_3, \ldots, z_{k+2}$, and (ii) induced by $x_1, x_2, \ldots, x_{k+1}$ and $y_1, y_2, \ldots, y_{k+1}$, (iii) $y_1, y_2, \ldots, y_{k+1}$ and $z_1, z_2, \ldots, z_{k+1}$) would lead to the contradiction $lev(z_{k+2}) = lev(x_{k+1}) = lev(y_{k+1}) = lev(z_{k+1})$.

However, we do have the following partial converse to Corollary 15, which, in particular, implies that all semi-orders are 1-weak.

Theorem 17 *If P is an ordered set with no induced $\underline{(k+2)}+\underline{1}$ then P is k-weak.*

Proof. Suppose for a contradiction that there are ordered sets that contain no $\underline{(k+2)}+\underline{1}$'s, yet are not k-weak. Let $P = (X, \prec)$ be such an ordered set with the least number of elements. Using Lemma 4, it is easy to see that P must be inseparable. Furthermore, P is not simply a chain or an antichain, because both chains and antichains are k-weak for all $k \geq 0$. Thus we may choose $x \in X$ to be a maximal non-isolate of P and choose $b \in X$ with $b \sim x$.

Since $P - x$ has 1 fewer element than P, it is k-weak. Take a k-leveling lev of $P - x$ with base b assigned $lev(b) = 1$. Define $lev(x) = \max[\{1 + lev(y) : y \in X, y \prec x\} \cup \{lev(z) - k : z \in X, z \sim x\}]$.

By the definition of $lev(x)$ and the fact that x is maximal in P, we know that the function lev satisfies Rule A for all pairs of elements in X. Thus Rule B must be violated for a pair of elements including x, that is, there exists $w \in X$ with $x \sim w$ but $|lev(x) - lev(w)| > k$. Again using the definition of $lev(x)$, we know $lev(x) - lev(w) \geq k+1$. Writing $L = lev(x) - lev(w)$, we have $L \geq k+1$.

Call a sequence of elements $C_{wd} : w = a_0, a_1, a_2, \ldots, a_r = d$ a *forced path from w to d* if for each $i \in \{0, 1, \ldots, r-1\}$ either $a_i \prec a_{i+1}$ and $lev(a_{i+1}) = lev(a_i) + 1$ or $a_i \sim a_{i+1}$ and $lev(a_i) - k = lev(a_{i+1})$.

Suppose $lev(w)$ is increased by 1. Then to preserve lev's being a k-leveling of $P - x$ we must increase $lev(d)$ by 1 iff there exists a forced path from w to d.

Claim: If there exists a forced path from w to x, then P has an induced $\underline{(k+2)} + \underline{1}$.

Proof of Claim: Let $C_{wx} : w = a_0, a_1, a_2, \ldots, a_r = x$ be a shortest forced path from w to x. Consider the sequence of differences: $S : lev(a_1) - lev(w), lev(a_2) - lev(a_1), lev(a_3) - lev(a_2), \ldots, lev(x) - lev(a_{r-1})$.

By the definition of a forced path, each term of S is either $+1$ or $-k$. Let t be the largest number of consecutive $+1$'s in S and let $a_j, a_{j+1}, a_{j+2}, \ldots, a_{j+t}$ be a subsequence of C_{wx} for which $a_{j+i} - a_{j+i-1} = 1$ for $i = 1, 2, \ldots, t$.

Note that the sum of the elements in S is $lev(x) - lev(w) \geq k+1$. If $t \leq k$ then every partial sum of S is less than or equal to k, a contradiction. Thus $t \geq k+1$.

Suppose $j = 0$. Then $w = a_0 \prec a_1 \prec \cdots \prec a_{k+1}$ is a chain induced in P with $lev(a_{k+1}) = lev(w) + k + 1 \leq lev(x)$. Because x is maximal, we know $x \not\prec a_{k+1}$. We are also given $w \sim x$ and therefore $a_{k+1} \not\prec x$. Thus $w \sim x$ and $a_{k+1} \sim x$ and we conclude that $w = a_0 \prec a_1 \prec \cdots \prec a_{k+1} || x$ is a $\underline{(k+2)} + \underline{1}$ induced in P, a contradiction.

