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Asymptotically Good Covers in Hypergraphs

Extended Abstract of the Dissertation

by

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ABSTRACT

In the early 1980's, V. Rödl proved the Erdős-Hanani Conjecture, sparking a remarkable sequence of developments in the theory of packing and covering in hypergraphs of bounded edge size. Generalizations were given by P. Frankl and Rödl, by N. Pippenger, and by others. In each case, an appropriate *semirandom* method was used to "construct" the desired optimal object (covering, matching, colouring) in several random stages, followed by a greedy stage.

The present paper provides an overview of the author's dissertation. The proof of the main theorem, which further generalizes some of the results mentioned above, is again probabilistic, and uses, in addition to earlier ideas, connections with so-called *normal* distributions on the set of matchings of a graph. For fixed $k \geq 2$, $\mathcal{H}$ a k-bounded hypergraph, and $t : \mathcal{H} \to \mathbf{R}^+$ a fractional cover, a sufficient condition is given to ensure that the edge cover number $\rho(\mathcal{H})$, *i.e.*, the size of a smallest set of edges of $\mathcal{H}$ with union $V(\mathcal{H})$, is asymptotically at most $t(\mathcal{H}) = \sum_{A \in \mathcal{H}} t(A)$. This settles a conjecture first publicized in Visegrád, June 1991.

1 Introduction

1.1 Integral v. fractional optima

One of the fundamental and most important problems in combinatorial theory is to understand the nature of and differences between integral optima and their fractional counterparts (see, *e.g.*, [31]). The latter are generally well-understood in theory, and in practice their determination is considered a tractable problem. The same cannot be said of the former, as "combinatorial explosion" frequently makes their determination possible only by resorting to prohibitively-long exhaustive searches. Witness the contrast between linear and integer programming (see [32]). The theory of linear programming is understood to the point where several polynomial-time algorithms are known, and even a famous non-polynomial algorithm, the simplex method, is usually very efficient. In contrast, no efficient algorithm is known for integer programming, and none are thought to exist. This raises the question: what separates these types of problems? An argument could even be made that better understanding of questions like this may be one of the keys to understanding the "$\mathcal{P} \overset{?}{\neq} \mathcal{NP}$" question in theoretical computer science (see [10]).

At any rate, the problem of understanding the differences between integral and fractional optima has fueled many research fires. Within discrete mathematics, this problem is certainly of fundamental importance in the theory of hypergraphs. As observed *e.g.* in [9], hypergraphs are such general objects that virtually every combinatorial problem can be phrased using their language. An overview of the author's dissertation [21], this paper presents an asymptotic connection between an integer-valued and a related fractional parameter for a natural class of hypergraphs. The result substantially generalizes work of V. Rödl [29], P. Frankl and Rödl [8], N. Pippenger (unpublished; see [27]), and J. Kahn [17] and settles a 1991 conjecture of the last author (see [15] and also [14]).

After introducing some terminology, we provide a brief survey starting with the above results.

1.2 Terminology

A *hypergraph* $\mathcal{H}$ is a collection, possibly with repeats, of subsets (called *edges*) of some finite set V, called the set of *vertices*. We write n for $|V|$ and m for $|\mathcal{H}|$. If $x \in V$, the *degree* of x, $d(x)$, is the number of edges containing x, while the *pairwise degree* of x and y, $d(x,y)$, is the number of edges containing both of $x, y \in V$. The maximum and minimum degrees in $\mathcal{H}$ are denoted, respectively, by $D(\mathcal{H})$ and $d(\mathcal{H})$. We say $\mathcal{H}$ is *k-bounded* (resp. *k-uniform*) if each of its edges contains at most (resp. exactly) k vertices.

