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Let c be the number of components. Each vertex in the directed graph encodes 1 or 2 vertices of T, and so $v \leq |T|$. Each component encodes at least one vertex of T, and so $c \leq |T|$. We have that $v + c \leq 2|T|$.

The vertex colored directed graph encoded by w embeds as an induced subgraph in the path on $v + c - 1$ vertices each of which is colored D. For example, $LDR \quad DR$ embeds in $DDDDDD$, though this is **not** a word embedding since the directed graph corresponds to a two letter word whereas the latter path is a single letter. This path $DDDDDD$ on $5 + 2 - 1$ vertices encodes the tournament $\widehat{S}'_{2\cdot 6} < \widehat{S}_{2\cdot 6+1}$. The tournament encoded by $LDR \quad DR$ therefore embeds in $\widehat{S}_{2\cdot 6+1}$. In general, the D colored path on $p = v + c - 1 < 2|T|$ vertices encodes the tournament $\widehat{S}'_{2p}$. Therefore $T \leq \widehat{S}'_{2p} < \widehat{S}_{4|T|+1}$. By Lemma 3.14, $T < \widehat{S}_{2n+1}$ for all $n \geq 2|T|$. ■

Theorem 4.3 *There is an algorithm which, given a 2-set of finite tournaments Γ, determines whether Γ is tight or loose.*

Proof:

This follows from the two previous Lemmas and Corollary 2.6. ■

We'd like to thank Stephen J. Greenfield for some useful comments on the exposition of this paper.

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2. $T < \widehat{N}_n$ *for some* n.

3. $T < \widehat{N}_{2|T|}$.

Proof:

$1 \Longrightarrow 3$
Clear.

$3 \Longrightarrow 2$
Clear.

$2 \Longrightarrow 1$
Suppose $T < \widehat{N}_{|T|+j}$ for some $j \geq 1$. Choose j minimal with this property. T is of the form $R_1 \ldots R_l$, where R_1 and R_l are strong tournaments with a single 3-arc, R_i is isomorphic to $N_{|R_i|}$, for $2 \leq i \leq l-1$, and we allow the case where R_1, $R_2 \ldots R_{l-1}$, or R_l may be empty. From the minimality of j we have that $j = l - 1$. For example, $N_7^l N_3 N_5 N_9^r < \widehat{N}_{24+3}$. If T is transitive then $\rho(i) = 2i$ is an embedding map of T into $\widehat{N}_{2|T|}$. Otherwise, we have that $l \leq |T| - 1$, and so $j \leq |T| - 2$. There is a vertex i of $\widehat{N}_{|T|+j}$ such that $T \leq \widehat{N}_{|T|+j}(i)$, where $\widehat{N}_{|T|+j}(i)$ is the induced subtournament on $V(\widehat{N}_{|T|+j}) - \{i\}$. By Lemma 3.4, $T < \widehat{N}_n$ for all $n \geq 2|T| \geq |T| + j$. ■

We now show we have an effective decision procedure for the embedding problem with respect to the other minimal antichain in Λ_1, namely $\mathcal{I}^O = \{\widehat{S}_{2n+1}\}_{4 \leq n}$.

Lemma 4.2 *Given a tournament* T *the following are equivalent:*

1. $T < \widehat{S}_{2n+1}$ *for all* $n \geq 2|T|$.

2. $T < \widehat{S}_{2n+1}$ *for some* n.

3. $T < \widehat{S}_{4|T|+1}$.

Proof:

$1 \Longrightarrow 3$
Clear.

$3 \Longrightarrow 2$
Clear.

$2 \Longrightarrow 1$
Suppose $T < \widehat{S}_{2n+1}$ for some n. Since $T \not\leq \widehat{S}_{2n+1}$ we have that $T \leq \widehat{S}'_{2n}$ and T can be encoded by a word w in $\Pi^\star$. The word w encodes a vertex colored directed graph whose components are directed paths. Let V be the vertex set of the directed graph. Let $v = |V|$.

We have that

$$V(\tilde{S}) = \{1, 2, 5, 7, 8, 10, 11, 13, 14, 15, 17, 18\}.$$

The arc relation on $\tilde{S}$ is given by the same rule as for $\widehat{S}_{2n+1} = \widehat{S}_{19}$.

The isomorphism g between $\tilde{S}$ and S is given by:

$$g(i) = \begin{cases} i, & \text{if } 1 \leq i \leq 5 \\ i+1, & \text{if } 7 \leq i \leq 15 \\ i+2, & \text{if } 17 \leq i \leq 18. \end{cases}$$

We have shown that there is a map from $\Pi^\star$ to the class of finite tournaments and $\mathcal{S}_{\mathcal{I}^O}$ is in the image. To complete the proof that $\mathcal{S}_{\mathcal{I}^O}$ is wqo, it remains to show that this map preserves embeddability. We check that if $S^1 \leq \widehat{S}'_{2m}$, $S^2 \leq \widehat{S}'_{2n}$ correspond to $w_1, w_2 \in \Pi^\star$ and $w_1 \leq w_2$, then $S^1 \leq S^2$. The relation $w_1 \leq w_2$ means that each component $P_i{}^1$ of G_{S^1} can be embedded into a component $P_{i'}{}^2$ of G_{S^2} in an order-preserving manner, with L or R-colored vertices corresponding to one element of $V(S^1)$ possibly mapping to D-colored vertices corresponding to two elements of $V(S^2)$. If we require that a vertex x of S^1 map to a vertex in the range $\{1, \ldots, n\}$ in S^2 if and only if x is in the range $\{1, \ldots, m\}$ in S^1, this induces a map of the vertices of S^1 into those of S^2 and this map preserves arcs.

For example, $D \cdot D \cdot R \leq DD \cdot R \cdot DR \cdot LDR$. Let $S^1 \leq \widehat{S}'_{2m}$, $S^2 \leq \widehat{S}'_{2n}$ be the tournaments encoded by $D \cdot D \cdot R$ and $DD \cdot R \cdot DR \cdot LDR$, respectively. We have already seen that $n = 9$ and $V(\tilde{S}^2) = \{1, 2, 5, 7, 8, 10, 11, 13, 14, 15, 17, 18\}$. Similarly, we have that $m = 5$ and $V(\tilde{S}^1) = \{1, 3, 6, 8, 10\}$. One of several possible embedding maps of S^1 into S^2 is given by

$$1 \mapsto 1; \quad 3 \mapsto 5; \quad 6 \mapsto 10; \quad 8 \mapsto 14; \quad 10 \mapsto 18.$$

■

Corollary 3.17 *$\mathcal{I}^O$ is a minimal infinite antichain.*

Proof:

Immediate from Lemma 2.4. ■

4 The case $k = 2$

In order to apply Corollary 2.6 to show an algorithm exists for the case of two constraints, we must continue the analysis of Λ_1. We must show that we can decide effectively whether a given finite tournament embeds as a proper subtournament in a tournament in an antichain of Λ_1. We show this first for $\mathcal{I}^L = \{\widehat{N}_n\}_{7 \leq n}$.

