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Abstract Deducibility and Domain Theory

by

Vladimir Yu. Sazonov[1,2]
Dept. of Computer Science
Rutgers University
New Brunswick, New Jersey 08903

Dmitri I. Sviridenko
Siberian Branch
of Russian Acad. of Sci.
Novosibirsk

[1]Permanent Member: Computer Logic Laboratory, Program Systems Institute of Russian Acad. of Sci., Pereslavl-Zalessky, 152140, Russia. E-mail: `sazonov@logic.botik.yaroslavl.su`
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ABSTRACT

[1] According to the thesis "computability = deducibility" [4] here are investigated intensional aspects of domain theory as mathematical theory of computability (cf. e.g. [1–4]).

A logistic system is any pair of sets $\mathcal{A} = \langle A, R\rangle$, where $R \subseteq \mathrm{Conf}(A) \rightleftharpoons \mathcal{P}(A)\times(A\cup\{\sharp\})$, $\sharp \notin A$. The intended interpretation: A is a set of sentences, R is a rule of inference, and $\sharp$ is a contradiction sign. As usually, R induces a relation $\vdash_R \subseteq \mathrm{Conf}(A)$ of (reflexive) deductive inference and also the classes $\mathrm{Cl}(\mathcal{A}) \subseteq \mathcal{P}(A)$ of sets closed under $\vdash_R$ and $\mathrm{Th}(\mathcal{A}) \subseteq \mathrm{Cl}(\mathcal{A})$ of consistent closed sets (theories) partially ordered by the inclusion relation. The following more general notion of deducibility $\Vdash_R$, which may be non-reflexive, plays an important role. Let $\Gamma \Vdash_R \varphi$ iff there exists a (well-founded) tree of inference $\Gamma \vdash_R \varphi$ which contains at least one configuration in R (i.e. is non-trivial). By imposing, if necessary, on deducibility notion suitable finitarity conditions (and others) it is possible to characterize rather naturally, from the point of view of the above mentioned thesis, various classes of domains, e.g. classes of all complete lattices with a base, conditionally complete partially ordered sets with a base, complete f_0-spaces (cf. also [4] where only finitary reflexive deducibility is considered), complete A_0-spaces, and continuous lattices. For example, $\mathrm{Th}(\langle A, \vdash\rangle)$ is an (arbitrary) complete A_0-space under $\subseteq$ if for some $R \subseteq \mathrm{Conf}(A)$ the following three conditions hold:

(1) $\Gamma/\varphi \in R \Rightarrow \Gamma$ is finite, (i.e. R is finitary),

(2) $\Gamma \Vdash_R \varphi \Rightarrow \hat{\Gamma} \Vdash_R \varphi$, where $\hat{\Gamma} \rightleftharpoons \cup\{\hat{\psi} : \psi \in \Gamma\}$ and $\hat{\psi} \rightleftharpoons \{\theta : \psi \Vdash_R \theta\}$, and

(3) $\Gamma \vdash \varphi \Leftrightarrow \Gamma \Vdash_R \hat{\varphi}$ and $\Gamma \vdash \sharp \Leftrightarrow \Gamma \Vdash_R \sharp$.

References

[1] Ju. L. Ershov, Computable functionals of finite types, *Algebra and Logic*, **11**, N4 (1972), 367–437 (in Russian; cf. also an English edition of the journal).

[2] Ju. L. Ershov, A theory of A-spaces, *Algebra and Logic*, **12**, N4 (1973), 369–414 (in Russian; cf. also an English edition of the journal).

[3] D. Scott, *Outline of a mathematical theory of computation*, Oxford Univ. Computing Lab., PRG-2, Oxford, 1970.

[4] D. Scott, Domains for denotational semantics, *LNCS* **140** (1982), 577–612.

[1]This page is a translation from the original Russian version of the abstract: V. Yu. Sazonov and D. I. Sviridenko, Abstract Deducibility and Domain Theory, *Seventh All Union Conference on Mathematical Logic, Abstracts*, Novosibitsk, 1984, p. 158.

Appendix

This Appendix was written by the first author in February 1996 when visiting DIMACS (Princeton and Rutgers Universities) due to persistence of Michael Bukatin and in connection with a related more recent paper:

> R. Hoofman, Continuous Information Systems, *Information and Computation* **105**, 42–71 (1993).