Otherwise $j \geq 1$. Now consider the chain $a_j \prec a_{j+1} \prec \cdots \prec a_{j+k+1}$ induced in P. We know $a_{j-1} \sim a_j$ with $lev(a_{j-1}) - k = lev(a_j)$ by the maximality of t. Now $lev(a_{j+k+1}) = lev(a_j) + k + 1 = lev(a_{j-1}) + 1$. If $a_{j-1} \prec a_{j+k+1}$, we would have a shorter forced path $(w = a_0, a_1, \ldots, a_{j-1}, a_{j+k-1}, \ldots, a_r = x)$ from w to x, a contradiction. If $a_{j+k+1} \prec a_{j-1}$, then $a_j \prec a_{j+k+1} \prec a_{j-1}$, contradicting $a_{j-1} \sim a_j$. Thus $a_{j-1} \sim a_{j+k+1}$ which together with our earlier result $(a_{j-1} \sim a_j)$ implies that $a_j \prec a_{j+1} \prec \cdots \prec a_{j+k+1} || a_{j-1}$ is a $\underline{(k+2)} + \underline{1}$ induced in P, a contradiction.

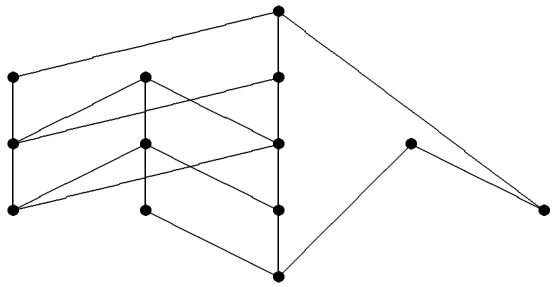

Figure 1: An ordered set that is $(6,2)$-free but not 2-weak.

This proves the claim.

Since we have assumed that P has no induced $\underline{(k+2)}+\underline{1}$, from the claim above, we know that there are no forced paths from w to x. Now increase $lev(w)$ by 1 and increase $lev(d)$ by 1 for all d for which there is a forced path from w to d. The resulting function is still a valid k-leveling of $P-x$. But now $lev(x)-lev(w)=L-1$. Continue the process until $lev(x)-lev(w)=k$. If this is now a valid k-leveling of P, we contradict the assumption that P is not k-weak. If not, find another w and proceed. Since values of lev never decrease in this procedure, and $lev(x)$ is unchanged, eventually we reach a contradiction. $\square$

Again, the converse to Theorem 17 is false, since Proposition 14 shows that $\underline{3}+\underline{1}$ is 1-weak. We conjecture that Theorem 17 is also best possible, that is, for each $k>0$ we conjecture that there exist $(k+4,2)$-free orders that are not k-weak. For $k=1$, Example 16 gives an ordered set which is (5,2)-free, yet not 1-weak. One can check that the order in Figure 1 is $(6,2)$-free (by inspection), but not 2-weak (using Algorithm k-Weak Leveling), thus providing an example for the case $k=2$.

4 Totally bounded 1-weak orders

4.1 Bitolerance orders

Tolerance graphs and orders were introduced in [7] and further studied by several authors, including [1],[2],[5],[8], and [9].

A *bounded bitolerance representation* of an ordered set $P=(X,\prec)$ is defined as follows. Each element $v\in X$ corresponds to an interval $I(v)=$

$[\ell e(v), re(v)]$ on the real line and two tolerant points $\ell t(v), rt(v) \in I(v)$ so that $x \prec y$ in P iff $re(x) < \ell t(y)$ and $rt(x) < \ell e(y)$. Such an ordered set is called a *bounded bitolerance order.* If in addition, $\ell t(v) \leq rt(v)$ for all $v \in X$, then the representation is called *totally bounded* and P is a *totally bounded bitolerance order.* In this case we write $C(v)$ to denote the central region $[\ell t(v), rt(v)]$ of v's interval.

The family of bounded bitolerance orders is one of the few classes of tolerance orders that has been completely characterized. An $O(|X|^3)$ recognition algorithm appears in [10].

In contrast, the class of totally bounded bitolerance orders has been characterized only in special cases. The most general of these is when the underlying ordered set is bipartite [4], that is, its longest chain has at most two elements. Clearly, a bipartite order is 1-weak – for example, assign all minimal elements a level of 1 and all remaining elements a level of 2.

