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Zero one laws for graphs with edge probabilities decaying with distance

by

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ABSTRACT

Let G_n be the random graph on $[n] = \{1, \ldots, n\}$ with the possible edge $\{i, j\}$ having probability being $p_{|i-j|} = 1/|i-j|^\alpha$, $\alpha \in (0, 1)$ irrational. We prove that the zero one law (for first order logic) holds.

0 Introduction

On 0-1 laws see expository papers e.g., Spencer [Sp]. In Łuczak, Shelah [LuSh 435] the following probabilistic context was investigated. Let $\bar{p} = \langle p_i : i \in \mathbb{N} \rangle$ be a sequence of probabilities, i.e. real numbers in the interval $[0,1]_{\mathbb{R}}$. For each n we draw a graph $G_{n,\bar{p}}$ with set of nodes $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$; for this we make the following independent drawing:

- for each (unordered) pair $\{i, j\}$ of numbers from $[n]$ we draw yes/no with probabilities $p_{|i-j|} /\ 1 - p_{|i-j|}$, and let

$$R_n = \{\{i, j\} : i < j \text{ are in } [n] \text{ and we draw yes}\}.$$

We consider R_n a symmetric irreflexive 2-place relation. So we have gotten a random model $\mathcal{M}^0_{n,\bar{p}} = ([n], R_n)$ (i.e. a graph), but we also consider the graph expanded by the successor relation $\mathcal{M}^1_{n,\bar{p}} = ([n], S, R_n)$ where $S = \{(\ell, \ell+1) : \ell \in \mathbb{N}\}$, (more exactly we use $S_n = S \restriction [n]$), and we also consider the graph expanded by the natural order on the natural numbers $\mathcal{M}^2_{n,\bar{p}} = ([n], <, R_n)$. Though we shall deal with random models, the reader can restrict himself to the case of graphs not losing comprehensibility.

Much information was gotten there, on when the 0-1 law holds (see Definition 1.1(1)) and when the convergence law holds (see Definition 1.1(2)), depending on conditions such as $\sum\limits_{i \in \mathbb{N}} p_i < \infty$ and $\sum\limits_{i \in \mathbb{N}} i p_i < \infty$.

The sequences $\bar{p}$ considered in [LuSh 435] were allowed to be quite chaotic, and in those circumstances the theorems were shown to be the best possible, e.g. counterexamples were gotten by replacing $\bar{p}$ by $\bar{p}'$ where for some fast increasing $\langle i_k : k \in \mathbb{N} \rangle$, $p'_j = \begin{cases} p_k & j = i_k \\ 0 & (\forall k) j \neq i_k. \end{cases}$

In [Sh 463] a new version of the 0-1 law was introduced, the very weak zero one law (see 0.1(3), the h version says that the difference between the probabilities for n and for m_n when $|n - m_n| \leq h(n)$, converges to zero) and it was proved for $\mathcal{M}^2_{n,p}$ when $\sum\limits_i p_i < \infty$ (we omit h when $m_n = n+1$ and investigate only the very weak 0-1 law). In [Sh 548] the very weak zero one law was proved for model with a random two place function and for graphs; Spencer has continued this determining the best h for which this holds.

Naturally arise the question what occurs if the p_i's are "well behaved". As in Shelah, Spencer [ShSp 304] this leads to considering $p_i = 1/i^a$. By the results of [LuSh 435], and (essentially) [ShSp 304], the "real" cases are (on the definition of $\mathcal{M}^\ell_{n,\bar{p}}$ see above):

(A) $\mathcal{M}^0_{n,\bar{p}}$ where $p_i = 1/i^\alpha$ for $i > 1$, $\alpha \in (0,1)_{\mathbb{R}}$ irrational and $p_1 = p_2$

(B) $\mathcal{M}^1_{n,\bar{p}}$ where $p_i = 1/i^\alpha$, $\alpha \in (0,1)_{\mathbb{R}}$ irrational

(C) $\mathcal{M}^2_{n,\bar{p}}$ where $p_i = 1/i^\alpha$, $\alpha \in (1,2)_{\mathbb{R}}$

The main aim of this work is to show that in the case **(A)** we have the 0-1 law, also in case **(B)** we prove the convergence law but at present we do not know the answer to problem **(C)** (note that the problem is whether there is a formula $\varphi(x)$ which holds in $\mathcal{M}^2_n$ for x small enough and fails for $n-x$, x small enough).

Note: if we let $p_i = 1/i^a$, surely $\{\ell, \ell+1\}$ is an edge, so it is fine, just essentially case **(A)** becomes case **(B)**. To preserve the distinction between **(A)** and **(B)** we set $p_1 = 1/2^\alpha$ in case **(A)**. This is one of many ways to preserve this distinction; the choice does not matter.

Main and original context

Random graph on $[n]$, with $p_i = 1/i^\alpha$ for $i > 1$ and $p_1 = p_2$; i.e. probability of the edge $\{i,j\}$ is $p_{|i-j|}$ and $\alpha \in (0,1) \setminus \mathbb{Q}$ i.e. is irrational.

But the proofs apply to wider family of cases. We can make a case such that both [ShSp 304] and [LuSh 435] are particular cases: the probability for $\{i,j\}$ being an edge of $\mathcal{M}_n$ for $i,j \in [n]$ is p^n_i. So in [ShSp 304], $p^n_i = p_n$ and in [LuSh 435], $p^n_i = p_i$. We then note that we shall get the same theory as in case **(A)** above in the limit, while simplyfying the probabilistic arguments, if we change the context to:

Second context

for $\mathcal{M}_n$ (graph on $\{1,\ldots,n\}$) with probability of $\{i,j\}$ being an edge is $p^n_i = \frac{1}{n^a} + \frac{1}{2^{|i-j|}}$.

So the probability basically has two parts

1) $(\frac{1}{2^{|i-j|}})$: depends only on the distance, but decays fast, so the average valency it contributes is bounded.

2) $(\frac{1}{n^\alpha})$: Does not depend on the distance, locally is negligible (i.e. for any particular $\{i,j\}$) but has "large integral". Its contribution for the valency of a node i is on the average "huge" (still $\ll n$).

We can think of this as two kinds of edges. The edges of the sort $n^{-\alpha}$ are like in the paper [ShSp 304]. The other ones still give large probability for some i to have valency with no *a priori* bound (though not compared to n, e.g. $\log n$).

Note: If we look at all the intervals $[i, i+k)$, and want that we get some ordered graph there and probability depends first on k, then the chance that for some i we get this graph (by "first kind edges") is ~ 1, essentially this behavior stops where $k \approx (\log(n))^b$ for some appropriate $b > 0$ (there is no real need here to calculate it). Now for any graph H on $[k]$ the probability that for a particular $i < n-k$ the mapping $\ell \mapsto i+\ell$ embeds H into $\mathcal{M}_n$ is $\geq (\frac{1}{k^\alpha})^{\binom{k}{2}}$ but is $\leq (\frac{1}{(k/3)^\alpha})^{(k/3)^2}$ (exactly $\prod\limits_{i<k} \frac{1}{i^{\alpha\cdot(k-i)}}$). Hence the probability that for $i < n/k$ the mapping $\ell \mapsto k\cdot i+\ell$ does embed H into $\mathcal{M}_n$ is $\leq \left(1-(\frac{1}{k^\alpha})^{\binom{k}{2}}\right)^{n/k}$. Hence if $\binom{k}{2} \prec (\log k)^2 = \beta^{\log n}$ then this probability is $\leq e^{-\beta}$. So this holds around $k = (\log n)^{1/2}$ and the bound for the other direction has the same order of magnitude. So with parameters, we can interpret, using a sequence of formulas $\bar\varphi$ and parameter $\bar a$, quite long initial segment of the arithmetic (see definition below). This is very unlike [ShSp 304] where first order formula $\varphi(\bar x)$ really says little on $\bar x$: normally that its cl^k–closure is $\bar x$ or something on the few elements which are in $\mathrm{cl}^k(\bar x)$ (so the first order sentences say not little on the model, but inside a model the first order formula says little). This had seemed like a sure sign of failure of the 0-1 law, but if one goes in this direction one fails to define $\bar a_0$ such that $\bar\varphi$ with the parameter $\bar a_0$ defines a maximal such initial segment of arithmetic, or at least are of size, say, $> \log\log\log n$. To interpret an initial segment of arithmetic of length k in $\mathcal{M}_n$ for $\bar\varphi$, meant that $\bar\varphi = \langle \varphi_0(\bar x^0, \bar y), \varphi_1(\bar x^1, \bar y), \varphi_2(\bar x^2, \bar y)\rangle$ is a sequence of (first order) formulas, and we can find a sequence $\bar a$ of length $\ell g(\bar y)$ such that:

the set $\{x : \mathcal{M}_n \vDash \varphi_0(x, \bar a)\}$ has k elements, say $\{b_0, \ldots, b_{k-1}\}$, satisfying:

$$\mathcal{M}_n \vDash \varphi_1(x_0, x_1, \bar a_0) \Leftrightarrow \bigvee_{\ell < m < k} (x_0, x_1) = (b_\ell, b_m),$$

$$\mathcal{M}_n \vDash \varphi_2(x_0, x_1, x_2, \bar a_0) \Leftrightarrow \bigvee_{\substack{\ell_0, \ell_1, \ell_2 > \ell \\ \ell_2 = \ell_0 + \ell_1}} (x_0, x_1, x_2) = (b_{\ell_0}, b_{\ell_1}, b_{\ell_2}),$$

$$\mathcal{M}_n \vDash \varphi_3(x_0, x_1, x_2, \bar a_0) \Leftrightarrow \bigvee_{\substack{\ell_0, \ell_1, \ell_2 < \ell \\ \ell_2 = \ell_0 \ell_1}} (x_0, x_1, x_2) = (b_{\ell_0}, b_{\ell_1}, b_{\ell_2}).$$

But it is not *a priori* clear whether our first order formulas distinguish large and small in such interpretation.

Note: all this not why 0-1 law holds, just explain the situation, and show we cannot prove the theory is too nice on the one hand but that this is not sufficient for failure of 0-1 law. Still what we say applies to both contexts, which shows that results are robust. A nice result would be if we can characterize $\langle p_i : i \in \mathbb{N}\rangle$ such that $\mathrm{Prob}\{i, j\} = p_i \Rightarrow 0 - 1$ holds (see below).

Our idea is that though the "algebraic closure" (suitably defined) is not bounded, it is small and we can show that a first order formula $\varphi(\bar x)$ is equivalent (in the limit case) to one speaking on the algebraic closure of $\bar x$.

Model theoretically we do not get in the limit a first order theory which is stable and generally "low in the stability hierarchy", see Baldwin, Shelah [BlSh 528], for cases with probability $\sim n^{-\alpha}$. However we get a variant of stability over a predicate: on "small" definable sets the theory is complicated, but for types with no small formulas we are in the stable situation. In fact the model theoretic setting is like [Sh 463] but we shall not pursue this.

Note that Baldwin, Shelah [BlSh 528] deal with random model with more relations R with probabilities $n^{\alpha(R)}$ (satisfying the parallel to irrationality of α). There, the almost sure theory is stable. In [Sh 550] we define a family of 0-1 contexts where further drawings of relations give us a new context in this family and in all such contexts, elimination of quantifiers to the algebraic closure (as in [ShSp 304], [BlSh 528]) holds, but the context is "almost nice" not nice, i.e. every $\bar a$ has a nontrivial closure. Here this is dealt with the general treatment of the elimination, but not used in the main case $\mathcal{M}^0_n$. We could deal with abstract version allowing further drawing also here.

We have choosen here quite extreme interpretation of "$\bar p$ is nice, nicely defined". It seems desirable to investigate the problem under more lenient conditions. A natural such family of $\bar p$'s is the family of monotonic ones. Can we in this family characterize

$$\{\bar p : \bar p \text{ monotonic}, \mathcal{M}^0_{n,\bar p} \text{ satisfies the } 0-1 \text{ law}\}?$$

This will be addressed and solved in [LeSh 581].

The two cases considered above are protypes of some families with the 0-1 law, but there are some others, for example with the value of the exponent α "in the neighbourhood" of a rational (and some degenerate ones of course).

Let us review the paper. In §1 we give the basic definitions, including $A <_i B$ (intended to mean: B is the algebraic closure of A but this closure has no *a priori* bound). The restriction to: $\mathcal{M}_n$ has set of elements $[n]$ (rather than some finite set) is not important for the proof. In §1 $A <_i B$ and $A <_s B$ are defined in terms of the number of embeddings of A into $\mathcal{M}_n$ in a sufficiently random model, and from $\leq_i$ we define $\mathrm{cl}^k(A, M)$.

In §2 a fundamental relation on structures M is cl^k. From it notions of $A <_i B$ and $A <_s B$ are defined in terms of embeddings $f \subseteq g$ of A, B into a sufficiently random $\mathcal{M}_n$ and the relations between $\mathcal{P}(B)$ and $\mathrm{cl}^k_{\mathcal{M}_n}(f(A))$. Then these definitions are reconciled with those in §1, where the closure is choosen as in §1. Two axiomatic frameworks for an abstract elimination of quantifiers argument are presented. (This generalizes [BlSh 528].) These frameworks and further conditions on cl^k provide suffiecient conditions for $0-1$ laws and convergence laws.

Note: in §2 we retain using "relation free amalgamation" (as in [BlSh 528], but in [Sh 550] we will use more general one). However we waive "random A has no non-trivial closure", hence use "almost nice" rather then nice.

In §3 we deal with the case the natural elimination of quantifiers is to monadic logic. This seems natural, although it is not used later.

The main point of §4 is to introduce a notion of weight $\mathbf{w}(A, B, \lambda)$ which depends on an equivalence relation λ on B. (Eventually such λ will be defined in terms of the "closeness" of images of points in B under embeddings into $\mathcal{M}_n$.) Relations $A \leq^*_i B$ and $A \leq^*_s B$ are defined in terms of $\mathbf{w}$. The intension is that $\leq^*_i$ is $\leq_i$ etc, thus we have direct characterization of the later.

§5 contains the major probability estimates. The appropriate λ is defined and thus the interpretations of $<^*_i$ and $<^*_s$ in the first context $(\mathcal{M}^0_n, p_i = \frac{1}{2^{|i-j|}})$. Many proofs are analogous to that in [ShSp 304] and [BlSh 528], so we treat them shortly. The new point is the dependence on distance, and hence the equivalence relations λ.

In §6 it is shown that the $<^*_i$ and $<^*_s$ of §5 agree with the $<_i$ and $<_s$ of §1. Further, if cl^k is defined from the weight function in §4, these agree with $<_i$, $<_s$ as in §2 and we prove the "simple almost niceness" of Definition 2.11, so the elimination of quantifiers result applies. This completes the proof of the $0-1$ law for the first context.

§7 indicates the changes needed for $\mathcal{M}^1_{n,\bar{p}}$ where only the convergence law is proved.

Note: our choice "$\mathcal{M}_n$ has set of element $[n]$" is just for simplicity (and tradition), we could have $\mathcal{M}_n$ has set of elements a finite set (not even fixed) and replace n^ε by $\|\mathcal{M}_n\|^\varepsilon$ as long as "for each k for every random enough $\mathcal{M}_n$ we have $\|\mathcal{M}_n\| > k$". Also the choice of n^ε in Definition 1.2 is the most natural but not unique case. We try to make the paper self contained.

Notation 0.1

- $\mathbb{N}$ is the set of natural numbers ($\{0, 1, 2, \ldots\}$)

- $\mathbb{R}$ is the set of reals
- $\mathbb{Q}$ is the set of rationals
- $i, j, k, \ell, m, n, r, s, t$ are natural numbers
- p, q are probabilities
- α, β, γ, δ are reals
- ε, ζ, ξ are positive reals (usually quite small) and also $\mathbf{c}$ (for constant in inequalities)
- λ is an equivalence relation
- M, N, A, B, C, D are graphs or more generally models (finite of fixed vocabulary, for notational simplicity with predicates only, if not said otherwise; the reader can restrict himself to graphs)
- $|M|$ is the set of nodes or elements of M, so $\|M\|$ is the number of elements.
- $A \subseteq B$ means A is a submodel of B i.e. A is B restricted to the set of elements of A (for graphs: induced subgraphs)
  We shall not always distinguish strictly between a model and its set of elements. If X is a set of elements of M, $M \restriction X$ is M restricted to X.
- a, b, c, d are nodes of graphs / elements of model
- $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ are finite sequences of nodes / elements
- x, y, z variable
- X, Y, Z are sets of elements
- $\mathcal{K}$ is a family of models of fixed vocabulary, usually τ
- $\bar{a}\hat{\ }\bar{b}$ or $\bar{a}\bar{b}$ is the concatanation of the two sequences, $\bar{a}\hat{\ }\bar{b}$ or $\bar{a}b$ is $\bar{a}\hat{\ }\langle b\rangle$

1 Weakly nice classes

We interprate here "few" by "for each ε for every random enough $\mathcal{M}_n$, there are (for each parameter) $< n^{\varepsilon}$". We could use other functions as well.

General Context 1.1 For $n \in \mathbb{N}$ let $\mathcal{M}_n$ be a random model in a fixed vocabulary τ, $\mathcal{M}_n$ varying on $\mathcal{K}_n$ with μ_n the distribution. Usually $\mathcal{M}_n$ is a model with universe $[n] = \{1, \dots, n\}$ (just for notational simplicity). $\mathcal{K}$ is a class of finite τ-models closed under isomorphisms and submodels and such that $\mathcal{K}_n \subseteq \mathcal{K}$ for each n. We omit μ_n when clear from context. The context (i.e. $(\mathcal{K}, \langle(\mathcal{K}_n, \mu_n) : n < \omega\rangle)$) is called $\mathfrak{K}$ and is considered fixed: we may "forget" to

mention $\mathcal{K}$.
The meaning of "for every random enough $\mathcal{M}_n$ we have Ψ" is

$$\langle \mathrm{Prob}(\mathcal{M}_n \models \Psi) : n < \omega \rangle \ \text{ converges to } 1;$$

alternatively, we write "almost surely $\mathcal{M}_n \models \Psi$".

Definition 1.2 *1. The* $0-1$ *law says: whenever* φ *is a f.o. sentence in vocabulary* τ*,*

$$\langle \mathrm{Prob}(\mathcal{M}_n \mathbf{ffl}\ \varphi) : n < \omega \rangle \ \textit{converges to } 0 \textit{ or } \textit{ to } 1.$$

2. *The convergence law says: whenever* φ *is a f.o. sentence in* τ*,*

$$\langle \mathrm{Prob}(\mathcal{M}_n \mathbf{ffl}\ \varphi) : n < \omega \rangle \ \textit{is a converging sequence.}$$

3. *The very weak* $0-1$ *law says: whenever* φ *is a f.o. sentence in* τ*,*

$$\lim_n [\mathrm{Prob}(\mathcal{M}_{n+1} \mathbf{ffl}\ \varphi) - \mathrm{Prob}(\mathcal{M}_n \mathbf{ffl}\ \varphi)] = 0.$$

4. *The* h*-very weak* $0-1$ *law for* $h : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ *say: whenever* φ *is a f.o sentence in* τ*,*

$$\lim_n \max_{\ell, k \in [0, h(n)]} |\mathrm{Prob}(\mathcal{M}_{n+k} \mathbf{ffl}\ \varphi) - \mathrm{Prob}(\mathcal{M}_{n+\ell} \mathbf{ffl}\ \varphi)| = 0$$

Notation 1.3 $f : A \hookrightarrow B$ means: f is an embedding of A into B (in the model theoretic sense, for graph: isomomorphism onto the induced subgraph).

Definition 1.4 *1. Let*

$$\mathcal{K}_\infty = \Big\{ A : \begin{array}{l} A \textit{ is a finite } \tau\textit{-model} \\ 0 < \limsup\limits_n [\mathrm{Prob}((\exists f)(f : A \hookrightarrow \mathcal{M}_n))] \end{array} \Big\}.$$

2. $A \le B$ *means:* $A, B \in \mathcal{K}_\infty$ *and* A *is a submodel of* B*.*

3. $A \le_i B$ *means:* $A \le B$ *and for each* $\varepsilon \in \mathbb{R}^+$ *we have:*

$$1 = \lim_n \left[\mathrm{Prob} \left(\begin{array}{l} \textit{if } f_0 : A \hookrightarrow \mathcal{M}_n \\ \textit{then the number of } f_1 \textit{ satisfying} \\ f_0 \subseteq f_1 : B \hookrightarrow \mathcal{M}_n \textit{ is } \le n^\varepsilon \end{array} \right) \right].$$

4. $A \le_s B$ *means:* $A \le B$ *and there is no* C *with* $A <_i C \le B$*.*

5. $A \le_{pr} B$ *means:* $A \le_s B$ *and there is no* C *with* $A <_s C <_s B$ *(pr abbreviates* primitive*).*

6. *$A <_a B$ means that $A \le B$ and,for some $\varepsilon \in \mathbb{R}^+$ for every random enough $\mathcal{M}_n$, for no $f : A \hookrightarrow \mathcal{M}_n$ do we have n^ε pairwise disjoint extensions g of f satisfying $g : B \hookrightarrow \mathcal{M}_n$.*

7. *$A \le^s_m B$ means $A \subseteq B$ are from $\mathcal{K}$ and for every $X \subseteq B$ with $\le m$ elements, we have $A \setminus (A \cap X) \le_s (B \setminus X)$.*

8. *$A \le^i_{k,m} B$ means $A \subseteq B$ are from $\mathcal{K}$ and for every $X \subseteq B$ with $\le k$ elements there is Y, $X \subseteq Y \subseteq B$ with $\le m$ elements such that $A \setminus (A \cap Y) \le_i (B \setminus Y)$.*

9. *For $h : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$, we define $A \le^h_i B$ as in part (3) replacing n^ε by $h(n, \varepsilon)$, and similarly $A \le^h_a B$ (in part (6)), hence $A \le^h_{pr} B$, $A <^h_a B$, $A \le^{s,h}_m B$, $A \le^{i,h}_{k,m} B$, $A \le^h_b B$.*

Remark 1.5 1. $A \le_i B$ is construed as $B \subseteq \mathrm{acl}(A)$.

2. Why do we restrict ourselves to $\mathcal{K}_\infty$ (in 1.4(1)-(8))? The relations in 1.4(1)-(6) describe situation in the limit. So why in 1.4(7), (8) do we not restrict ourselves to $A, B \in \mathcal{K}_\infty$? As for $A \in \mathcal{K}_\infty$, for quite random $\mathcal{M}_n$, and $f : A \hookrightarrow \mathcal{M}_n$ the set $\mathrm{cl}^k(f(A), \mathcal{M}_n)$ may be quite large, say with $\log(n)$ elements, so it (more exactly the restriction of $\mathcal{M}_n$ to it) is not necessarily in $\mathcal{K}_\infty$, this is a major point here.

Let us expand.
If $A \in \mathcal{K}$ has a copy in a random enough $\mathcal{M}_n$ and we have $0 - 1$ law then T_∞ says that copies of A occur. But if $\mathcal{M}_n$ is random enough, $A = \{a_1, a_2, a_3\} < \mathcal{M}_n$, and $B = \mathcal{M}_n \setminus \mathrm{cl}^k(\{a_1, a_2, a_3\}, \mathcal{M}_n)$ has, say, $\log(n)$ elements *then* it does not follow that $T_\infty \vdash$"a copy of B occurs", as $\mathcal{M}_n$ may not be random enough for B. Still for the statement like

$$(\exists x_1, x_2, x_3)(\mathrm{cl}^k(\{x1, x_2, x_3\}) \models \varphi)$$

the model $\mathcal{M}_n$ will be random enough. The point is that the size of B could be computed only after we have $\mathcal{M}_n$.
Another way to look at it: models M_∞ of T_∞ are very random in a sense, but $\mathrm{cl}(\{a_1, a_2, a_3\}, M_\infty)$ is infinite even uncountable, so randomness concerning it becomes meaningless.

Definition 1.6 *For $A \subseteq M$ and $k < \omega$ define*

(a) $\mathrm{cl}^k(A, M) = \bigcup\{B : B \subseteq M,\ B \cap A \le_i B,$ *and* $|B| \le k\}$,

(b) $\mathrm{cl}^{k,0}(A, M) = A$,

(c) $\mathrm{cl}^{k,m+1}(A, M) = \mathrm{cl}^k(\mathrm{cl}^{k,m}(A, M))$.

Observation 1.7 *1) For all $\ell, k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$ we have*

$$1 = \lim_n\, [\mathrm{Prob}(A \subseteq \mathcal{M}_n, |A| \le \ell \Rightarrow |\mathrm{cl}^k(A, \mathcal{M}_n)| < n^\varepsilon)].$$

2) Moreover, for every $k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}^+$ for some $\zeta \in \mathbb{R}^+$ (actually, any $\zeta < \varepsilon/(k + 1)$ will do) we have

$$1 = \lim_n\, [\mathrm{Prob}(|A| \le \mathcal{M}_n, |A| \le n^\zeta \Rightarrow |\mathrm{cl}^k(A, \mathcal{M}_n)| < n^\varepsilon)].$$

Remark 1.8 True for $\mathrm{cl}^{k,m}$ too, but can use 1.6.