A set of edges with union V is called an *(edge) cover*; the size of a smallest cover is called the *cover number* and denoted $\rho(\mathcal{H})$. A related notion is that of a *matching*, a set of pairwise disjoint edges. The size of a largest matching is the *matching number*, denoted by $\nu(\mathcal{H})$. A *colouring* of $\mathcal{H}$ is a function $\sigma : \mathcal{H} \to S$, where S is any set; the colouring σ is *proper* if $\sigma(A) \neq \sigma(B)$ for all distinct, nondisjoint $A, B \in \mathcal{H}$. The *chromatic index*, $\chi'(\mathcal{H})$, is the least size of an S admitting a proper colouring. A generalization of $\chi'(\mathcal{H})$ is the *list-chromatic index*, $\chi'_\ell(\mathcal{H})$, defined to be the least r such that whenever $S(A)$ is a set of size r for each $A \in \mathcal{H}$, there exists a proper colouring σ of $\mathcal{H}$ with $\sigma(A) \in S(A)$ for all $A \in \mathcal{H}$. Considering the case when all the "lists" $S(A)$ are identical shows that the inequality $\chi'_\ell(\mathcal{H}) \geq \chi'(\mathcal{H})$ always holds. One natural reason for studying list-colourings is that an ordinary colouring problem in which some colours have already been assigned is a list-colouring problem.

The notions of an edge cover and a matching both have fractional counterparts, which we now define. In each of the following definitions, t is a function from $\mathcal{H}$ to $\mathbf{R}^+$. The sum $\sum_{A\in\mathcal{H}} t(A)$ is always denoted by $t(\mathcal{H})$. We call t a *fractional cover* in case $\sum_{A\ni x} t(A) \geq 1$ for all $x \in V$ and a *fractional matching* in case $\sum_{A\ni x} t(A) \leq 1$ for all $x \in V$. Note that if the range of t in these definitions is restricted to $\{0, 1\}$, then alternate (and equivalent) definitions of edge cover and matching are obtained.

A common theme underlying the results of [21] and those they generalize is that barring "local obstructions", "good asymptotic behaviour" is achieved. Suppose $t : \mathcal{H} \to \mathbf{R}^+$ is a fractional cover. The next two definitions are aimed at capturing in technical terms what it means to have nice local behaviour of t in $\mathcal{H}$. Define $\bar{t} : 2^V \to \mathbf{R}^+$ by

$$\bar{t}(A) \triangleq \sum_{\substack{B\supseteq A\\ B\in\mathcal{H}}} t(B)$$

for $A \subseteq V$, and, for $i \geq 2$, set

$$\alpha_i(t) \triangleq \max_{\substack{W\subseteq V\\ |W|=i}} \bar{t}(W).$$

One measure of nice local behaviour of t in $\mathcal{H}$ is $\alpha_i(t) \to 0$ for some fixed i.

The promised survey is presented after our asymptotic notation is defined. We use $f \sim g$, $f \lesssim g$ and $f = o(g)$, respectively, for $f/g \to 1$, $\limsup f/g \leq 1$ and $f/g \to 0$, these limits taken as some parameter(s), always indicated, approach(es) some limiting value(s). We adopt the "uniformity convention" of [27], *viz.*: any limiting statement involving one or more free variables ranging over vertices, edges or hypergraphs is understood to hold uniformly with respect to all possible choices of these variables.

1.3 History

For a similar and more detailed discussion of what follows, the reader is referred to §5–§8, in particular §6, of the paper [15] by Kahn. Füredi's comprehensive survey [9, §8] also provides more details than we give here.

Our discussion begins with a 1963 conjecture of Erdős and Hanani [7] concerning the asymptotic behaviour of ν and ρ for a particular family of hypergraphs. Fix positive integers t, r, v, with $1 < t < r < v$, let V be a v-set, and define the hypergraph $\mathcal{H} = \mathcal{H}(v, r, t)$ on the vertex set $\binom{V}{t}$ by

$$\mathcal{H} \triangleq \left\{ \binom{R}{t} : R \in \binom{V}{r} \right\}.$$

The matching and edge cover numbers for this hypergraph are often denoted, respectively, by $P(v,r,t)$ and $C(v,r,t)$ to suggest the terms *packing* and *covering*. Since $\mathcal{H}$ is $\binom{r}{t}$-uniform on $\binom{v}{t}$ vertices, the inequalities

$$P(v,r,t) \le \frac{\binom{v}{t}}{\binom{r}{t}} \le C(v,r,t) \tag{1}$$

are a case of the trivial inequalities

$$\nu \le \frac{n}{k} \le \rho, \tag{2}$$

which hold for every k-uniform hypergraph on n points.