The goal of the present paper is to give an English extended version of the above Abstract originally in Russian. All lacking definitions and proofs are given. The material of this Appendix was never published. Also some comparisons with the approach of Hoofman are presented.

According to the above abstract, the symbol $\vdash$ always denotes some reflexive, possibly infinitary deductive inference relation. More precisely, given any set $R \subseteq \mathrm{Conf}(A) \rightleftharpoons \mathcal{P}(A) \times (A \cup \{\sharp\})$, $\sharp \notin A$, of inference *rules* or *configurations*, we write $\Gamma \vdash_R \varphi$ if there exists a (proof) tree with the nodes labelled by elements in $A \cup \{\sharp\}$ so that φ is in the root, all leaves are labelled by elements of Γ and each branching in the tree corresponds to some rule in R in the evident sense. Note that $\sharp$ may occur only in the root.

In general, using *infinitary* rules Γ/φ (we write Γ/φ or $\frac{\Gamma}{\varphi}$ for $\langle \Gamma, \varphi \rangle$ and $\varphi_1, \varphi_2, \ldots$ for $\Gamma = \{\varphi_1, \varphi_2, \ldots\}$) with infinite Γ results in proof trees with infinite branching and with infinitely many vertices. As usually, we consider only *well-founded* proof-trees, i.e. trees with all chains finite.

Let $\Vdash_R$ be defined by using nontrivial R-inference trees, involving at least one inference rule from R. Therefore, $\Vdash_R$ may be not reflexive if R does not contain rules of the kind φ/φ.

A trivial proof tree contains only one vertex $\varphi \in A$, the root. Such a tree is considered as a proof $\Gamma \vdash_R \varphi$ of φ from any $\Gamma \subseteq A$ such that $\varphi \in \Gamma$. So, $\Gamma \vdash_R \varphi$ iff $\Gamma \Vdash_R \varphi$ or $\varphi \in \Gamma$. We also write $\Gamma \vdash \Pi$ if $\Gamma \vdash \varphi$ for all $\varphi \in \Pi$, and the same for $\Vdash$.

It is this seemingly insignificant difference between $\vdash_R$ and $\Vdash_R$ which allows us and R. Hoofman to give (actually in somewhat different ways) a natural description of Ju. L. Ershov's complete A_0-spaces or, equivalently, of D. Scott's continuous domains in terms of an abstract deducibility notion.

> Very informally, each step of the "strict" deducibility $\Vdash$ may be considered as an "action" whose performing may result in a (finitely) nonrecoverable loss of "information". The above condition (2) on $\Vdash$ means that this loss is happening "gradually" or "continuously". Also, such $\Vdash$ may be called "dense" (like the standard linear ordering of the set of rational numbers). We may proceed so gradually that just a "little" information may be lost in one step. Then the condition (3) means that by some infinitary way the information, nevertheless, may be recovered from *all* its strict consequences. The latter seems reasonable due to continuity of $\Vdash$. We postulate that φ and the whole set of its strict consequences $\hat{\varphi}$ contain, in a sense (just in the sense of $\vdash$), the "same" information.

Inference relations of the type $\vdash$ and $\Vdash$ may be evidently characterized by the following axioms. (Note, that transfinite induction on well-founded inference trees is used here.)

(*Weakening*) $\Gamma \vdash (\Vdash)\ \varphi \ \&\ \Gamma \subseteq \Pi \Rightarrow \Pi \vdash$ (resp., $\Vdash$) φ;

(*Transitivity*) $\Gamma \vdash (\Vdash)\ \Pi \vdash (\Vdash)\ \varphi \Rightarrow \Gamma \vdash$ (resp., $\Vdash$) φ;

(*Reflexivity*) $\varphi \in \Gamma \Rightarrow \Gamma \vdash \varphi$

for any $\Gamma, \Pi \subseteq A$ and $\varphi \in A \cup \{\sharp\}$.

