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Very weak zero one law for random graphs with order and random binary functions

by

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0 Introduction

In Compton Henson Shelah [CHSh245] it was proved that the zero one law fail for first order logic, for $([n], F_n)$, F_n the random two place function (on $[n] = \{1, 2, \ldots, n\}$) where

Definition 0.1 *The zero-one law for the logic $\mathcal{L}$ and random model M_n means that for a fixed vocabulary τ, for each natural number n, μ_n is a probablity measure on a set K_n of τ-models, such that for each sentence $\psi \in \mathcal{L}$, the sequence $\langle \mathrm{Prob}_{\mu_n}(M \models \psi) : n < \omega \rangle$ converge to zero or to one. Usually K_n is the set of τ-models with universe $[n] = \{1, 2, 3, \ldots, n\}$ with equal probability if not said otherwise. We will let M_n be a random model chosen by μ_n and say $\mathrm{Prob}_{\mu_n}(M_n \models \psi)$ converge ... (as a function of n). $\mathcal{L}$ is first order logic if not said otherwise.*

It was not clear at the time whether we can really interprate arithmetic up to n on the random model $([n], F_n)$ (i.e. with large probability) so that we can e.g. find a sentence ψ, almost surely true if n even and almost surely false if n is odd (as in Kaufman Shelah [KfSh201]) or not.

The situation was similar concerning the random graph of probability p with order , (i.e. we flip a coin with probability p for each possible edge) so we get $(G_{n,p}, <) = ([n], R_n, <)$, so we put also the linear order on $[n]$ in the model's vocabulary; this question is probably more attractive to graph theorist, the first to logicians.

Two years ago, looking at a similar situation (from Luczak Shelah [LuSh435]) we succeed in [Sh463] to give it a reasonable answer in its context : we cannot push the result to the maximum, because the very weak zero-one law hold where:

Definition 0.2 *The very weak zero one law hold for the logic $\mathcal{L}$ and $\langle (K_n, \mu_n) : n < \omega \rangle$ if for every $\psi \in \mathcal{L}$ we have:*

$\otimes$ $\lim[\mathrm{Prob}_{\mu_{n+1}}(M_n \models \psi) \ - \ \mathrm{Prob}_{\mu_n}(M_n \models \psi)] = 0$

But in [Sh463] I was not able to find if that law holds for the two context discused above, which look much simpler and immediate to formulate than the one solved there. However in summer 93 (in a meeting in Poznan) Spencer and Luczak have speculated on the subject and Spencer asked me whether the very weak zero-one law holds for the graph theoretic version, i.e. $(G_{n,p}, <)$ (i.e. for random graph on $\{1, 2, ..., n\}$ but with the model expanded by the order,with constant edge probability $p = 1/2$).

We prove here that the answer is yes , even under considerably general condition, but first we concetrate on those two examples. We relay on a theorem (1.4) quoted form Ajtai [A].

1 Random 2-place function and random graph with order.

Theorem 1.1 *For a constant $p \in (0, 1)$ we have : $(G_{n,p}, <)$ satisfies the very weak 0-1 law*

Theorem 1.2 *$([n], F_n)$ satisfies the very weak 0-1 law where F_n is the random 2-place function from $[n]$ to $[n]$.*

The proof are similar and relay on the uniformity of the situation so we try to write in a way stressing this. To prove the nearness of the values for n and $n+1$, we produce both in an indirect way:

PROOF For Theorem 1.1
STAGE A:

First chooses a random graph $G_{2n+1,p}$ on $[2n+1]$ with set of edges called $R_{2n+1,p}$,

second chooses subsets P_n^0, P_n^1 of $[2n+1]$ of size n, size $n+1$ respectively (all subsets of this size has the same probability). Next for $\ell = 0, 1$ let N_n^ℓ be $(P_n^\ell, R_{n,p} \restriction P_n^\ell, < \restriction P_n^\ell)$. Lastly let $M_n^\ell = ([n+\ell], R_n^\ell, <)$ be the uniques model of this form isomorphism to N_n^ℓ.