Lemma 4.1 *Given a tournament T, $|T| \geq 4$, the following are equivalent:*

1. *$T < \widehat{N}_n$ for all $n \geq 2|T|$.*

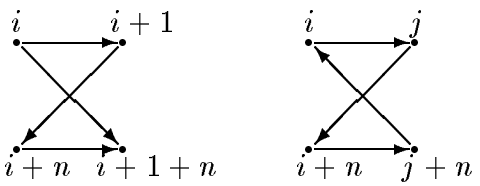

Figure 3: The arcs between V_i and V_j

An arc joins V_i to V_j in G_S if $j = i + 1$ and both i and $i + 1 + n$ are present in $V(S)$. We associate to $S \leq \widehat{S}'_{2n}$ the word $w = P_1 \dots P_r \in \Pi^\star$, where $P_1, \dots, P_r$ are the components of G_S, in their natural order.

For example, suppose $n = 10$ and $S \leq \widehat{S}'_{20}$ has vertex set

$$\{1, 2, 5, 8, 9, 11, 12, 14, 15, 16, 19, 20\}.$$

The sets $V_i \cap V(S)$ and the arcs of the directed graph G_S are as follows:

$$(1, 11) \to (2, 12) \quad (14) \quad (5, 15) \to (16) \quad (8) \to (9, 19) \to (20).$$

The connected components of G_S are the vertex colored directed paths

$$DD \quad R \quad DR \quad LDR.$$

Observe that in each component of G_S, only the first vertex can be colored L and only the last can be colored R. G_S also corresponds to a word w_S in $\Pi^\star$. In our example, the letters of w_S are DD, R, DR, and LDR.

We now verify that the isomorphism type $\tilde{S}$ of S can be recovered from w_S, or equivalently, from G_S together with the induced ordering on its components. Let v be the number of vertices in G_S. Let c be the number of components in G_S whose terminal vertex is colored L or D, except we do not count the last component regardless of the color of its terminal vertex. Let $n = c + v$. $\tilde{S}$ will be a proper subtournament of $\widehat{S}_{2n+1}$. We use the rules given above to determine the vertices, namely:

- If an arc joins V_i to V_{i+1} in G_S then both i and $i + 1 + n$ are in $V(\tilde{S})$.
- If V_i is colored L, R, or D then i, $i + n$, or both are in $V(\tilde{S})$, respectively.

The arc relation on $\tilde{S}$ is given by the same rule as for $\widehat{S}_{2n+1}$.

Continuing with our example, we will reconstruct the isomorphism type $\tilde{S}$ of S from the word $DD \cdot R \cdot DR \cdot LDR$. The directed graph associated with this word has eight vertices and one component with terminal vertex color D. We take n to be 9. The first vertex in the directed graph is colored D, so without loss of generality, we may assume $V_1 = (1, 10)$. Since $V_1 \to V_2$ and V_2 is colored D, we have $V_2 = (2, 11)$. $V_2 \not\to V_i$, for all i, so the next vertex does not contain $3 + n$. Since the next vertex is colored R, we can take it to be $V_4 = (4 + n) = (13)$. Continuing in this manner we obtain

$$(1, 10) \to (2, 11) \quad (13) \quad (5, 14) \to (15) \quad (7) \to (8, 17) \to (18).$$

Corollary 3.15 $\mathcal{S}_{\mathcal{I}^O} = \mathcal{A}_{\mathcal{I}^O}$.

Proof: $\mathcal{A}_{\mathcal{I}^O} \subseteq \mathcal{S}_{\mathcal{I}^O}$ by the definitions of these classes of tournaments. By the previous lemma, any proper subtournament of $\widehat{S}_{2n+1}$ embeds in $\widehat{S}_{2m+1}$ for all $m \geq n$, and so $\mathcal{S}_{\mathcal{I}^O} \subseteq \mathcal{A}_{\mathcal{I}^O}$. ■

We will now present an encoding of $\mathcal{S}_{\mathcal{I}^O}$ by finite words in a wqo alphabet, from which the wqo property of $\mathcal{S}_{\mathcal{I}^O}$ is derived using Fact 1 (Higman). The minimality of $\mathcal{I}^O$ then follows from Lemma 2.4. First we will discuss the alphabet and its ordering relation by giving several examples.

Let Π be the set of $\{L, R, D\}$-colored directed paths P in which only the initial vertex of P may be colored L, and only the terminal vertex may be colored R. Such a path can be represented by its sequence of colors. For example, each of $LDDR$, $DDDR$, $LDDD$, and $DDDD$ is a path of length four of the type under consideration. Note that each of these paths is a **single** letter in our alphabet Π.

We write $P_1 \leq P_2$ for $P_1, P_2 \in \Pi$ if there is an embedding of $V(P_1)$ into $V(P_2)$ which respects arcs, and which respects colors apart from the possibility that vertices of any color may map to vertices of color D. For example, $LDDR$ embeds in $DDDR$ but $DDDD$ does not. Π is wqo under this relation. Note, however, that if we do not allow L or R to map to D then $\{LD^kR : k \geq 0\}$ would encode an antichain, since the successor relation defined by the arc relation of a directed path must be preserved.

Lemma 3.16 *$\mathcal{S}_{\mathcal{I}^O}$ is wqo.*

Proof:

It is enough to show that there is a map from $\Pi^\star$ to the class of finite tournaments such that $\mathcal{S}_{\mathcal{I}^O}$ is in its image and the map preserves embeddability. It is more natural, however, to exhibit the inverse map. We begin with a tournament S in $\mathcal{S}_{\mathcal{I}^O}$ and construct a word in $\Pi^\star$ which encodes S, or more specifically, the isomorphism type of S.

The vertex set of $\widehat{S}'_{2n}$ is $\{1, \ldots, 2n\} \subseteq \mathbb{Z}/(2n+1)\mathbb{Z}$; we now think of this set as a subset of $\mathbb{N}$, equipped with the natural order. For $1 \leq i \leq n$ let $V_i = \{i, i+n\} \subseteq V(\widehat{S}'_{2n})$. Observe that the arcs in $\widehat{S}'_{2n}$ are determined by the following rules:

- Within V_i: $i + n \to i$
- On $V_i \times V_j$, $j > i + 1$: $i \to j \to i + n \to j + n \to i$
- On $V_i \times V_j$, $j = i + 1$: As in the previous case, except $i \to j + n$.

The arcs between V_i and V_j are illustrated in Figure 3 for two cases. On the left we see the arcs when $j = i + 1$ and on the right we see the arcs when $j > i + 1$. The reversal of the arc $(j + n, i)$ when $j = i + 1$ defines a successor relation on the set of vertex pairs.

For $S \leq \widehat{S}'_{2n}$, we associate a vertex-colored directed graph G_S to S as follows. The vertices of G_S are the sets V_i for $1 \leq i \leq n$ such that $V_i \cap V(S) \neq \emptyset$, and the color of V_i is L, R or D ("left", "right", or "double") according to whether $V_i \cap V(S)$ is $\{i\}$, $\{i + n\}$, or V_i.