Definition 1.9 *$\mathfrak{K} = \langle \mathcal{M}_n : n < \omega \rangle$ is* weakly nice *if whenever $A <_s B$ (so $A \neq B$), there is $\varepsilon \in \mathbb{R}^+$ with*

$$1 = \lim_n \left[\mathrm{Prob} \left(\begin{array}{l} \textit{if } f_0 : A \hookrightarrow \mathcal{M}_n \quad \textit{then there is } F \textit{ with } |F| \geq n^\varepsilon \quad \textit{and} \\ \quad \textit{(i) } f_1 \in F \Rightarrow f_0 \subseteq f_1 : B \hookrightarrow \mathcal{M}_n \\ \quad \textit{(ii) } f' \neq f'' \in F \Rightarrow \mathrm{Rang}(f') \cap \mathrm{Rang}(f'') = \mathrm{Rang}(f_0) \end{array} \right) \right].$$

If clause (ii) holds we say the $f \in F$ are pairwise disjoint over f_0 or over A. In such circumstances we say that ε witnesses $A <_s B$.

Remark 1.10 Being weakly nice means there is a gap between being algebraic and non-algebraic, so we have a strong dichotomy.

Fact 1.11 *For every A, B, C in $\mathcal{K}_\infty$:*

1. $A \leq_i A$,
2. $A \leq_i B,\ B \leq_i C \Rightarrow A \leq_i C$,
3. $A \leq_s A$,
4. *if* $A_1 \leq B_1$, $A_2 \leq B_2$, $A_1 \leq A_2$, $B_1 \leq B_2$, $B_1 \setminus A_1 = B_2 \setminus A_2$ *then* $A_2 \leq_s B_2 \Rightarrow A_1 \leq_s B_1$ *and* $A_1 \leq_i B_i \Rightarrow A_2 \leq_i B_2$,
5. $A <_i B$ *iff for no C do we have* $A \leq C < B$ & $\neg C <_a B$.

Proof Easy (e.g. 1.11(5) by the Δ-system argument (note $|B|$ is constant).

Claim 1.12 ($\mathfrak{K}$ *weakly nice) If $A <_s B <_s C$ then $A <_s C$*

Proof Let $\varepsilon > 0$ witness $A <_s B$ in Definition 1.9 and let $\zeta > 0$ wittness $B <_s C$ in Definition 1.9. Choose $\xi = \min\{\varepsilon/2, \zeta/2\}$. Let n be large enough in particular $n^\varepsilon > |C|$ and $f_0 : A \hookrightarrow \mathcal{M}_n$. So we have (almost surely) $\{f_1^i : i < i^*\}$, where $i^* \geq n^\varepsilon$, and $f_0 \subseteq f_1^i$ and $f_1^i : B \hookrightarrow \mathcal{M}_n$ and the f_1^i's are pairwise disjoint over A.

Now, almost surely for every i we have $\{f_2^{i,j} : j < j_i^*\}$ with $f_1^i \subseteq f_2^{i,j}$ and $f_2^{i,j} : C \hookrightarrow \mathcal{M}_n$ and, fixing i, the $f_2^{i,j}$'s are pairwise disjoint over B and $j_i^* \geq n^\zeta$.
Clearly (when the above holds) for $\ell^* = n^\xi$ we can find $\{j_k : k \leq \ell^*\}$ such that $\{f_2^{k,j_k} : k < \ell^*\}$ are pairwise disjoint over A (just choose j_k by induction on k such that: $\mathrm{Rang}(f_2^{k,j_k} {}^- (C \setminus B))$ is disjoint to

$$\bigcup \{\mathrm{Rang}(f_i {}^- (B \setminus A)) : i < \ell^*\} \cup \bigcup \{\mathrm{Rang}(f^{i,j_i} {}^- (C \setminus B)) : i < k\};$$

at stage k, the number of inappropriate $j < n^\zeta$ is

$$\leq |C \setminus A| \times i + |B \setminus A| \times \ell^* \leq |C| \times \ell^* = |C| \times n^\xi.)$$

$\blacksquare_{1.12}$

Fact 1.13 *Suppose $A \leq B \leq C$.*

1. *If $A \leq_i C$* then *$A \leq_i B$ and $B \leq_i C$.*
2. *If $A \leq_s C$* then *$A \leq_s B$.*
3. *If $A <_{pr} C$ and $A \leq_s B \leq_s C$* then *either $B = A$ or $B = C$.*

PROOF Reread the definitions.

Fact 1.14 1. *If $A \leq_s B$* then *there is some $n < \omega$ and a sequence $A = A_0 <_{pr} A_1 <_{pr} \ldots <_{pr} A_n = B$ (possibly $n = 0$).*

2. *If $A <_{pr} C$ and $A < B < C$ then $B <_i C$*

PROOF For proving (2), choose a maximal B' such that $B \leq_i B' \leq C$, it exists as C is finite (being in $\mathcal{K}_\infty$), and as $B \leq_i B$ (by 1.11(1)). It follows that if $B' < B'' \leq C$ then $\neg B' \leq_i B''$ (by 1.11(2)). Hence $B' \leq_s C$. But $A <_{pr} C$ hence by the Definition 1.4(5) we have $A <_s C$ so by 1.13(2) $A <_s B'$ so by the definition of $<_{pr}$ we have $B' = C$, so $B \leq_i B' = C$ as required. Part (1) is clear as C is finite (being in $\mathcal{K}_\infty$) and the definition of $\leq_{pr}$. $\blacksquare_{1.14}$

Claim 1.15 *$\mathcal{K}$ is weakly nice iff whenever $A <_{pr} B$ there is $\varepsilon \in \mathbb{R}^+$ such that*

$$1 = \lim_n \left[\mathrm{Prob} \left(\begin{array}{c} \textit{if} \;\; f_0 : A \hookrightarrow \mathcal{M}_n \; \textit{then there is } F \textit{ with } |F| \geq n^\varepsilon \textit{ and} \\ f_1 \in F \;\; \Rightarrow \;\; f_0 \subseteq f_1 : B \hookrightarrow \mathcal{M}_n \end{array} \right) \right]$$

PROOF $\Rightarrow$ is obvious (as $A <_{pr} B$ implies $A <_s B$).

Let us prove $\Leftarrow$: we have $A \leq_s B$ and by fact 1.14(1) there is a sequence $A = A_0 <_{pr} A_1 <_{pr} \cdots <_{pr} A_k = B$. The proof is by induction on k. The induction step for $k > 1$ is by 1.12 and $k = 0$ is 1.11(3). So assume $k = 1$, hence $A <_{pr} B$. By fact 1.14(2) if $A < B' \leq B$ then $B' \leq_i B$. Fix $p \in (0,1)_{\mathbb{R}}$. If n is large enough then the probability of having both

(a) for every $f_0 : A \hookrightarrow \mathcal{M}_n$ there are at least n^ε different extensions f_1^i satisfying $f_0 \subseteq f_1^i : B \hookrightarrow \mathcal{M}_n$ and

(b) for every $a \in B \setminus A$ and $f_0^+ : A \cup \{a\} \hookrightarrow \mathcal{M}_n$ there are at most $n^{\varepsilon/2}$ different extensions f_2^i satisfying $f_0^+ \subseteq f_2^i : B \hookrightarrow \mathcal{M}_n$

is $\geq 1 - p$ (for clause (b) use $A \cup \{a\} <_i B$ for every $a \in B \setminus A$ which holds by 1.14(2)). Let $f_0 : A \hookrightarrow \mathcal{M}_n$, and let $\langle f_1^j : j < j^* \rangle$ be a maximal family of pairwise disjoint extensions of f_0 to an embedding of B into $\mathcal{M}_n$. By (b) we have

$$n^\varepsilon \leq j^* \times (B \setminus A) \times (B \setminus A) \times n^{\varepsilon/2}.$$

Hence if n is large enough, $j^* > n^{\varepsilon/3}$ (with probability $\geq 1 - p$), and this is enough. $\blacksquare_{1.15}$

Claim 1.16 $\mathrm{cl}^{k,m}(A,M) \subseteq \mathrm{cl}^{k^*}(A,M)$ *where* $k^* = \sum\limits_{i<m} k^i$.

Proof Define $k(\ell)$ by induction on $\ell \le m$: $k(0)=1$ and for $\ell < m$, $k(\ell+1) := (k(\ell)+1)k$. For $\ell \le m$ define $A_\ell = \mathrm{cl}^{k,\ell}(A,M)$. Now if $x \in A_m$ then there is some $\ell < m$ such that $x \in A_{\ell+1} \setminus A_\ell$. Let us prove by induction on $\ell \le m$ that $x \in A_\ell \Rightarrow x \in \mathrm{cl}^{k(\ell)}(A,M)$. For $\ell = 0$ this is clear. If $x \in A_{\ell+1} \setminus A_\ell$ then there is C with $|C| \le k$ such that $x \in C$ and $C \cap A_\ell <_i C$. By the induction hypothesis, for $y \in C \cap A_\ell$ we have $y \in \mathrm{cl}^{k(\ell)}(A,M)$ hence there is C_y with $|C_y| \le k(\ell)$ such that $y \in C_y$ and $C_y \cap A <_i C_y$. Let $C^0 = \bigcup\limits_{y \in C \cap A_\ell} C_y \cap A$, $C^1 = \bigcup\limits_{y \in C_y \cap A_\ell} C_y$ and $C^2 = C^1 \cup C$. We get

$$|C^2| \le k(\ell) \cdot |C \cap A_\ell| + |C \setminus A_\ell| \le (k(\ell)+1) \cdot k \le k(\ell+1),$$

so (as $x \in C^2$) it suffices to show that $C^0 \le_i C^2$ and by transitivity (i.e. by 1.11(2)) it suffices to show that $C^0 \le_i C^1$ and that $C^1 \le_i C^2$. Why $C^1 \le_i C^2$? Because $C \cap A_\ell \le_i C$ and $C \cap A_\ell \subseteq C^1 \subseteq A_\ell$ and hence $C^1 \le_i C^1 \cup C = C^2$ by 1.11(4). Why $C^0 \le_i C^1$? Let $C \cap A_\ell = \{y_i : i < r\}$. Now $C^0 \le_i C^0 \cup C_{y_0}$ because $A \cap C_{y_0} \le_i C_{y_0}$ and by induction

$$C^0 \le_i C^0 \cup C_{y_0} \le_i C^0 \cup C_{y_0} \cup C_{y_1} \le_i \ldots \le_i C^0 \cup \bigcup_{i<r} C_{y_i} = C^1.$$

$\blacksquare_{1.16}$

Claim 1.17 *For every* $\varepsilon \in \mathbb{R}^+$ *and* ℓ, k, m *we have*

$$1 = \lim_n \left[\mathrm{Prob} \left(\begin{array}{l} \text{if } A \in \mathcal{K}_\infty, |A| \le \ell \text{ and } f : A \hookrightarrow \mathcal{M}_n \\ \text{then } |\mathrm{cl}^{k,m}(f(A), \mathcal{M}_n)| < n^\varepsilon \end{array} \right) \right].$$

Proof By the previous claim 1.16, w.l.o.g. $m = 1$. This holds by Definition 1.4(3) and Definition 1.6. $\blacksquare_{1.17}$

Fact 1.18 *1. For every A and m, k, for any $M \in \mathcal{K}$ if $f : A \hookrightarrow M$ then*

- (α) $\mathrm{cl}^{k,m}(f(A),M) \le^i_{k,m} \mathrm{cl}^{k,m+1}(f(A),M)$,
- (β) *for some $m' = m'(k,m)$ we have*

$$A_i \le^i_{k,m'} \mathrm{cl}^{k,m}(f(A),M)$$

(we can get more),

- (γ) $A_i \le_i \mathrm{cl}^{k,m}(f(A),\mathcal{M}_n)$ *or the second is not in* $\mathcal{K}_\infty$.

2. For every m, k, ℓ for some r we have:

for any $A \in \mathcal{K}_\infty$,

$$1 = \lim_n \left[\mathrm{Prob} \left(\text{if } f : A \hookrightarrow \mathcal{M}_n \text{ then } A \le^i_{\ell,r} \mathrm{cl}^{k,m}(f(A), \mathcal{M}_n) \right) \right].$$

Remark 1.19 In our main case $\mathcal{K} = \mathcal{K}_\infty$.
Note for 1.18(1)(γ) that $\mathrm{cl}^{k,m}(f(A), \mathcal{M}_n)$ is in general not neccessarily in $\mathcal{K}_\infty$.

Proof 1) Let $A_0 = f(A)$ and for $\ell \le n$ let $A_\ell = \mathrm{cl}^{k,\ell}(f(A), M)$, and assume $A_n \in \mathcal{K}_\infty$. So for $\ell < m$ we have $A_{\ell+1} = A_1 \cup \bigcup_{j<m_\ell} C_{\ell,j}$ with $|C_{\ell,j}| \le k$ and $A_{\ell+1} \cap C_{\ell,j} \le_i C_{\ell,j}$. It follows by 1.11 that $\langle A_\ell \cup \bigcup_{i<j} C_{\ell,i} : j \le m_\ell \rangle$ is $\le_i$-increasing and $A_\ell \le_i A_{\ell+1}$. By induction we get $A_0 \le_i A_m$.
2) Read the proofs of 1.18(1) + 1.16. $\blacksquare_{1.18}$

Remark 1.20 In a more general context the previous conclusion is part of the definition of "$\mathcal{K}$ is nice" and also $\bigcup\!\!\!\!\cup$ of 1.23 is a basic property below.

Fact 1.21 *$\mathcal{K}_\infty$ is closed under isomorphisms and taking submodels.*

Fact 1.22 *For every ℓ, k, m there is a first order formula $\varphi(y, x_0, \ldots, x_{\ell-1})$ such that for every $M \in \mathcal{K}$ and $a_0, \ldots, a_{\ell-1}$ in M*

$$M \models \varphi(b, a_0, \cdots, a_{\ell-1}) \quad \textit{iff} \quad b \in \mathrm{cl}^{k,m}(\{a_0, \cdots, a_{\ell-1}\}, M).$$

Proof By finiteness of τ.

Definition 1.23 *$C_1 \underset{B}{\bigcup\!\!\!\!\cup} C_2$ means: they are all submodels of some $A \in \mathcal{K}$ and for every relation symbol R in τ, if $\bar{a} \subseteq C_1 \cup B \cup C_2$ and $R(\bar{a})$ holds then either $\bar{a} \subseteq C_1 \cup B$ or $\bar{a} \subseteq C_2 \cup B$.*

2 Abstract Closure Context.

Here we are inside the 0-1 context but without $\le_i$ and $\le_s$. The main result is a sufficient condition for having 0-1 law or at least convergence. We have here some amount of freedom, so we give two variants of the main result of this section: 2.14, 2.17, we shall use 2.17. Thus on a first reading one may skip Definitions 2.3(2)–(4) ("possible"), 2.8 and 2.9, Remark 2.10 and Lemma 2.14 in favour of the alternative development in Definition 2.11 and 2.15 through 2.16. Lemma 2.13 is needed in both cases. We want to "eliminate quantifiers" in a restricted sense: in the simple form we quantify only on the closure so each $\varphi(\bar{x})$ is equivalent to some ψ_φ in which quantifiers are over $\mathrm{cl}^{k,m}(\bar{x})$; all this is for a random enough model where $\mathrm{cl}^{k.m}$ is small, still it is not necessarily tiny. The closure does not need to be in $\mathcal{K}_\infty$ (though in our application it does). The quantifier elimination result generalizes the result of [BlSh 528]. The chief additional ingredient in the proof here is the use of the addition (= Feferman–Vaught) theorem to analyze a pair of models in stable amalgamation; this is necessary as we do not have *a priori* bound on the size of the closure, whereas there we have. Moreover, the argument in [BlSh 528] is simpler because $<_i$ is defined concretely from a dimension function.

Note that the assumption "cl is f.o. definable" (2.2 clause (d)) is not serious: if it fails we have all "$y \in \mathrm{cl}^k(\bar{x})$" as atomic formulas in ψ_φ.

Context 2.1 In this context in addition to $\mathfrak{K}$ (defined in 1.1) we have an additional basic operation cl which is a closure operation for $\mathcal{K}$ (see 2.2), so cl is in general not defined in Definition 1.6 and $\leq_i$, $\leq_s$, $\leq_a$ are defined in Definition 2.5 and in general are not the ones defined in Definition 1.4. However, we use $\mathcal{K}_\infty$ (from 1.4(1)). Lastly $\bigcup$ is as in 1.23 (can be axiomatized too and moreover generalize to the case of non–uniqueness, as in [Sh 550]). For simplicity assume $\tau_{\mathcal{K}}$ (the vocabulary of $\mathcal{K}$) is finite with no function symbols. In later sections saying $\mathfrak{K}$ means cl is from §1.

Definition 2.2 *1) We say* cl *is a* closure operation *for $\mathcal{K}$ if for $M \in \mathcal{K}$ and $k \in \mathbb{N}$ the operation $\mathrm{cl}^k(X, M)$ is defined if and only if $X \subseteq M$ and the operation satisfies:*

(a) $X \subseteq \mathrm{cl}^k(X, M) \subseteq M$, *and* $X \subseteq Y \subseteq M \Rightarrow \mathrm{cl}^k(X) \subseteq \mathrm{cl}^k(Y) \subseteq M$,

(b) *if* $\mathrm{cl}^k(X, M) \subseteq N \subseteq M$ *then* $\mathrm{cl}^k(X, N) = \mathrm{cl}^k(X, M)$,

(c) *for* $k \leq \ell$, $\mathrm{cl}^k(X, M) \subseteq \mathrm{cl}^\ell(X, M)$.

2) We say that the closure operation cl *is* f.o. definable *if (d) below holds true (and we assume this when not said otherwise)*

(d) *the assertion "$b \in \mathrm{cl}^k(\{a_0, \dots a_{\ell-1}\}, M)$" is f.o. definable in $\mathcal{K}$.*

3) We say cl *is* transitive *if for every k for some m, for every $X \subseteq M \in \mathcal{K}$ we have $\mathrm{cl}^k(\mathrm{cl}^k(X, M), M) \subseteq \mathrm{cl}^m(X, M)$.*

Definition 2.3 *1. For $X \subseteq M$ and $k, m \in \mathbb{N}$ we define $\mathrm{cl}^{k,m}(X, M)$ by induction on m:*

$$\mathrm{cl}^{k,0}(X, M) = X$$
$$\mathrm{cl}^{k,1}(X, M) = \mathrm{cl}^k(X, M)$$
$$cl^{k,m+1}(X, M) = cl^{k,1}(cl^{k,m}(X, M), M)$$

(if we write $\mathrm{cl}^{k,m-0}(X, M)$ and $m = 0$ we mean $\mathrm{cl}^{k,0}(X, M) = X$)

2. We say $(N, B, \bar{B}, k)$ is possible for $(\mathfrak{K}, \mathrm{cl})$ if:

(a) $B_i \subseteq N \in \mathcal{K}_\infty$, $B \subseteq N$ *and* $\mathrm{cl}^k(B_i, N) \subseteq B_{i+1}$ *for* $i < \ell g(\bar{B}) - 1$

(b) *it is not true that:*
for every random enough $\mathcal{M}_n$, for no embedding $f : N \hookrightarrow \mathcal{M}_n$, do we have:

$$\text{for } i < \ell g(\bar{B}) - 1,\ \mathrm{cl}^k(f(B_i), \mathcal{M}_n) \subseteq f(\mathrm{cl}^k(B_i), N) \cup \mathrm{cl}^k(f(B), \mathcal{M}_n).$$

3. *If we write* (N, C, B, k) *we mean* $(N, C, \langle B, \mathrm{cl}^k(B, N)\rangle, k)$.

4. *We say* $(N, B, \bar{a}, k, m)$ *is possible for* $\mathfrak{K}$ *if* $(N, B, \bar{B}, k)$ *is possible for* $\mathfrak{K}$ *where* $\bar{B} = \langle \mathrm{cl}^{k,i}(\bar{a}, N) : i \leq m\rangle$.

5. *We say the closure operation* cl^k *is* (ℓ, r)*-local when:*

   for $M \in \mathcal{K}$, $X \subseteq M$ *and* $Z \subseteq M$ *if* $Z \subseteq \mathrm{cl}^k(X, M)$, $|Z| \leq \ell$ *then for some* Y, $Z \subseteq Y$, $|Y| \leq r$ *and* $\mathrm{cl}^k(Y \cap X, M \restriction Y) = Y$.

6. *We say the closure operation* cl *is local* if *for every* k*, for some* r, cl^k *is* $(1, r)$*-local. We say that* cl *is simply local if* cl^k *is* $(1, k)$*-local for every* k.

7. *We say that* cl^k *is* r*-transparent if* cl^k *is* $(1, r)$*-local and*

   $$A \leq_i B \ \&\ |B| \leq r \qquad \Rightarrow \qquad \mathrm{cl}^k(A, B) = B.$$

   We say that cl^k *is transparent if it is* r*-transparent for some* r.

Remark 2.4 1) Concerning "possible in $\mathfrak{K}$", in the main case $\mathcal{M}^0_{n,\bar{p}}$, it is degenerated, i.e. if $\bar{a} \subseteq N \in \mathcal{K}_\infty$ then $(N, \bar{a}, k, m)$ is possible. But for the case with the successor relation it has a real role.
2) Note: *if* cl^k is $(1, r)$-local and "$y \in \mathrm{cl}^k(\{x_1, \ldots, x_r\}, M)$" is f.o. definable *then* for every m, s we have "$y \in \mathrm{cl}^{k,m}(\{x_1, \ldots, x_s\}, M)$" is f.o. definable.

Definition 2.5 (For our 0-1 context with cl **as a basic operation)**

1. $A \leq_i B$ *if and only if* $A \subseteq B \in \mathcal{K}_\infty$ *and for some* $k, m \in \mathbb{N}$ *and every random enough* $\mathcal{M}_n$ *and for every embedding* $f : B \hookrightarrow \mathcal{M}_n$ *we have* $f(B) \subseteq \mathrm{cl}^{k,m}(f(A), \mathcal{M}_n)$.

2. $A <_s B$ *if and only if* $A \leq B \in \mathcal{K}_\infty$ *and for every* $k, m \in \mathbb{N}$ *and random enough* $\mathcal{M}_n$ *and* $f : A \hookrightarrow M_n$ *there is* g *such that* $f \subseteq g$*, and* $g : B \hookrightarrow \mathcal{M}_n$ *with* $g(B) \cap \mathrm{cl}^{k,m}(f(A), \mathcal{M}_n) = f(A)$.

3. $(\mathfrak{K}, \mathrm{cl})$ *is* weakly nice *if for every* $A \subseteq C \in \mathcal{K}_\infty$*, for some* B *we have* $A \leq_i B \leq_s C$.

4. *We say* $\mathfrak{K}$ *(more exactly* $(\mathfrak{K}, \mathrm{cl})$*) is* smooth *when:*

   $$\text{if } A \subseteq B \subseteq N \in \mathcal{K}_\infty,\ A \subseteq C \subseteq N,\ B \underset{A}{\overset{N}{\biguplus}} C,$$
   $$\text{then } B <_i B \cup C \Leftrightarrow A <_i C$$

   (note $\Leftarrow$ *is always true).*

Fact 2.6 $\leq_i, \leq_s$ *and* cl *from section 1 satisfy definitions 2.2(1)-(3), 2.3(5), (6), and 2.5(1)-(2) and, in particular* cl *is transitive and* cl *is simply local. The notions of weakly nice defined in §1 and in 2.5(3) are the same.*

Remark 2.7 Note that the assumption "$\mathfrak{K}$ is weakly nice" is very natural in the application we have in mind *but* is not natural in this context.