Theorem 1.2
STAGE A

First choose a random THREE place function F_n^* from $[2n+1]$ into itself (with all functions of the same probability)

Second choose for $\ell = 0, 1$, subsets P_n^ℓ of $[2n+1]$ of size n , size $n+1$ respectively (all subsets of this size has the same probability). Now for $\ell = 0, 1$ from P_n^ℓ and F_n^* we will produce together a model N_n^ℓ: it is $(P_n^\ell, F_n^{*,\ell})$ (so the points are the elements of P_n^ℓ) and $F_n^{*,\ell}$ is a (partial) two place function from P_n^ℓ into itself defined by:

> $F_n^{*,\ell}(i,j)$ is $F^*(i, j, k_n^\ell(i,j))$ where $k = k_n^\ell(i,j)$ is the minimal $k \in [2n+1]$ for which this value is in P_n^ℓ

(this seem lengthy but works for all i, with probablity close to 1). Lastly let M_n^ℓ be the unique model $([n+\ell], F_n^\ell)$ isomorphism to N_n^ℓ by an isomorphism preserving the order.

<u>continuation of the proof of Theorem 1.1</u>
STAGE B:

Fact 1.2.1 *1. N_n^ℓ has the same distribution as $(G_{n+\ell,p}, <)$.*

2. For a given first order ψ there is a first order sentence φ_ψ in the vocabulary $\{<, R, P\}$ ($<$, R two place predicates, P a unary predicates) such that for $\ell = 0, 1$:

$$N_n^\ell \models \psi \textit{ iff } ([2n+1], <, R, P_n^\ell) \models \varphi_\psi$$

3. For proving the theorem it sufice to prove

$$\mathrm{Prob}(N_n^0 \models \psi \leftrightarrow N_n^1 \models \neg\psi) \textit{ converge to zero.}$$

PROOF : 1) Because we can first draw the subsets P_n^ℓ, and only then draw say the graph - this is totally trivial. (We use the M_n^ℓ's)
2) Immediate
3) Reflect

continuation of the proof of Theorem 1.2
STAGE B

Fact 1.2.2 *0) With probability converging to 1, F_n^ℓ is a total two place function from $[n+\ell]$ to itself (i.e. not really partial).*

1) Modulo the assumption that F_n^ℓ is total, the distribution of M_n^ℓ is the same as that of $([n+\ell], F_n)$

2) For any given first order ψ there is a first order sentence φ_ψ in the vocabulary $\{F, P\}$ (F three place function,P one place predicate) such that for $\ell = 0, 1$:

$$N_n^\ell \models \psi \textit{ iff } ([2n+1], F_n, P_n^\ell) \models \psi_\varphi$$

3) For proving the theorem it sufice , for each first order sentence in the right vocabulary, to prove

$$\text{Prob}(N_n^0 \models \psi \leftrightarrow N_n^1 \models \neg\psi) \textit{ converge to zero.}$$

PROOF

0),1) We can first draw P_n^ℓ. Next, concerning F^* we can think of drawing the values of $F^*(i,j,k)$ by induction on k so we have a possibility of getting a "bad" value i.e. not in P, and then we look at the next k, of course there are values for later $k's$ but they do not affect us. The main point is that all values have the same probability and for different pairs (i,j) from P_n^ℓ, and the decisions are independent. Also what about the possibility that for some i,j for no k will $F^*(i,j,k) \in P_n^\ell$? The probability for fixed i,j is $\leq (\frac{n+\ell}{2n+\ell})^{n+\ell} \leq \frac{1}{1.5^n}$, so the probability that this occurs for some $i,j \in P_n^\ell$ is neglible, proving clause (0) (well, for $F_n^{*,\ell}$, hence for F_n^ℓ)

2), 3) Immediate.

Discussion 1.3 Why does this work? The point was the uniformity in the edge distribution etc.

* * *

Continuation of the proofs of Theorems 1.1, 1.2:

STAGE C:

We have actually proved (in clause (2) of the Fact above), that the following suffice:

Theorem 1.4 *For a given vocabulary τ and sentence ψ in vocabulary $\tau + P$, for every $\epsilon > 0$, for every n large enough and model M with universe $[2n+1]$ and vocabulary τ , the following two probabilities differ by less than ϵ:*

a. *the probability of* (M, P) *satisfies* ψ *where we draw a subset* P *of* M *with* $n+1$ *elements (all subsets of the same probability)*

b. *the probability of* (M, P) *satisfies* ψ *where we draw a subset* P *of* M *with* n *elements (all subsets of the same probability).*

PROOF By Ajtai [A], 5.7 (i.e. by its proof), provided that throghout we use "balanced cilinders" were

Definition 1.5 *we call a subset* A *of* ${}^n2$ *a balanced cilinder if it has the form* $\{f \in {}^n2 : f$ *extend* $f_0\}$ *, with* f_0 *balanced , which means:* $|\{i : f(i) = 0\}| = |\{i : f(i) = 1\}|$.