We will show that if $m < n$ then $\widehat{S}_{2m+1} \not\leq \widehat{S}_{2n+1}$, thus establishing that $\mathcal{I}^O$ is an antichain. The crucial idea is that $\widehat{S}_{2n+1}$ has a cyclic successor relation.

We observe that the automorphism group of $\widehat{S}_{2n+1}$ is transitive. The automorphism $\rho(x) = x - i + j$ maps the vertex i to the vertex j for any $i, j \in \mathbb{Z}/(2n+1)\mathbb{Z}$.

Definition 3.12 *Given a tournament T and a vertex a in T, let $'a$ and a' be the induced subtournaments on the vertex sets $\{b \in T : b \to a\}$ and $\{b \in T : a \to b\}$ respectively.*

Let $C_3(x, y, A)$ denote the tournament obtained by replacing the vertex z of the 3-cycle on $\{x, y, z\}$ with the tournament A so that $x \to y \Longrightarrow V(A) \to x$. We note that in $\widehat{S}_{2n+1}$, $0' = C_3(n+1, 1, \{2, \ldots, n-1\})$, where the induced subtournament on $\{2, \ldots, n-1\}$ is isomorphic to L_{n-2}.

Lemma 3.13 [17] *$\mathcal{I}^O$ is an antichain.*

Proof:

Suppose for some $m \leq n$ we have $\widehat{S}_{2m+1}$ embedded in $\widehat{S}_{2n+1}$. We study the mapping of the vertices of $\widehat{S}_{2m+1}$ to the vertices of $\widehat{S}_{2n+1}$. Let $x \in V(\widehat{S}_{2m+1})$.

We know that $x' = C_3(x+m+1, x+1, \{x+2, \ldots, x+m-1\})$. The arc $x+(m+1) \to x+1$ is the unique $(m-2)$-arc in x'. Note that this requires $m \geq 4$. If x maps to the vertex y in $V(\widehat{S}_n)$ then the $(m-2)$-arc in x' maps to the $(n-2)$-arc in y', so necessarily $(x+1) \mapsto (y+1)$. As the embedding respects this successor relation, $m = n$. ■

We now deal with minimality. We define $\widehat{S}'_{2n}$ to be the induced subtournament on $V(\widehat{S}_{2n+1}) - \{0\}$. Since the automorphism group of $\widehat{S}_{2n+1}$ is transitive, $\widehat{S}'_{2n}$ is isomorphic to the induced subtournament on $V(\widehat{S}_{2n+1}) - \{i\}$, for any $i \in \{0, \ldots, 2n\}$.

Lemma 3.14 *If $m \leq n$ then $\widehat{S}'_{2m} < \widehat{S}_{2n+1}$.*

Proof:

We will exhibit an embedding. Define a mapping $f : V(\widehat{S}'_{2m}) \to V(\widehat{S}_{2n+1})$ as follows:

$$f(j) = \begin{cases} j, & \text{if } j \leq m \\ n - m + j, & \text{if } j > m. \end{cases}$$

It is straightforward to check that f is an embedding. ■

It may seem that the embedding function f in the previous lemma would give an embedding of $\widehat{S}_{2m+1}$ into $\widehat{S}_{2n+1}$, for $m < n$. We illustrate with an example that this is not so. Suppose $f : V(\widehat{S}_9) \to V(\widehat{S}_{11})$. The problem, of course, is with the vertex 0. In $\widehat{S}_9$ we have $4 \to 0$ but $f(0) \to f(4)$ in $\widehat{S}_{11}$.

Since any proper subtournament of $\widehat{S}_{2m+1}$ embeds in $\widehat{S}'_{2m}$, it also embeds in $\widehat{S}_{2n+1}$ for all $n \geq m$. In the remainder of this section we will be studying $\mathcal{S}_{\mathcal{I}^O}$, the class of tournaments which properly embed in some element of $\mathcal{I}^O$. Recall that $\mathcal{A}_{\mathcal{I}^O}$ is the class of tournaments which embed in all but finitely many elements of $\mathcal{I}^O$.

For example, if we remove vertices 7 and 15 from $\widehat{N}_{24}$, the resulting induced subtournament is $N_6^l N_7 N_9^r$. In general, the subtournament induced by removing vertices from the tournament $\widehat{N}_n$ is of the form $R_1 R_2 \dots R_l$, where R_i is strong, and $R_i \in \{N_n\}_{n\geq 1} \cup \{N_n^l\}_{n\geq 4} \cup \{N_n^r\}_{n\geq 4}$, for $1 \leq i \leq l$. If we encode T by the word $R_1 \dots R_l$, then an embedding of words induces an embedding of the corresponding tournaments.

Lemma 3.9 *$\mathcal{S}_{\mathcal{I}^L}$ is wqo.*

Proof:

Let $\Sigma = \{N_n\}_{n\geq 1} \cup \{N_n^l\}_{n\geq 4} \cup \{N_n^r\}_{n\geq 4}$. Associate to any word $R_1 \dots R_l$ in the alphabet Σ, the corresponding tournament $T = R_1 \dots R_l$. Every proper subtournament of any $\widehat{N}_n$ corresponds to such a word, and an embedding of words induces an embedding of the corresponding tournaments. Thus Fact 1 of §1 applies. ■

Corollary 3.10 *$\mathcal{I}^L$ is a minimal antichain.*

Proof:

Immediate from Lemma 2.4. ■

3.2 An Antichain of Modified Local Orders

We introduce the second of our two antichains, $\mathcal{I}^O$, which is made up of slightly modified local orders. We will sketch the proof that it is an antichain and then prove that it is minimal.

Definition 3.11 *For $n \geq 1$, the* local order *S_{2n+1} is the tournament realized by $\mathbb{Z}/(2n+1)\mathbb{Z}$, with the arc relation*

$$a \to b \text{ if and only if } b - a \in \{1, 2, \dots, n\}.$$

For $n \geq 4$, $\widehat{S}_{2n+1}$ is the tournament obtained from S_{2n+1} by reversing the arcs $(i, i+n)$ and $(i+n+1, i)$, for $0 \leq i \leq n-1$. We define $\mathcal{I}^O = \{\widehat{S}_{2n+1}\}_{4\leq n}$.

To understand the structure of the local order on $2n+1$ vertices, imagine that the vertices are equally spaced around a circle and that from vertex 0 the other vertices are numbered clockwise in increasing order. For any vertex i, there will be n vertices to the left of i, each dominated by i. There will be n vertices to the right of i, each dominating i. In particular, 0 dominates the vertices $1, \dots, n$ and is dominated by the vertices $n+1, \dots, 2n$. In $\widehat{S}_{2n+1}$ the arcs $(0, n)$ and $(n+1, 0)$ are reversed. Similarly, for each vertex $i \leq n$, the arcs $(i, n+i)$ and $(n+1+i, i)$ are reversed.