The following Proposition 1 shows that the above conditions (2) and (3) on $\Vdash_R \subseteq \mathrm{Conf}(A)$ and on (reflexive) inference relation $\vdash = \vdash_S$ for some $S \subseteq \mathrm{Conf}(A)$ determine some appropriate version of S as a natural extension $\hat{R}$ of R. Let $\hat{\psi} \rightleftharpoons \{\theta | \psi \Vdash_R \theta\}$ and $\hat{R} \rightleftharpoons R + \hat{\psi}/\psi$ for all $\psi \in A$. We define $\hat{\Gamma} \rightleftharpoons \cup\{\hat{\psi} | \psi \in \Gamma\}$ in a *distributive* way, and also $[\![\Gamma]\!] = [\![\Gamma]\!]_R \rightleftharpoons \{\theta | \Gamma \Vdash_R \theta\}$ and $[\Gamma] = [\Gamma]_R \rightleftharpoons \{\theta | \Gamma \vdash_R \theta\}$ for any $\Gamma \subseteq A$. Evidently, $\hat{\Gamma} \subseteq [\![\Gamma]\!]$, $\hat{\psi} = [\![\psi]\!]$ and $\hat{\Gamma} \vdash_{\hat{R}} \Gamma$. Note, that (each case of) the rule $\hat{\psi}/\psi$ is infinitary one if $\hat{\psi}$ is an infinite set, while R contains only finitary rules by the condition (1) which will be used later. (In general, it is possible that infinitary S may have an equivalent finitary form S_0 such that $\vdash_S = \vdash_{S_0}$.)

Proposition 1 *Conditions (2) and (3) on relations* $\Vdash_R$ *and* $\vdash$ *imply*

$$\vdash = \vdash_{\hat{R}}.$$

Proof Let $\Gamma \vdash \varphi$. If $\varphi \neq \sharp$ then this is equivalent to $\Gamma \Vdash_R \hat{\varphi}$ by (3). This gives $\Gamma \vdash_{\hat{R}} \varphi$, as required. The case of $\varphi = \sharp$ follows from the second part of (3).

Conversely, show that $\hat{R} \subseteq \vdash$. The inclusion $R \subseteq \vdash$ follows from the first part of (3) by

$$\Gamma/\varphi \in R \Rightarrow \Gamma \Vdash_R \varphi \Rightarrow \Gamma \Vdash_R \hat{\varphi} \overset{(3)}{\Longleftrightarrow} \Gamma \vdash \varphi$$

for $\varphi \neq \sharp$, and from the second part of (3) for $\varphi = \sharp$. Finally, we have $\hat{\varphi} \vdash \varphi$ for all $\varphi \neq \sharp$ because, again by (3), this is equivalent to $\hat{\varphi} \Vdash_R \hat{\varphi}$, i.e. to the following special case of (2): $\forall \theta \in A(\varphi \Vdash_R \theta \Rightarrow \hat{\varphi} \Vdash_R \theta)$.

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It follows from Proposition 1 and the last statement of the Abstract (cf. also Theorem 7 below) that to define any complete A_0-space (up to some order isomorphism; cf. the definition below) we may start with an appropriate finitary R, satisfying the continuity condition (2) above, and then take infinitary $\hat{R}$ and reflexive $\vdash_{\hat{R}}$ which, in turn, defines $\mathrm{Th}(\langle A, \vdash_{\hat{R}}\rangle)$. The last family of subsets of A under the subset relation $\subseteq$ will be the complete A_0-space we need. In fact, any R satisfying (2) together with corresponding reflexive deducibility relation defined by $\vdash\ \rightleftharpoons\ \vdash_{\hat{R}}$ also satisfy the condition (3).

Proposition 2 *Condition (2) on* R *implies*

$$\Gamma \vdash_{\hat{R}} \varphi \Leftrightarrow \Gamma \Vdash_R \hat{\varphi} \quad \textit{and} \quad \Gamma \vdash_{\hat{R}} \sharp \Leftrightarrow \Gamma \Vdash_R \sharp.$$

Therefore, it is sufficient to apply the rule $\hat{\varphi}/\varphi$ only on the last step of a $\vdash_{\hat{R}}$-inference.

Proof Both converse implications are trivial. We prove direct implications by transfinite induction on well-founded proofs. Let $\Gamma \vdash_{\hat{R}} \varphi$. The following cases are possible (whether φ is $\sharp$ or not).