Definition 2.8 *The 0-1 context $\mathfrak{K}$ with closure* cl *(or the pair $(\mathfrak{K}, \mathrm{cl})$ or $\mathfrak{K}$ when* cl *is understood) is* almost nice *if*

(A) the universal demand:

for every k, m_0 and ℓ, ℓ' there are

$$m^* = m^*(k, m_0, \ell, \ell') > m_0, \;\; k^* = k^*(k, m, \ell, \ell') \geq k \;\; \text{and } t = t(k, m_0, \ell, \ell')$$

such that for every random enough $\mathcal{M}_n$ we have:

if $\bar{a} \in {}^{\ell}|\mathcal{M}_n|$ *and* $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n)$

then *there are $m_1 \leq m^* - m_2$ and $m_2 \in [m_0, m^*]$ and $B \subseteq \mathrm{cl}^{k, m_1}(\bar{a}, \mathcal{M}_n)$ and $B^* \subseteq \mathcal{M}_n$ such that:*

(α) $|B| \leq t$ *and* $\bar{a} \subseteq B$,

(β) $B^* = [\mathrm{cl}^{k, m_0}(\bar{a}b, \mathcal{M}_n) \setminus \mathrm{cl}^{k, m_2}(B, \mathcal{M}_n)] \cup B$ *so neccessarily* $b \in B^*$ *and* $\bar{a} \subseteq B^*$,

(γ) $B <_s B^*$ *or at least for every first order formula $\varphi = \varphi(\ldots, x_a, \ldots)_{a \in B}$ of quantifier depth $\leq \ell'$ there is B' such that $B <_s B'$ (so $B' \in \mathcal{K}_\infty$) and:*

$$B^* \;\mathbf{ffl}\; \varphi(\ldots, a, \ldots)_{a \in B} \quad \textit{iff} \quad B' \;\mathbf{ffl}\; \varphi(\ldots, a, \ldots)_{a \in B},$$

(δ) $\mathcal{M}_n {}^{-} B^* \underset{\mathcal{M}_n {}^{-} B}{\overset{\mathcal{M}_n}{\Cup}} \mathcal{M}_n {}^{-} \mathrm{cl}^{k, m_2}(B, \mathcal{M}_n)$,

(ε) $(B^*, B, \langle \bar{a}b, B^* \cap \mathrm{cl}^{k, m_0 - 1}(\bar{a}b, \mathcal{M}_n)\rangle, k)$ *is possible for $\mathfrak{K}$ or at least for each φ, as in (γ) for some B', B'' we have*

(i) $(B', B, \langle \bar{a}b, B''\rangle, k)$ *is possible for $\mathfrak{K}$ and*

(ii) $(B^*, \mathrm{cl}^{k, m_0 - 1}(\bar{a}b, \mathcal{M}_n) \cap B^*), b, c)_{c \in B} \equiv_{\ell'} (B', B'', b, c)_{c \in B}$,

(ζ) *for $m \leq m_{\varphi_1}$ we have*

$$\mathrm{cl}^{k_{\varphi_1}, m}(\bar{a}b, B^*) = B^* \cap \mathrm{cl}^{k_{\varphi_1}, m}(\bar{a}b, \mathcal{M}_n).$$

(B) the existential demand:

If $\ell, k, m_0, m_1, m_2 \in \mathbb{N}$ *and* B, B^*, $B^{\otimes}$, $\bar{a}$ *and* b *satisfy* $\bar{a} \subseteq B$, $B^* \subseteq N \in \mathcal{K}_\infty$, $B <_s B^*$ *and* $(B^*, B, \langle ab, B^{\otimes}\rangle, k)$ *is possible for $\mathfrak{K}$*

then *for every random enough $\mathcal{M}_n$ we have:*

if $\bar{a}' \in {}^{\ell}|\mathcal{M}_n|$ *and* $f : B \hookrightarrow \mathcal{M}_n$ *and* $\mathrm{Rang}(f) \subseteq \mathrm{cl}^{k, m_1}(\bar{a}, \mathcal{M}_n)$ *and* $f(\bar{a}') = \bar{a}$

then *there is an extension g of f satisfying*

$$g : N {}^{-} B^* \hookrightarrow \mathcal{M}_n$$

and such that

(α) $g(B^*) \cap \mathrm{cl}^{k,m_2}(f(B), \mathcal{M}_n) = f(B)$,

(β) *if $m_0 > 0$ then*

$$\mathrm{cl}^k(g(B^{\otimes}), \mathcal{M}_n) \subseteq g(B^*) \cup \mathrm{cl}^{k,m_2}(g(B), N)$$

(hence, by clause (b) of Definition 2.2 and (γ) below, equality holds),

(γ) $\mathcal{M}_n \restriction g(B^*) \underset{\mathcal{M}_n \restriction f(B)}{\overset{\mathcal{M}_n}{\Cup}} \mathcal{M}_n \restriction \mathrm{cl}^{k,m_2}(f(B), \mathcal{M}_n)$,

Definition 2.9 *If in 2.8, above $k^* = k$ we add "k preserving".*

Remark 2.10 1. Note that if $\mathcal{K} = \mathcal{K}_\infty$ and cl is local (or just cl^k is $(l_k - r_k)$-local for each k) (which holds in the cases we are interested in) then in clauses (γ), (ε) of (A) above the two possibilities are equivalent.

2. Why in 2.8(A)(β) we have "neccessarily $b \in B^*$"? Because

$$b \in \mathrm{Rang}(\bar{a}b) \subseteq \mathrm{cl}^{k,m_0}(\bar{a}b, \mathcal{M}_n) \quad \text{and}$$

$$\begin{aligned} \mathrm{cl}^{k,m_2}(B, \mathcal{M}_n) &\subseteq \mathrm{cl}^{k,m_2}(\mathrm{cl}^{k,m_1}(\bar{a}, \mathcal{M}_n)) \subseteq \mathrm{cl}^{k,m_1+m_2}(\bar{a}, \mathcal{M}_n) \\ &\subseteq \mathrm{cl}^{k,m^*}(\bar{a}, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n) \end{aligned}$$

and b does not belong to the later.

3. Why do we use $\mathrm{cl}^{k,m_2}(B, \mathcal{M}_n)$? Part of our needs is that this set is definable from B without b.

4. In clause (γ), Definition 2.8(A), there is one B' for all such φ (as the set of f.o. formulas of quantifier depth ℓ is closed under Boolean combinations) so for some $B' \in \mathcal{K}_\infty$ we have $B \leq_s B'$, and $(B'', c)_{c \in B} \equiv_\ell (B^*, c)_{c \in B}$

In our main case, also the following variant of the property applies (see 2.19 below).

Definition 2.11 *1) $(N, B, \langle B_0, B_1 \rangle, k)$ is* simply good *for $(\mathfrak{K}, \mathrm{cl})$ if for every random enough $\mathcal{M}_n$, for every embedding $f : B \hookrightarrow \mathcal{M}_n$ there is an extension g of f satisfying $g : N \hookrightarrow \mathcal{M}_n$ such that:*

(i) $g(N) \cap \mathrm{cl}^k(f(B), \mathcal{M}_n) = f(A)$,

(ii) $g(N) \underset{f(B)}{\Cup} \mathrm{cl}^k(f(B), \mathcal{M}_n)$,

(iii) $\mathrm{cl}^k(g(B_0), \mathcal{M}_n) \subseteq g(B_1) \cup \mathrm{cl}^k(g(B), \mathcal{M}_n)$

(natural but not used is $\mathrm{cl}^k(g(B_0), \mathcal{M}_n) \cap g(N) = g(\mathrm{cl}^k(B_0, N))$). If we write B_0 instead $\langle B_0, B_1 \rangle$, we mean $B_1 = N$.
2) The 0-1 context with closure $(\mathfrak{K}, \mathrm{cl})$ is simply almost nice *if* cl *is transitive, smooth, local, transparent and*

(A) the universal demand:

for every k and ℓ, ℓ' there are

$$m^* = m^*(k, \ell, \ell'), \quad k^* = k^*(k, \ell, \ell') \geq k \quad \text{and } t = t(k, \ell, \ell')$$

such that for every random enough $\mathcal{M}_n$ we have:
if $\bar{a} \in {}^{\ell}|\mathcal{M}_n|$ *and* $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$
then *there are* $B \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$ *and* $B^* \subseteq \mathcal{M}_n$ *such that:*

(α) $|B| \leq t$ *and* $\bar{a} \subseteq B$ *and* $\mathrm{cl}^k(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$,

(β) $B^* = [\mathrm{cl}^k(\bar{a}b, M_n) \setminus \mathrm{cl}^k(B, \mathcal{M}_n)] \cup B$,

(γ) $B <_s B^*$ *(so $B^* \in \mathcal{K}_\infty$) or at least for every f.o. formula $\varphi = \varphi(x_b, \ldots, x_a, \ldots)_{a \in B}$ of quantifier depth $\leq \ell'$ there is B' such that $B <_s B'$ (so $B' \in \mathcal{K}_\infty$) and:*

$$B^* \models \varphi(b, \ldots, c, \ldots)_{c \in B} \quad \textit{iff} \quad B' \models \varphi(b, \ldots, c, \ldots)_{c \in B}$$

(in short $(B^*, b, \ldots, c, \ldots)_{c \in B} \equiv_{\ell'} (B', b, \ldots, c, \ldots)_{c \in B}$),

(δ) $\mathcal{M}_n \restriction B^* \underset{\mathcal{M}_n \restriction B}{\overset{\mathcal{M}_n}{\Cup}} \mathcal{M}_n \restriction \mathrm{cl}^k(B, \mathcal{M}_n)$

(ε) $B^* \in \mathcal{K}_\infty$ *and $(\mathcal{M}_n \restriction B^*, B, \bar{a}b, k)$ is simply good for $(\mathfrak{K}, \mathrm{cl})$ or at least for some B', b' we have:*

(i) *$(B', B, \bar{a}b', k)$ is good for $\mathfrak{K}$ and*

(ii) $(B^*, b, \ldots, c, \ldots)_{c \in B} \equiv_{\ell'} (B', b', \ldots, c, \ldots)_{c \in B}$.

3) If above always $k^* = k$ we say: $\mathfrak{K}$ is simply almost nice depth preserving.
4) We say that $(\mathfrak{K}, \mathrm{cl})$ is simply nice *(i.e. omitting the almost) if 2.11(2) holds but we omit (ε) and add*

(B) *if $B <_s B^*$ and $B_0 \subseteq B^*$ and $k \in \mathbb{N}$ then (B, A, B, k) is simply good (which is equivalent to (B, A, B, k) is simply nice).*

Similarly in Definition 2.8.

Remark 2.12 In 2.11(2) we can weaken the demands: get also $k^{\otimes} \in \mathbb{N}$ and replace, i.e.

$$\mathrm{cl}^{k^{\otimes}}(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*}(\bar{a}, \mathcal{M}_n)$$

(ε') $(B', B, \bar{a}b, k, k^{\otimes})$ is simply good

which means that in 2.11(1) we replace

$$\mathrm{cl}^k(g(B_0), \mathcal{M}_n) \subseteq g(B_1) \cup \mathrm{cl}^{k^{\otimes}}(g(B), \mathcal{M}_n).$$

Lemma 2.13 below (the addition theorem, see [CK] or [Gu]) is well known; this is the point where $\Cup$ is used.

Lemma 2.13 *For finite vocabulary τ and f.o formula (in τ) $\psi(\bar z, \bar z^1, \bar z^2)$, $\bar z = \langle z_1, \ldots, z_s\rangle$, there are $i^* \in \mathbb{N}$ and τ- formulas $\theta^1_i(\bar z, \bar z^1) = \theta^1_{i,\psi}(\bar z, \bar z^1)$, $\theta^2_i(\bar x, \bar z) = \theta^2_{i,\psi}(\bar z, \bar z^2)$ for $i < i^*$ each of quantifier depth at most that of ψ such that:*

if $N_1 \underset{N_0}{\overset{N}{\Cup}} N_2$, $N_1 \cap N_2 = N_0$, $N_1 \cup N_2 = N$ *and the set of elements of* N_0 *is* $\{c_1, \ldots, c_s\}$, $\bar c = \langle c_1, \ldots, c_s\rangle$ *and* $\bar c^1 \in {}^{\ell g \bar z^1}(N_1)$ *and* $\bar c^2 \in {}^{\ell g \bar z^2}(N_2)$
then

$$N \models \psi[\bar c, \bar c^1, \bar c^2] \quad \textit{iff} \quad \textit{for some } i < i^*, N_1 \models \theta^1_i[\bar c, \bar c^1] \textit{ and } N_2 \models \theta^2_i[\bar c, \bar c^2].$$

Main Lemma 2.14 (Context as above) *Assume* $(\mathfrak{K}, \mathrm{cl})$ *is almost nice (and* cl *is definable).*
1) Let $\varphi(\bar x)$ be a f.o. formula. Then *for some $m_\varphi \in \mathbb{N}$ and $k = k_\varphi \geq \ell g(\bar x) + q.d.(\varphi(\bar x))$ and $\psi_\varphi(\bar x)$ we have:*
for every random enough $\mathcal{M}_n$ and $\bar a \in {}^{\ell g(\bar x)}|\mathcal{M}_n|$

$(*)$ $\mathcal{M}_n \models \varphi(\bar a)$ *if and only if* $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi, m_\varphi}(\bar a, \mathcal{M}_n) \models \psi_\varphi(\bar a)$.

2) Moreover, if $(\mathfrak{K}, \mathrm{cl})$ *is almost nice in k-preserving way (see 2.8(A)) and for simplicity we will consider "$y \in \mathrm{cl}^{k,m}(\bar x, M)$" as an atomic formula when computing the q.d. of ψ_φ* then *we can demand: $q.d.(\psi_\varphi) \leq q.d.(\varphi)$.*
3) The quantifier depth of ψ_φ in (1) is $\leq q.d.(\varphi)$ if we consider "$y \in \mathrm{cl}^{k,m}(\bar x, M)$" as atomic.

Proof We shall ignore (2), (3) (which are not used). We prove the statement by induction on $r = q.d.(\varphi(\bar x))$ and first note (by clause (d) of 2.2)

$(*)^+$ in $(*)$, possibly changing ψ_φ one can replace $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi, m_\varphi}(\bar a, \mathcal{M}_n)$ by any N with $\mathrm{cl}^{k_\varphi, m_\varphi}(\bar a, \mathcal{M}_n) \subseteq N \subseteq \mathcal{M}_n$.

Case 1 $r = 0$. Trivial.

Case 2 $r > 0$. Let $\varphi(\bar x) = (\exists y)\varphi_1(\bar x, y)$ and let

$$m^* = m^*(k_{\varphi_1}, \ell g(\bar x), \ell'), \quad k_\varphi = k^*(k_{\varphi_1}, \ell g(\bar x), \ell'), \quad t = t(k_{\varphi_1}, m_{\varphi_1}, \ell g(\bar x), \ell')$$

with ℓ' suitable (see its use below) and let $m_\varphi := m^* + m_{\varphi_1}$. Let ψ_{φ_1} be such that $(*)$ holds for φ_1, and let $\psi^2_{\varphi_1}$ be such that $(*)^+$ holds for it (for φ_1). It is enough to prove the following two statements:

Statement 1: There is $\psi^1_\varphi(\bar x)$ (f.o) such that:

$(*)_1$ for every random enough $\mathcal{M}_n$, for every $\bar a \in {}^{\ell g(\bar x)}|\mathcal{M}_n|$ we have $(\alpha)_1 \Leftrightarrow (\beta)_1$ where:

$(\alpha)_1$ $\mathcal{M}_n {}^- \operatorname{cl}^{k_\varphi,m_\varphi}(\bar a,\mathcal{M}_n) \models \psi^1_\varphi(\bar a)$

$(\beta)_1$ $\mathcal{M}_n \models$ "there is $b \in \operatorname{cl}^{k_\varphi,m^*}(\bar a,\mathcal{M}_n)$ such that $\varphi_1(\bar a,b)$ holds."

Statement 2: There is $\psi^2_\varphi(\bar x)$ (f.o) such that:

$(*)_2$ for every random enough $\mathcal{M}_n$ and for every $\bar a \in {}^{\ell g(\bar x)}|\mathcal{M}_n|$ we have $(\alpha)_2 \Leftrightarrow (\beta)_2$ where:

$(\alpha)_2$ $\mathcal{M}_n {}^- \operatorname{cl}^{k,m^*}(\bar a,\mathcal{M}_n) \models \psi^2_\varphi(\bar a)$

$(\beta)_2$ $\mathcal{M}_n \models$ "there is $b \in \mathcal{M}_n \setminus \operatorname{cl}^{k_\varphi,m^*}(\bar a,\mathcal{M}_n)$ such that $\varphi_1(\bar a,b)$ holds"

(note: $(\beta)_1$, $(\beta)_2$ are complementary, but it is enough that always at least one holds).
Note that as "$y \in \operatorname{cl}^{k_\varphi,m^*}(\bar x)$" is f.o definable, by 2.2, clause (d) we can in $(\alpha)_2$ replace m^* by m_φ, changing ψ^2_φ to $\psi^{2.5}_\varphi$.
Clearly these two statements are enough and $\psi^1_\varphi(\bar x) \vee \psi^{2.5}_\varphi(\bar x)$ is as required.

Proof of statement 1:
Easily, by the induction hypothesis as

$$\operatorname{cl}^{k_{\varphi_1},m_{\varphi_1}}(\bar a b, M_n) \subseteq \operatorname{cl}^{k_\varphi,m^*+m_{\varphi_1}}(\bar a,\mathcal{M}_n) = \operatorname{cl}^{k_\varphi,m_\varphi}(\bar a,\mathcal{M}_n) \subseteq \operatorname{cl}^{k_\varphi,m_\varphi}(\bar a,\mathcal{M}_n)$$

and by the fact that the closure is sufficiently definable.

Proof of statement 2:
We will use a series of equivalent statements $\otimes_\ell$.

$\otimes_1$ is $(\beta)_2$

$\otimes_2$ there are $m_1 \le m^* - m_2$, $m_2 \in [m_{\varphi_1}, m^*]$, b, B and B^*, B' such that:
$b \in \mathcal{M}_n$, $b \notin \operatorname{cl}^{k_\varphi,m^*}(\bar a,\mathcal{M}_n)$, $\bar a \subseteq B \subseteq \operatorname{cl}^{k_{\varphi_1},m_1}(\bar a,\mathcal{M}_n)$, $|B| \le t$,
$B^* = B \cup [\operatorname{cl}^{k_{\varphi_1},m_{\varphi_1}}(\bar a b,\mathcal{M}_n) \setminus \operatorname{cl}^{k_{\varphi_1},m_2}(B,\mathcal{M}_n)]$ and $B \le_s B' \in \mathcal{K}_\infty$
and $(B^*,b,c)_{c\in B} \equiv_{\ell'} (B',b,c)_{c\in B}$ (see 2.10(4)) and

$$B^* \underset{B}{\overset{\mathcal{M}_n}{\Cup}} \operatorname{cl}^{k_{\varphi_1},m_2}(B,\mathcal{M}_n)$$

and so $B = B^* \cap \operatorname{cl}^{k_{\varphi_1},m_2}(B,\mathcal{M}_n)$ and

$\oplus_2$ $\mathcal{M}_n \models \varphi_1(\bar a,b)$

$(*)_2$ $\otimes_1 \Leftrightarrow \otimes_2$

Why? The implication $\Leftarrow$ is trivial, the implication $\Rightarrow$ holds by clause (A) in the definition of almost nice.

$\otimes_3$ like $\otimes_2$ but replacing $\oplus_2$ by

$\oplus_3$ $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \models \psi_{\varphi_1}(\bar{a}, b)$.

$(*)_3$ $\otimes_2 \Leftrightarrow \otimes_3$

Why? By the induction hypothesis.

$\otimes_4$ like $\otimes_3$ replacing $\oplus_3$ by

$\oplus_4$ $\mathcal{M}_n \restriction [\mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \cup \mathrm{cl}^{k_{\varphi_1}, m_2}(B, \mathcal{M}_n)] \models \psi^2_{\varphi_1}(\bar{a}, b)$.

$(*)_4$ $\otimes_3 \Leftrightarrow \otimes_4$

Why? By $(*)^+$ in the beginning of the proof, the definition of B^* and the choice of $\psi^2_{\varphi_1}$.

For notational simplicity we assume $B \neq \emptyset$, and similarly assume $\bar{a}$ is with no repetition and apply the lemma 2.13 several times.

First for $m \leq m_{\varphi_1}$ we apply 2.13 to the case $s = t$, $\bar{z} = \langle z_1, \ldots, z_t \rangle$, $\bar{z}^1 = \langle z^1_1, z^1_2 \rangle$, $\bar{z}^2$ empty and the formula "$z^1_2 \in \mathrm{cl}^{k_{\varphi_1}, m}(\bar{z}, z^1_1)$" and get $i^*_{1,m} \in \mathbb{N}$ and formulas $\theta^1_{1,m,i}(\bar{z}, z^1_1, z^1 - 2)$ and $\theta^2_{1,m,i}(\bar{z})$ for $i < i^*_{1,m}$. Let

$$u^*_1 = \{(m, i) : m \leq m_{\varphi - 1}, i < i^*_{1,m}\}.$$

Second for $m \leq m_{\varphi_1}$ we apply it to the case $s = t$, $\bar{z}^2 = \langle z^2_1 \rangle$, $\bar{z}^1 = \langle z^1_1 \rangle$, $\bar{z} = \langle z_1, \ldots, z_t \rangle$ and the formula "$z^2_1 \in \mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\bar{z}, z^1_1)$" get $i^*_{2,m} \in \mathbb{N}$ and formulas $\theta^1_{2,m,i}(\bar{z}, \bar{z}^1)$ and $\theta^2_{2,m,i}(\bar{z}, z^2_1)$, for $i < i^*_{2,m}$.

Let $\tau^1 = \tau_k \cup \{P_1, P_2\}$, with P_1, P_2 unary predicates: for $\theta \in \mathcal{L}[\tau]$ let $\theta^{[P_\ell]}$ be θ speaking on P_ℓ. Let $\psi^* = \psi^*_1 \wedge \psi^*_2 \wedge \psi^*_3$ where

$$\begin{aligned} \psi^*_1 =: \ & \psi^2_{\varphi_1}(z_1, \ldots, z_{\ell g(\bar{x})}, z^1_1) \\ \psi^*_2 =: \ & \bigwedge_{m \leq m_{\varphi_1}} (\forall y)\Big[y \in \mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\{z_1, \ldots, z_{\ell g(\bar{x})}, z^1_1) \\ & \qquad \equiv (\psi^{*,1}_{2,m}(z_1, \ldots, z_t, z^1_1, y) \vee \psi^{*,2}_{2,m}(z_1, \ldots, z_t, z^1_1, y))\Big], \end{aligned}$$

where

$$\psi^{*,1}_{2,m}(z_1, \ldots, z_t, z^1_1, y) =: \bigvee_{i < i^*_{1,m}} (\theta^1_{1,m,i}(z_1, \ldots, z_t, z^1_1, y)^{[P_1]} \wedge \theta^2_{1,m,i}(z_1, \ldots, z_t)^{[P_2]})$$

$$\psi^{*,2}_{2,m}(z_1, \ldots, z_t, z^1_1, y) =: \bigvee_{i < i^*_{2,m}} (\theta^2_{1,m,i}(z_1, \ldots, z_t, z^1_1)^{[P_1]} \wedge \theta^2_{2,m,i}(z_1, \ldots, z_t, y)^{[P_2]})$$

$$\psi^*_3 =: \bigwedge_{m \leq m_{\varphi_1}} (\forall y)\Big(P_1(y) \to [y \in \mathrm{cl}^{k_{\varphi_1}, m_{\varphi_1}}(\{z_1, \ldots, z_{\ell g(\bar{x})}, z^1_1\})]\Big).$$

So we have defined ψ^*. Now we apply 2.13 the third time, with the vocabulary $\tau_{\mathcal{K}} \cup \{P_1, P_2\}$ to the case $s = t$, $\bar{z}^2$ empty, $\bar{z}^1 = \langle z^1_i \rangle$, $\bar{z} = \langle z_1, \ldots, z_\ell \rangle$, and $\psi(\bar{z}, \bar{z}^1, \bar{z}^2) = \psi(\bar{z}, z^1_1) =$

$\psi^*(\langle z_1,\dots,z_{\ell g \bar{x}}\rangle, z^1_1)$ and get i^*, $\theta^1_{3,i}(\bar{z},\bar{z}^1)$ and $\theta_{3,i}(\bar{z},\bar{z}^2)$ as there. W.l.o.g. there are $u^{1,j}_m \subseteq i^*_{1,m}$ such that:

$$\theta^2_{3,i}(\bar{z}) \vdash \bigwedge_{i\in u^{1,j}_m} \theta^2_{1,m,i}(\bar{z}) \ \wedge \bigwedge_{\substack{i<i^*_{1,m}\\ i\notin u^{1,j}_m}} \neg\theta^2_{1,m,i}(\bar{z}),$$

and $u^{2,j}_m \subseteq i^*_{2,m}$ such that:

$$\theta^1_{3,i}(\bar{z},z^1) \vdash \bigwedge_{i\in u^{2,j}_m} \theta^1_{2,m,i}(\bar{z},z^1) \ \wedge \bigwedge_{\substack{i<i^*_{2,m}\\ i\notin u^{2,j}_m}} \neg\theta^1_{2,m,i}(\bar{z},z^1).$$

Let

$\otimes_5$ like $\otimes_4$ but replacing $\oplus_4$ by

> $\oplus_5$ letting $c_1,\dots,c_t$ list B possibly with repetitions but such that $\langle c_1,\dots,c_{\ell g(\bar{x})}\rangle = \bar{a}$ and letting
> $$P^*_1 = B^* \text{ and } P^*_2 = \mathrm{cl}^{k_{\varphi_1},m_2}(\{c_1,\dots,c_t\},\mathcal{M}_n)$$
> *there is* $i<i^*$ such that:
> **(i)** $(\mathcal{M}_n{}^-(P^*_1\cup P^*_2),P^*_1,P^*_2)\models\psi^*$ (the model is a τ'-model).

Now

$(*)_5$ $\otimes_4 \Leftrightarrow \otimes_5$.