Erdős and Hanani [7] proved that the inequalities (1) are asymptotically tight for $t = 2$, r fixed, and for $t = 3$, $r = q$ or $q + 1$ (q a prime power), where we take $v \to \infty$ in each case. They conjectured this result to be true for every fixed t and r, with $1 < t < r$ and $v \to \infty$. Progress on this conjecture was incremental (see [4, 25]) until it was proved by V. Rödl in the early 1980's (see also J. Spencer's account [33] and — for a recent "continuous" proof — [35]). We state Rödl's result as

(1.1) Theorem ([29]) *For every fixed t and r, with $1 < t < r$,*

$$P(v,r,t) \sim \frac{\binom{v}{t}}{\binom{r}{t}} \sim C(v,r,t) \quad (v \to \infty).$$

The packings and coverings for which the respective limiting statements in Theorem 1.1 hold are called *asymptotically good*.

The first instance known to the author of a method of proof related to Rödl's appeared in a pioneering sequence of papers [2, 23, 1] by J. Komlós, E. Szemerédi and co-authors. For background on these results, see Kahn's survey [15, §7]. The proof technique, dubbed the "Rödl-nibble" by Spencer [34, 3], has always been considered beautiful and is now viewed as seminal, for it led relatively quickly to some remarkable generalizations.

First, P. Frankl and Rödl [8] gave sufficient conditions on a k-uniform hypergraph with n vertices to ensure that the inequalities (2) are asymptotically tight. It was not long before N. Pippenger simplified the conditions and proved the following result, unpublished by him, but appearing in [34] and [9]. The version we state here was given by Pippenger and Spencer in [27].

(1.2) Theorem *Let $k \ge 2$ be fixed and $\mathcal{H}$ a k-uniform hypergraph satisfying*

$$D(\mathcal{H}) - d(x) < o(D(\mathcal{H})) \text{ for all vertices } x, \tag{3}$$

and

$$d(x,y) < o(D(\mathcal{H})) \text{ for all distinct vertices } x, y. \tag{4}$$

Then

$$\nu(\mathcal{H}) \sim \frac{n}{k} \sim \rho(\mathcal{H}) \quad (D(\mathcal{H}) \to \infty).$$

Pippenger's improvement was the replacement of an explicit bound (approximately $D/(\log n)^3$) on pairwise degrees, required by Frankl and Rödl, by the weaker assumption (4).

To see that these are generalizations of Rödl's theorem, note that the hypergraph $\mathcal{H}(v, r, t)$ defined at the bottom of page 2 satisfies the hypotheses of the Frankl-Rödl theorem and hence of Theorem 1.2. The latter result forms a branching point in the path of generalizations discussed so far. In one direction, Pippenger and Spencer [27] proved that the hypotheses (3) and (4) of Theorem 1.2 actually guarantee that the obvious relationship between $\chi'(\mathcal{H})$ and $D(\mathcal{H})$, namely that $\chi'(\mathcal{H}) \geq D(\mathcal{H})$, is asymptotically tight. Their result was further generalized by Kahn [16], who replaced $\chi'(\mathcal{H})$ by $\chi'_\ell(\mathcal{H})$.

We now turn to the other branch of developments, the one extended in [21]. A few years ago, Kahn generalized Theorem 1.2 with the next result.

(1.3) Theorem ([17]) *Let $k \geq 2$ be fixed, $\mathcal{H}$ a k-bounded hypergraph, and $t : \mathcal{H} \to \mathbf{R}^+$ a fractional cover. Then $\rho(\mathcal{H}) \lesssim t(\mathcal{H})$ as $\alpha_2(t) \to 0$.*

Kahn [17] also proved an analogous result for matchings, but it is interesting to note that both parts of Theorem 1.2 may be recovered from Theorem 1.3 alone. A routine exercise shows that taking $t : \mathcal{H} \to \mathbf{R}^+$ to be identically $1/d$ gives the cover portion of Theorem 1.2. The equivalence of the two asymptotic statements $\rho \sim n/k$ and $\nu \sim n/k$, noted in [27], then gives the matching portion of Theorem 1.2.