- $\varphi \in \Gamma$. Then $\Gamma \Vdash_R \hat{\varphi}$ because this means that $\varphi \Vdash_R \theta \Rightarrow \Gamma \Vdash_R \theta$ for all $\theta \in A$.
- The proof considered (ends by $\hat{\varphi} \vdash_{\hat{R}} \varphi$ and) begins with $\Gamma \vdash_{\hat{R}} \hat{\varphi}$. Then the induction hypothesis gives $\Gamma \Vdash_R \hat{\hat{\varphi}}$. We have seen that (2) yields $\hat{\varphi} \Vdash_R \hat{\varphi}$. Then (2) also implies $\hat{\hat{\varphi}} \Vdash_R \hat{\varphi}$ and hence $\Gamma \Vdash_R \hat{\varphi}$ holds, by transitivity of $\Vdash$.
- Let $\Gamma \vdash_{\hat{R}} \Pi \Vdash_R \varphi$. Then it suffices to note that $\Gamma \Vdash_R \hat{\Pi}$, by induction hypothesis, and $\hat{\Pi} \Vdash_R \varphi$, by (2).

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Define a *conditionally (or consistently, or bounded) complete partially ordered set (poset) with a subbase* $\langle X, X_0, \sqsubseteq \rangle$ as any non-empty poset $\langle X, \sqsubseteq \rangle$, with $X_0 \subseteq X$ the subbase, such that the following two conditions hold:

(i) any upper bounded subset $S \subseteq X$ has a least upper bound (lub) $\sqcup S$ in X (e.g., $\perp \rightleftharpoons \sqcup\emptyset$ is the least element of X) and

(ii) $x = \sqcup\{x_0 \in X_0 | x_0 \sqsubseteq x\}$ for all $x \in X$.

Note, that $\tilde{X}_0 \rightleftharpoons \{x_1 \sqcup \ldots \sqcup x_n \in X | x_i \in X_0,\ n \geq 0\}$ (for existing unions) is also a subbase which is called a *base* because, moreover, $\{\tilde{x}_0 \in \tilde{X}_0 | \tilde{x}_0 \sqsubseteq x\}$ is directed for each $x \in X$. We could additionally postulate that the given subbase X_0 is actually a base. The partial order $\sqsubseteq$ is also called *approximation relation.*

Theorem 3 *Any logistic system* $\mathcal{A} = \langle A, \vdash \rangle$ *defines a conditionally complete poset* $X = \mathrm{Th}(\mathcal{A})$ *wrt* $\sqsubseteq \rightleftharpoons \subseteq$. *It has a subbase* $X_0 \rightleftharpoons \{[\varphi] | \varphi \in A, \varphi \nvdash \sharp\}$. *Conversely, any conditionally complete poset with a subbase* $\langle X, X_0, \sqsubseteq \rangle$ *is isomorphic to some* $\mathrm{Th}(\mathcal{A})$ *with* X_0 *corresponding to* $\{[\varphi] | \varphi \in\ A\}$.

Proof The first part is evident. Conversely, let $A \rightleftharpoons X_0 \subseteq X$. Define an inference relation $\vdash$ by taking $\Gamma \vdash \varphi$ iff $\varphi \sqsubseteq \sqcup\Gamma$, for $\Gamma \subseteq A$ and $\varphi \in A$, and $\Gamma \vdash \sharp$ iff Γ is unbounded in X. Weakening, transitivity and reflexivity properties of $\vdash$ are easily verified. Then the required isomorphism is given by the mapping $x \mapsto \{\varphi \in A | \varphi \sqsubseteq x\} : X \to \mathrm{Th}(\mathcal{A})$ and its inverse $T \mapsto \sqcup T : \mathrm{Th}(\mathcal{A}) \to X$.