Why? Look at what the statements mean. Next let

$\otimes_6$ like $\otimes_5$ but replacing $\oplus_5$ by

> $\oplus_6$ letting $c_1,\dots,c_t$ list B possibly with repetitions but such that $\langle c_1,\dots,c_{\ell g(\bar{x})}\rangle = \bar{a}$ and letting
> $$P^*_1 = B^* \text{ and } P^*_2 = \mathrm{cl}^{k_{\varphi_1},m_1}(\{c_1,\dots,c_t\},\mathcal{M}_n)$$
> *there is* $i<i^*$ such that:
> **(i)** $(\mathcal{M}_n{}^- P^*_1, P^*_1, P^*_2\cap P^*_1)\models\theta^1_{3,i}[\langle c_1,\dots,c_t\rangle, b]$,
> **(ii)** $(\mathcal{M}_n{}^- P^*_2, P^*_1\cap P^*_2, P^*_2)\models\theta^2_{3,i}[\langle c_1,\dots,c_t\rangle]$.

Now

$(*)_6$ $\otimes_5 \Leftrightarrow \otimes_6$

Why? By the choice of $\theta^1_{3,i}$, $\theta^2_{3,i}$ $(i<i^*)$.

However in the two τ'-models appearing in $\oplus_6$, the predicates P_1, P_2 are interpreated in a trivial way: as the whole universe of the model or as $\{c_1,\dots,c_t\}$. So let:

(a) $\theta^1_{4,i}(z_1,\dots,z_t,y)$ be $\theta^1_{3,i}(z_1,\dots,z_t,y)$ with each atomic formula of the form $P_1(\sigma)$ or $P_2(\sigma)$ being replaced by $\sigma=\sigma$ or $\bigvee^t_{u=1}\sigma=z_u$ respectively,

(b) $\theta^2_{4,i}(z_1,\dots,z_t)$ be $\theta^2_{3,i}(z_1,\dots,z_t)$ with each atomic formula of the form $P_1(\sigma)$ or $P_2(\sigma)$ being replaced by $\bigvee^t_{u=1}\sigma=z_u$ or $\sigma=\sigma$ respectively.

So let

$\otimes_7$ like $\otimes_6$ but replacing $\oplus_6$ by

$\oplus_7$ letting $c_1,\dots,c_t$ list B possibly with repeatitions but such that $\langle c_1,\dots,c_{\ell g\bar{x}}\rangle=\bar{a}$, *there is* $i<i^*$ such that

(i) $\mathcal{M}_n{}^-\, B'\models\theta^1_{4,i}[\langle c_1,\dots,c_t\rangle,b]$ and

(ii) $\mathcal{M}_n{}^-\,\mathrm{cl}^{k,m_1}(\langle c_1,\dots,c_t\rangle,\mathcal{M}_n)\models\theta^2_{4,i}(\langle c_1,\dots,c_t\rangle)$.

$(*)_7$ $\otimes_6\Leftrightarrow\otimes_7$

Why? By the choice of the $\theta^1_{4,i}$, $\theta^2_{4,i}$ and the property of B' (stated in $\otimes_2$).

Let $\mathcal{P}=\{(N,c_1,\dots,c_t): N\in\mathcal{K}_\infty$, with the set of elements $\{c_1,\dots,c_t\}\}$. Let

$$\psi^4_j(\bar{z},z',y)=\bigvee_{i\in u^{1,j}_{m_0-1}}\theta^1_{2,m_0-1,i}(z_1,\dots,z_t,z^1_1,y).$$

Let $\{(N_j,c^j_1,\dots,c^j_t): j<j^*\}$ list the members of $\mathcal{P}$ up to isomorphism, so with no two isomorphic. For every $j<j^*$ and $i<i^*$ choose if possible $(N_{j,i},c^j_1,\dots,c^j_t,b^j_i)$ such that:

(i) $N_j\le_s N_{j,i}$ (in $\mathcal{K}_\infty$),

(ii) $b^j_i\in N_{j,i}\setminus N_j$,

(iii) $N_{i,j}=\mathrm{cl}^{k,m_{\varphi_1}}(\{c^j_1,\dots,c^j_{\ell g\bar{x}},b^j_i\},N_{i,j})$ and

(iv) $N_{j,i}\models\theta^1_{4,i}(\langle c^j_1,\dots,c^j_t\rangle,b^j_i)$ and

(v) $(N_{j,i},B,\langle\{c^j_i: i=1,\dots,\ell g(\bar{x})\}\cup\{b^j_i\}$, $\{d\in N_{j,i}: N_{j,i}\models\psi^4_0(c_1,\dots,c_t,b^j_i,d]\}\rangle,k)$ is possible for $\mathfrak{K}$.

Let

$$w=\{(i,j): i<i^*, j<j^* \text{ and } (N_{i,j},c^j_1,\dots,c^j_t,b^j_i) \text{ is well defined}\}.$$

Let

$\otimes_8$ there are $m_1\le m^*-m_2$, $m_2\le m^*$, $m_2\ge m_{\varphi_1}$, b, B such that:
$\bar{a}\subseteq B\subseteq\mathrm{cl}^{k^*,m_2}(\bar{a},\mathcal{M}_n)$, $|B|\le t(k_{\varphi_1},m_{\varphi_1},\ell g(\bar{x}))$, $b\notin\mathrm{cl}^{k^*,m^*}(\bar{a},\mathcal{M}_n)$, $b\in\mathcal{M}_n$, and

$\oplus_8$ for some $c_1,\dots,c_t$ listing B such that $\bar{a}=\langle c_1,\dots,c_{\ell g\bar{x}}\rangle$ *there are* $i<i^*$, $j<j^*$ such that:

(i) $(\mathcal{M}_n \restriction B, c_1, \dots, c_t) \cong (N_j, c^j_1, \dots, c^j_t)$ i.e. the mapping $c^j_1 \mapsto c_1$, $c^j_2 \mapsto c_2$ embed N_j into $\mathcal{M}_n$,

(ii) $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1}, m_1}(B, \mathcal{M}_n) \models \theta^2_{4,i}(\langle c_1, \dots, c_t\rangle)$

$(*)_8$ $\otimes_7 \Leftrightarrow \otimes_8$

Why? For proving $\otimes_7 \Rightarrow \otimes_8$ let $c_1, \dots, c_t$ as well as $i < i^*$ be as in $\oplus_7$, let $j < j^*$ be such that $(\mathcal{M}_n \restriction B, c_1, \dots, c_t) \cong (N_j, c^j_1, \dots, c^j_t)$. The main point is that B' exemplies that $(i, j) \in w$. For proving $\otimes_8 \Leftarrow \otimes_7$ use part (B) of Definition 2.8.

We now have finished as $\otimes_8$ can be expressed as a f.o formula straightforwardly. So we have carried the induction hypothesis on the quantifier depth thus finishing the proof. $\blacksquare_{2.14}$

Claim 2.15 *Assume* $(\mathfrak{K}, \mathrm{cl})$ *is simply almost nice. Then in Definition 2.11(2) clause (A) we can add (possibly increasing* m^*, k^*, t*)*

(ζ) $\quad \mathrm{cl}^k(\bar{a}b, B^*) = B^* \cap \mathrm{cl}^k(\bar{a}b, \mathcal{M}_n)$

Remark 2.16 In our main case we choose

$(\otimes)$ $\quad \mathrm{cl}^k(A, N) = \bigcup\{B : B \subseteq N \text{ and } (A \cap B) \leq_i B\}$.

In this case clause (ζ) follows without changing m^*, k^*, t; it just follows from (α)–(ε) there. Clearly $(\otimes)$ implies $(\mathfrak{K}, \mathrm{cl})$ is local and transparent with $r = k$.

Proof Let k, ℓ, ℓ' be given and let r be such that cl^k is $(1, r)$-local. Let $k' \geq k$ be such that

$(*)_1$ $B \subseteq N \in \mathcal{K}$, $|B| \leq r \times r \quad \Rightarrow \quad \mathrm{cl}^k(\mathrm{cl}^k(B, N)) \subseteq \mathrm{cl}^{k'}(B, N)$

(note: as cl^k is $(1, r)$-local, this applies to any B). As $(\mathfrak{K}, \mathrm{cl})$ is transparent (and by 2.2(1)(c)) without loss of generality

$(*)_2$ if $B \leq_i N \in \mathcal{K}$ and $\|N\| \leq r$ then $\mathrm{cl}^{k'}(B, N) = N$.

Let $m^* = m^*(k', \ell, \ell')$, $k^* = k^*(k', \ell, \ell')$ and $t = t(k', \ell, \ell')$ (the functions from clause (A) of definition 2.11).

So let $\mathcal{M}_n$ be random enough, and $\bar{a} \in {}^\ell|\mathcal{M}_n|$, $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n)$. Hence, by clause (A) of Definition 2.11, we can find $B \subseteq \mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n)$ and $B^* \subseteq \mathcal{M}_n$ and B' satisfying clauses (α)–(ε) of (A) of Definition 2.11. For each $d \in \mathrm{cl}^k(\bar{a}b, \mathcal{M}_n)$ as cl^k is $(1, r)$-local, we can find a set $C_d \subseteq \mathrm{cl}^k(\bar{a}b, \mathcal{M}_n)$ such that $d \in C_d$, $|C_d| \leq r$ and $C_d \restriction (\bar{a}b) <_i C_d$. Let $C^0_d = C_d \cap B$, $C^1_d = C_d \cap B^*$ and $C^2_d = C_d \cap \mathrm{cl}^k(B, \mathcal{M}_n)$.

Now clearly $C_d = C^1_d \cup C^2_d$ also $C^0_d = C^1_d \cap C^2_d$ and by clause (δ) of 2.11(A) we have $C^1_d \underset{C^0_d}{\overset{\mathcal{M}_n}{\Cup}} C^2_d$ and by momotonicity (2.2(1)(a)) $(C^0_d \cup \{b\}) \leq_i C_d$.

By smoothness it follows that $C^0_d <_i C^2_d$ and $C^0_d \cup \{b\} <_i C^1_d$. Now if $d \in B^* \cap \mathrm{cl}^k(\bar{a}b, \mathcal{M}_n)$ then $d \in C^1_d$, by the uniqueness of $\Cup$ amalgamation and preservation of cl^k by isomorphism, w.l.o.g.:

$(*)_3$ if $d_1,\, d_2 \in B^* \cap \mathrm{cl}^k(\bar{a}b, \mathcal{M}_n)$ and

$$(C_{d_1}, d_1, b, \bar{a}, c, C^0_{d_1}, C^1_{d_1}, C^2_{d_1})_{c\in C^0_{d_1}} \cong (C_{d_2}, d_2, b, \bar{a}, c, C^0_{d_2}, C^1_{d_2}, C^2_{d_2})_{c\in C^0_{d_2}}$$

then $C^2_{d_1} = C^2_{d_2}$.

Let $B^\oplus = B \cup \{C^2_d : d \in B^*\}$. So $B^\oplus \subseteq \mathrm{cl}^k(B, \mathcal{M}_n)$ hence $\mathrm{cl}^k(B^\oplus, \mathcal{M}_n) \subseteq \mathrm{cl}^{k'}(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k'}(\bar{a}, \mathcal{M}_n)$ (see remark to $(*)_1$ and 2.11(2)(A)(α), resp.) Now as $B^* \underset{B}{\bigcup\!\!\!\!\bigcup} \mathrm{cl}^k(B, \mathcal{M}_n)$ we get, letting $B^{**} = B^* \cup B^\oplus$

$(*)_4$ $B^* \underset{B}{\bigcup\!\!\!\!\bigcup} \mathrm{cl}^k(B^\oplus, \mathcal{M}_n)$, $B \subseteq B^\oplus \subseteq \mathrm{cl}^k(B^\oplus, \mathcal{M}_n)$ hence $B^{**} \underset{B^\oplus}{\bigcup\!\!\!\!\bigcup} \mathrm{cl}^k(B^\oplus, \mathcal{M}_n)$.

By the choice of $B^\oplus$

$(*)_5$ $\mathrm{cl}^k(\bar{a}b, B^{**}) = B^{**} \cap \mathrm{cl}^k(\bar{a}b, \mathcal{M}_n)$.

Now B^{**}, $B^\oplus$, $B \cup B^\oplus$ (free amalgamation) are as required, i.e. satisfy clauses (α)-(ζ). $\blacksquare_{2.15}$

Lemma 2.17 *1) Assume* $(\mathfrak{K}, \mathrm{cl})$ *is simply almost nice. Let* $\varphi(\bar{x})$ *be a f.o. formula.* Then *for some* $k = k_\varphi \geq \ell g(\bar{x}) + q.d.(\varphi(\bar{x}))$ *and* $\psi_\varphi(\bar{x})$ *we have:*

for every random enough $\mathcal{M}_n$ *and* $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$

(∗) $\mathcal{M}_n \models \varphi(\bar{a})$ *if and only if* $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi}(\bar{a}, \mathcal{M}_n) \models \psi_\varphi(\bar{a})$

2) The quantifier depth of ψ_φ *in (1) is* $\leq q.d.(\varphi)$ *if we consider "*$y \in \mathrm{cl}^{k,m}(\bar{x}, M)$*" as atomic.*

Remark 2.18 Moreover if $\mathfrak{K}$ is simply almost nice in depth-preserving way (see 2.8(A)) and for simplicity we will consider "$y \in \mathrm{cl}^k(\bar{x}, M)$" as atomic formula *then* for any f.o. $\varphi(\bar{x})$, for $k = \ell g(\bar{x}) + q.d.(\varphi(\bar{x}))$ for some m and ψ_φ with $q.d.(\psi_\varphi)$ at most $q.d.(\varphi)$, we have, for $\mathcal{M}_n$ random enough and $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$:

$(*)$ $\mathcal{M}_n \models \varphi(\bar{a})$ if and only if $\mathcal{M}_n \restriction \mathrm{cl}^{k,m}(\bar{a}, \mathcal{M}_n) \models \psi_\varphi$.

Proof We prove the statement by induction on $r = q.d.(\varphi(\bar{x}))$. First note (by clause (d) of 2.2)

$(*)^+$ in $(*)$, possibly changing ψ_φ one can replace $\mathcal{M}_n \restriction \mathrm{cl}^{k_\varphi}(\bar{a}, \mathcal{M}_n)$ by any N with $\mathrm{cl}^{k_\varphi}(\bar{a}, \mathcal{M}_n) \subseteq N \subseteq \mathcal{M}_n$.

Case 1 $r = 0$ Trivial.

Case 2 $r > 0$ Let $\varphi(\bar{x}) = (\exists y)\varphi_1(\bar{x}, y)$ and let (the functions are from 2.11(2) with the stronger demand of 2.15)

$$m^* = m^*(k_{\varphi_1}, \ell g(\bar{x}), \ell'), \quad k^* = k^*(k, \ell g(\bar{x}), \ell'), \quad t = t(k, \ell g(\bar{x}), \ell')$$

with ℓ' suitable (just the quantifier depth of $\psi^2_{\varphi_1}$ defined below) and let k_φ be[1] such that:

$$|A| \leq \ell g(\bar{x}) + 1, \quad A \subseteq N \Rightarrow \mathrm{cl}^{k_{\varphi_1}}(\mathrm{cl}^{k^*, m^*}(a, N), N) \subseteq \mathrm{cl}^{k_\varphi}(A, N).$$

Let $\psi_{\varphi_1}(\bar{x}, y)$ be such that $(*)$ holds for φ_1, and let $\psi^2_{\varphi_1}(\bar{x}, y)$ be such that $(*)^+$ holds for it (for φ_1).

It is enough to prove the following two statements (see below):

Statement 1: There is $\psi^1_\varphi(\bar{x})$ (f.o.) such that:

$(*)_1$ for every random enough $\mathcal{M}_n$, for every $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ we have $(\alpha)_1 \Leftrightarrow (\beta)_1$ where:

- $(\alpha)_1$ $\mathcal{M}_n{}^- \mathrm{cl}^{k_\varphi}(\bar{a}, \mathcal{M}_n) \models \psi^1_\varphi(\bar{a})$
- $(\beta)_1$ $\mathcal{M}_n \models$ "there is $b \in \mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n)$ such that $\varphi_1(\bar{a}, b)$ holds."

Statement 2: There is $\psi^2_\varphi(\bar{x})$ (f.o) such that:

$(*)_2$ for every random enough $\mathcal{M}_n$ and for every $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ we have $(\alpha)_2 \Leftrightarrow (\beta)_2$ where:

- $(\alpha)_2$ $\mathcal{M}_n{}^- \mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n) \models \psi^2_\varphi(\bar{a})$
- $(\beta)_2$ $\mathcal{M}_n \models$ "there is $b \in \mathcal{M}_n \setminus \mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n)$ such that $\varphi_1(\bar{a}, b)$ holds"

(note: $(\beta)_1$, $(\beta)_2$ are complementary, but it is enough that always at least one holds).
Note that as "$y \in \mathrm{cl}^{k^*, m^*}(\bar{x})$" is f.o. definable, by 2.2, clause (d) and the choice of k_φ we can in $(\alpha)_2$ replace cl^{k^*, m^*} by cl^{k_φ}, changing ψ^2_φ to $\psi^{2.5}_\varphi$.
Clearly these two statements are enough and $\psi^{1.5}_\varphi(\bar{x}) \vee \psi^{2.5}_\varphi(\bar{x})$ is as required.

Proof of statement 1:
Easily, by the induction hypothesis as

$$\mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, M_n) \subseteq \mathrm{cl}^{k_{\varphi_1}}(\mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n), \mathcal{M}_n) \subseteq \mathrm{cl}^{k_\varphi}(\bar{a}, \mathcal{M}_n)$$

and by the fact that the closure is sufficiently definable. So in this case $\psi_\varphi(\bar{a})$ is $(\exists y)\psi_{\varphi_1}(\bar{a}, y)$.

Proof of statement 2:
We will use a series of equivalent statements $\otimes_\ell$.

[1] if we have changed 2.11(A) a little, $k_\varphi = k^*$ will be O.K.: instead assuming $b \notin \mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n)$ assume $\mathrm{cl}^k(\bar{a}b, \mathcal{M}_n) \not\subseteq \mathrm{cl}^{k^*, m^*}(\bar{a}, \mathcal{M}_n)$. Allowing to increase m^*, the two versions are equivalent.

$\otimes_1$ is $(\beta)_2$

$\otimes_2$ there are b, B and B^*, B' such that:

$b \in \mathcal{M}_n$, $b \notin \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$, $\bar a \subseteq B \subseteq \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$,
$\mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar a, \mathcal{M}_n)$, $|B| \le t(k_{\varphi_1}, \ell g(\bar x), \ell')$,
$B^* = B \cup [\mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n) \setminus \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)]$ and $B \le_s B' \in \mathcal{K}_\infty$ and
$(B^*, b, c)_{c\in B} \equiv_{\ell'} (B', b, c)_{c \in B}$ (see 2.10(4)) and $B^* \underset{B}{\overset{\mathcal{M}_n}{\Cup}} \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)$ and so
$B = B^* \cap \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)$ and $\mathrm{cl}^k(\bar a b, B^*) = \mathrm{cl}^k(\bar a b, \mathcal{M}_n) \cap B^*$ and $(B', B, \bar a b, k)$ is good and

$\oplus_2$ $\mathcal{M}_n \models \varphi_1(\bar a, b)$

$(*)_2$ $\otimes_1 \Leftrightarrow \otimes_2$

Why? The implication $\Leftarrow$ is trivial, the implication $\Rightarrow$ holds by clause (A) in the definition of almost nice and 2.15.

$\otimes_3$ like $\otimes_2$ but replacing $\oplus_2$ by

$\oplus_3$ $\mathcal{M}_n \,{}^-\, \mathrm{cl}^{k_{\varphi_1}}(\bar a b, M_n) \models \psi_{\varphi_1}(\bar a, b)$.

$(*)_3$ $\otimes_2 \Leftrightarrow \otimes_3$

Why? By the induction hypothesis.

$\otimes_4$ like $\otimes_3$ replacing $\oplus_3$ by

$\oplus_4$ $\mathcal{M}_n \,{}^-\, [\mathrm{cl}^{k_{\varphi_1}}(\bar a b, \mathcal{M}_n) \cup \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)] \models \psi^2_{\varphi_1}(\bar a, b)$.

$(*)_4$ $\otimes_3 \Leftrightarrow \otimes_4$

Why? By $(*)^+$ in the beginning of the proof, the requirements on B^* and the choice of $\psi^2_{\varphi_1}$.

For notational simplicity we assume $B \ne \emptyset$, and similarly assume $\bar a$ has no repetitions and apply the lemma 2.13 with the vocabulary $\tau_{\mathfrak{K}}$ to the case $s = t$, $\bar z^2$ empty, $\bar z^1 = \langle z^1_i \rangle$, $\bar z = \langle z_1, \dots, z_\ell \rangle$, and $\psi(\bar z, \bar z^1, \bar z^2) = \psi(\bar z, z^1_1) = \psi^2_{\varphi_1}(\langle z_1, \dots, z_{\ell g \bar x}\rangle, z^1_1)$ and get i^*, $\theta^1_i(\bar z, \bar z^1)$ and $\theta^2_i(\bar z)$ for $i < i^*$ as there; in particular the quantifier depth of θ^1_i, θ^2_i for $i < i^*$ is at most the quantifier depth of $\psi^2_{\varphi_1}$.
Next let

$\otimes_5$ like $\otimes_4$ but replacing $\oplus_4$ by

$\oplus_5$ letting $c_1, \dots, c_t$ list B possibly with repetitions but such that $\langle c_1, \dots, c_{\ell g(\bar x)}\rangle = \bar a$ *there is* $i < i^*$ such that:

(i) $B^* \models \theta^1_i[\langle c_1, \ldots, c_t\rangle, b]$
(ii) $\mathrm{cl}^k(B, \mathcal{M}_n) \models \theta^2_i[\langle c_1, \ldots, c_t\rangle]$

Now

$(*)_5$ $\otimes_4 \Leftrightarrow \otimes_5$

Why? By the choice of θ^1_i, θ^2_i for $i < i^*$, so lemma 2.13.
Let $\mathcal{P} = \{(N, c_1, \ldots, c_t) : N \in \mathcal{K}_\infty$, with the set of elements $\{c_1, \ldots, c_t\}\}$. Let $\{(N_j, c^j_1, \ldots, c^j_t) : j < j^*\}$ list the members of $\mathcal{P}$ up to isomorphism, so with no two isomorphic. For every $j < j^*$ and $i < i^*$ choose if possible $(N_{j,i}, c^j_1, \ldots, c^j_t, b^j_i)$ such that:

(i) $N_j \leq_s N_{j,i}$ (in $\mathcal{K}_\infty$),

(ii) $b^j_i \in N_{j,i} \setminus N_j$,

(iii) $N_{j,i} = \mathrm{cl}^{k_{\varphi_1}}(\{c^j_1, \ldots, c^j_{\ell g \bar{x}}, b^j_i\}, N_{j,i})$ and

(iv) $N_{j,i} \models \theta^1_i(\langle c^j_1, \ldots, c^j_t\rangle, b^j_i)$ and

(v) $(N_{j,i}, \{c^j_1, \ldots, c^j_t\}, \{c^j_1, \ldots, c^j_{\ell g \bar{x}}, b^j_i\}, k)$ is good for $\mathfrak{K}$.

Let
$$w = \{(i, j) : i < i^*, j < j^* \text{ and } (N_{i,j}, c^j_1, \ldots, c^j_t, b^j_i) \text{ is well defined}\}.$$

Let

$\otimes_6$ there are b, B such that: $b \in \mathcal{M}_n$, $b \notin \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $\bar{a} \subseteq B \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $\mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $|B| \leq t(k_{\varphi_1}, \ell g(\bar{x}), \ell')$, and

$\oplus_6$ for some $c_1, \ldots, c_t$ listing B such that $\bar{a} = \langle c_1, \ldots, c_{\ell g \bar{x}}\rangle$
there are $i < i^*$, $j < j^*$ such that:
(i) $(\mathcal{M}_n {}^- B, c_1, \ldots, c_t) \cong (N_j, c^j_1, \ldots, c^j_t)$ i.e. the mapping $c^j_1 \mapsto c_1$, $c^j_2 \mapsto c_2$ embeds N_j into $\mathcal{M}_n$,
(ii) $\mathcal{M}_n {}^- \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \models \theta^2_i(\langle c_1, \ldots, c_t\rangle)$

$(*)_6$ $\otimes_5 \Leftrightarrow \otimes_6$

Why? For proving $\otimes_5 \Rightarrow \otimes_6$ let $c_1, \ldots, c_t$ as well as $i < i^*$ be as in $\oplus_5$, let $j < j^*$ be such that $(\mathcal{M}_n {}^- B, c_1, \ldots, c_t) \cong (N_j, c^j_1, \ldots, c^j_t)$. The main point is that B' exemplifies that $(i, j) \in w$ (remember: B' is from $\otimes_2$, and if $B^* \in \mathcal{K}_\infty$, we normally could have choosen $B' = B^*$).
For proving $\otimes_6 \Leftarrow \otimes_5$ use definition of good tuples in Def. 2.11.

We now have finished as $\otimes_6$ can be expressed as a f.o formula straightforwardly. So we have carried the induction hypothesis on the quantifier depth thus finishing the proof. $\blacksquare_{2.17}$

Conclusion 2.19 *1. Assume* $(\mathfrak{K}, \mathrm{cl})$ *is almost nice or simply almost nice.*

Then*:* $\mathfrak{K}$ *satisfies the 0-1 law* iff *for any* k*,* m *we have*

$$\langle \mathcal{M}_n \,{}^{-}\, \mathrm{cl}^{k,m}(\emptyset) : n < \omega \rangle \text{ satisfies the 0-1 law.}$$

2. Similarly with convergence.

Proof 1) As for every sentence ψ there is a sentence θ_ψ such that for every random enough $\mathcal{M}_n$ we have

$$\mathcal{M}_n \models \theta_\psi \Leftrightarrow \mathcal{M}_n \,{}^{-}\, \mathrm{cl}^{k,m}(\emptyset) \models \psi,$$

which holds as $\mathrm{cl}^{k,m}$ is f.o definable.