In this section we generalize the characterization of totally bounded bitolerance orders to the case where the underlying order is 1-weak. Indeed, this was our original motivation for studying 1-weak and subsequently k-weak orders.

Lemma 18 *Let $P = (X, \prec)$ be a totally bounded bitolerance order with representation $(I(v), C(v) : v \in X)$. If $C(x) \cap C(y) \neq \emptyset$ then $x \sim y$ in P.*

Proof. The proof follows immediately from the definition of totally bounded bitolerance orders.

Theorem 19 *If P is a 1-weak ordered set, the following are equivalent.*

1. *P is a totally bounded bitolerance order.*
2. *P is a bounded bitolerance order.*
3. *For each pair of adjacent levels of P, there is an indexing $x_1, x_2, \ldots, x_n$ of the elements on the lower level and an indexing $y_1, y_2, \ldots, y_m$ of the elements on the upper level so that whenever $x_i \sim y_j$ in P then either $x_i \sim y_k$ for all $k < j$ or $x_\ell \sim y_j$ for all $\ell < i$.*

Proof. **(1) $\Longrightarrow$ (2)** This follows immediately from the fact that totally bounded bitolerance orders are bounded bitolerance orders.

(2) $\Longrightarrow$ (3) Suppose P is a bounded bitolerance order. Then the suborder of P induced by two adjacent levels is also a bounded bitolerance order. Now by Theorem 2.12 of [4], the desired indexing exists.

(3) $\Longrightarrow$ (1) Now suppose that P is a 1-weak ordered set with 1-leveling function lev so that each pair of adjacent levels can be indexed as in the hypothesis of (3). Note that this means each level (except the top and bottom) gets indexed twice. Given these two sets of indices, we construct a totally bounded bitolerance representation of P.

Choose M large enough so that each level of P contains fewer than $M/2$ elements. Consider the elements on adjacent levels i and $i+1$. Index the elements x with $lev(x) = i$: $x_1, x_2, \ldots, x_n$ and those y with $lev(y) = i + 1$: $y_1, y_2, \ldots, y_m$ as given by the hypothesis of (3). These indexings will determine the right tolerant points and endpoints of the elements on level i and the left tolerant points and endpoints of the elements on level $i+1$ as follows.

$*$ Let $rt(x_j) = iM - j$, and

$*$ let

$$re(x_j) = \begin{cases} iM & \text{if } x_j \prec y_1, \\ iM + k & \text{if } x_j \prec y_{k+1} \text{ and } x_j \sim y_\ell \text{ for all } \ell : 1 \leq \ell \leq k. \end{cases}$$

$**$ Let $\ell t(y_j) = iM + j$, and

$**$ let

$$\ell e(y_j) = \begin{cases} iM & \text{if } x_1 \prec y_j, \\ iM - k & \text{if } x_{k+1} \prec y_j \text{ and } x_\ell \sim y_j \text{ for all } \ell : 1 \leq \ell \leq k. \end{cases}$$

Thus for an element z on level i, $re(z)$ and $rt(z)$ are determined by the items marked with a $(*)$ using z's indexing as an element of the lower level of adjacent levels i and $i+1$ and $\ell e(z)$ and $\ell t(z)$ are assigned by the items marked with a $(**)$ using z's indexing as an element of the upper level of adjacent levels $i-1$ and i. It remains to assign left tolerant and endpoints to the elements on level 1 and right tolerant and endpoints to the elements on level T, the top level. We do this as follows.

For each element x on level 1, let $\ell e(x) = \ell t(x) = 0$ and for elements y on the top level T, let $re(y) = rt(y) = (T+1)M$.

Now we need only show that (1) our intervals and tolerances give a valid totally bounded bitolerance representation, and (2) the order represented is indeed P.

To show (1), let z be an element on level p. Then using $(*)$ with $i = p$ we have $re(z) \geq rt(z) \geq pM - M/2$. And using $(**)$ with $p = i+1$ we have $\ell e(z) \leq \ell t(z) \leq (p-1)M + M/2 = pM - M/2$. Thus the representation is totally bounded.