Summary

The implications between the results discussed above are summarized in the following diagram.

IMPLICATIONS

$$\begin{tikzcd} & & (\textbf{[27]}) \arrow[d, Rightarrow] & (\textbf{[16]}) \arrow[l, Rightarrow] \\ (1.1) & (\textbf{[8]}) \arrow[l, Rightarrow] & (1.2) \arrow[l, Rightarrow] & (1.3) \arrow[l, Rightarrow] \end{tikzcd}$$

The main result of [21] further generalizes Theorem 1.3.

1.4 Outline

We devote §2 to stating our main theorem, discussing how it should be interpreted, and placing it in context with the earlier results mentioned above. In §3, we attempt to give some idea of its proof. A few "post-production" notes are presented in §4.

2 Main results

Before we can state the main result of the dissertation [21], we need another definition, which extends the conditions for "nice local behaviour" of the fractional cover t in $\mathcal{H}$. For $X \subseteq V$, define $t|_X : 2^X \to \mathbf{R}^+$ by

$$t|_X(A) \triangleq \begin{cases} \displaystyle\sum_{\substack{B \cap X = A \\ B \in \mathcal{H}}} t(B) & \text{for } A \subseteq X, |A| \geq 2 \\ 0 & \text{for } A \subseteq X, |A| < 2. \end{cases}$$

Denote by $b(t)$ the largest b such that for any $X \subseteq V$ with $|X| \leq b$, the function $t|_X$ may be expressed as a convex combination of matchings of the hypergraph 2^X. Another measure of "nice local behaviour" is that $b(t) \to \infty$.

A version of the following statement first appeared as Conjecture 8.4 in [15] and another as Conjecture 5.6 of [14]. It is the main result of the author's dissertation [21], and a proof will also appear in [20].

(2.1) Theorem *Let $k \geq 2$ be fixed, $\mathcal{H}$ a k-bounded hypergraph, and $t : \mathcal{H} \to \mathbf{R}^+$ a fractional cover. Then $\rho(\mathcal{H}) \lesssim t(\mathcal{H})$ as $\alpha_3(t) \to 0$ and $b(t) \to \infty$.*

As in Theorem 1.3, we see that $\rho(\mathcal{H})$ is asymptotically at most $t(\mathcal{H})$, now under more general hypotheses on nice local behaviour.

To see that this generalizes Theorem 1.3, we simply need to check that $\alpha_3(t) \to 0$ and $b(t) \to \infty$ as $\alpha_2(t) \to 0$. The first of these is, of course, trivial; the second is the content of an easy lemma whose proof is omitted. Thus, Theorem 2.1 indeed generalizes Theorem 1.3, and so extends the summary diagram of §1.3 and contains as corollaries many of the results discussed there. However, we should stress that the proof of Theorem 2.1 requires — basically takes as its starting point — Theorem 1.3, so we are not giving a new derivation of any of the earlier results.

We emphasize that the hypotheses "$\alpha_3(t) \to 0$" and "$b(t) \to \infty$" are a concrete expression of the notion of "nice local behaviour" of t in $\mathcal{H}$. Theorem 2.1 gives an elegant asymptotic relationship between covering behaviour (reflected by $\rho(\mathcal{H})$) and fractional covering behaviour (reflected by $t(\mathcal{H})$), provided the local behaviour is nice in this sense.

It should be noted that, unlike Theorem 1.3 where $\alpha_2(t) \to 0$ sufficed, *some* hypothesis other than $\alpha_3(t) \to 0$ is certainly needed if one hopes to obtain the conclusion $\rho(\mathcal{H}) \lesssim t(\mathcal{H})$. For consider the case when $\mathcal{H}$ is a graph (with fractional cover t): here, $\alpha_3(t) \equiv 0$, while it is certainly *not* the case that every graph $\mathcal{H}$ has $\rho(\mathcal{H}) \leq t(\mathcal{H})$ (Exercise: why has "$\lesssim$" been replaced by "$\leq$" in this remark?).