$S_{2m+1} \leq S_{2n+1}$ whenever $m \leq n$ under the embedding:

$$f(j) = \begin{cases} j, & \text{if } j \leq m \\ n - m + j, & \text{if } j > m. \end{cases}$$

It is straightforward to check in each case that f is an embedding. ■

Since any proper subtournament of $\widehat{N}_n$ embeds in some $\widehat{N}_n(i)$, it also embeds in $\widehat{N}_m$ for all $m \geq n$. In the remainder of this section we will be studying $\mathcal{S}_{\mathcal{I}^L}$, the class of tournaments which properly embed in some element of $\mathcal{I}^L$. Recall that $\mathcal{A}_{\mathcal{I}^L}$ is the class of tournaments which embed in all but finitely many elements of $\mathcal{I}^L$. To establish that $\mathcal{I}_L$ is minimal we must show that $\mathcal{S}_{\mathcal{I}^L} = \mathcal{A}_{\mathcal{I}^L}$ and is wqo.

Corollary 3.5 $\mathcal{S}_{\mathcal{I}^L} = \mathcal{A}_{\mathcal{I}^L}$.

Proof:
$\mathcal{A}_{\mathcal{I}^L} \subseteq \mathcal{S}_{\mathcal{I}^L}$ by the definition of each of these classes. By the previous lemma, any proper subtournament of $\widehat{N}_n$ embeds in $\widehat{N}_m$ for all $m \geq n$, and so $\mathcal{S}_{\mathcal{I}^L} \subseteq \mathcal{A}_{\mathcal{I}^L}$. ■

To show $\mathcal{S}_{\mathcal{I}^L}$ is wqo we establish an encoding of $\mathcal{S}_{\mathcal{I}^L}$ with words in a wqo alphabet. The letters in a word encoding a tournament will be the strongly connected components of the tournament. The strongly connected components will be elements of the union of three wqo classes of tournaments. Fact 1 (Higman) then implies that the class of tournaments $\mathcal{S}_{\mathcal{I}^L}$ is wqo. The minimality of $\mathcal{I}^L$ then follows from Lemma 2.4.

Definition 3.6 *For $n \geq 4$, let N_n^l be the tournament obtained from N_n by reversing the arc $(1,3)$ and let N_n^r be the tournament obtained from N_n by reversing the arc $(n-2,n)$.*

The tournament N_n^l can be visualized as a path of 2-arcs anchored on the "left" by a 3-arc. Similarly, the tournament N_n^r can be visualized as a path of 2-arcs anchored on the "right" by a 3-arc. The tournament N_n can be visualized as a path of 2-arcs with neither end anchored.

Lemma 3.7 *Each of the sets $\{N_n\}_{n\geq 1}$, $\{N_n^l\}_{n\geq 4}$, and $\{N_n^r\}_{n\geq 4}$ is wqo with respect to the tournament embeddability relation.*

Proof:
In each case, given two distinct tournaments in the class, the one with the smaller vertex set embeds in the one with the larger vertex set by mapping the 3-arc to the 3-arc, if there is such an arc, and then mapping successive vertices to successive vertices, where the successor relation is defined by the path of 2-arcs. ■

Definition 3.8 *By T_1T_2 we mean the tournament with vertex set $V(T_1) \dot{\cup} V(T_2)$ which has subtournaments T_1, T_2 with $T_1 \Longrightarrow T_2$. A tournament is* strong *if it is not of the form T_1T_2 with $V(T_1) \neq \emptyset$ and $V(T_2) \neq \emptyset$. Equivalently, a tournament is strong if it is strongly connected.*

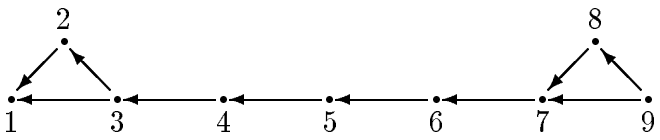

Figure 2: Some arcs of $\widehat{N}_9$

Definition 3.1 *L_n is a transitive tournament on the vertex set $\{1,\ldots,n\}$ with the arc relation given by $i \to j$ if $i < j$. N_n is the tournament obtained from L_n by reversing the arcs $(i, i+1)$, for $1 \le i \le n-1$. For $n \ge 7$, $\widehat{N}_n$ is the tournament obtained from N_n by reversing the arcs $(1,3)$ and $(n-2,n)$. We define $\mathcal{I}^L = \{\widehat{N}_n\}_{7 \le n}$.*

A representation of $\widehat{N}_9$ is given in Figure 2 with only the right-to-left arcs shown.

Definition 3.2 *An arc (x,y) in a tournament T is said to be a* p-arc *if (x,y) occurs in at least p distinct copies of C_3.*

Note that in the tournament $\widehat{N}_9$ the arcs $(4,5)$ and $(5,6)$ are 2-arcs and $(3,4)$ and $(6,7)$ are 3-arcs.

Lemma 3.3 [17] *$\mathcal{I}^L$ is an antichain.*

Proof:

The 2-arcs in $\widehat{N}_n$ define a path on the vertices $\{3,\ldots,n-2\}$ while the 3-arcs anchor each end of this path. We can think of this path as defining a successor relation.

Suppose $\widehat{N}_m \le \widehat{N}_n$ for $m \le n$. The pattern of 3-arcs and 2-arcs in $\widehat{N}_m$ and $\widehat{N}_n$ must be preserved by the embedding, and we conclude that $m = n$. ■

We now deal with minimality. First we show that any proper subtournament of $\widehat{N}_n$ embeds in $\widehat{N}_m$ for all $m \ge n$.

Lemma 3.4 *For $i \in \{1,\ldots,n\}$, let $\widehat{N}_n(i)$ be the induced subtournament of $\widehat{N}_n$ on $V(\widehat{N}_n) - \{i\}$. If $n \le m$ then $\widehat{N}_n(i) < \widehat{N}_m$.*

Proof:

We will exhibit an embedding which depends on the value of i. Define a mapping $f : V(\widehat{N}_n(i)) \to V(\widehat{N}_m)$ as follows:

$$\text{If } i \ne 2, n-1 \quad f(j) = \begin{cases} j, & \text{if } j < i \\ m-n+j, & \text{if } j > i. \end{cases}$$

$$\text{If } i = 2 \qquad f(j) = \begin{cases} m-n+2, & \text{if } j = 1 \\ m-n+j, & \text{if } j > 2. \end{cases}$$

$$\text{If } i = n-1 \qquad f(j) = \begin{cases} j, & \text{if } j < n-1 \\ n-1 \quad , & \text{if } j = n. \end{cases}$$

Given a loose $(k+1)$-set $\Gamma = \{T_1, \ldots, T_{k+1}\}$, let $\Gamma_i = \Gamma - \{T_i\}$. Then Γ_i is loose for every $1 \leq i \leq k+1$.

By induction, for each such k-subset Γ_i, $1 \leq i \leq k+1$, there is a witness $\mathcal{J}_{j_i}$ in Λ_k. If T_i embeds in only finitely many elements of $\mathcal{J}_{j_i}$ for some i then $\mathcal{J}_{j_i}$ is a witness for Γ. Suppose this is not the case. We have that $\Gamma \in \mathcal{A}_{j_1} \times \ldots \times \mathcal{A}_{j_{k+1}}$. Let $I = (j_1, \ldots, j_{k+1})$.