If $\mathfrak{K}$ satisfies 0-1 law then for some truth value $\mathfrak{t}$ for random enough $\mathcal{M}_n$

$$\mathcal{M}_n \models \text{“}\theta_\psi \equiv \mathfrak{t}\text{”}$$

hence

$$\mathcal{M}_n \,{}^{-}\, \mathrm{cl}^{k,m}(\emptyset) \models \text{“}\psi = \mathfrak{t}\text{”}.$$

The other direction is similar by the main theorem 2.13.
2) Similar, so left to the reader. $\blacksquare_{2.19}$

3 Further abstract closure context

The context below is not used later so it can be skipped but it seems natural.

Looking again at Definition 2.8 or 2.11(2), clause (A), we note that there is an asymmetry: we try to represent $\mathrm{cl}^{k,m}(\bar{a}b, \mathcal{M}_n)$ and some $C \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$ as free amalgamation over some B small enough (with *a priori* bound depending on $\ell g(\bar{a})$ and k only, there $C = \mathrm{cl}^k(B, \mathcal{M}_n)$). So this basis of free amalgamation is in $\mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$ without elements from $\mathrm{cl}^{k,m}(\bar{a}b, \mathcal{M}_n) \setminus \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$. Suppose we allow this and first we deal with the case $\mathcal{M}_n$ is a graph. So a member d of $\mathrm{cl}^{k,m}(\bar{a}b, \mathcal{M}_n)$ may code a subset of $\mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$: the set

$$\{c \in cl^{k^*,m^*}(\bar{a}, \mathcal{M}_n) : \text{ the pair } \{c, d\} \text{ is an edge}\}.$$

So though we are interested in f.o. formulas $\varphi(\bar{x})$ speaking on $\mathcal{M}_n$, we are drawn into have $\psi_\varphi(\bar{x})$, the formula speaking on $\mathrm{cl}^{k_\varphi,m_\varphi}(\bar{x})$, being a monadic formula. Once we allow also three place relations and more, we have to use second order logic (still can say which quantifiers we need). For this elimination, thinking of an $\mathcal{M}_n$, we need that any possible extension of $\mathrm{cl}^{k,m}(\bar{a}, \mathcal{M}_n)$ occurs; so in the most natural cases, $|\mathrm{cl}^{k,m+1}(\bar{a}, \mathcal{M}_n)|$ may be with $2^{|\mathrm{cl}^{k,m}(\bar{a}, \mathcal{M}_n)|}$ elements, so in the natural case $\mathrm{cl}^{k,m}(\bar{a}, \mathcal{M}_n) < \log_*(|\mathcal{M}_n|)$. Still possibly $\mathrm{cl}^{k,m+1}(\bar{a}, \mathcal{M}_n)$ is not larger than $\mathrm{cl}^{k,m}(\bar{a}, \mathcal{M}_n)$.

However there is a big difference between the monadic case and the more general case. For monadic logic addition theorems like 2.13 are known, but those are false for second order logic.

So we have good enough reason to separate the two cases. We choose here to generalize the “simply almost nice with $\mathcal{K} = \mathcal{K}_\infty$” case only for readability.

Context 3.1 *As in §2 for* $(\mathfrak{K}, \mathrm{cl})$.

Definition 3.2 *1) The 0-1 context with the closure operation* $(\mathfrak{K}, \mathrm{cl})$ *is* s.m.a. (simply monadically almost) nice *if* $\mathcal{K} = \mathcal{K}_\infty$, $\mathfrak{K}$ *is transitive smooth local and*

(A) *for every* k *and* ℓ, *there are* $m^* = m^*(k,\ell)$, $k^* = k^*(k,\ell)$ *and* $t_1 = t_1(k,\ell)$, $t_2 = t_2(k,\ell)$ *such that:*

for every $\mathcal{M}_n$ *random enough we have:*

if $\bar{a} \in {}^{\ell}(\mathcal{M}_n)$, $\mathrm{cl}^k(\bar{a}b, \mathcal{M}_n) \not\subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$

then *there are* B^*, B^1, B^2 *such that:*

(α) $\bar{a} \subseteq B^1$ *and* $\mathrm{cl}^k(B^1, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$ *and* $|B^1| \le t_1$,

(β) $B^1 \subseteq B^2$, $B^2 \cap \mathrm{cl}^k(B^1, \mathcal{M}_n) = B^1$, $|B^2| \le t_2$, $b \in B^2$,

(γ) $B^* = [\mathrm{cl}^k(\bar{a}b, \mathcal{M}_n) \setminus \mathrm{cl}^k(B^1, \mathcal{M}_n)] \cup B^1 \cup B^2$ *and* $\mathrm{cl}^k(\bar{a}b, B^*) = B^* \cap \mathrm{cl}^k(\bar{a}b, \mathcal{M}_n)$,

(δ) $\mathcal{M}_n \restriction B^* \underset{\mathcal{M}_n \restriction B^2}{\overset{\mathcal{M}_n}{\Cup}} \mathcal{M}_n \restriction (B^2 \cup \mathrm{cl}^k(a, \mathcal{M}_n))$ *(also here* $\Cup$ *is relation free amalgamation),*

(ε) *if* Q *is a predicate from* $\tau_{\mathcal{K}}$ *and* $\mathcal{M}_n \models Q(\bar{b})$, $\bar{b} \subseteq \mathrm{cl}^k(B^1, \mathcal{M}_n) \cup B^2$ then: $\mathrm{Rang}(\bar{b}) \cap B^2 \subseteq B^1$ *or* $\mathrm{Rang}(\bar{b}) \setminus B^2$ *has at most one member; if this holds (including* $\mathrm{cl}(B^1, \mathcal{M}_n \restriction (B^2 \cup \mathrm{cl}^k(B^1, \mathcal{M}_n)) = \mathrm{cl}^k(B^1, \mathcal{M}_n))$*) we say* B^2 *is monadic over* $\mathrm{cl}^k(B^1, \mathcal{M}_n)$ *inside* $\mathcal{M}_n$,

(ζ) $(B^*, B^1, B^2, \bar{a}b, k)$ *is m. good (see below).*

2) We say $(B^*, B^1, B^2, \bar{a}b, k)$ *is* m. good *when:* for every *random enough* $\mathcal{M}_n$, *and* $f : B^1 \hookrightarrow \mathcal{M}_n$, *and* $C^1 \in \mathcal{K}$ *such that* $\mathcal{M}_n \restriction \mathrm{cl}^k(f(B^1), \mathcal{M}_n) \le C^1$, *and* $f^+ : B^2 \hookrightarrow C^1$ *extending* f *such that* $C^1 = f^+(B^2) \cup \mathrm{cl}^k(f(B^1), \mathcal{M}_n)$ *and* $f^+(B^2)$ *is monadic over* $\mathrm{cl}^k(\bar{a}, \mathcal{M}_n)$ *inside* C^1 there are $g^+ : C^1 \hookrightarrow \mathcal{M}_n$ *and* $g : B^* \hookrightarrow \mathcal{M}_n$ *such that* $g \restriction B^2 = (f^+ \circ g^+) \restriction B^2$ *and*

$$g(B^*) \underset{g(B^2)}{\Cup} g^+(C^1) \quad \text{and} \quad \mathrm{cl}^k(g(\bar{a}b), \mathcal{M}_n) \subseteq g(B^*).$$

3) Assume $\mathbf{E} \subseteq \{(C, B^1, B^2) : B^1 \le B^2 \le C \in \mathcal{K}\}$ *is closed under isomorphism. We say* B^2 *is* $\mathbf{E}$*-over* D *inside* N *if* $B^2 \le N \in \mathcal{K}$, $D \le N$ *and* $(N \restriction (B^2 \cup D), B^2 \cap D, B^2) \in \mathbf{E}$. *We say* $\mathfrak{K}$ *is s.*$\mathbf{E}$*.a nice if in 3.2(1) we replace* (ε) *by*

$(\varepsilon)'$ B^2 *is* $\mathbf{E}$*-over* $\mathrm{cl}^k(B^1, \mathcal{M}_n)$ *inside* $\mathcal{M}_n$,

and replace m-good by $\mathbf{E}$*-good, which is gotten similarly. We say* $\mathbf{E}$ *is monadic if* $(C, B^1, B^2) \in \mathbf{E}$ *implies*

$$\bar{a} \in Q^C \Rightarrow \mathrm{Rang}(\bar{a}) \cap B^2 \subseteq B^1 \ \vee \ |\mathrm{Rang}(\bar{a}) \setminus B^2| \le 1.$$

4) We say $\mathbf{E}$ *as in 3.2(3) is simply monadic if it is monadic and for any* $B^1 \leq B^2 \in \mathcal{K}$, *letting*

$$\Gamma = \Big\{\theta(y,\bar{b}) : \quad \bar{b} \subseteq B^2 \textit{ is with no repetition } \theta(y,\bar{x}) \textit{ is an atomic formula,} \\ \textit{each variable actually appearing}\Big\}$$

we have

$$\Big\{(C, R_{\theta(y,\bar{b})}, c)_{\theta(y,\bar{b})\in\Gamma, c\in B^1} : \begin{array}{l} C \in \mathcal{K}, \textit{ such that:} \\ B^1 \leq C,\ R_{\theta(y,\bar{b})} \textit{ is a subset of } C \setminus B^1 \textit{ and} \\ \textit{there are } C^1, f \textit{ such that:} \\ C \leq C^1 \in \mathcal{K},\ f : B^2 \hookrightarrow C^1,\ f(B^2) \cap C = B^1, \\ f \restriction B^1 = \mathrm{id}_{B^1}, \textit{ and for } \theta(y,\bar{b}) \in \Gamma \textit{ we have} \\ R_{\theta(y,\bar{b})} = \{d \in C \setminus B^1 : C^1 \models \theta[d, g(\bar{b})]\}\Big\} \end{array}$$

is definable by a monadic formula[2].

Lemma 3.3 *Assume* $(\mathfrak{K}, \mathrm{cl})$ *is s.*$\mathbf{E}$*.a. nice and* $\mathbf{E}$ *is simply monadic.* Then *for every f.o. formula* $\varphi(\bar{x})$ *there are* k *and a monadic formula* $\psi_\varphi(\bar{x})$ *such that:*

$(*)$ *for every random enough* $\mathcal{M}_n$, *for every* $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ *we have*

$$\mathcal{M}_n \models \varphi(\bar{a}) \quad \Leftrightarrow \quad \mathcal{M}_n \restriction \mathrm{cl}^k(\bar{a}, \mathcal{M}_n) \models \psi_\varphi(\bar{a}).$$

Remark 3.4 Some of the assumption of 3.3 are open to manipulations; others are essential.
1) As said above, the "monadic" needed in order to use an addition theorem, the price of removing it is high: essentially above we need that after finding the copy $g(B^2)$ realizing the required type over $\mathrm{cl}^k(B^1, \mathcal{M}_n)$, we need to find $g(B^*)$, or a replacement like B' in the proofs in §2 but only the holding of some formula $\varphi(\ldots, b, \ldots)_{b\in B^1}$ in B^* is important. Now this formula speaks on $\mathrm{cl}^k(B, \mathcal{M}_n)$ too, and we need that all possibilities occur so this includes cases where B' has to be of cardinality much larger than $\mathrm{cl}^k(\bar{a}, \mathcal{M}_n)$. So we do not formulate such lemma (nor the parallel of 2.14). Of course some specific information may help to control the situation.
2) If $\mathbf{E}$ is monadic but not simply monadic, not much is changed: we should allow new quantifiers in ψ_φ. Let $C^1 <^{\mathbf{E}}_B C^2$ if $B \leq C^1 \leq C^2$ and $(C^2, B, B \cup (C^2 \setminus C^1)) \in E$. We want the quantifier to say for $(C^1, R_{\theta(y,\bar{b})}, c)_{\theta(\bar{y},\bar{b})\in\Gamma, c\in B}$ that it codes C^2 with $C^1 \leq^{\mathbf{E}}_B C^2$, but then we should be able to iterate.

The situation is similar to the case that in §2, we have: cl is not f.o. definable.
3) In 3.3 we essentially demand

$(*)$ for each t, for random enough $\mathcal{M}_n$, for every $B \subseteq \mathcal{M}_n$, $|B| \leq t$, if $\mathcal{M}_n \restriction \mathrm{cl}^k(\bar{a}, \mathcal{M}_n) <^{\mathbf{E}}_{\bar{a}} C$ then C is embeddable into $\mathcal{M}_n$ over $\mathrm{cl}^k(\bar{a}, \mathcal{M}_n)$.

[2] We can restrict ourselve to the cases $C = \mathrm{cl}^k(B, C)$.

Of course we need this just for a dense set of such C's, dense in the sense that a monadic sentence is satisfied.
4) As we have done in 2.14(3), 2.17(2), we can add that the quantifier depth of φ and the monadic quantifier depth of ψ_φ are equal as long as the depth of the formulas from "simply monadic" is not counted.
5) Can we find a reasonable context where the situation from 3.3 and 3.4(1) above holds? Suppose we draw edges as here and redraw in the neighborhood of each edge. Let us describe drawing of a model on $[n]$. For each $i < j$ from $[n]$ we flip a coin $E_{i,j}$ on whether we have $\{i,j\}$ as a pre-edge with probability $p^n_{i,j}$. If we succeed then for any pair (i',j') from $[n]$ we flip a coin $E_{i,j,i',j'}$ with probability $p^n_{i,j,i',j'}$. The flippings are independent and finally (i',j') is an edge if and only if for some $i < j$, (i,j) is a pre-edge and we also succeed in $E_{i,j,i',j'}$. For our case let ($\alpha \in (0,1)_{\mathbb{R}}$) irrational):

Distribution 1

$$p^n_{i,j} = p_{|i-j|} = \begin{cases} 1/|i-j|^\alpha & \text{when } |i-j| > 1 \\ 1/2 & \text{if } \ |i-j| = 1 \end{cases}$$

and $p^n_{i,j,i',j'} = \frac{1}{2^{|i-i'|+|j-j'|}}$;

Distribution 2

$p^n_{i,j}$ is as above and

$$p^n_{i,j,i',j'} = \begin{cases} \frac{1}{2^{|i-i'|+|j-j'|}} & \text{if } i = i' \ \vee \ j = j' \\ 0 & \text{if otherwise.} \end{cases}$$

Now distribution 2 gives us an example as in Lemma 3.3, distribution 1 fits the non-monadic case. We may wonder whether actually the 0–1 law holds. It is intuitively clear that for distribution 2 the answer is "yes", for distribution 1 the answer is "no".
6) Why in distribution 1 from (5) the 0–1 law fails (in fact fails badly)? I.e. it tells me that for distribution 1 we can find $A \subseteq B$ such that for every random enough $\mathcal{M}_n$, for some $f : A \hookrightarrow \mathcal{M}_n$, the number of $g : B \hookrightarrow \mathcal{M}_n$ extending f is quite large, and on the set of such g we can interprate an initial segment N_f of arithmetic. Now we can compare such N_{f_1}, N_{f_2} (with some parameters).
The case of distribution 2 should be similar to this paper.

Proof of 3.3:

We prove the statement. By induction on $r = q.d.(\varphi(\bar{x}))$ and first note (by clause (d) of 2.2)

$(*)^+$ in $(*)$, possibly changing ψ_φ one can replace $\mathcal{M}_n$ $\restriction$ $\mathrm{cl}^{k_\varphi}(\bar{a}, \mathcal{M}_n)$ by any N with $\mathrm{cl}^{k_\varphi}(\bar{a}, \mathcal{M}_n) \subseteq N \subseteq \mathcal{M}_n$.

Case 1 $r = 0$. Trivial.

Case 2 $r > 0$. Let $\varphi(\bar{x}) = (\exists y)\varphi_1(\bar{x}, y)$ and let $m^* = m^*(k_{\varphi_1}, \ell g(\bar{x}))$, $k^* = k^*(k, \ell g(\bar{x}))$, $t = t(k, \ell g(\bar{x}))$. Let k_φ be k'. Let ψ_{φ_1} be such that $(*)$ holds for φ_1, and let $\psi^2_{\varphi_1}$ be such that $(*)^+$ holds for it (for φ_1).

It is enough to prove the following two statements:

Statement 1: There is $\psi^1_\varphi(\bar{x})$ (f.o) such that:

$(*)_1$ for every random enough $\mathcal{M}_n$, for every $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ we have $(\alpha)_1 \Leftrightarrow (\beta)_1$ where:

$(\alpha)_1$ $\mathcal{M}_n \restriction \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n) \models \psi^1_\varphi(\bar{a})$

$(\beta)_1$ $\mathcal{M}_n \models$"there is b satisfying $\mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$ and such that $\varphi_1(\bar{a}, b)$ holds."

Statement 2: There is $\psi^2_\varphi(\bar{x})$ (f.o.) such that:

$(*)_2$ for every random enough $\mathcal{M}_n$ and for every $\bar{a} \in {}^{\ell g(\bar{x})}|\mathcal{M}_n|$ we have $(\alpha)_2 \Leftrightarrow (\beta)_2$ where:

$(\alpha)_2$ $\mathcal{M}_n \restriction \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n) \models \psi^2_\varphi(\bar{a})$

$(\beta)_2$ $\mathcal{M}_n \models$ "there is b satisfying $\mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \nsubseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$ and such that $\varphi_1(\bar{a}, b)$ holds"

(note: $(\beta)_1$, $(\beta)_2$ are complementary, but it is enough that always at least one holds).
Note that as "$y \in \mathrm{cl}^{k^*,m^*}(\bar{x})$ is f.o. definable, by 2.2, clause (d) and by the choice of k_φ we can in $(\alpha)_2$ replace cl^{k^*,m^*} by cl^{k_φ}, changing ψ^2_φ to $\psi^{2.5}_\varphi$, and similarly in $(\alpha)_1$ replace cl^{k^*,m^*} by $\mathrm{cl}^{k^*_\varphi}$ changing ψ^1_φ to $\psi^{1.5}_\varphi$.
Clearly these two statements are enough and $\psi^{1.5}_\varphi(\bar{x}) \vee \psi^{2.5}_\varphi(\bar{x})$ is as required.

Proof of statement 1:
Easily, by the induction hypothesis and by the fact that the closure is sufficiently definable.

Proof of statement 2:
We will use a series of equivalent statements $\otimes_\ell$.

$\otimes_1$ is $(\beta)_2$,

$\otimes_2$ there are b, B and B^*, B_1, B_2 such that:

$b \in \mathcal{M}_n$, $\mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \nsubseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $\bar{a} \subseteq B_1 \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$,

$\mathrm{cl}^{k_{\varphi_1}}(B_1, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $|B_1| \leq t_1(k_{\varphi_1}, \ell g(\bar{x}))$, $B_1 \leq B_2 \leq B^*$,

$b \in B^*$, $B^* \setminus B_1$ disjoint to $\mathrm{cl}^{k_{\varphi_1}}(B_1, \mathcal{M}_n)$ and $B_1 \leq_s B^* \in \mathcal{K}_\infty$ and

$B^* \underset{B_2}{\overset{\mathcal{M}_n}{\Cup}} \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)$ and $\mathrm{cl}^k(\bar{a}b, B^*) = \mathrm{cl}^k(\bar{a}b, \mathcal{M}_n) \cap B^*$ and

$(B^*, B_1, B_2, \bar{a}b, k)$ is $\mathbf{E}$–good and

$\oplus_2$ $\mathcal{M}_n \models \varphi_1(\bar{a}, b)$.

$(*)_2$ $\otimes_1 \Leftrightarrow \otimes_2$

Why? The implication $\Leftarrow$ is trivial, the implication $\Rightarrow$ holds by clause (A) in the definition 3.2.

$\otimes_3$ like $\otimes_2$ but replacing $\oplus_2$ by

$\oplus_3$ $\mathcal{M}_n \restriction \mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, M_n) \models \psi_{\varphi_1}(\bar{a}, b)$.

$(*)_3$ $\otimes_2 \Leftrightarrow \otimes_3$

Why? By the induction hypothesis.

$\otimes_4$ like $\otimes_3$ replacing $\oplus_3$ by

$\oplus_4$ $\mathcal{M}_n \restriction [\mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \cup \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n)] \models \psi^2_{\varphi_1}(\bar{a}, b)$.

$(*)_4$ $\otimes_3 \Leftrightarrow \otimes_4$

Why? By $(*)^+$ in the beginning of the proof, the definition of B^* and the choice of $\psi^2_{\varphi_1}$.

For notational simplicity we assume $B \neq \emptyset$, and similarly assume $\bar{a}$ is with no repetition and apply Lemma 3.5 below with the vocabulary $\tau_{\mathcal{K}}$ to the case $s = t$, $\bar{z}^2$ empty, $\bar{z}^1 = \langle z^1_i \rangle$, $\bar{z} = \langle z_1, \ldots, z_\ell \rangle$, and $\psi(\bar{z}, \bar{z}^1, \bar{z}^2) = \psi(\bar{z}, z^1_1) = \psi^2_\varphi(\langle z_1, \ldots, z_{\ell g \bar{x}} \rangle, z^1_1)$ and get i^*, $\theta^1_i(\bar{z}, \bar{z}^1)$ and $\theta^2_i(\bar{z})$ for $i < i^*$ as there.
Next let

$\otimes_5$ like $\otimes_4$ but replacing $\oplus_5$ by

$\oplus_5$ letting $c_1, \ldots, c_{t_2}$ list B_2 possibly with repetitions but such that $\{c_1, \ldots, c_{t_1}\} = B_1$ and $\langle c_1, \ldots, c_{\ell g(\bar{x})} \rangle = \bar{a}$ and *there is* $i < i^*$ such that:

(i) $B^* \models \theta^1_i[\langle c_1, \ldots, c_{t_2} \rangle, b]$

(ii) $\mathcal{M}_n \restriction (B_2 \cup \mathrm{cl}^k(B_1, \mathcal{M}_n)) \models \theta^2_i[\langle c_1, \ldots, c_t \rangle]$.

Now

$(*)_5$ $\otimes_4 \Leftrightarrow \otimes_5$

Why? by the choice of θ^1_i, θ^2_i $(i < i^*)$.

Let $\mathcal{P} = \{(N, c_1, \ldots, c_{t_2}) : N \in \mathcal{K}_\infty$, with the set of elements $\{c_1, \ldots, c_{t_2}\}\}$. Let $\{(N_j, c^j_1, \ldots, c^j_{t_2}) : j < j^*\}$ list the members of $\mathcal{P}$ up to isomorphism, so with no two isomorphic. For every $j < j^*$ and $i < i^*$ choose if possible $(N_{j,i}, c^j_1, \ldots, c^j_t, b^j_i)$ such that:

(i) $N_j \leq_s N_{j,i}$ (in $\mathcal{K}_\infty$),

(ii) $b^j_i \in N_{j,i} \setminus N_j$,

(iii) $N_{i,j} = \mathrm{cl}^{k,m_{\varphi_1}}(\{c^j_1, \ldots, c^j_{\ell g \bar{x}}, b^j_i\}, N_{i,j})$ and

(iv) $N_{j,i} \models \theta^1_i(\langle c^j_1, \ldots, c^j_t\rangle, b^j_i)$ and

(v) $(N_{j,i}, \{c^j_1, \ldots, c^j_{t_1}\}, \{c^j_1, \ldots, c^j_{t_2}\}, \{c^j_1, \ldots, c^j_{\ell g \bar{x}}, b^j_i\}, k)$ is $\mathbf{E}$–good.

Let

$$w = \{(i,j) : i < i^*, j < j^* \text{ and } (N_{i,j}, c^j_1, \ldots, c^j_t, b^j_i) \text{ is well defined}\}.$$

Let $\{\theta(y, \bar{x}) : \theta \text{ is a basic formula, } \bar{x} \subseteq \{x_1, \ldots, x_{t_2}\}\}$.