Kahn's paper [15] also mentioned two special cases of Theorem 2.1 which are stated as corollaries below. They give appealing illustrations of the "point" of the theorem by making explicit some cases where Theorem 1.3 does not apply while Theorem 2.1 does. We extend the α-notation by defining, for hypergraphs $\mathcal{H}$ and Γ on a common vertex set V and for

$t : \mathcal{H} \to \mathbf{R}^+$, the parameter $\alpha(t,\Gamma)$ by

$$\alpha(t,\Gamma) \triangleq \max_{\substack{W \subseteq V \\ W \notin \langle \Gamma \rangle}} \bar{t}(W),$$

where $\langle \Gamma \rangle$ denotes the *hereditary closure* of Γ, *i.e.*, $\langle \Gamma \rangle \triangleq \{A : \exists B \in \Gamma, B \supseteq A\}$. Asking for $\alpha(t,\Gamma) \to 0$ means asking for nice local behaviour, with some exceptions governed by Γ.

(2.2) Corollary *Let $k \geq 2$ be fixed, $\mathcal{H}$ a k-bounded hypergraph, Γ a bipartite graph on $V(\mathcal{H})$, and $t : \mathcal{H} \to \mathbf{R}^+$ a fractional cover. Then $\rho(\mathcal{H}) \lesssim t(\mathcal{H})$ as $\alpha(t,\Gamma) \to 0$.*

When Γ is a 2-bounded hypergraph (*e.g.* a graph), we have $\alpha(t,\Gamma) \leq \alpha_2(t)$, with equality when Γ is empty, so Corollary 2.2 is itself a generalization of Theorem 1.3. Using this observation as a starting point, one can readily construct examples satisfying the hypotheses of Theorem 2.1 but not those of Theorem 1.3.

Recall for a graph Γ, the *girth*, $g(\Gamma)$, is the length of a shortest cycle in Γ. The second corollary substitutes the assumption that Γ has large girth for the hypothesis that Γ is bipartite.

(2.3) Corollary *Let $k \geq 2$ be fixed, $\mathcal{H}$ a k-bounded hypergraph, Γ a graph on $V(\mathcal{H})$, and $t : \mathcal{H} \to \mathbf{R}^+$ a fractional cover. Then $\rho(\mathcal{H}) \lesssim t(\mathcal{H})$ as $\alpha(t,\Gamma) \to 0$ and $g(\Gamma) \to \infty$.*

These corollaries express the idea that we may relax $\alpha_2(t) \to 0$ provided the graph Γ on which we allow $\bar{t}$ to be large does not itself contain obstructions to good asymptotic covering behaviour.

Having stated our results, we move toward giving an idea of how the main proof goes.

3 Proof highlights

3.1 Strategy

Avoiding technical details, here we discuss the strategy used in the proof of Theorem 2.1. Fix $k \geq 2$, let $\mathcal{H}$ be k-bounded hypergraph, and let $t : \mathcal{H} \to \mathbf{R}^+$ be a fractional cover. To prove the theorem, we must show that for any given $\varepsilon > 0$, we can achieve

$$\rho(\mathcal{H}) < (1+\varepsilon)t(\mathcal{H}), \tag{5}$$

provided $\alpha_3(t)$ is sufficiently small and $b(t)$ is sufficiently large. The fact that (5) is identical to the conclusion of Theorem 1.3 is exploited as follows. We construct a new hypergraph $\mathcal{H}^*$ and fractional cover $t^* : \mathcal{H}^* \to \mathbf{R}^+$ satisfying the hypotheses of Theorem 1.3, with the extra property that the conclusion $\rho(\mathcal{H}^*) \lesssim t^*(\mathcal{H}^*)$ implies our desired conclusion (5). The assumption that $\alpha_3(t) \to 0$ for $\mathcal{H}$ is mainly used in obtaining the hypothesis $\alpha_2(t^*) \to 0$ for $\mathcal{H}^*$.