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A family $x_i \in X$, $i \in I$, is called *directed* in X if $\forall ij \in I \exists k \in I(x_i, x_j \sqsubseteq x_k)$. A poset X is called *directly complete* if any directed family of its elements has a lub in X. A subset $U \subseteq X$ is called Scott's *open* (with respect to $\sqsubseteq$) if $x \in U$ and $x \sqsubseteq y$ always imply $y \in U$ and for all directed lubs $\sqcup_i x_i \in U$ implies $x_i \in U$ for some i. This class of open sets is closed under arbitrary unions and finite intersections and, therefore, constitutes a (T_0-)topology. Let Interior(S) for any $S \subseteq X$ be the largest open subset of S wrt Scott's topology on X. Let $x \prec y \rightleftharpoons y \in \mathrm{Interior}(\{z|x \sqsubseteq z\})$. A *complete A_0-space* $\langle X, \sqsubseteq\rangle$ *with a subbase* $X_0 \subseteq X$ is defined as conditionally and directly complete poset such that each $x \in X$ is represented as the lub $x = \sqcup\{x_0 \in X_0 | x_0 \prec x\}$.

If X contains the largest element, it will be denoted as $\top$. Otherwise, fix any $\top \notin X$ and let formally $x \sqsubseteq \top$, $x \prec \top$ and $\sqcup Y = \top$ for all $x \in X$ and for all unbounded $Y \subseteq X$. For example, for unbounded Y we have $x \sqsubseteq \sqcup Y$ iff $x \in X$.

Proposition 4 $\mathrm{Th}(\mathcal{A})$ *is directly complete iff* $\mathcal{A}$ *satisfies finite consistency property: each contradictory set* $\Gamma \subseteq A$ *has a finite contradictory subset.*

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Note, that (1) and the second part of (3) evidently entail finite consistency property of $\mathcal{A} = \langle A, \vdash\rangle$.

Proposition 5 *Conditions (1)–(3) imply that the set* $\check{F} \rightleftharpoons \{T \in \mathrm{Th}(\mathcal{A}) | T \Vdash F\}$ *is Scott's open in* $\langle Th(\mathcal{A}), \subseteq\rangle$ *for any finite* $F \subseteq A$.

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Therefore, $T \Vdash F$ implies $[F] \prec T$ for any $T \in \mathrm{Th}(\mathcal{A})$ and consistent finite F.

Proof Evidently, $\check{F}$ is closed upward wrt $\subseteq$. Let $T = [\cup T_i] = \sqcup T_i$ be a directed lub of consistent theories T_i and $T \Vdash F$. Then $\cup T_i \vdash T \Vdash F$ and, by (3), (2) and transitivity, we have $\cup T_i \Vdash \hat{T} \Vdash F$ and $\cup T_i \Vdash F$. Hence $T_i \Vdash F$ for some i because, by (1), $\Vdash$ is finitary.

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Proposition 6 *Conditions (2) and (3) yield equivalences*

$$\Gamma \vdash \varphi \Leftrightarrow \hat{\Gamma} \vdash \varphi \Leftrightarrow [\![\Gamma]\!] \vdash \varphi,$$

$$\Gamma \Vdash \varphi \Leftrightarrow \hat{\Gamma} \Vdash \varphi \Leftrightarrow [\![\Gamma]\!] \Vdash \varphi.$$

Proof

$$\Gamma \vdash \varphi \stackrel{(3)}{\Longleftrightarrow} \Gamma \Vdash \hat{\varphi} \stackrel{(2)}{\Longrightarrow} \hat{\Gamma} \Vdash \hat{\varphi} \stackrel{(3)}{\Longleftrightarrow} \hat{\Gamma} \vdash \varphi \Rightarrow [\![\Gamma]\!] \vdash \varphi \stackrel{(3)}{\Longleftrightarrow} [\![\Gamma]\!] \Vdash \hat{\varphi} \Rightarrow \Gamma \Vdash \hat{\varphi} \stackrel{(3)}{\Longleftrightarrow} \Gamma \vdash \varphi,$$

$$\Gamma \vdash \sharp \stackrel{(3)}{\Longleftrightarrow} \Gamma \Vdash \sharp \stackrel{(2)}{\Longrightarrow} \hat{\Gamma} \Vdash \sharp \stackrel{(3)}{\Longleftrightarrow} \hat{\Gamma} \vdash \sharp \Rightarrow [\![\Gamma]\!] \vdash \sharp \stackrel{(3)}{\Longleftrightarrow} [\![\Gamma]\!] \Vdash \sharp \Rightarrow \Gamma \Vdash \sharp \stackrel{(3)}{\Longleftrightarrow} \Gamma \vdash \sharp,$$

$$\Gamma \Vdash \varphi \stackrel{(2)}{\Longrightarrow} \hat{\Gamma} \Vdash \varphi \Rightarrow [\![\Gamma]\!] \Vdash \varphi \Rightarrow \Gamma \Vdash \varphi.$$