As $\mathbf{E}$ is simply monadic (see Definition 3.2(4)) we have: for some formula θ^3_i such that

(*) if $\{d_1, \ldots, d_{t_1}\} \leq C \in \mathcal{K}$ and $R_{\varphi(y,\bar{x})} \subseteq C$ and $C \leq C_1 \in \mathcal{K}$, $d_{t_1+1}, \ldots, d_{t_2} \in C_2$, and

$$R_{\theta(y,\ldots,x_i,\ldots)} = \{e \in C : C_1 \models \theta[e, \ldots, d_i, \ldots]\} \quad \text{for } \theta(y, \ldots, x_i, \ldots) \in \Gamma$$

then:

$$C \models \theta^2_i[d_1, \ldots, d_{t_1}] \quad \Leftrightarrow \quad (C_1, \ldots, R_{\theta(y,\bar{x})}, \ldots) \models \theta^3_i[d_1, \ldots, d_{t_1}].$$

Let

$\otimes_6$ there are b, B_1 such that: $b \in \mathcal{M}_n$, $\mathrm{cl}^{k_{\varphi_1}}(\bar{a}b, \mathcal{M}_n) \nsubseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $\bar{a} \subseteq B_1 \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $\mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \subseteq \mathrm{cl}^{k^*,m^*}(\bar{a}, \mathcal{M}_n)$, $|B| \leq t_1(k_{\varphi_1}, \ell g(\bar{x}))$, and

$\oplus_6$ for some $c_1, \ldots, c_{t_1}$ listing B_1 such that $\bar{a} = \langle c_1, \ldots, c_{\ell g \bar{x}}\rangle$ *there are* $i < i^*$, $j < j^*$ such that:

(i) $(\mathcal{M}_n{}^- B, c_1, \ldots, c_{t_1}) \cong (N_j, c^j_1, \ldots, c^j_{t_1})$ i.e. the mapping $c^j_1 \mapsto c_1$, $c^j_2 \mapsto c_2$ embeds N_j into $\mathcal{M}_n$,

(ii) $\mathcal{M}_n{}^- \mathrm{cl}^{k_{\varphi_1}}(B, \mathcal{M}_n) \models \theta^3_i(\langle c_1, \ldots, c_{t_1}\rangle)$.

$(*)_6$ $\otimes_5 \Leftrightarrow \otimes_6$.

Why? For proving $\otimes_5 \Rightarrow \otimes_6$ let $c_1, \ldots, c_t$ as well as $i < i^*$ be as in $\oplus_7$, let $j < j^*$ be such that $(\mathcal{M}_n{}^- B, c_1, \ldots, c_t) \cong (N_j, c^j_1, \ldots, c^j_t)$. A main point is that B^* exemplifies that $(i,j) \in w$. For proving $\otimes_6 \Leftarrow \otimes_5$ use part (B) of Definition 2.8.

Now we have finished as $\otimes_6$ can be expressed as a monadic formula straightforwardly. So we have carried the induction hypothesis on the quantifier depth thus finishing the proof. $\blacksquare_{3.3}$

The following is the parallel of 2.13 for monadic logic.

Lemma 3.5 *For finite vocabulary τ and f.o formula (in τ) $\psi(\bar{z}, \bar{z}^1, \bar{z}^2)$, $\bar{z} = \langle z_1, \ldots, z_s\rangle$, there are $i^* \in \mathbb{N}$ and τ- formulas $\theta^1_i(\bar{z}, \bar{z}^1) = \theta^1_{i,\psi}(\bar{z}, \bar{z}^1)$, $\theta^2_i(\bar{x}, \bar{z}) = \theta^2_{i,\psi}(\bar{z}, \bar{z}^2)$ for $i < i^*$ each of quantifier depth at most that of ψ such that:*

if $N_1 \underset{N_0}{\overset{N}{\biguplus}} N_2$, *$N_1 \cap N_2 = N_0$, $N_1 \cup N_2 = N$ and the set of elements of N_0 is $\{c_1, \ldots, c_s\}$, $\bar{c} = \langle c_1, \ldots, c_s\rangle$ and $\bar{c}^1 \in {}^{\ell g \bar{z}^1}(N_1)$ and $\bar{c}^2 \in {}^{\ell g \bar{z}^2}(N_2)$*
then

$$N \models \psi[\bar{c}, \bar{c}^1, \bar{c}^2] \quad \textit{iff} \quad \textit{for some } i < i^*, N_1 \models \theta^1_i[\bar{c}, \bar{c}^1] \textit{ and } N_2 \models \theta^2_i[\bar{c}, \bar{c}^2].$$

4 Applications

How do we apply the general theorem to our problem? So here our irrational number $\alpha \in (0,1)_{\mathbb{R}}$ is fixed. We work in Main Context (the other one would work out as well).

Context 4.1 $p_i = 1/i^\alpha$ for $i > 1$, $p_1 = p_2$ ($\alpha \in (0,1)$ irrational) and $\mathcal{M}_n = \mathcal{M}_n^0$ (i.e. only the graph).

Fact 4.2 *For any graph A*

$$1 = \lim_n \operatorname{Prob}(A \text{ embeddable into } \mathcal{M}_n).$$

Moreover for every $\varepsilon > 0$

$$1 = \lim_n \operatorname{Prob}(A \text{ has } \geq n^{1-\varepsilon} \text{ disjoint copies in } \mathcal{M}_n).$$

This is easy still, before proving note:

Conclusion 4.3 $\langle \mathrm{cl}^{m,k}_{\mathcal{M}_n}(\emptyset) : n < \omega \rangle$ *satisfies the 0–1 law (being a sequence of empty models).*

Hence (see 2.19)

Conclusion 4.4 $\mathcal{K}_\infty = \mathcal{K}$ *and for our main theorem it suffices to prove* simple almost niceness *of* $\mathfrak{K}$.

(This is in fact what we always meant by "random enough" in previous discussions.)
PROOF of 4.2:
Let the nodes of A be $\{a_0, \ldots, a_{k-1}\}$. Let the event E^n_r be:

$$a_\ell \mapsto 2rk + 2\ell \quad \text{is an embedding of } A \text{ into } \mathcal{M}_n.$$

The point of this is that for various values of m these tries are going to speak on pairwise disjoint sets of nodes, so independent.

Now

Subfact 4.5 $\operatorname{Prob}(E^n_r) = q > 0$ *(i.e. > 0 but it does* not *depend on n).*

(Note: this is not true in the second context i.e. $1/n^a + 1/2^{|i-j|}$, as here the probability depends on n. But still, we can have $\geq q > 0$ which suffices.)
MAIN CASE

$$q = \prod_{\substack{\ell<m<k \\ \{\ell,m\} \text{ edge}}} 1/[2(m-\ell)]^a \times \prod_{\substack{\ell<m<k \\ \{\ell,m\} \text{ not an edge}}} \left(1 - \frac{1}{[2(m-\ell)]^a}\right).$$

(What we need is that all the relevant edges have probability > 0, < 1. So $p = 1/i^a$ is not O.K., so we have changed p_1. Anyway, in our case we multiplied by 2 to avoid this (in the definition of the graph).)

For the second case,

$$q \geq \prod_{\substack{\ell<m<k \\ \{\ell,m\} \text{ edge}}} \frac{1}{e^{|m-e|}} \times \prod_{\substack{\ell<m<k \\ \{\ell,m\} \text{ not an edge}}} (1 - \frac{1}{e^{|m-e|}}).$$

So these $\mathrm{Prob}(E^n_r)$ have a positive lower bound which does not depend on r. Also $E_0, \ldots, E_{[\frac{n}{2k}]-1}$ are independent. So the probability that they all fail is

$$\prod_{i<[\frac{n}{2k}]} (1 - \mathrm{Prob}(E_i)) \leq \prod_i (1-q) \leq (1-q)^{\frac{n}{2k}}$$

which goes to 0 quite fast. $\blacksquare_{4.2}$

Definition 4.6 *1) Let*

$$\mathcal{T} = \{(A,B,\lambda) : \ \textit{$A \subseteq B$ graphs (generally models) and} \\ \textit{λ an equivalence relation on $B \setminus A$}\}$$

We may write (A,B,λ) instead $(A,B,\lambda \restriction (B \setminus A))$.
2) We say that $X \subseteq B$ is λ-closed if:

$$x \in X \ \textit{(and $x \in B \setminus A$)} \qquad \textit{implies} \qquad x/\lambda \subseteq X.$$

Let

$$\Xi(A,B) =: \{\lambda : \ \textit{$(A,B,\lambda) \in \mathcal{T}$, and if $C \subseteq B \setminus A$ is a nonempty λ-closed} \\ \textit{set then $\mathbf{w}_\lambda(A,B) > 0$}\}$$

Story

We want to ask for any given copy of A in $\mathcal{M}_n$, is there a copy of B above it, and how many, we hope for dichotomy: i.e. usually none, always few *or* always many. The point of λ is to take distance into account. But here being near is important, $b_1 \lambda b_2$ will indicate that b_1 and b_2 are near. Note that being near is not transitive. But "luck" helps us. We will succeed to "pretend" it is. We will look at many candidates for $B \setminus A$ and compute the expected value. We want to show that saying "variance small" says that the truth is near expectation.

Definition 4.7 *For $(A,B,\lambda) \in \mathcal{T}$, for our given irrational $\alpha \in (0,1)_{\mathbb{R}}$ let*

$$\mathbf{v}(A,B,\lambda) = \mathbf{v}_\lambda(A,B) = |(B \setminus A)/\lambda|$$

be the number of λ equivalence classes ($\mathbf{v}$ stands for vertices) in $B \setminus A$.

(This measures degrees of freedom in choosing candidates for B and A.)

Let

$$\begin{aligned} \mathbf{e}(A,B,\lambda) &= \mathbf{e}_\lambda(A,B) \\ &= |\{e : e \text{ an edge of } B, e \not\subseteq A, \text{ and } e \not\subseteq X/\lambda \text{ for } X \in B \setminus A\}| \end{aligned}$$

This measures the number of "expensive", "long" edges ($\mathbf{e}$ stands for edges).

Story
$\mathbf{v}$ larger means that there are more candidates for B,
$\mathbf{e}$ large means that the probability per candidate is smaller.

Definition 4.8 *1) For $(A, B, \lambda) \in \mathcal{T}$ we define ($\mathbf{w}$ stands for weight)*

$$\mathbf{w}(A, B, \lambda) = \mathbf{w}_\lambda(A, B) = \mathbf{v}_\lambda(A, B) - \alpha \mathbf{e}_\lambda(A, B)$$

2) Let

$$\Xi(A, B) =: \{\lambda : \ (A, B, \lambda) \in \mathcal{T}, \text{ and if } C \subseteq B \setminus A \text{ is a nonempty } \lambda\text{-closed set then } \mathbf{w}_\lambda(A, B) > 0\}.$$

Observation 4.9 *1. $(A, B, \lambda) \in \mathcal{T}$ & $A \neq B \ \Rightarrow \ \mathbf{w}_\lambda(A, B) \neq 0$.*

2. *If $A \subseteq B \subseteq C$, $(A, C, \lambda) \in \mathcal{T}$ and B is λ-closed then*
   - **(a)** *$(A, B, \lambda \restriction (B \setminus A)) \in \mathcal{T}$, and*
   - **(b)** *$(B, C, \lambda \restriction (C \setminus B)) \in \mathcal{T}$ and*
   - **(c)** *$\mathbf{w}_\lambda(A, C) = \mathbf{w}_{\lambda \restriction (B \setminus A)}(A, B) + \mathbf{w}_{\lambda \restriction (C \setminus B)}(B, C)$ (see proof of 4.16)*
   - **(d)** *similarly for $\mathbf{v}$ and $\mathbf{e}$*
3. *Note that 4.9(2) legitimizes our writing λ instead of $\lambda \restriction C/A$ or $\lambda \restriction (B \setminus (C \cup A))$ when $(A, B, \lambda) \in \mathcal{T}$ and C is a λ-closed subset of B/A. Thus we may write, e.g., $\mathbf{w}_\lambda(A \cup C, B)$ for $\mathbf{w}(A \cup C, \lambda \restriction (B \setminus A \setminus C))$.*

PROOF 1) As α is irrational.
2) Clauses **(a)**, **(b)** are totally immediate, and for a proof of clauses **(c)**, **(d)** see the proof of 4.16.
3) Left to the reader.

Discussion 4.10 Note: $\mathbf{w}_\lambda(A, B)$ measures in a sense the expected value of having a copy of B over a given copy of A with λ saying "near to". Of course when λ is the identity this degenerates to the definition in [ShSp 304].
We want to characterize $\leq_i$ and $\leq_s$, using $\mathbf{w}$ and to prove that they are O.K. (meaning form a nice context). Looking at the expected behaviour, we attempt to give "effective" definition (depending on α only).
All of this, of course, just says what the intention of these functions is (i.e. $<^*_i$, $<^*_s$, $<^*_{pr}$ via $\mathbf{v}$, $\mathbf{e}$, $\mathbf{w}$), we still did not prove anything.

Definition 4.11 **(1)** *$A \leq B$ means A is a submodel of B and $A \leq^* B$ means $A \leq B$,*

(2) $A <^*_c B$ if $A \leq^* B$ *and for every* λ,

$$(A, B, \lambda) \in \mathcal{T} \Rightarrow \mathbf{w}_\lambda(A, B) < 0,$$

(3) $A \leq^*_i B$ if $A \leq^* B$ *and for every* A' *we have*

$$A \leq^* A' <^* B \Rightarrow A' <^*_c B,$$

(4) $A \leq^*_s B$ if $A \leq^* B$ *and for no* A'

$$A <^*_i A' \leq B,$$

(5) $A \leq^*_a B$ if $A \leq^* B$, $\neg(A <^*_s B)$,

(6) $A <^*_{pr} B$ if $A \leq^* B$ *and* $A <^*_s B$ *but for no* C

$$A <^*_s C <^*_s B.$$

Remark 4.12 We *intend* to prove that usually $\leq^*_x = <_x$ but it will take time.

Lemma 4.13 *Suppose* $A \leq A' < B$, $(A', B, \lambda) \in \mathcal{T}$ *and* $\mathbf{w}_\lambda(A', B) > 0$. *Then there is* $A'' \leq B \setminus A$, A'' *is* λ*–closed and*

$(*), (A', \lambda)$ $\mathbf{w}_\lambda(A'', B) > 0$ *and* if $C \subseteq B/A$, $C \neq \{\emptyset, B/A'\}$ *and* C *is* λ*–closed* then $w_\lambda(A'', A'' \cup C) > 0$ *and* $\mathbf{w}_\lambda(A'' \cup C, B) < 0$.

Proof Let C' be a maximal λ–closed subset of $B \setminus A'$ such that $\mathbf{w}_\lambda(A' \cup C', B) > 0$. Such a C' exists since $C' = \emptyset$ is as required and B is finite. Let $A'' = A' \cup C'$. Since C' is λ–closed, $(A'', B, \lambda^- (B \setminus A'')) \in \mathcal{T}$. Now suppose $D \subseteq B \setminus A''$ is λ–closed, $D \notin \{\emptyset, B \setminus A''\}$. By the maximality of C', $\mathbf{w}_\lambda(A'' \cup D, B) < 0$. Since (by 4.9(2)(c))

$$0 < \mathbf{w}_\lambda(A'', B) = \mathbf{w}_\lambda(A'', D) + \mathbf{w}_\lambda(A'' \cup D, B)$$

(but the last term is negative) we conclude $\mathbf{w}_\lambda(A'', D) > 0$. $\blacksquare_{4.13}$

Claim 4.14 *Assume* $A <^* B$. *The following statements are equivalent:*

(i) $A <^*_i B$,

(ii) *for no* A' *and* λ *do we have:*

$(*)_2$ $A \leq^* A' <^* B$, $(A', B, \lambda) \in \mathcal{T}$ *and* $\mathbf{w}_\lambda(A', B) > 0$,

(iii) *for no* A', λ *do we have:*

$(*)_3$ $A \leq^* A' <^* B$, $(A', B, \lambda) \in \mathcal{T}$, $\mathbf{w}_\lambda(A', B) > 0$ *and* $(*)_1$ *of 4.13.*

PROOF For the equivalence of the first and the second clauses read Definition 4.11(2), 4.11(4) (remembering 4.9(1)). Trivially $(*)_3 \Rightarrow (*)_2$ and hence the second clause implies the third one. If the second clause fails let this be exemplified by A', λ satisfying $(*)_2$ and then just use 4.13. $\blacksquare_{4.14}$

Remark 4.15 1) In 4.14(ii) we get: if $C \subseteq B \setminus A'$ is λ-closed nonempty then $\mathbf{w}(A', A' \cup C, \lambda \restriction C) > 0$.
[Why? If $C \neq B \setminus A'$ this is stated explicitily, otherwise this means $\mathbf{w}(A', B, \lambda) > 0$ which holds.]
2) In 4.14(iii), i.e. 4.13(∗), we can allow any λ–closed $C \subseteq B \setminus A$ if we make the inequalities non-strict.
[Why? Check.]

Claim 4.16 $A \leq_s^* B$ if and only if *either* $A = B$ *or for some* λ *we have:*
$(A, B, \lambda) \in \mathcal{T}$ *and* $\mathbf{w}_\lambda(A, B) > 0$, *and*
for every nonempty λ*-closed* $C \subseteq B \setminus A$, *we have* $\mathbf{w}(A, A \cup C, \lambda \restriction C) > 0$.

PROOF The *only if* direction:
So we have $A \leq_s^* B$. If $A = B$ we are done, so assume $A <_s^* B$. Let C be minimal such that $A \leq C \leq B$ and for some λ_0 the triple (C, B, λ_0) satisfies: for every non empty λ_0-closed $C' \subseteq B \setminus C$ we have $\mathbf{w}(C, C \cup C', \lambda_0 \restriction C') \geq 0$ (exists as $C = B$ is O.K.). If $A = C$ we have finished. Otherwise, by hypothesis $A \leq_s^* B$ which means that $\neg(A <_i^* C)$, by 4.14 (see the third clause), for some C', λ_1 we have $A \leq C' <^* C$, $(C', C, \lambda_1) \in \mathcal{T}$ and for every λ_1-closed $D \subseteq C \setminus C'$ satisfying $D \neq \{\emptyset, C \setminus C'\}$ we have

$$\mathbf{w}(C', C' \cup D, \lambda_1 \restriction D) > 0, \quad \text{and} \quad \mathbf{w}(C' \cup D, C, \lambda_1 \restriction (C \setminus C' \setminus D)) < 0.$$

Define an equivalence relation λ on $B \setminus C'$: an equivalence class of λ is an equivalence class of λ_0 or an equivalence class of λ_1.

We prove that (C', B, λ) is as required, thus contradicting the minimality of C. So let $D \subseteq B \setminus C'$ be λ-closed and we define $D_0 = D \cap (B \setminus C)$, $D_1 = D \cap (C \setminus C')$. Clearly D_0 is λ_0-closed so $\mathbf{w}(C, C \cup D_0, \lambda \restriction D_0) \geq 0$ (see 4.15(2)), and D_1 is λ_1-closed so $w(C', C' \cup D_1, \lambda \restriction D_1) \geq 0$ by 4.15(2). Now

$$\begin{aligned}\mathbf{v}(C', C' \cup D, \lambda) = \; & |D/\lambda| = |D_1/\lambda_1| + |D_0/\lambda_0| \\ & = \mathbf{v}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{v}(C', C' \cup D_0, \lambda \restriction D_0) \\ & = \mathbf{v}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{v}(C, C \cup D_0, \lambda \restriction D_0),\end{aligned}$$

$$\begin{aligned}& \mathbf{e}(C', C' \cup D, \lambda) = \\ & \quad = \mathbf{e}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{e}(C' \cup D_1, C' \cup D_1 \cup D_0, \lambda \restriction D_0) \\ & \quad \leq \mathbf{e}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{e}(C, C \cup D_1, \lambda \restriction D_1),\end{aligned}$$

and hence

$$\begin{aligned}
&\mathbf{w}(C', C' \cup D, \lambda) = \\
&\quad = \mathbf{v}(C', C' \cup D, \lambda) - \alpha \mathbf{e}(C', C' \cup D, \lambda) \\
&\quad = \mathbf{v}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{v}(C, C \cup D_0, \lambda \restriction D_0) \\
&\qquad -\alpha \mathbf{e}(C', C' \cup D_1, \lambda \restriction D_1) - \alpha \mathbf{e}(C' \cup D_1, C' \cup D_1 \cup D_0, \lambda \restriction D_0) \\
&\quad \geq \mathbf{v}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{v}(C, C \cup D_0, \lambda \restriction D_0) \\
&\qquad -\alpha \mathbf{e}(C', C' \cup D_1, \lambda \restriction D_1) - \alpha \mathbf{e}(C, C \cup D_0, \lambda \restriction D_0) \\
&\quad = \mathbf{w}(C', C' \cup D_1, \lambda \restriction D_1) + \mathbf{w}(C, C \cup D_0, \lambda \restriction D_0) \geq 0,
\end{aligned}$$

and the inequality holds by the irrationality of α, i.e. by 4.9(1).
The *if* direction:
So let λ be as required in the second clause. Suppose $A <^* C \leq^* B$, and we shall prove that $\neg(A <^*_i C)$. We use $A' = C$ and $\lambda' = \lambda \restriction A'$ and so by 4.14 it suffices to show $\mathbf{w}_\lambda(A, C, \lambda') > 0$. Let $D = \bigcup\{x/\lambda : x \in C \setminus A\}$, so D is non empty λ-closed. Hence by the present assumption on A, B, λ we have $\mathbf{w}(A, D, \lambda \restriction D) > 0$. Now

$$\mathbf{v}(A, C, \lambda \restriction C) \geq |C/\lambda| = \mathbf{v}(A, D, \lambda \restriction D)$$

and

$$\mathbf{e}(A, C, \lambda \restriction C) \leq \mathbf{e}(A, D, \lambda \restriction D)$$

so $\mathbf{w}(A, C, \lambda \restriction C) \geq \mathbf{w}(A, D, \lambda \restriction D) > 0$ as requested. $\blacksquare_{4.16}$

Fact 4.17 *If $A <^*_c B$ and $A <^* B' \leq^* B$* then *$A <^*_c B'$.*

PROOF If the conclusion fails, then for some λ' we have $(A, B', \lambda') \in \mathcal{T}$ and $\mathbf{w}(A, B', \lambda') > 0$. There is an equivalence relation λ on $B \setminus A$ such that $\lambda \restriction (B' \setminus A) = \lambda'$, and

$$(\exists x \in B' \setminus A)(\forall y \in B \setminus B')(y \lambda x).$$

So

$$\mathbf{v}(A, B, \lambda) = \mathbf{v}(A, B', \lambda'), \qquad \mathbf{e}(A, B, \lambda) \geq \mathbf{e}(A, B', \lambda')$$

and hence

$$\mathbf{w}_\lambda(A, B, \lambda) \leq \mathbf{w}_{\lambda'}(A, B') < 0,$$

contradicting the assumption $A <^*_c B$.

Discussion 4.18 We can now explain our intensions.

First

Intention: We can prove that these $<^*_x$ have the formal properties of $<_x$, like $<^*_i$ is a partial order etc.
Let $A <_b B \Leftrightarrow$ for random enough $\mathcal{M}_n$ and $f : A \to \mathcal{M}_n$, the maximal number of pairwise disjoint $g \supseteq f$ satisfying $g : B \to \mathcal{M}_n$ is $< n^\varepsilon$ (for every fixed ε)

Second

We intend to prove:

$(*)$ $A <^*_b B \quad \Rightarrow \quad A <_b B$

For this it suffices that for every $f : A \to \mathcal{M}_n$ and ε, the expected value of $g : B \to \mathcal{M}_n$ satisfying $\mathrm{Rang}(g \restriction B \setminus A)$, $f(A)$ are with distance $\geq n^\varepsilon$; is $\leq 1/n^\varepsilon$ for some $\varepsilon > 0$. Then, the expected value of k such disjoint g is $\frac{1}{n^{k\varepsilon}}$. So if $k\varepsilon > |A|$, the expected value of f with k g's is $< \frac{1}{n^{k\varepsilon - |A|}}$, so for random enough $\mathcal{M}_n$, there are no such k-tuples of pairwise disjoint g's. But, if $k \sim \log(n)$ we get that there are $\ll n^\varepsilon$ etc. So $<_i = <^*_i$ is also O.K.

The next step
is the universal part from niceness. This does not involve any probability, just weight computations (and previous stages). By the "universal part of nice" we mean 2.11(A):

if $\bar{a} \in {}^k(\mathcal{M}_n)$, $b \in \mathcal{M}_n$ then there are $m_1 < m$, $B \subseteq \mathrm{cl}^{k,m_1}(\bar{a})$ such that $\bar{a} \subseteq B$,

$$\mathrm{cl}^{k,m}(\bar{a}) \underset{B}{\overset{\mathcal{M}_n}{\bigcup\!\!\!\!\bigcup}} (\mathrm{cl}^{k,m}(\bar{a}, b) \setminus \mathrm{cl}^{k,m}(B)) \cup B$$

Lastly, again probability; the existential part of nice (i.e. clause (B) of Definition 2.8).

5 The probabilistic inequalities

Context 5.1 As in §4, so $p_i = 1/i^\alpha$, for $i > 1$, $p_1 = p_2$ (where $\alpha \in (0,1)_\mathbb{R}$ irrational) and $\mathcal{M}_n = \mathcal{M}^0_n$ (only the graph).