Now to show (2), let Q be the totally bounded bitolerance order given by our representation. We will show $P = Q$ by considering the possible comparabilities/incomparabilities between pairs of elements x and y. Without loss of generality, assume $lev(x) \leq lev(y)$.

- **Case 1:** $lev(x) \leq lev(y) - 2$.

  Since lev is a 1-leveling, by the second part of Proposition 3 we know that $x \prec y$ in P. Thus we must show that $x \prec y$ in Q. Let $p = lev(x)$, thus $lev(y) \geq p+2$. Then by $(*)$ and $(**)$ $re(x) \leq pM + M/2 < (p+1)M \leq \ell t(y)$, and $rt(x) \leq pM < (p+1)M - M/2 \leq \ell e(y)$. Thus $x \prec y$ in Q as desired.

- **Case 2:** $lev(x) = lev(y)$.

  Let $p = lev(x) = lev(y)$. Since lev is a 1-leveling, we know $x \sim y$ in P by the first part of Proposition 3. Thus we must show $x \sim y$ in Q. Note that $pM - M/2 \in [\ell t(z), rt(z)]$ for all elements z on level p. Thus $C(x) \cap C(y) \neq \emptyset$, and Lemma 18 implies $x \sim y$ in Q.

- **Case 3:** $lev(x) = lev(y) - 1$.

  Let $lev(x) = i$ and $lev(y) = i+1$. Index the elements of levels i and $i+1$ as given in the hypothesis. Let $x_1, x_2, \ldots, x_n$ be the indexing of level i and $y_1, y_2, \ldots, y_m$ be the indexing of level $i+1$. Thus $x = x_r$ and $y = y_s$ for some $r : 1 \leq r \leq n$ and $s : 1 \leq s \leq m$.

  - $x \prec y$ in P (i.e., $x_r \prec y_s$ in P)

    If $x_r \prec y_1$ then $re(x) = iM < \ell t(y)$. Otherwise, $re(x) = re(x_r) = iM + k$ where $x_r \prec y_{k+1}$ and $x_r \sim y_\ell$ for all $\ell : 1 \leq \ell \leq k$. Thus $k < s$ and therefore, $re(x_r) < iM + s$. But $\ell t(y) = \ell t(y_s) = iM + s$, thus again $re(x) < \ell t(y)$.

Similarly, if $x_1 \prec y_s$ then $\ell e(y) = iM > rt(x)$. Otherwise, $\ell e(y) = \ell e(y_s) = iM - k$ where $x_{k+1} \prec y_s$ and $y_s \sim x_\ell$ for all $\ell : 1 \leq \ell \leq k$. But $x_r \prec y_s$, thus $k < r$ and so $\ell e(y) > iM - r$. But $rt(x) = rt(x_r) = iM - r$. Thus again, $le(y) > rt(x)$.

Together these results imply that $x \prec y$ in Q.

– $x \sim y$ in P (i.e., $x_r \sim y_s$ in P)

First note that $re(y) \geq \ell t(y) > iM > rt(x) \geq \ell t(x)$, so $y \not\prec x$ in Q. Thus it suffices to show $x \not\prec y$ in Q.

By the indexing condition (3), either (i) $x_r \sim y_\ell$ for all $\ell : 1 \leq \ell \leq s$, or (ii) $x_\ell \sim y_s$ for all $\ell : 1 \leq \ell \leq r$. In case (i), $re(x) = re(x_r) \geq iM + s$ and $\ell t(y) = \ell t(y_s) = iM + s$, so $x \not\prec y$ in Q. In case (ii), $rt(x) = rt(x_r) = iM - r$ and $\ell e(y) = \ell e(y_s) \leq iM - r$, so again $x \not\prec y$ in Q. $\Box$

Together with Theorem 19, Langley's recognition algorithm [10] for bounded bitolerance orders immediately implies a recognition algorithm for totally bounded bitolerance orders in the 1-weak case. The equivalence of (1) and (2) in Theorem 19 is false in general. A separating example (which is 2-weak) appears in [4].

5 Future Directions

In future work [in preparation] we study the *weakness* of an ordered set, where the weakness $wk(P)$ is the minimum k such that P is k-weak. We prove $wk(P) \leq \lceil \frac{n-2}{2} \rceil$ for an n element ordered set P, which results in an $O(n^6)$ algorithm for computing weakness.

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