Since Γ is loose, there is a $\mathcal{M} \in \mathcal{L}_I$ such that $\mathcal{M} \leq \Gamma$, therefore $\mathcal{J}_{\mathcal{M}} \in \Lambda_{k+1}$ is a witness for Γ. ■

Corollary 2.6 *If Λ_k can be chosen so that for each antichain $\mathcal{I}$ in Λ_k there is an algorithm for determining whether a tournament T embeds in some element of $\mathcal{I}$, then there is an algorithm for determining whether a $(k+1)$-set Γ of tournaments is loose.*

Proof:

Each element $\mathcal{I}$ of Λ_k is minimal so if a tournament T properly embeds in some tournament in $\mathcal{I}$, T embeds in all but finitely many tournaments in $\mathcal{I}$ and $\mathcal{I}$ is not a witness for T. Otherwise, T is isomorphic to one or embeds in no element of $\mathcal{I}$ and $\mathcal{I}$ is a witness. In other words, we can effectively determine if $\mathcal{I}$ is a witness for T.

Given a $(k+1)$-set $\Gamma = \{T_1, \ldots, T_{k+1}\}$, we can effectively check whether there is an element $\mathcal{I}$ of Λ_k that is a witness for T_i for all $1 \leq i \leq k+1$. If there is such an $\mathcal{I}$, Γ is loose.

On the other hand, if there is no such element $\mathcal{I}$ of $\Lambda_k = \{\mathcal{J}_1, \ldots, \mathcal{J}_n\}$, then, using the notation of the Finiteness Theorem, we compare Γ with each of the finitely many minimal loose $(k+1)$-sequences in $\mathcal{A}_{j_1} \times \ldots \times \mathcal{A}_{j_{k+1}}$ for each sequence of indices $(j_1, \ldots, j_{k+1}) \in [n]^{k+1}$. If we find a minimal loose sequence $\mathcal{M}$ with $\mathcal{M} \leq \Gamma$ then Γ is loose. Otherwise, Γ is tight.■

We will show in the next two sections that Λ_1 can be chosen as required by Corollary 2.6. We then deduce the existence of the desired algorithm for $k = 2$.

3 Λ_1

In this section we will describe two infinite antichains $\mathcal{I}^L$ and $\mathcal{I}^O$ such that for any loose tournament T either $\mathcal{I}^L$ or $\mathcal{I}^O$ is a witness for T. These antichains were introduced in [17] and proofs that these are the only two witnesses required for the case $k = 1$ are completed in [18, 19]. The issue of minimality did not arise in that analysis. We will show here that these antichains are in fact minimal. Thus we may take $\Lambda_1 = \{\mathcal{I}^L, \mathcal{I}^O\}$ in the Finiteness Theorem.

3.1 An Antichain of Modified Linear Orders

We introduce the first of our two antichains, $\mathcal{I}^L$, which is made up of slightly modified transitive tournaments. We will sketch the proof that it is an antichain and then prove that it is minimal.

1. $\mathcal{I}$ is minimal.

2. $\mathcal{S}_\mathcal{I} = \mathcal{A}_\mathcal{I}$ and is wqo.

Proof:

$1 \Longrightarrow 2$

$\mathcal{A}_\mathcal{I} \subseteq \mathcal{S}_\mathcal{I}$ by definition. Suppose $T \in \mathcal{S}_\mathcal{I} - \mathcal{A}_\mathcal{I}$ and $\mathcal{I}'$ is an infinite subset of $\mathcal{I}$ which omits T in the sense that T does not embed in any element of $\mathcal{I}'$. We claim that $\{T\} \cup \mathcal{I}'$ is an antichain. $\mathcal{I}'$ is an antichain and T embeds in no element of $\mathcal{I}'$. Suppose some element $T' \in \mathcal{I}'$ embeds in T. Let T'' be the element of $\mathcal{I}$ into which T properly embeds. Then $T' < T < T''$. We have that $T' < T''$ and both tournaments are element of $\mathcal{I}$, a contradiction. Now $\{T\} \cup \mathcal{I}'$ is an infinite antichain and $\{T\} \cup \mathcal{I}' \ll \mathcal{I}$, a contradiction.

$\mathcal{A}_\mathcal{I}$ is wqo since for any antichain $\mathcal{J}$ in $\mathcal{A}_\mathcal{I}$, $\mathcal{J} \leq \mathcal{I}$ and $\mathcal{J} \cap \mathcal{I} = \emptyset$, and thus, if $\mathcal{J}$ were infinite we would have $\mathcal{J} \ll \mathcal{I}$, a contradiction.

$2 \Longrightarrow 1$

Suppose $\mathcal{I}'$ is an infinite antichain with $\mathcal{I}' \ll \mathcal{I}$. Let $T \in \mathcal{I}'$ properly embed in some element $S \in \mathcal{I}$. Then S is comparable to T and so is not in $\mathcal{I}'$. Since T embeds in all but finitely many elements of $\mathcal{I}$, we have that only finitely many elements of $\mathcal{I}$ are in $\mathcal{I}'$ and $\mathcal{I}'' = \mathcal{I}' - \mathcal{I}$ is an infinite antichain. Every element of $\mathcal{I}'$ is in $\mathcal{I}$ or properly embeds in an element of $\mathcal{I}$. $\mathcal{I}''$ contains no element of $\mathcal{I}$, and so $\mathcal{I}'' \subseteq \mathcal{S}_\mathcal{I}$, a contradiction. ■

We now state and prove the Finiteness Theorem in a form which is more precise than that given in the introduction. Recall that an infinite antichain $\mathcal{I}$ is called a *witness* for a loose constraint Γ if a cofinite subset of $\mathcal{I}$ is an antichain in $\mathcal{Q}_\Gamma$. If $\mathcal{I}$ is a witness for Γ and $\mathcal{J} \leq \mathcal{I}$ is a minimal antichain, then $\mathcal{J}$ is also a witness for Γ. We can always choose a minimal antichain as a witness.

Theorem 2.5 (Finiteness Theorem) *For each positive integer k there is a finite set Λ_k of minimal infinite antichains such that if Γ is loose and $|\Gamma| = k$ then Λ_k contains a witness for Γ.*

Proof:

We proceed by induction on $k = |\Gamma|$.

The case $k = 0$ is trivial. $\mathcal{Q}_\emptyset$ is the class of all tournaments. Let Λ_0 be a set containing one minimal infinite antichain.

Assume the result holds for k. We will show that it holds for $k + 1$.

Let $\Lambda_k = \{\mathcal{J}_1, \ldots, \mathcal{J}_n\}$. Let $\mathcal{A}_i = \mathcal{A}_{\mathcal{J}_i}$, the class of tournaments which embed in all but finitely many elements of $\mathcal{J}_i$, for $1 \leq i \leq n$. By Fact 2 above and Lemma 2.4, $\mathcal{A}_{i_1} \times \ldots \times \mathcal{A}_{i_{k+1}}$ is wqo for any sequence of subscripts $I = (i_1, \ldots, i_{k+1})$. In particular, the set $\mathcal{L}_I$ of minimal loose $(k+1)$-sequences in $\mathcal{A}_{i_1} \times \ldots \times \mathcal{A}_{i_{k+1}}$ is finite. For $\mathcal{M} \in \mathcal{L}_I$, let $\mathcal{J}_\mathcal{M}$ be a witness for $\mathcal{M}$. Let $\Lambda_{k+1} = \Lambda_k \cup \{\mathcal{J}_\mathcal{M} : \mathcal{M} \in \mathcal{L}_I, I \text{ a } (k+1)\text{-sequence of subscripts}\}$.