Basically, $\mathcal{H}^*$ is constructed from $\mathcal{H}$ by contracting edges of a *nearly-perfect* matching M in a certain 2-bounded hypergraph Γ on $V(\mathcal{H})$. This means the vertex set of $\mathcal{H}^*$ is the set V^* of elements of M, while the edges of $\mathcal{H}^*$ correspond to (a subset of) the edges of $\mathcal{H}$. The matching M is chosen at random according to a certain probability distribution on $\mathcal{M}$, the set of all matchings contained in Γ, and the map $t^* : \mathcal{H}^* \to \mathbf{R}^+$ is obtained from t by scaling each $t(A)$ according to probabilities of certain events associated with the edge $A \in \mathcal{H}$.

Now we pose the question: what conditions on the pair $(\mathcal{H}^*, t^*)$ will suffice to give (5) from

$$\rho(\mathcal{H}^*) \lesssim t^*(\mathcal{H}^*) \; ? \tag{6}$$

Suppose we establish the additional properties

$$t^*(\mathcal{H}^*) \lessapprox t(\mathcal{H}) \tag{7}$$

and

$$|\{x \in V : x \prec M\}| \geq (1 - \zeta)n, \tag{8}$$

where $x \prec M$ means x is contained in some element of M, and $\zeta > 0$ is as small as we need. Then, given a cover $\mathcal{G}^* \subseteq \mathcal{H}^*$ of $\mathcal{H}^*$, we obtain a cover of $\mathcal{H}$ by using the edges in $\mathcal{H}$ corresponding to $\mathcal{G}^*$ (whence covering all vertices $x \prec M$), together with an edge of $\mathcal{H}$ for each vertex $x \not\prec M$. Inequality (8) shows that this greedy stage requires only at most ζn extra edges; using this fact, then (6) followed by (7) gives

$$\rho(\mathcal{H}) < \rho(\mathcal{H}^*) + \zeta n \lesssim t^*(\mathcal{H}^*) + \zeta n \lessapprox t(\mathcal{H}) + \zeta n. \tag{9}$$

Finally, the basic inequality $n \leq kt(\mathcal{H})$ (valid for any fractional cover t of a k-bounded hypergraph on n vertices) and the fact that ζ is sufficiently small may be used to obtain (5) from (9).

The remainder of the proof consists of arranging for the hypotheses of Theorem 1.3 to be satisfied for (a modified version of) the pair $(\mathcal{H}^*, t^*)$, proving precise versions of the properties (7) and (8), and making the estimate (9) precise. These sometimes difficult and technical steps are facilitated mainly through repeated application of the inequalities of Chebyshev and Markov.

3.2 "Normal" distributions[†]

Here we say a little more about the objects Γ, $\mathcal{M}$, and the probability distribution p on $\mathcal{M}$, all mentioned in the preceding section. Starting with the hypergraph $\Sigma \triangleq V \cup \binom{V}{2}$ on $V(\mathcal{H})$, we pare this rather abundant object down to a subhypergraph $\Gamma \subseteq \Sigma$ that is in a certain technical sense *compatible* with $\mathcal{H}$. Some definitions are needed before much more can be said.

Recall that $\mathcal{M}$ denotes the set of matchings contained in Γ. A probability distribution p on $\mathcal{M}$ is said to have *marginals* f if, for M a matching chosen at random according to p

[†] *Normal* has been superseded by *hard-core*; see the remark under **Nomenclature** in §4.

and any $F \in \Gamma$, we have $\Pr(F \in M) = f(F)$. The distribution p is called *normal* if it can be represented as

$$p(M) = \frac{\prod_{F\in M} \lambda(F)}{\sum_{M'\in\mathcal{M}} \prod_{E\in M'} \lambda(E)}$$

for each $M \in \mathcal{M}$, where λ is a function from Γ to $\mathbf{R}^+$. This rather unfortunate choice of terminology is inherited from its connection with genetic algorithms, where the term *normal population* arises; see, *e.g.*, [28]. Some authors early to consider normal distributions were H. Kunz [24], and O.J. Heilmann and E.H. Lieb [12, 13]. Each of these papers considered properties of normal distributions connected with the so-called *matching polynomial* of a graph and approached the subject from the point of view of theoretical physics. As there is an extensive literature on the matching polynomial (see [11] for just a trace), we do not attempt here to give a complete bibliography on the subset related to normal distributions, and content ourselves with the references already given.