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Theorem 7 *Under conditions (1)–(3),* $\langle \mathrm{Th}(\mathcal{A}), \subseteq\rangle$ *is a complete* A_0*-space with a subbase* $\{[\alpha] | \alpha \in A, \alpha \nvdash \sharp\}$. *Conversely, if* $\langle X, X_0, \sqsubseteq\rangle$ *is a complete* A_0*-space with a subbase then isomorphic* $\mathrm{Th}(\langle A, \vdash\rangle)$ *with corresponding subbase* $\{[\alpha] | \alpha \in A,\ \alpha \nvdash \sharp\}$ *and with some* $\Vdash$ *satisfying (1)–(3) also exist.*

Proof It follows from Propositions 6 and 5 that, under (1)–(3), the equalities $T = [[\![T]\!]] = [\{\alpha | T \Vdash \alpha\}] = \sqcup\{[\alpha] | T \Vdash \alpha\} = \sqcup\{[\alpha] | [\alpha] \prec T\}$ hold for any consistent theory T. This gives the first part of the theorem.

Conversely, consider any complete A_0-space $\langle X, X_0, \sqsubseteq\rangle$ with a subbase. Then some $\mathcal{A}$ with an isomorphic $\mathrm{Th}(\mathcal{A})$ and with corresponding subbase $\{[\alpha] | \alpha \in A,\ \alpha \not\vdash \sharp\}$ may be obtained by Theorem 3:

$$A \rightleftharpoons X_0,$$

$$\Gamma \vdash \varphi \quad \text{iff} \quad \varphi \sqsubseteq \sqcup\Gamma \quad \text{and}$$

$$\Gamma \vdash \sharp \quad \text{iff} \quad \Gamma \text{ is unbounded in } X.$$

Let us also define

$$\Gamma \Vdash \varphi \quad \text{iff} \quad \varphi \prec \sqcup\Gamma \quad \text{and}$$

$$\Gamma \Vdash \sharp \quad \text{iff} \quad \Gamma \text{ is unbounded in } X$$

for any $\Gamma \subseteq X_0$ and $\varphi \in X_0$. This relation $\Vdash$ trivially satisfies weakening property. By using continuity properties of complete A_0-space X we easily obtain that $\Vdash$ is (essentially) finitary: $\Gamma \Vdash \varphi$ implies $\Gamma_0 \Vdash \varphi$ for some finite $\Gamma_0 \subseteq \Gamma$. Its transitivity $\Gamma \Vdash \Pi \Vdash \varphi \Rightarrow \Gamma \Vdash \varphi$ follows by $\varphi \prec \sqcup\Pi \sqsubseteq \sqcup\Gamma \Rightarrow \varphi \prec \sqcup\Gamma$. The conditions (2) and (3) are translated, respectively, as $\varphi \prec \sqcup\Gamma \Rightarrow \varphi \prec \sqcup\hat{\Gamma}$ and $\varphi \sqsubseteq \sqcup\Gamma \Leftrightarrow \hat{\varphi} \prec \sqcup\Gamma$ and also follow from the definition of a complete A_0-space by $\varphi = \sqcup\hat{\varphi}$ and $\sqcup\hat{\Gamma} = \sqcup\Gamma$.

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This finishes proof of the main result on representation of complete A_0-spaces by infinitary reflexive inference relation $\vdash$ satisfying (1)–(3) for some $\Vdash$.

A Comparison with Hoofman's Paper

Note, that Hoofman also uses finitary non-reflexive inference relations to describe in somewhat different way arbitrary Scott's continuous domains (which are the same as complete A_0-spaces). Instead of our continuity condition (2) he uses a weaker one which we write as

(2′) $\Gamma \Vdash_R \varphi \Rightarrow [\![\Gamma]\!] \Vdash_R \varphi$.