More fully, choose numbers $2n + 1 = n_0 > n_1 > n_2, \ldots n_k$ such that n_ℓ is very near to $n^{\epsilon(\ell)}$ for appropriate $\epsilon(\ell)$'s and $n_\ell - n_{\ell+1}$ is even (so each n_ℓ is odd). Now we choose the $P's$ in the following process (most of it common to both) (all choices are with equal distribution)

choose a subset A_0 of $[2n + 1]$ of size $n_0 - n_1$, choose a balaced function f_0 from A_0 to $\{0, 1\}$,

choose a subset A_ℓ of $B_\ell =^{df} [2n + 1] \setminus \cup_{m<\ell} A_m$ of size $n_\ell - n_{\ell+1}$ and choose a balanced function f_ℓ from A_ℓ to $\{0, 1\}$.

Lastly choose P_n^ℓ , a subset of $[2n + 1]$ of size $n + \ell$ such that for each $\ell \in \{1, \ldots, k\}$, we have $x \in A_\ell$ and

$$x \in P_n^\ell \text{ iff } f_\ell(x) = 1.$$

(Note : in $[2n + 1] \setminus \cup_{m<\ell} A_m$ there are exactly n_ℓ elements and the free choice for P_n^ℓ is only in $B =^{df} [2n+1] \setminus \cup_{m<k} A_m$, which has n_k elements, n_k is odd so as each f_ℓ is balanced , we will choose $[n_k/2] + \ell$ elements from B_k.

Note that this is equivalent to the previous random choice of P_n^ℓ.

<u>End of the pf of Theorem 1.1,1.2</u>

By Theorem 1.4 and clause (2) of the Fact from stage B above. $\blacksquare_{1.1}$ $\blacksquare_{1.2}$

* * *

Discussion 1.6 **(1)** Note that the results of [Sh463] cannot be gotten in this way as the proof here use high symmetry

(2) We may want to phrase the result generally;

One way: just say that M_n, M_{n+1} can be gotten as above : draw a model on $[2n + 1]$ (i.e. with this universe), then choose randomly subsets P_n^ℓ with $n + \ell$ elements and restrict yourself to it.

Another formulation, more specific: the random model for n is (for simplicity) on $[n]$, has relations : $<$ and the relations induced by random function (unary or k-place) into some fixed (= not depending on n) structures $\mathfrak{A}$ (finite or a probability space) and random functions from n to itself , the probability distribution uniform.

(3) Two random linear order satisfies the very weak 0-1 zero law (mean: take two random functions from $[n]$ to $[0,1]_{\mathbb{R}}$). Easily fall under 1.6(2).

(4) All this is for fixed probabilities; we then can allow probabilities depending on n so that each model $\mathfrak{A}$ is replaced by an $\mathfrak{A}_n$ i.e. it vary with n.

(5) As said in [Sh463] we can also consider $\lim(\mathrm{Prob}_{h(n)}(M_{h(n)} \models \psi) - \mathrm{Prob}_n(M_n \models \psi)) = 0$, i.e. characterize the function h for which this holds. Hopefully there is a threshold phenomena. Probably this family of problems will appeal to mathematicians with an analytic background.

(6) If we use logic stronger than first order , it cannot be too strong (on monadic logic see [KfSh201]), but we may allow quantification over subsets of size k_n, e.g. $\log(n)$ there are two issues:

- **(A)** when for both n and $n+1$ we quantify over subsets of size k_n, we should just increase M by having the set $[n]^{k_n}$ as a set of extra elements, so in (*), P is chosen as a random subset of $\{1, 2, 3, \ldots, 2n, 2n+1\}$ with n or $n+1$ elements but the model has about $(2n+1)^{k_n}$ elements; this require stronger theorem, still true (up to very near to exponentiation)
- **(B)** if $k_n \neq k_{n+1}$ we need to show it does not matter we may choose to round $k_n = \log_2(n)$ so only for rare n the value change so we weaken a little the theorem *or* we may look at sentences for which this does not matter . Or we can choose k_n radomly with not too large expectation . E.g. if for a random graph on n (probability 0.5) we ask on the property "the size of maximal clique of size at most $[\log_2 n]^2$ is even" it satisfies the very weak zero one law

Of course we know much more on this still it show that this old result (more exactly - a weakened version) can be put in our framework.

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