Proposition 2.2 *Let $\mathcal{I}$ be an infinite antichain. Then there is a minimal antichain $\mathcal{J}$ with $\mathcal{J} \leq \mathcal{I}$.*

Proof:

We will apply König's Lemma to a tree whose vertices are finite sequences of finite tournaments. In this context, we will denote tournaments by lower case letters t_i.

Let T_0 be the tree whose vertices are all finite antichains $\mathcal{I}_0$ such that $\mathcal{I}_0 \subset \mathcal{I}'$ for some infinite antichain $\mathcal{I}' \leq \mathcal{I}$. For example, each finite subset of $\mathcal{I}$ will be a node in the tree. Also, a finite set of subtournaments of elements of $\mathcal{I}$ will be a node if such a set together with an infinite subset of $\mathcal{I}$ is an antichain.

The tree ordering is inclusion, and hence at the n^{th} level of T_0 we have sets of cardinality n. The union of the nodes in an infinite branch is an infinite antichain. In particular, $\mathcal{I}$ is the union of the nodes in an infinite branch. We will prune the tree so that the union of the nodes in an infinite branch is a minimal antichain $\mathcal{J}$ with $\mathcal{J} \leq \mathcal{I}$.

If $\mathcal{I}_0 \in T_0$ let $|\mathcal{I}_0|$ be the sequence $(|t| : t \in \mathcal{I}_0)$, arranged in nondecreasing order. For example, if $\mathcal{I}_0 = \{t_1, t_2, t_3, t_4\}$ then $|\mathcal{I}_0| = (|t_1|, |t_2|, |t_3|, |t_4|)$. Let $T_1 = \{\mathcal{I}_0 \in T_0 : |\mathcal{I}_0|$ is lexicographically minimal among all vertices at the same level$\}$. We claim that T_1 is a subtree of T_0. It is enough to show that any element $\{t_1, \ldots, t_n\}$ of T_1 has an immediate predecessor in T_0 which is also an element of T_1. Since the sequence of cardinalities associated with $\{t_1, \ldots, t_n\}$ is lexicographically minimal, necessarily the sequence of cardinalities associated with $\{t_1, \ldots, t_{n-1}\}$ is also lexicographically minimal and $\{t_1, \ldots, t_{n-1}\} \in T_1$.

For all vertices $\mathcal{I}_0$ of T_1, the sequence $|\mathcal{I}_0|$ is determined by the level of $\mathcal{I}_0$. Thus T_1 has finitely many vertices at each level. Furthermore T_0, and hence also T_1, has vertices at every level. So König's Lemma applies and T_1 contains an infinite branch B. Let the infinite antichain $\mathcal{J}$ be the union of the nodes in B. By construction $\mathcal{J} \leq \mathcal{I}$.

We claim that $\mathcal{J}$ is minimal. Suppose that $\mathcal{J}' \ll \mathcal{J}$. Arrange $\mathcal{J}$ as a sequence $(t_1, t_2, t_3, \ldots)$, where $\{t_1, \ldots, t_n\}$ is the n^{th} vertex of B. Choose n minimal so that some tournament t' in $\mathcal{J}'$ properly embeds in t_n. Let $\mathcal{J}^\star = \{t_1, \ldots, t_{n-1}\} \cup \mathcal{J}'$. We claim that $\mathcal{J}^\star$ is an infinite antichain. No element of $\mathcal{J}'$ embeds in an element of $\{t_1, \ldots, t_{n-1}\}$ by our choice of n. Suppose $t_i < t''$ for some $1 \leq i \leq n-1$ and some $t'' \in \mathcal{J}'$. Since $\mathcal{J}' \leq \mathcal{J}$, there is a $\tilde{t} \in \mathcal{J}$ with $t'' \leq \tilde{t}$ but then $t_i < t'' \leq \tilde{t}$ and $t_i < \tilde{t}$ with $\{t_i, \tilde{t}\} \subset \mathcal{J}$, a contradiction. $\mathcal{J}^\star \leq \mathcal{I}$, $\mathcal{J}^\star$ contains $\mathcal{J}_0^\star = \{t_1, \ldots, t_{n-1}, t'\}$, and $|\mathcal{J}_0^\star|$ is lexicographically strictly less than $|\{t_1, \ldots, t_n\}|$. We conclude that $\{t_1, \ldots, t_n\} \notin T_1$, a contradiction. ■

Minimal antichains are useful because they are associated naturally with wqo classes, as we will now show. This allows us to use wqo theory to prove minimality.

Definition 2.3 *Let $\mathcal{I}$ be an infinite antichain. $\mathcal{A}_\mathcal{I}$ is the class of tournaments which embed in all but finitely many elements of $\mathcal{I}$. (This should not be confused with $\mathcal{Q}_\mathcal{I}$, which is the class of tournaments which omit every element of $\mathcal{I}$). $\mathcal{S}_\mathcal{I}$ is the class of tournaments that properly embed in some element of $\mathcal{I}$.*

Lemma 2.4 *If $\mathcal{I}$ is an infinite antichain the following are equivalent:*

A word is a string of letters. In the word ordering we have that $cat \leq crate$ but $act \not\leq crate$. Second, suppose our alphabet Σ is the set of digits $\{1,2,3,4,5,6,7,8,9\}$ with the usual ordering of the natural numbers. Words in this alphabet are natural numbers but note that the ordering on the words is not the usual one. For example, $23 \not\leq 322$ in the word ordering.

Fact 2 The product of finitely many well-quasi-orders is a well-quasi-order, under the product ordering $\bar{a} \leq \bar{b}$ if and only if $a_i \leq b_i$ for all i.

Fact 2 is a special case of Fact 1 in which the conclusion of Fact 1 is applied to some of the words of fixed length in the alphabet Σ consisting of the disjoint sums of the given well-quasi-orders. More specifically, if we have n wqo's, we restrict our attention to words of length n where the i^{th} letter is from the i^{th} wqo. In this case the function f of Fact 1 can only be the identity.

Continuing with our second example above, if we take the product of four copies of Σ, the elements of the resulting wqo would be natural numbers with four digits. Again, the word ordering and the natural number ordering are not the same. For example, $2323 \not\leq 3232$.

2 The Finiteness Theorem

In this report we will consider only finite tournaments. An "antichain" means an antichain of finite tournaments, that is, a collection $\mathcal{I}$ of finite tournaments which are pairwise incomparable in the sense that none embeds in any other. Note that antichains may be finite or infinite.

The finiteness theorem states that for each k, there is a finite set of infinite antichains which are witnesses for loose k-sets. Our proof that the set of witnesses is finite depends on the fact that we can always choose a witness to be minimal in the following sense: each tournament in the witnessing antichain is as "small" as possible, i.e., replacing tournaments in the antichain with proper subtournaments results in an infinite set which is not an antichain. We now make this precise.