Definition 5.2 *Let $\varepsilon > 0$, $k \in \mathbb{N}$, $\mathcal{M}_n \in \mathcal{K}$ and $A <^* B$ be in $\mathcal{K}_\infty$. Assume $f\colon A \hookrightarrow \mathcal{M}_n$ is an embedding. Define*

$$\begin{aligned} G^{\varepsilon,k}_{A,B}(f, \mathcal{M}_n) := \Big\{ \bar{g} : \ & (1)\ \bar{g} = \langle g_\ell : \ell < k \rangle, \quad (2)\ \bigwedge_\ell f \subseteq g_\ell, \quad (3)\ g_\ell\colon B \hookrightarrow M_n \\ & (4)\ \ell_1 \neq \ell_2 \Rightarrow \mathrm{Rang}(g_{\ell_1}) \cap \mathrm{Rang}(g_{\ell_2}) = \mathrm{Rang}(f) \\ & (5)\ [\ell < k \ \&\ x \in B \setminus A \ \&\ y \in A] \Rightarrow |g_\ell(x) - g_\ell(y)| \geq n^\varepsilon \Big\}. \end{aligned}$$

The size of this set has natural connection with the number of pairwise disjoint extensions $g : B \hookrightarrow \mathcal{M}_n$ of f, hence with the holding of $A <_s B$, see 5.3 below.

Fact 5.3 *For every ε and k and $A \leq B$ we have: for every n and $M \in \mathcal{K}_n$ and $f : A \hookrightarrow M_n$ we have: if $G^{\varepsilon,k}_{A,B}(f, M_n) = \emptyset$* then

$$\begin{aligned} \max\{\ell : \ & \textit{there are } g_m : B \hookrightarrow M_n \textit{ for } m < \ell \textit{ such that } f \subseteq g_m, \\ & [m_1 < m_2 \Rightarrow \mathrm{Rang}(g_{m_1}) \cap \mathrm{Rang}(g_{m_2}) \subseteq \mathrm{Rang}(f)]\} \leq \\ & \leq 2|A|n^\varepsilon + (k-1) \end{aligned}$$

PROOF Among them, for each $x \in A$, there are $\leq 2|A|n^\varepsilon$ functions g with $\min\{|g(x) - g(y)| : y \in B \setminus A\} < n^\varepsilon$ and at most $k-1$ others. Note that in a family of functions whose ranges intersect in $f[A]$, each is determined by its value on any $x \in B \setminus A$. $\blacksquare_{5.3}$

The following is central, it does not yet prove almost niceness by here the paralel from [ShSp 304], [BlSh 528] were immediate, and here we see the main additional difficulties we have.

Theorem 5.4 *Assume $A <^* B$ (both in $\mathcal{K}_\infty$).* Then *a sufficient condition for*

$\otimes_1$ *for every $\varepsilon > 0$, for some $k \in \mathbb{N}$, for every random enough $\mathcal{M}_n$ we have:*

$(*)$ *if $f\colon A \hookrightarrow \mathcal{M}_n$ then $G^{\varepsilon,k}_{A,B}(f, \mathcal{M}_n) = \emptyset$*

is the following:

$\otimes_2$ *$A <^*_c B$ (i.e. if $(A, B, \lambda) \in \mathcal{T}$ then $\mathbf{w}_\lambda(A, B) < 0$).*

Remark 5.5 From $\otimes_1$ we can conclude: for every $\varepsilon \in \mathbb{R}^+$ we have: for every random enough $\mathcal{M}_n$, for every $f : A \hookrightarrow \mathcal{M}_n$, there cannot be $\geq n^\varepsilon$ extensions $g : B \hookrightarrow \mathcal{M}_n$ of f pairwise disjoint over f.
For this, first choose $\varepsilon_1 < \varepsilon$ such that $2|A|n^{\varepsilon_1} < n^\varepsilon$. Note that for any k, $G^{\varepsilon,k}_{A,B}(f, \mathcal{M}_n) \subseteq G^{\varepsilon_1,k}_{A,B}(f, \mathcal{M}_n)$. Choose k_1 for ε_1 by 5.3. Then the number of pairwise disjoint extensions of f is $\leq 2|A|n^{\varepsilon_1} + (k_1 - 1)$. For sufficiently large n this is $< n^\varepsilon$.

Remark 5.6 We think of g extending f such that $g\colon B \hookrightarrow \mathcal{M}_n$ that satisfies, for some constants c_1 and c_2:

$$x\lambda y \Rightarrow |g(x) - g(y)| < c_1$$

and

$$[\{x, y\} \in \mathrm{edge}(B) \;\&\; \{x, y\} \not\subseteq A \;\&\; \neg x\lambda y] \Rightarrow |g(x) - g(y)| \geq c_2.$$

So the number of such $g \supset f$ is $\sim n^{|(B\setminus A)/\lambda|}$, the probability of each being an embedding, assuming f is one, is $\sim n^{-\alpha\mathbf{e}(A,B,\lambda)}$ and the number of such extensions g is $\sim n^{\mathbf{v}(A,B,\lambda)}$, hence the expected value is $\sim n^{\mathbf{w}_\lambda(A,B)}$ ($\sim$ means "up to a constant"). So $A <^*_i B$ says that there are few such copies of B over any copy of A, i.e. the expected value is < 1. In [ShSp 304], λ is equality, here things are somewhat more complicated.

Proof of 5.4:

Step A.

W.l.o.g B is minimal i.e. $A < B' < B \Rightarrow \neg(A <^*_c B')$.
This is because if there exists such a B' then $\otimes_1$ for A and B' implies $\otimes_1$ for A and B.

Step B.

It suffices to prove that for some $\zeta > 0$ we have:

$(*)_\zeta$ $\mathrm{Exp}(|G^{\varepsilon,1}_{A,B}(f, \mathcal{M}_n)| : f : A \hookrightarrow \mathcal{M}_n) \leq \frac{1}{n^\zeta}$

(remember that Exp stands for "expectation").

To see that choose k such that $k \cdot \zeta > |A|$ (an overkill). Now assuming $f: A \hookrightarrow \mathcal{M}_n$ we have:

$$\begin{aligned}
&\mathrm{Exp}(|G^{\varepsilon,k}_{A,B}(f, \mathcal{M}_n)|) = \\
&\sum \Big\{\mathrm{Prob}(\bigwedge_{\ell<k} g_\ell: B \hookrightarrow \mathcal{M}_n) : \ \bar{g} \text{ satisfies clauses (1), (2), (4) and (5)}\Big\} = \\
&\qquad \text{(by disjointness and independence)} \\
&\sum \Big\{ \prod_{\ell<k} \mathrm{Prob}(g_\ell: B \hookrightarrow \mathcal{M}_n) : \ \bar{g} \text{ satisfies clauses (1), (2), (4) and (5)}\Big\} \leq \\
&\qquad \text{(as the sum has just more terms, and all terms are non negative)} \\
&\sum \Big\{ \prod_{\ell<k} \mathrm{Prob}(g_\ell: B \hookrightarrow \mathcal{M}_n) : \ \bar{g} \text{ satisfies clauses (1), (2) and (5)}\Big\} = \\
&\sum \Big\{ \prod_{\ell<k} \mathrm{Prob}(g_\ell: B \hookrightarrow \mathcal{M}_n) : \ \bigwedge_{\ell<k} [g_\ell \text{ satisfies clauses (2) and (5)}]\Big\} = \\
&\qquad \text{(since now all } k\text{-element sequences of extensions of } f \text{ are allowed)} \\
&\prod_{\ell<k} \Big\{ \sum \mathrm{Prob}(g_\ell: B \hookrightarrow \mathcal{M}_n) : \ g_\ell \text{ satisfies clauses (2) and (5)}\Big\} = \\
&\prod_{\ell<k} \mathrm{Exp}(|G^{\varepsilon,1}_{A,B}(f, \mathcal{M}_n)|) \leq (\tfrac{1}{n^\zeta})^k.
\end{aligned}$$

By a well known easy lemma, for each $f: A \rightarrow \{1, \dots, n\}$, it follows that

$$\mathrm{Prob}\Big(|G^{\varepsilon,k}_{A,B}(f, M_n)| \neq \emptyset : f: A \hookrightarrow \mathcal{M}_n\Big) < \frac{1}{n^{k\cdot\zeta}}.$$

Therefore

$$\begin{aligned}
&\mathrm{Prob}\Big(\exists f: A \hookrightarrow M_n \text{ such that } |G^{\varepsilon,k}_{A,B}(f, M_n)| \neq \emptyset\Big) < \\
&\qquad \tfrac{1}{n^{k\cdot\zeta}} |\{f : f: A \rightarrow [n]\}| \leq \tfrac{n^{|A|}}{n^{k\cdot\zeta}} \longrightarrow 0.
\end{aligned}$$

(So if A has many edges there is a waste and if A has no edges then this is the right choice of k).

Step C.

We now show, with some effort, that without loss of generality $\mathrm{Rang}(f) = [|A|](= \{1, 2, \dots, |A|\})$ – this is certainly helpful though it isn't really necessary.
To see that let f_0 be a bijection between A and $[|A|]$. Divide the possible g's to finitely many classes by their types (defined below) ("finite" means here: with a bound depending on A and B but not on n). For each type we will prove that the expected value for f under the assumption $f : A \hookrightarrow \mathcal{M}_n$ is $\leq \mathrm{Const}(A, B) \times$[the expected value for f_0 as required under the assumption $f_0 : A \hookrightarrow \mathcal{M}_n$]. We tend to ignore "$f$, f_0 is an embedding of A".
Notation: Define a linear order $<^*_g$ on B: $x <^*_g y$ iff $g(x) < g(y)$.
Let $A = \{a_1, \dots, a_{m-1}\}$ where $0 < i < j < m \Rightarrow a_i <^*_g a_j \equiv g(a_i) < g(a_j)$ (actually depend on f only). We add two "pseudoelements" a_0 and a_m to B and stipulate $f(a_0) = 0$, $f(a_m) = 1$ so $a_0 <_g b <_g a_m$ for all $b \in B$. For each $\ell < m$, the set $\{b \in B \setminus A : a_\ell <^*_g b <^*_g a_{\ell+1}\}$ is listed by $b_{\ell,1} <^*_g b_{\ell,2} \dots <^*_g b_{\ell,m_\ell-1}$ and we stipulate $b_{\ell,0} = a_\ell$, $b_{\ell,m_\ell} = a_{\ell+1}$.

For f, n , ε, A, B define

$$\begin{aligned}
G^{\varepsilon,1}_{A,B}(f, [n]) := \{g : \ & g: B \hookrightarrow [n], \text{ i.e. } g \text{ is one-to-one and} \\
& g \text{ satisfies clauses (2) and (5)}\}.
\end{aligned}$$

We define $\mathrm{tp}^0(g) = \mathrm{tp}^0_{A,B}(g)$ to be the following information (where the number of possibilities has a bound which depends only on $|B|$, and remember $[\beta]$ is the largest integer $\le \beta$):

(a) $\{(x,y) : x, y \in B,\ g(x) < g(y)\}$,

(b) $\{\langle \ell, j, \left[\frac{g(b_{\ell,j+1}) - g(b_{\ell,j})}{f(a_{\ell+1}) - f(a_\ell)} \cdot (m_\ell + 8)^3\right]\rangle : \ell < m \text{ and } j < m_\ell\}$

(note[3]: the number of possibilities in clause (b) is at most $\prod\limits_{\ell<m} \prod\limits_{j<m_\ell} (m_\ell+8)^3$).

Fixing the type τ, we know m, $\langle m_\ell : \ell < m\rangle$, of course f is given hence we know $\langle a_\ell : \ell = 1, 2, \ldots, m\rangle$. In addition for each $\ell < m$, when $m_\ell > 1$, we can compute some $x^*_\ell < x^{**}_\ell$ and $j_\ell < m_\ell$, depending only on f, n and τ, such that if $g \in G^{\varepsilon,1}_{A,B}(f,[n])$ and $\mathrm{tp}^0(g) = \tau$ then

$(*)$ $f(a_\ell) \le g(b_{\ell,j_\ell}) \le x^*_\ell < x^{**}_\ell \le g(b_{\ell,j_\ell+1}) \le f(a_{\ell+1})$ and $x^{**}_\ell - x^*_\ell \ge \frac{1}{2m_\ell} \cdot (a_{\ell+1} - a_\ell)$.

Next define a permutation $\sigma = \sigma_{f,\tau}$ of $[n]$ by:

(a) $\sigma(a_\ell) = \ell$ for $\ell = 1, \ldots, m-1$,

(b) $\sigma(a_\ell + i) = m + 2mi + 2\ell$ when $1 \le i \le x^*_\ell - a_\ell$ and $\ell \in \{1, \ldots, m-1\}$,

(c) $\sigma(a_{\ell+1} - i) = m + 2mi + 2\ell + 1$ when $1 \le i \le a_{\ell+1} - x^{**}_\ell$ and $\ell \in \{1, \ldots, m-1\}$,

(d) complete σ to a permutation in whatever way.

Fact 5.7 *1) There is some* $\mathbf{c} = \mathbf{c}_{f,\tau}$ *such that:*

if $i, j \in \{a_1, \ldots, a_{m-1}\} \cup \bigcup\limits_{\ell=0}^{m-1} (a_\ell, x^*_\ell] \cup \bigcup\limits_{\ell=0}^{m-1} [x^{**}_\ell, a_{\ell+1})$
then $|\sigma(i) - \sigma(j)| \le \mathbf{c} \times |i - j|$.

2) If $j \in [n] \setminus \{a_1, \ldots, a_{m-1}\}$ *and* $k \le n$ *and* $\bigwedge\limits_{\ell=1}^{m-1} |j - a_\ell| \ge k$

then $\bigwedge\limits_{\ell=1}^{m-1} |\sigma(j) - \sigma(a_\ell)| \ge k$.

Proof 1) By symmetry w.l.o.g. $i < j$, and by the triangle inequality w.l.o.g. j is the successor of i in the set of possibilities (in 5.7). If for some $\ell < m$ we have $\{i, j\} \subseteq (a_\ell, x^*_\ell]$, then $|\sigma(i) - \sigma(j)| \le 2m \times |i - j|$, so if $\mathbf{c} \ge 2m$ we are O.K. by clause (b) in the Definition. The same holds if $\{i, j\} \subseteq [x^{**}_\ell, a_{\ell+1})$ for some $\ell < m$. The case $i = a_\ell$ for some $\ell \in \{1, \ldots, m-1\}$ gives $\sigma(i) < m$, $\sigma(j) = \sigma(i+1) = \sigma(a_\ell + 1) = m + 2m + 1 + 2\ell \le 5m$ hence if $\mathbf{c} \ge 5m$ we are O.K. and similarly if $j = a_\ell$ for some $\ell \in \{1, \ldots, m-1\}$. We are left with the case $i = x^*_\ell$, $j = x^{**}_\ell$ for some ℓ. But then $\sigma(i) = m + 2m(x^*_\ell - a_\ell) + 2\ell$ and $\sigma(j) = m + 2m(a_{\ell+1} - x^{**}_\ell) + 2\ell + 1$, so $|\sigma(i) - \sigma(j)| = 2m[(a_{\ell+1} - a_\ell) - (x^{**}_\ell - x^*_\ell)] \pm 1 \le 2m(a_{\ell+1} - a_\ell) + 1 \le (2m+1)(a_{\ell+1} - a_\ell) \le$

[3]We did not care about the exact bound

$(2m+1)\times 2m_\ell \times (x^{**}_\ell - x^*_\ell) = (2m+1)\cdot 2m_\ell \cdot (j-i)$. Thus if $\mathbf{c} \ge 2m_\ell(2m+1)$ we are O.K., so we have finished.
2) Check. $\blacksquare_{5.7}$

Now σ induces a 1-to-1 function from

$$G^{\varepsilon,1,\tau}_{A,B}(f,[n]) := \{g \in G^{\varepsilon,1}_{A,B}(f,[n]) : \mathrm{tp}^0(g) = \tau\}$$

into $G^{\varepsilon,1}_{A,B}(f_0,[n])$ that is $g \mapsto g\circ\sigma^{-1}$. [Why? Because σ is a permutation of $[n]$ and $f\circ\sigma^{-1} = f_0$ and 5.7(2).] We note

$(*)$ $\mathrm{Prob}(g \in G^{\varepsilon,1}_{A,B}(f,\mathcal{M}_n)) \le \mathbf{c}^1 \cdot \mathrm{Prob}(g\circ\sigma^{-1} \in G^{\varepsilon,1}_{A,B}(f_0,\mathcal{M}_n))$

for some constant $\mathbf{c}^1$. (By 5.7(1) of course we should consider also the probability of non edges but this is easier as there is a constant upper bound < 1). So

$$\begin{aligned}\mathrm{Exp}(|G^{\varepsilon,1,\tau}_{A,B}(f,\mathcal{M}_n)|) = \sum\{\mathrm{Prob}(g \in G^{\varepsilon,1}_{A,B}(f,\mathcal{M}_n)) : g \in G^{\varepsilon,1,\tau}_{A,B}(f,[n])\} \le \\ \mathbf{c}^1 \sum\{\mathrm{Prob}(g\circ\sigma^{-1} \in G^{\varepsilon,1}_{A,B}(f,\mathcal{M}_n)) : g \in G^{\varepsilon,1,\tau}_{A,B}(f,[n])\} \le \\ \mathbf{c}^1 \sum\{\mathrm{Prob}(g' \in G^{\varepsilon,1}_{A,B}(f_0,\mathcal{M}_n)) : g' \in G^{\varepsilon,1,\tau}_{A,B}(f_0,[n])\} \le \\ \mathbf{c}^1 \mathrm{Exp}(|G^{\varepsilon,1}_{A,B}(f_0,\mathcal{M}_n)|).\end{aligned}$$

Also

$$G^{\varepsilon,1}_{A,B}(f,[n]) = \bigcup\{G^{\varepsilon,1,\tau}_{A,B}(f,[n]) : \tau \text{ is a } \mathrm{tp}^0\text{–type}\}$$

and the number of tp^0–types has a bound not depending on n, say $\mathbf{c}^2$. Hence

$$\begin{aligned}&\mathrm{Exp}(|G^{\varepsilon,1}_{A,B}(f,\mathcal{M}_n)|) = \sum_\tau \mathrm{Exp}(|G^{\varepsilon,1,\tau}_{A,B}(f,\mathcal{M}_n)|) \le \\ &\le \sum_\tau \mathbf{c}^1 \mathrm{Exp}(|G^{\varepsilon,1}_{A,B}(f_0,\mathcal{M}_n)|) = \mathbf{c}^2\mathbf{c}^1\mathrm{Exp}(|G^{\varepsilon,1}_{A,B}(f_0,\mathcal{M}_n)|).\end{aligned}$$

So the expected value of $G^{\varepsilon,1}_{A,B}(f,\mathcal{M}_n)$ assuming $f : A \hookrightarrow \mathcal{M}_n$ is $\le$ a constant times the expected value of $G^{\varepsilon,1}_{A,B}(f_0,\mathcal{M}_n)$ assuming $f_0 : A \hookrightarrow \mathcal{M}_n$. Therefore it is enough to prove that the expected value of the second value is small (i.e $\le n^{-\zeta}$).
Note: we are using heavily that we are dealing with expected values that are additive, so dependence for various g's are irrelevant.

Step D.
First we will define a new type of g, $\mathrm{tp}^1(g)$ (assuming $g \supseteq f_0$). Let $B\backslash A = \{b_1, b_2, \dots, b_m\}$. Stipulate $b_0 \in {}^B A$ with $f_0(b_0)$ maximal in $\mathrm{Rang}(f_0)$, i.e. m and stipulating $b_{m+1} \in B$ with $g(b_{m+1}) = n+1$, we let $\mathrm{tp}^1(g)$ includes the following information:

(α) $\{(x,y) : g(x) < g(y)\}$ (so w.l.o.g. $g(b_0) < g(b_1) < g(b_2) < \dots$),

(β) the order between $\gamma_\ell = \gamma_\ell[g] := g(b_{\ell+1}) - g(b_\ell)$ for $\ell \le m$, i.e. the truth value of each statement $\gamma_{\ell_1} < \gamma_{\ell_2}$.

Let $\sigma = \sigma[g] = \sigma(\tau) = \sigma(\mathrm{tp}^1(g))$ be a permutation on $\{0, 1, \ldots, m-1\}$ (without m, as $\gamma_m = n - \sum\limits_{\ell<m} \gamma_\ell - (m-1))$ such that $\gamma_{\sigma(0)} \leq \gamma_{\sigma(1)} \leq \ldots \leq \gamma_{\sigma(m-1)}$. Let $\bar{\gamma}[g] = \langle \gamma_\ell[g] : \ell < m \rangle$. Denote for type τ (for tp^1):

$$\begin{aligned}\Gamma = \Gamma[\tau] = \Gamma^n_m[\tau] := \{\bar{\gamma} : \ \ & \bar{\gamma} = \langle \gamma_0, \ldots, \gamma_{m-1} \rangle, \ \gamma_{\sigma(0)} \leq \gamma_{\sigma(1)} \leq \ldots \leq \gamma_{\sigma(m-1)}, \\ & \gamma_\ell > 0 \ \text{ and } \ \gamma_0 \geq n^\varepsilon \ \text{ and } \ |A| + \sum_{\ell<m} \gamma_\ell \leq n\}.\end{aligned}$$

Note, for each $\bar{\gamma} \in \Gamma$ there is a unique $g = g_{\bar{\gamma}} \in G^{\varepsilon,1}_{A,B}(f_0, [n])$ such that $\bar{\gamma}(g) = \bar{\gamma}$ and $G^{\varepsilon,1}_{A,B}(f_0, [n]) = \{g_{\bar{\gamma}} : \bar{\gamma} \in \Gamma\}$. [Why? Let $g_{\bar{\gamma}}(b_\ell) = |A| + \sum\{\gamma_m : m < \ell\}$.]

In order to compute the probability that g is a graph embedding, we introduce the following notation. Let

$$\begin{aligned} E_B &= \{(c,d) : c \in B, d \in B, \{c,d\} \not\subseteq A \text{ and: } c \in A \ \& \ d \in B \setminus A \text{ or} \\ &\qquad \bigvee\{c = b_{\ell_1} \wedge d = b_{\ell_2} : 1 \leq \ell_1 < \ell_2 \leq m\}\}, \\ E^0_B &= \{(c,d) \in E_B : \{c,d\} \in \mathrm{edge}(B)\}, \\ E^1_B &= E_B \setminus E^0_B. \end{aligned}$$

For $(c,d) \in E_B$ let $w(c,d)$ be defined as follows: if $(c,d) = (b_{\ell_1}, b_{\ell_2})$ and $1 \leq \ell_1 < \ell_2 \leq m$ then $w(c,d) = \{\ell_1, \ell_1+1, \ldots, \ell_2-1\}$ and if $c \in A$, $d \in b_{\ell_2}$ then $w(c,d) = \{0, 2, \ldots, \ell_2-1\}$. So $w(c,d)$ is a non empty subset of $\{1, \ldots, m-1\}$, in fact an interval of it. Thus $\sum\limits_{\ell \in w(c,d)} \gamma_\ell$ is the distance between $g(c)$ and $g(d)$ (used below). Let $k(c,d)$ be the minimal $k \in w(x,y)$ such that $\gamma_k = \max\{\gamma_\ell : \ell \in w(c,d)\}$. So $k(c,d)$ is computable from $\mathrm{tp}^1(g)$.