It is natural to try to characterize the marginals f for which there exists a normal distribution p on $\mathcal{M}$. Such a characterization was given recently by Rabinovich *et al.* [28] and is a generalization of Edmonds' matching polytope theorem [6].

(3.1) Theorem ([28]) *There exists a normal distribution on $\mathcal{M}$ with marginals f if and only if the following inequalities hold for all $x \in V$ and all $W \subseteq V$:*

$$\sum_{F\ni x} f(F) < 1 \quad \text{and} \quad \sum_{\substack{F\subseteq W\\ |F|=2}} f(F) < \left\lfloor \frac{|W|}{2} \right\rfloor .$$

The notion of compatibility between $\mathcal{H}$ and Γ, mentioned above, ensures an intimate connection between t and a normal distribution p on $\mathcal{M}$ with marginals f: for each $F \in \Gamma$, we have $\sum_{F'\supseteq F} f(F') = \bar{t}(F)$. This interpretation of t as giving certain probabilities is crucial in obtaining the fractional cover hypothesis for the scaled version t^* of t. After some preliminary reductions have been performed, the existence of such a p is guaranteed by Theorem 3.1, the hypotheses of which are satisfied with the help of the assumption that $b(t) \to \infty$.

Finally, we remark on the importance of obtaining the normal distribution p. The simple form of such distributions allows us to prove what came to be called our *Essential Proposition*, stating, roughly, that edges $F_1, \ldots, F_\ell \in \Gamma$ which are pairwise far apart in Γ behave approximately independently, in the sense that $\Pr(F_1, \ldots, F_\ell \in M) \approx \prod_{i=1}^{\ell} f(F_i)$. This "near-independence" is used as a substitute for true stochastic independence in each application of Chebyshev's inequality.

4 Notes added for technical report

Applications

The approximate independence properties of normal distributions, mentioned in the preceding paragraph, have already been useful in several settings beyond that of proving Theorem 2.1, for which they were originally developed. Kahn [18] proved a "central limit"-type theorem for matchings drawn uniformly from the set of all matchings in a simple graph. He also showed (in [19]) that for multigraphs G, the chromatic index $\chi'(G)$ is asymptotically the *fractional* chromatic index of G as the latter grows large. Both results rely to some extent on the near-independence properties of normal distributions. See the papers cited above for details (and definitions) omitted here; see [22] for a mini-survey of these applications.

Nomenclature

Partly due to its overburdened use, the term "normal" will be replaced by "hard-core" in future discussions (*e.g.* [18, 19, 20]) of the probability distributions of the type introduced in §3.2. The latter term is used for these distributions in statistical physics (see, *e.g.*, [5]), where the values $\lambda(E)$ are often called *activities.*

Activity

In closing, we note that at present, there remains considerable liveliness in the area of asymptotic packing and covering of hypergraphs. In [26], N. Kuzjurin studied the functions $P(v, r, t)$ and $C(v, r, t)$ of Rödl's theorem (Theorem 1.1) and investigated the question: what is a threshold function $r(v)$ for the existence of asymptotically good packings and coverings when t increases slowly? Spencer [35] discovered a technique using branching processes that yields "continuous" proofs of Theorems 1.1, 1.2 and 1.3 (the author learned of the implication for Theorem 1.3 from [36]). His technique shows, moreover, that a "random greedy" algorithm almost surely finds an "efficient" cover for each relevant hypergraph. An alternate proof of this last fact as it relates to Pippenger's theorem (Theorem 1.2) was then given by Rödl and L. Thoma [30]. Their proof is based on the Rödl-nibble.

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