However, (2) would be also sufficient to get the same class of domains because his proof is based on the same non-reflexive inference relation as in the proof of Theorem 7 ($\Gamma \Vdash \varphi \Leftrightarrow \varphi \prec \sqcup\Gamma$) which actually satisfies (2).

Another minor difference consists in his requirement (going from the original paper of D. Scott on information systems) that each $\varphi \in A$ must be consistent ($\varphi \nVdash \sharp$).

Note, that for many real examples of logistic systems known from logic (independently on any domain theory) it is algorithmically *undecidable* whether a formula is consistent or not. We consider logistic systems as a reasonable approach to *computability* and *logical* aspects of domain theory and *denotational semantics* of programming languages. Also we induce any inference relation by giving a set R of proof rules because usually it is this natural way how logistic systems appear. Despite the present very abstract consideration of logistic systems, a reasonable application of this approach, initiated by D. Scott, may be connected with some logics having more usual form, say, with logical connectives, quantifiers, modalities. These may be some program logics, so that axiomatic, operational and denotational semantics of programs could be investigated in a unifying "logical" way and in a very general framework which does not restrict us a priori to this or that special kind of logistic systems.

We use non-reflexive finitary $\Vdash$ satisfying (2) to define its infinitary extension $\vdash$ whose $\mathrm{Th}(\vdash)$ gives the domain we need. Of course, this can be done directly in terms of $\Vdash$, without any mentioning $\vdash$, by admitting respective *restriction* on theories in $\mathrm{Th}(\Vdash)$: take only such $\Vdash$-consistent sets $T \subseteq A$ closed under $\Vdash$ which satisfy the following additional condition

$$\hat{\varphi} \subseteq T \Rightarrow \varphi \in T.$$

So defined class of $\Vdash$-theories coincides with $\{[\Gamma] | \Gamma \subseteq A,\ \Gamma \text{ consistent}\}$.

Hoofman also considers a restriction of $\mathrm{Th}(\Vdash)$ for *his* $\Vdash$ satisfying (2′), but in a quite different way. He imposes the following requirement on $T \in \mathrm{Th}(\Vdash)$, instead of ours,

$$\varphi \in T \Rightarrow \exists \Gamma \subseteq T . \Gamma \Vdash \varphi.$$

Like above, this class of $\Vdash$-theories coincides with $\{[\![\Gamma]\!] | \Gamma \subseteq A,\ \Gamma \text{ consistent}\}$. Remember, that consistency notions both in the sense of $\vdash$ and $\Vdash$ coincide by the condition (3).

Then each element x in the given complete A_0-space $\langle X, X_0, \sqsubseteq \rangle$ is represented as a theory wrt $\Vdash$ defined by $\Gamma \Vdash \varphi \Leftrightarrow \varphi \prec \sqcup \Gamma$; just take $\{\varphi \in A = X_0 | \varphi \sqsubseteq x\}$ in our approach and $\{\varphi \in A = X_0 | \varphi \prec x\}$ in Hoofman's.

Also, the following Proposition shows that both these approaches give (different) descriptions of actually *isomorphic* domains.

Proposition 8 *Under (2) and (3), for all* $\Gamma, \Pi \subseteq A$

$$[\Gamma] \subseteq [\Pi] \Leftrightarrow [\![\Gamma]\!] \subseteq [\![\Pi]\!].$$

Proof ($\Rightarrow$) Let $\Pi \vdash \Gamma \Vdash \varphi$. Then $\Pi \Vdash \hat{\Gamma} \Vdash \varphi$ by (3) and (2). Therefore $\Pi \Vdash \varphi$, as required.

($\Leftarrow$) Let $[\![\Gamma]\!] \subseteq [\![\Pi]\!]$ and $\Gamma \vdash \varphi$. Then $\Gamma \Vdash [\![\varphi]\!]$, by (3). Hence $[\![\varphi]\!] \subseteq [\![\Gamma]\!] \subseteq [\![\Pi]\!]$, $\Pi \Vdash [\![\varphi]\!]$ and again $\Pi \vdash \varphi$, by (3). This gives $\Pi \vdash \varphi$, as required.

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