Definition 2.1 *Let $\mathcal{I}, \mathcal{J}$ be two antichains. We say $\mathcal{I}$ is less than or equal to $\mathcal{J}$, written $\mathcal{I} \leq \mathcal{J}$, if every tournament in $\mathcal{I}$ embeds in some tournament in $\mathcal{J}$. We say $\mathcal{I}$ is much less than $\mathcal{J}$, written $\mathcal{I} \ll \mathcal{J}$, if $\mathcal{I} \leq \mathcal{J}$ and some tournament in $\mathcal{I}$ embeds properly in a tournament of $\mathcal{J}$. We call an antichain $\mathcal{I}$* minimal *if $\mathcal{I}$ is infinite and there is no infinite antichain $\mathcal{J}$ with $\mathcal{J} \ll \mathcal{I}$.*

For example, if $\mathcal{I} \subseteq \mathcal{J}$ then $\mathcal{I} \leq \mathcal{J}$. The relation $\leq$ is easily seen to be a partial ordering. Observe that for $\mathcal{I} \leq \mathcal{J}$, either $\mathcal{I} \subseteq \mathcal{J}$ or $\mathcal{I} \ll \mathcal{J}$. If $\mathcal{I}$ is minimal, than any infinite subset of $\mathcal{I}$ is also minimal.

The existence of minimal antichains will provide the key to the proof of our Finiteness Theorem. Examples of minimal antichains will be given in §3. First we show that any infinite antichain can be pruned down to a minimal antichain.

possible a priori, we have constructed infinitely many minimal antichains which can probably be used to prove that finitely many such antichains cannot suffice. To prove this rigorously will require further analysis of these infinitely many minimal antichains.

Our main problem is closely connected with other algorithmic and model theoretic problems. Though the wqo property is not in itself as strong as an explicit structure theorem, it does imply the polynomial-time solvabilty of a number of problems, cf. [1, 2, 7, 8, 10, 11, 14, 16, 21]. Also, in model theory, the problem of determining whether a finitely constrained class of directed graphs contains uncountably many homogeneous directed graphs reduces to a finite set of wqo problems for an associated class $\mathcal{Q}_\Gamma$ of tournaments [4, 17].

The fact that the class of all tournaments is not wqo is not in itself a triviality. The earliest published example of an infinite antichain of tournaments occurs in [12], though the existence of such antichains is noted in [15]. We will exhibit two such infinite antichains in §3.

We now discuss the basic facts of wqo theory which we will be using.

Definition 1.3 *1. A* quasi-ordering *is a set together with a reflexive, transitive binary relation, the order relation.*

2. An antichain *in a quasi-ordering is a subset whose elements are pairwise incomparable.*

3. A well-quasi-ordering (wqo) *is a quasi-ordering containing no infinite antichain and no infinite strictly descending sequence.*

A quasi-ordering is only slightly more general than a partial ordering. In a partial order, $a \leq b$ and $b \leq a$ implies $a = b$ whereas in a quasi-ordering this defines an equivalence relation. This is a technicality that will play no role in our discussion.

Remarks

1. The class of finite tournaments with the embeddability relation is a quasi-ordering.

2. A collection of finite tournaments cannot contain an infinite strictly descending sequence, so the notion of wqo simplifies to the one given earlier.

The following is one of the main tools for determining whether a class of structures is wqo.

Fact 1 (Higman [13]) If Σ is a wqo alphabet then the set $\Sigma^\star$ of finite words in the alphabet is a wqo under the relation of monotone word embeddability: $w \leq w'$ if and only if $w_i \leq w'_{f(i)}$, for some strictly increasing function f from the set of indices of the letters of w to the set of indices of the letters of w'. In this context, $w = (w_1, \ldots, w_n)$ and $w' = (w'_1, \ldots, w'_{n'})$.

We give two examples. First, suppose our alphabet Σ is the set of letters $\{a, b, \ldots, z\}$ with the order relation given by equality. In other words, two distinct letters are incomparable.

Actually we focus on a more limited problem, inasmuch as we restrict the problem to a set Γ of fixed size k. It is certainly possible that there is a separate algorithm for each fixed k and no uniform algorithm in general.

This problem was solved for $k = 1$ in [17, 18, 19]. It is fairly easy to show that the problem has an algorithmic solution by showing that the set of minimal loose tournaments is finite. In order to give an explicit algorithm, it is necessary to identify the minimal loose tournaments explicitly. It turns out that there are two delicate cases, which are resolved by showing that the constraints involved are both tight, in other words, two structure theorems are required which make fairly explicit how to encode the tournaments in the class defined by the given constraint. As the analysis shows that those are the "last" structure theorems given by single constraints, it is not surprising that the analysis in these two cases is quite intricate [18, 19].

The finiteness theorem proved here provides some support for the view that the main problem is algorithmically decidable for fixed k, though because the proof is itself quite ineffective, it does not lead directly to such a result. But the explicit solution for $k = 1$ can be combined with the finiteness theorem below to yield a positive result for the case of two constraints.

Theorem 2.5 (Finiteness Theorem) *Let $k \geq 0$ be given. There is a finite set Λ_k of infinite antichains of finite tournaments such that any loose constraint Γ with $|\Gamma| = k$ is witnessed by an antichain in Λ_k.*

If Γ is loose then $\mathcal{Q}_\Gamma$ contains an infinite antichain. Intuitively, we want a witness for Γ to be an infinite antichain in $\mathcal{Q}_\Gamma$. For technical reasons, we only require that all but finitely many elements of a witness are contained in $\mathcal{Q}_\Gamma$. In particular, if $\mathcal{I}$ is an infinite antichain, it serves as a witness for any finite subset $\Gamma \subset \mathcal{I}$.

Definition 1.2 *An infinite antichain $\mathcal{I}$ of finite tournaments is a* witness *for the looseness of Γ if $\mathcal{I} - \mathcal{Q}_\Gamma$ is finite; in other words, a cofinite subset of $\mathcal{I}$ is an antichain in $\mathcal{Q}_\Gamma$.*

A sufficiently good description of Λ_k leads to an effective procedure for deciding whether a constraint Γ of order k is loose or tight. In fact, one gets a little more. Our proof of Theorem 2.5 shows that given a sufficiently effective description of Λ_k, there is an algorithm which determines whether or not a constraint of order $k + 1$ is loose or tight. However, only the existence of **some** algorithm can be shown, not the existence of an algorithm which guarantees that we also have a sufficiently good description of Λ_{k+1}.

The solution for the case $k = 1$ in [17] does in fact deliver an adequate description of Λ_1 to show there is an algorithm for $k = 2$, though in order to see this is so, some additional work will need to be done here. To show there is an algorithm for the case $k = 3$ would require the characterization of Λ_2, a problem which seems rather difficult, but can at least be approached in a systematic way.

One naturally wonders if a **global** finiteness theorem is also true, namely, there may be a single finite set Λ_∞ which contains witnesses for all loose constraints Γ. Though this is

The proof is an application of the Pigeon Hole Principle. Take any vertex in a tournament with at least four vertices. Surely that vertex either dominates two vertices or is dominated by two vertices. In either case, the induced subtournament on the set consisting of the original vertex together with those two vertices is transitive. ■

By Ramsey's Theorem, if $\Gamma = \{L_n\}$, then $\mathcal{Q}_\Gamma$, the set of tournaments omitting L_n contains finitely many tournaments. Indeed, it is straightforward to show by induction on n that if a tournament T has at least 2^{n-1} vertices, then L_n embeds in T. In the induction step, one considers any vertex a in a tournament and partitions the vertices in $V(T) - \{a\}$ into the set of those which dominate a and the set of those which are dominated by a. One of these sets contains at least half the vertices and by induction the induced tournament on the set embeds L_{n-1}. The induced subtournament on $V(L_{n-1}) \cup \{a\}$ is L_n.

There are a number of useful structure theorems for less trivial classes $\mathcal{Q}_\Gamma$, beginning with the archetypal case in which Γ consists of a single oriented 3-cycle, C_3. $\mathcal{Q}_\Gamma$ consists of the transitive finite tournaments, or equivalently, the finite linear orderings. This is an infinite but highly structured class.

Given a finite set Γ of finite tournaments, the class $\mathcal{Q}_\Gamma$ can, in general, be quite "wild". One reasonable way to formalize the distinction between "structured/unstructured" or, alternatively, "tame/wild" is via the notion of well-quasi-ordering (wqo). An *antichain* in $\mathcal{Q}_\Gamma$ is a set $\mathcal{I}$ of tournaments in $\mathcal{Q}_\Gamma$ such that no element of $\mathcal{I}$ embeds in any other element of $\mathcal{I}$. For example, the set $\{L_3, C_3\}$ forms an antichain with two elements. $\mathcal{Q}_\Gamma$ is *wqo* if it contains no infinite antichain.

The theory of well-quasi-orderings provides us with methods for detecting whether a class of tournaments is wqo. Any class whose elements have a natural encoding by words in a suitable alphabet, or more generally by labelled trees, is a wqo class. This is formalized and illustrated in [17]. We call a finite set Γ of finite tournaments *tight* if $\mathcal{Q}_\Gamma$ is wqo, and *loose* otherwise. This terminology reflects the fact that the set Γ serves as a constraint. If Γ consists of a single tournament T, we will say that T is loose or tight, correspondingly, and write $\mathcal{Q}_T$ for $\mathcal{Q}_{\{T\}}$. We have seen that any transitive tournament T is tight since $\mathcal{Q}_T$ contains only finitely many tournaments. The tournament C_3 is also tight. $\mathcal{Q}_{C_3}$ consists of the finite transitive tournaments. A transitive tournament on m vertices embeds in a transitive tournament on n vertices whenever $m \leq n$ and so an antichain in $\mathcal{Q}_{C_3}$ has at most one element.

We have not yet given an example of a tournament T which is loose. That requires demonstrating that $\mathcal{Q}_T$ is **not** wqo, that is, we must prove the existence of an infinite antichain that "witnesses" the looseness of T. One method of demonstrating that such an antichain exists is by exhibiting it. Some particularly useful examples of such antichains are presented in §3.

The problem that concerns us here is the following:

Main Problem: Is there an algorithm which will determine whether a given finite set Γ of finite tournaments is loose or tight?

1 Introduction

A tournament is a directed graph in which each pair of vertices is linked by a single arc. If x and y are vertices of a tournament T and (x, y) is an arc, we say that x *dominates* y and use the notation $x \to y$. Figure 1 illustrates two tournaments, an oriented 3-cycle which we call C_3 and a tournament with four vertices. We let $V(T)$ denote the vertex set of T. By $|T|$ we mean the cardinality of $V(T)$. A tournament is finite if its vertex set is finite.

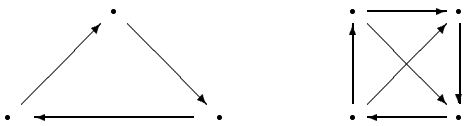

Figure 1: Some Tournaments

We say that two tournaments T_1 and T_2 are *isomorphic*, $T_1 \simeq T_2$, if there is a bijection between their vertex sets which preserves the arc relation. A *subtournament* of a tournament T has a vertex set which is a subset of $V(T)$ and the arcs are those induced from T. A tournament T_1 *embeds* in a tournament T_2, written $T_1 \leq T_2$, if T_1 is isomorphic with a subtournament of T_2. In other words, we can find a copy of T_1 in T_2. The tournament on three vertices in Figure 1 embeds in the tournament shown on four vertices. Note that there are two possible embedding maps and there is one arc which is in the image of both maps.

In this paper we will generally identify a tournament with its isomorphism class. In particular, we will use T_1 embeds in T_2, $T_1 \leq T_2$, and T_1 is a subtournament of T_2 interchangeably.

If Γ is a finite set of finite tournaments, let $\mathcal{Q}_\Gamma$ denote the set of finite tournaments T which *omit* Γ in the sense that no element of Γ embeds in T; in other words, Γ is taken as a set of *forbidden subtournaments* or *obstructions*. For example, if Γ consists of the tournament on two vertices then $\mathcal{Q}_\Gamma$ consists of the tournament on just one vertex.

A transitive tournament has the property that for all vertices i, j, k, if $i \to j \to k$ then $i \to k$. Alternatively, the vertices of a transitive tournament can be numbered in such a way that $i < j$ if and only if $i \to j$. In other words, the arc relation defines a linear order on the vertex set. A tournament is transitive if and only if it omits C_3. We use L_n to denote the transitive tournament on n vertices.

Based on the following lemma, if Γ consists of the transitive tournament on three vertices then $\mathcal{Q}_\Gamma$ consists of the tournament on one vertex, the tournament on two vertices, and C_3. Equivalently, $\mathcal{Q}_\Gamma$ consists of the tournaments with at most three vertices except, of course, for the transitive tournament on three vertices.

Lemma 1.1 *Any tournament with at least four vertices has a transitive subtournament on three vertices.*

Proof:

ABSTRACT

A class of finite tournaments determined by a set of "forbidden subtournaments" is well-quasi-ordered if and only if it contains no infinite antichain (a set of incomparable elements). It is not known if there is an algorithm which decides whether or not a class of finite tournaments determined by a finite set of forbidden subtournaments has this property. We prove a noneffective finiteness theorem bearing on the problem. We show that for each fixed k, there is a finite set of infinite antichains, Λ_k, with the following property: if any class defined by k forbidden subtournaments contains an infinite antichain, then a cofinite subset of an element of Λ_k must be such an antichain. By refining this analysis and using an earlier result giving an explicit algorithm for the case $k = 1$, we show that there exists an algorithm which decides whether or not a class of finite tournaments determined by two forbidden subtournaments is well-quasi-ordered.

Keywords: antichain, decidability, forbidden, Higman, Kruskal, local order, polynomial time, tournament, well-quasi-order

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A Decision Problem Involving Tournaments

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