Fixing the (new) type τ, hence $\sigma = \sigma(\tau)$, we have: (where we assume $f_0 : A \hookrightarrow \mathcal{M}_n$)

$$\begin{aligned} \beta^* &=: \mathrm{Exp}|\{g \in G^{\varepsilon,1}_{A,B}(f_0, \mathcal{M}_n) : \mathrm{tp}^1(g) = \tau \text{ hence } \bar{\gamma}[g] \in \Gamma[\tau]\}| \\ &= \sum_{\bar{\gamma} \in \Gamma[\tau]} \mathrm{Exp}(|\{g_{\bar{\gamma}} \in G^{\varepsilon,1}_{A,B}(f_0, \mathcal{M}_n) : \mathrm{tp}^1(g) = \tau \text{ and } \bar{\gamma}[g] = \bar{\gamma}\}| \\ &= \sum_{\bar{\gamma} \in \Gamma[\tau]} \mathrm{Prob}(g_{\bar{\gamma}} \in G^{\varepsilon,1}_{A,B}(f_0, \mathcal{M}_n)) \end{aligned}$$

and this equals:

$$\begin{aligned} \beta^* &= \sum_{\bar{\gamma} \in \Gamma[\tau]} \Big(\prod_{(c,d) \in E^0_B} \Big(\frac{1}{(\sum_{\ell \in w(c,d)} \gamma_\ell)^\alpha}\Big) \cdot \prod_{(c,d) \in E^1_B} \Big(1 - \frac{1}{(\sum\limits_{\ell \in w(c,d)} \gamma_\ell)^\alpha}\Big)\Big) \leq \\ &\leq \sum_{\bar{\gamma} \in \Gamma[\tau]} \Big(\prod_{(c,d) \in E_B} \frac{1}{(\gamma_{k(c,d)})^\alpha} \Big). \end{aligned}$$

This equals (where $j(\ell) =: |\{(c,d) \in E^1_B : k(c,d) = \ell\}|$ so neccessarily $j(m) = 0$; actually we should write $j(\ell, \tau)$):

$$= \sum_{\bar{\gamma} \in \Gamma[\tau]} \Big(\prod_{\ell \in [0,m)} (\frac{1}{\gamma_\ell^{\alpha \cdot j(\ell)}}) \Big) = \sum_{\bar{\gamma} \in \Gamma[\tau]} \prod_{\ell \in [0,m-1)} \frac{1}{\gamma_\ell^{\alpha \cdot j(\ell)}}.$$

For $\bar\gamma \in \Gamma$ let $\bar s(\bar\gamma) := \langle s_\ell(\gamma_\ell) : \ell < m\rangle$ where $s_\ell(\gamma_\ell) = [\log_2(\gamma_\ell)]$. Define

$$\begin{aligned} S =: \{\bar s :\ & \bar s = \langle s_\ell : \ell < m\rangle,\ 0 \le s_{\sigma(0)} \le s_{\sigma(1)} \le \ldots \le [\log_2(n)] \\ & \text{and}\ \ s_0 \ge [\log_2(n^\varepsilon)]\}. \end{aligned}$$

So the above is

$$\beta^* \le \sum_{\bar s \in S} \sum_{\bar\gamma \in \Gamma[\tau], \bar s(\bar\gamma) = \bar s} \Big(\prod_{\ell \in [0,m)} (\frac{1}{\gamma_\ell^{\alpha \cdot j(\ell)}}) \Big) = \sum_{\bar s \in S} \beta_{\bar s}$$

where $\beta_{\bar s} := \sum\limits_{\bar\gamma \in \Gamma[\tau], \bar s(\bar\gamma) = \bar s} \Big(\prod\limits_{\ell \in [0,m)} (\frac{1}{\gamma_\ell^{\alpha \cdot j(\ell)}}) \Big)$. Now

$$[\log_2(\gamma_\ell)] = s_\ell(\gamma_\ell) \le \log_2(\gamma_\ell) < s_\ell(\gamma_\ell) + 1$$

and hence

$$\frac{1}{\gamma_\ell^{\alpha \cdot j(\ell)}} = \frac{1}{2^{(\log_2(\gamma_\ell)) \cdot \alpha \cdot j(\ell)}} \le \frac{1}{2^{s_\ell(\gamma_\ell) \cdot \alpha \cdot j(\ell)}}$$

and

$$[\bar\gamma \in \Gamma[\tau] \ \ \&\ \ \bar s(\bar\gamma) = \bar s] \quad \Rightarrow \quad \prod_{\ell \in [0,m)} \frac{1}{(\gamma_\ell)^{\alpha \cdot j(\ell)}} \le \prod_{\ell \in [0,m)} \frac{1}{2^{s_\ell(\gamma) \cdot \alpha \cdot j(\ell)}}.$$

Note that it is a common bound to all $\bar\gamma \in \{\bar\gamma \in \Gamma[\tau] : \bar s(\bar\gamma) = \bar s\}$. Hence

$$\beta_{\bar s} \le |\{\bar\gamma \in \Gamma : \bar s(\bar\gamma) = \bar s\}| \cdot \prod_{\ell \in [0,m)} (\frac{1}{2^{s_\ell \cdot \alpha \cdot j(\ell)}}).$$

Now

$$\begin{aligned} &|\{\bar\gamma \in \Gamma[\tau] : \bar s(\bar\gamma) = \bar s\}| \le |\{\bar\gamma : \bar\gamma = \langle \gamma_\ell : \ell < m\rangle \text{ and } [\log_2(\gamma_\ell)] = s_\ell\}| = \\ &\prod_{\ell < m} |\{\gamma : \log_2(\gamma_\ell) = s_\ell\}| = \prod_{\ell < m} |\{\gamma : 2^{s_\ell} \le \gamma < 2^{s_\ell + 1}\}| = \prod_{\ell < m} (2^{s_\ell + 1} - 2^{s_\ell}) \end{aligned}$$

and hence

$$\begin{aligned} &\beta_{\bar s} \le \prod_{\ell \in [0,m)} (2^{s_\ell + 1} - 2^{s_\ell}) \cdot \prod_{\ell \in [0,m)} (\tfrac{1}{2^{s_\ell \cdot \alpha \cdot j(\ell)}}) = \\ &\prod_{\ell \in [0,m)} (2^{s_\ell}) \cdot \prod_{\ell \in [0,m)} (\tfrac{1}{2^{s_\ell \cdot \alpha \cdot j(\ell)}}) = \prod_{\ell \in [0,m)} (2^{s_\ell (1 - \alpha \cdot j(\ell))}) = 2^{\sum\limits_{\ell \in [0,m)} s_\ell (1 - \alpha \cdot j(\ell))}. \end{aligned}$$

Choose $\bar s^* \in S$ for which the expression is maximal. Note that for each $\bar s \in S$, hence also for this $\bar s^*$, every s^*_ℓ is in $[0, [\log_2(n^\varepsilon)]]$ or in $[[\log_2(n^\varepsilon)], [\log_2(n)]]$. We claim that every s^*_ℓ belongs to the set $\{0, [\log_2(n^\varepsilon)], [\log_2(n)]\}$. Why? Let E be an equivalence relation on $[0,m)$ such that $\ell E k$ *if and only if* $s^*_\ell = s^*_k$. For $\langle \sigma(\ell) : \ell < m\rangle$ this equivalence relation E is convex. Now take the last equivalence class (block) with a value that is different from our desired values and move this block up or down to get a better expression. Is it possible that the change makes no difference? No, as α is irrational.

So the number we are estimating is bounded from above by (more see 5.8 below)

$$\beta^* \le \sum_{\bar s \in S} \beta_{\bar s} \le |S| \cdot \beta_{\bar s} \le (\log_2(n))^m \beta_{\bar s^*}.$$

Step E.
Now we show that for the $\bar{s}^*$ (choosen above) $\beta_{\bar{s}^*} \leq \frac{1}{n^\zeta}$ for appropriate ζ.

Define two equivalence relations on $B \setminus A = \{b_1 \dots b_m\}$:

(1) $b_i \lambda_1 b_j$ if and only if
for a suitable $g \in G^{\varepsilon,1}_{A,B}(f_0,[n])$ (i.e. with $\mathrm{tp}^1(g) = \tau$) we have
$i = j$ or $i < j$ & $\bigwedge\limits_{i \leq \ell \leq j-1} s^*_\ell \in \{0, [\log_2(n^\varepsilon)]\}$ or
$j < i$ & $\bigwedge\limits_{j \leq \ell \leq i-1} s^*_\ell \in \{0, [\log_2(n^\varepsilon)]\}$;
this is an equivalence relation, convex for the natural ordering on $\{b_1 \dots b_m\}$ according to $\sigma[\tau]$.

(2) $b_i \lambda_0 b_j$ if and only if
for a suitable $g \in G^{\varepsilon,1}_{A,B}(f_0,[n])$ (i.e. with $\mathrm{tp}^1(g) = \tau$) we have
$i = j$ or $i < j$ & $\bigwedge\limits_{i \leq \ell \leq j-1} s^*_\ell = 0$ or $j < i$ & $\bigwedge\limits_{j \leq \ell \leq i-1} s^*_\ell = 0$;
this is an equivalence relation, convex for the natural ordering on $\{b_1 \dots b_m\}$.

(3) Let $C_1 < C_2 < \dots < C_t$ be the λ_1-equivalence classes, ordered naturally. Let $\lambda_2 = \lambda_1 {}^- (\bigcup\limits_{\ell=2}^{t} C_\ell)$.

Case 1 $t > 1$, $\mathbf{w}\Big(A \cup C_1, B, \lambda_1 {}^- (\bigcup\limits_{\ell=2}^{t} C_\ell)\Big) < 0$ and $s^*_0 = [\log_2(n^\varepsilon)]$.

W.l.o.g. ε is small enough (for our given A, B) so that for some $\zeta \in \mathbb{R}^+$,

$$\begin{aligned}
&\beta_{\bar{s}^*} \leq \mathbf{c} \cdot 2^{\sum_\ell s_\ell \cdot (1-\alpha \cdot j(\ell))} \\
&= \mathbf{c} \cdot 2^{\sum_\ell \{s_\ell \cdot (1-\alpha \cdot j(\ell)) : s_\ell = [\log_2(n)]\}} \times 2^{\sum_\ell \{s_\ell \cdot (1-\alpha \cdot j(\ell)) : s_\ell \in \{0, [\log_2(n^\varepsilon)]\}\}} \\
&\leq n^{\mathbf{c}\varepsilon} \times n^{(t-1) - \alpha \mathbf{e}_{\lambda_2}(A \cup C_1, B)} = n^{\mathbf{c} \cdot \varepsilon} \times n^{\mathbf{w}_{\lambda_2}(A \cup C_1, B)} \leq \tfrac{1}{n^\zeta},
\end{aligned}$$

and we are done (by the assumption of the case $\mathbf{w}_{\lambda_2}(A \cup C_1, B) < 0$, hence as ε is small enough we have $-\mathbf{w}_{\lambda_2}(A \cup C_1, B) > \mathbf{c} \cdot \varepsilon$).

Case 2 $s^*_0 = [\log_2(n)]$.

Now $\mathbf{w}_{\lambda_1}(A, B) < 0$ (as $A <^*_c B$ by the assumption $\bigotimes_2$ of 5.4) and the proof is similar to that of case 1.

Case 3 $s^*_0 = [\log_2(n^\varepsilon)]$, $t > 1$ and $\mathbf{w}_{\lambda_2}(A \cup C_1, B) \geq 0$.

We will try to get a contradiction to the minimality of B from stage A, and let us try $B' = A \cup C_1$.
Let λ be an equivalence relation on C_1, we have to prove that $\mathbf{w}_\lambda(A, A \cup C_1) < 0$.

Let λ^* be an equivalence relation on $B \setminus A$ defined by: $C_2, \ldots, C_t$ are equivalence classes of λ^*, no $x \in C_1$ and $y \in \bigcup\limits_{\ell=2}^{t} C_\ell$ are equivalent, and $\lambda^* \upharpoonright C_1 = \lambda$. Now (first inequality as $A <_c^* B$) we have

$$\begin{aligned}
&0 > \mathbf{w}_{\lambda^*}(A, B) = |(B \setminus A)/\lambda^*| - \alpha \cdot \mathbf{e}_{\lambda^*}(A, B) \\
&= |(B \setminus C_1 \setminus A)/\lambda^*| + |C_1/\lambda^*| - \alpha \cdot \mathbf{w}_{\lambda^*}(A, B) \\
&= |(B \setminus C_1 \setminus A)/\lambda^*| + |C_1/\lambda^*| - \alpha \cdot (\mathbf{e}_\lambda(A, A \cup C_1) + \mathbf{w}_{\lambda_2}(A \cup C_1, B)) \\
&= [|(B \setminus (C_1 \cup A))/\lambda_1| - \alpha \cdot \mathbf{e}_{\lambda_1}(A \cup C_1, B)] + [|C_1/\lambda| - \alpha \cdot \mathbf{e}_\lambda(A, A \cup C_1)] \\
&= \mathbf{w}_{\lambda_1}(A \cup C_1, B) + \mathbf{w}_\lambda(A, A \cup C_1).
\end{aligned}$$

Therefore $0 > \mathbf{w}_{\lambda_1}(A \cup C_1, B) + \mathbf{w}_\lambda(A, A \cup C_1)$. By the assumption of the present case 3 we have $0 \le \mathbf{w}_{\lambda_1}(A \cup C_1, B)$ so $0 > \mathbf{w}_\lambda(A, A \cup C_1)$. As λ was any equivalence relation on C_1 we get $A <_c^* A \cup C_1$, a contradiction as desired (for this case).

Case 4 $s_0^* = [\log_2(n^\varepsilon)]$ and $t = 1$.

This is like case 2, using λ_0 and replacing n by n^ε.

The four cases cover all possibilities. So we have finished getting a bound on $\beta_{\bar{s}^*}$ of the form $1/n^\zeta$ ($\zeta \in \mathbb{R}^+$) hence to β^* (see end of stage D) and as said in stages B, C this suffices. $\blacksquare_{5.4}$

Remark 5.8 1) Actually the situation is even better as $\sum\limits_{\bar{s} \in S} \beta_{\bar{s}} \le \mathbf{c}\beta_{\bar{s}^*}$ by the formula for the sum of a geometric series (induction on m).
2) We can get logarithmic bound instead n^ε.

Lemma 5.9 *Assume*

(A) $(A, B, \lambda) \in \mathcal{T}$,

(B) $(\forall B')[A \subset B' \subseteq B,\ B'$ *is* λ*-closed* $\Rightarrow$ $\mathbf{w}_{\lambda \upharpoonright B'}(A, B') > 0]$
(where "B' is λ-closed" means $x\lambda y$ & $x \in B'$ $\Rightarrow$ $y \in B'$).

Then *there is* $\zeta \in \mathbb{R}^+$*, in fact* $\zeta = \min\{\mathbf{w}_{\lambda \upharpoonright B'}(A, B') : A \subseteq B' \subseteq B,\ B'$ *is* λ*–closed*$\}$*, such that:*

> *for every small enough* $\varepsilon > 0$*, random enough* $\mathcal{M}_n$*, for every* $f : A \hookrightarrow \mathcal{M}_n$ *and* k *with* $0 < k < k + n^{1-\varepsilon} < n$*, there are* $n^{(1-\varepsilon)\cdot\zeta}$ *pairwise disjoint extensions* g *of* f *with*
>
> **(i)** $g : B \hookrightarrow \mathcal{M}_n$,
>
> **(ii)** $g(B \setminus A) \subseteq [k, k + n^{1-\varepsilon})$.

PROOF Let the λ-equivalence classes be $B_1 \dots B_{\ell(*)}$, where
$B_\ell = \{b_{\ell,1} \dots b_{\ell,m_\ell}\}$.
W.l.o.g. $\mathrm{Rang}(f) \cap [k, k + \frac{1}{|A|+1} n^{1-\varepsilon}) = \emptyset$.
For $\ell = 1 \dots \ell(*)$ let $k_\ell = k + \frac{n^{1-\varepsilon}}{2\ell(*)+1} \cdot \frac{2\ell-1}{|A|+1}$ and $I_\ell = [k_\ell, k_\ell + \frac{n^{1-\varepsilon}}{2\ell(*)+1} \cdot \frac{2\ell}{|A|+1})$ and $J_\ell = \{i \in I_\ell :$ m_ℓ divides $(i - k_\ell)\}$.

Now if $\bar{j} = \langle j_1 \dots j_{\ell(*)} \rangle \in \prod_\ell J_\ell$ and $g_{\bar{j}}$ is the function with domain B satisfying $g_{\bar{j}}(b_{\ell,i}) = j_\ell + i$ and $g_{\bar{j}} \restriction_A = f$, then from the random $\mathcal{M}_n$ we can compute the set $Y = \{\bar{j} : g_{\bar{j}}$ is an embedding$\}$. Note that

(Rang$(g_{\bar{j}_1}) \setminus$ Rang(f)) is disjoint from (Rang$(g_{\bar{j}_2}) \setminus$ Rang(f))
if and only if
Rang$(\bar{j}_1)$ is disjoint from Rang$(\bar{j}_2)$.

Reflecting, this is similar to [ShSp 304], [BlSh 528] and is a case of [Sh 550]; informally the transformation of [ShSp 304], [BlSh 528] corresponding to non λ-equivalent pairs consists of

(i) the probability of an edge is $\in [\mathbf{c}_1 \frac{1}{(n^{1-\varepsilon})^\alpha}, \mathbf{c}_2 \frac{1}{(n^{1-\varepsilon})^\alpha}]$ here for some positive constant $\mathbf{c}_1 < \mathbf{c}_2$ (and not as there $\frac{1}{n^\alpha}$),
as $|J_\ell|$ is $\sim n^{1-\varepsilon}$,

(ii) the number of nodes is now $n^{1-\varepsilon}$ (and not n),
factoring by λ,

(iii) here we have several candidates for edges for each pair j_1, j_2, as we are thinking of equivalence classes as nodes,

(iv) we have random unary predicates coming from edges inside a λ–equivalence class.

However, (i) and (iii) are insignificant and (ii) is o.k. if ε is small enough.

Formally, for every graph $M \in \mathcal{K}_n$ we define a model $N[M]$ (this depends of f and the k etc). Its set of elements is

$$\bigcup_{\ell=1}^{\ell(*)} R_\ell^{N[M]},$$

where

(a) $R_\ell^{N[M]} = R_\ell^n$ is a unary predicate, $R_\ell^{N[M]}$ is

$$\{i : 1 \le i \le \frac{n^{1-\varepsilon}}{2\ell(*)+1} \cdot \frac{2\ell}{|A|+1} \cdot \frac{1}{m_\ell}\} \times \ell$$

(so is constant),

(b) if $e = \{b_1, b_2\}$, $b_1 = b_{\ell_1, j_1}$, $b_2 = b_{\ell_2, j_2}$ with $\ell_1 < \ell_2$ from $[\ell(*)]$ (so $j_1 \in [m_1]$, $j_2 \in [m_2]$) then $Q_e^{N[M]}$ is a binary predicate:

$$Q_e^{N[M]} = \{\{(i_1, \ell_1), (i_2, \ell_2)\} \colon \begin{array}{l} \{k_{\ell_1} + m_{\ell_1} \times i_1 + j_1, k_{\ell_2} + m_{\ell_2} \times i_2 + j_2\} \\ \text{is an edge of } M\} \end{array}$$

(i.e. it is symmetric and trivially irreflexive),

(c) if $e = \{a, b\}$, $a \in A$, $\ell_1 \in [\ell(*)]$, $j \in [m_1]$, $b = b_{\ell_1, j}$ then Q_e is a unary predicate of $N[M]$:

$$Q_e^{N[M]} = \{(i, \ell) : \{f(a), k_\ell + m_\ell \times i + j\} \text{ is an edge of } M\},$$

(d) if $e = \{b_{\ell, j_1}, b_{\ell, j_2}\}$, $\ell \in [\ell(*)]$, $j_1 < j_2$ in $[m_\ell]$, then Q_e is a unary predicate

$$Q_e^{N[M]} = \{(i, \ell) : \{k_\ell + m_\ell \times i + j_1, k_\ell + m_\ell \times i + j_2\} \text{ is an edge of } M\}.$$

Now let $\mathcal{K}_n^* = \{N[M] : M \in \mathcal{K}_n\}$ and let $\mathcal{N}_n$ vary on $\mathcal{K}_n^*$ with

$$\mathrm{Prob}_{\mu_n^*}(\mathcal{N}_n = N) = \mathrm{Prob}_{\mu_n}(N[\mathcal{M}_n] = N).$$

Now reflecting, the relations Q_e are drawn independently, moreover all the instances are drawn independently and the probabilities are essentially constant. More formally, for $e = \{b_{\ell_1, j_1}, b_{\ell_2, j_2}\} \subseteq B$, $\ell_1 < \ell_2$ in $[\ell(*)]$ and $x \in R_{\ell_1}$, $y \in R_e$ the event

$$E(x, y, e) \stackrel{\text{def}}{=} \mathcal{N}_n \models R_e(x, y)$$

has probability $p^e_{(x,y)}$ which satisfies

$$\begin{aligned} p^e_{x,y} = \mathrm{Prob}_{\mu_n}(\{k_{\ell_1} + m_{\ell_1} \times x + j_1, k_{\ell_2} + m_{\ell_2} \times y + j_1\} \text{ an edge of } \mathcal{M}_n) = \\ (|(k_{\ell_1} + m_{\ell_1} \times x + j_1) - (k_{\ell_2} + m_{\ell_2} \times y + j_2)|)^{-\alpha}. \end{aligned}$$

Now

$$\frac{n^{1-\varepsilon}}{(2\ell(*) + 1) \times (|A| + 1)} \leq |k_{\ell_1} - m_{\ell_1} \times x + j_1 - (k_{\ell_2} + m_{\ell_2} \times y + j_2)| \leq \frac{n^{1-\varepsilon}}{2(|A| + 1)}.$$

Hence for some constants $\mathbf{c}_1 < \mathbf{c}_2$ (i.e. numbers depending on A, B but not on n)

$$\mathbf{c}_1 \cdot (n^{1-\varepsilon})^\alpha \leq p^e_{x,y} \leq \mathbf{c}_2 \cdot (n^{1-\varepsilon})^\alpha.$$

Similarly, if $e = \{a, b_{\ell, j}\}$, $a \in A$, $\ell \in [\ell(*)]$, $j \in [m_\ell]$ and $x \in R^n_\ell$ the event

$$E(x, e) \stackrel{\text{def}}{=} \mathcal{N}_n \models Q_e(x)$$

has probability p^e_x which, for some $\varepsilon(e) \in [0, \varepsilon]_{\mathbb{R}}$ and constants $\mathbf{c}_3$, $\mathbf{c}_4$, satisfies

$$\mathbf{c}_3 \cdot (n^{1-\varepsilon(e)})^\alpha \leq p^e_x \leq \mathbf{c}_4 \cdot (n^{1-\varepsilon(e)})^\alpha.$$

Lastly, for $e = \{b_{\ell,j_1}, b_{\ell,j_2}\}$, $\ell \in [\ell(*)]$, $j_1 < j_2$ in $[m_\ell]$ and $x \in R^n_\ell$ the event

$$E(x,e) \stackrel{\rm def}{=} \mathcal{N}_n \models Q_e(x)$$

has probability p^e_x which is constant $\in (0,1)_{\mathbb{R}}$: it is

$$p_{|j_1-j_2|} = \begin{cases} |j_1-j_2|^{-\alpha} & \text{if} \quad |j_1-j_2|>1 \\ \frac{1}{2} & \text{if} \quad |j_1-j_2|=1. \end{cases}$$

Now these events $E(x,y,e)$, $E(x,e)$ are independent. This is a particular case of [Sh 550], but to draw it from [BlSh 528], first draw the unary predicates. There we know with large probability good lower and upper bounds to the number of elements in

$$\begin{array}{rl} R^{n,*}_\ell = \{x \in R^n_\ell : & \text{if } e = \{b_{\ell,j_1}, b_{\ell,j_2}\},\ j_1 < j_2 \text{ in } [m_\ell] \\ & \quad \text{or } e = \{a, b_{\ell,j}\},\ j \in [m_\ell] \\ & \text{then} \quad e \in \text{edge}(B) \quad \Rightarrow \quad \mathcal{N}_n \models Q_e(x)\} \end{array}$$

Now the rest of the drawing and what we need is like [BlSh 528] only having $\ell(*)$ sort of elements, which are not exactly of the same size: we can throw some nodes to equalize. $\blacksquare_{5.9}$

Now, 5.4, 5.9 are enough for proving $<^*_i = <_i$, $<^*_s = <_s$, weakly nice and similar things. But we need more.

Lemma 5.10 *Assume*

(A) $(A,B,\lambda) \in \mathcal{T}$,

(B) $\xi = \mathbf{w}_\lambda(A,B) > 0$,

(C) *if* $A <^* C <^* B$, *and* C *is* λ*-closed then* $\mathbf{w}_\lambda(C,B) < 0$ *(hence neccessarily* $\xi \in (0,1)_{\mathbb{R}}$ *and* $C \neq \emptyset \quad \Rightarrow \quad \mathbf{w}_\lambda(A,C) > 0$*).*

Then *for every* $\varepsilon \in \mathbb{R}^+$*, for every random enough* $\mathcal{M}_n$*, for every* $f : A \hookrightarrow \mathcal{M}_n$ *we have*

(a) *the number of* $g : B \hookrightarrow \mathcal{M}_n$ *extending* f *is at least* $n^{\xi-\varepsilon}$.

(b) *also the maximal number of pairwise disjoint extension* $g : B \hookrightarrow \mathcal{M}_n$ *of* f *is at least this number.*

Remark 5.11 We can get reasonably much better bound (see [ShSp 304], [BlSh 528] and [Sh 550]) but this suffices.

Proof Repeat the proof of 5.9 noting that ($\bar{j}, g_j$ as there for random enough $\mathcal{M}_n$):

$(*)$ for some $k^* = k^*(A,B)$, for every x

$$|\{\bar{j} \in J : x \in \text{Rang}(g_j) \setminus \text{Rang}(f)\}| < k^*.$$

Claim 5.12 *Assume*

(A) $(A, B, \lambda) \in \mathcal{T}$ *and*

(B) *if* $C \subseteq B \setminus A$ *is non empty and* λ*–closed then* $\mathbf{w}_\lambda(A, A \cup C) > 0$.

Then *for some* $\varepsilon_0 \in \mathbb{R}^+$ *for every* $\varepsilon \in (0, \varepsilon_0)_\mathbb{R}$ *every random enough* $\mathcal{M}_n$ *for every* $f : A \hookrightarrow \mathcal{M}_n$ *we have*

(a) *the number of* $g : B \hookrightarrow \mathcal{M}_n$ *extending* f *is at least* $n^{\mathbf{w}_\lambda(A,B)-\varepsilon}$*; moreover*

(b) *for every* $X \subseteq [n]$, $|X| \leq n^{\varepsilon_0-\varepsilon}$*, the number of* $g : B \hookrightarrow \mathcal{M}_n$ *extending* f *with* $\mathrm{Rang}(g) \cap X \subseteq \mathrm{Rang}(f)$ *is at least* $n^{\mathbf{w}_\lambda(A,B)-\varepsilon}$.

Remark 5.13 By 4.16 the statement

"for some λ the hypothesis of 5.12 holds"

is equivalent to "$A \leq^*_s B$".

PROOF Choose a sequence $\bar{B} = \langle B_\ell : \ell < k \rangle$ such that:

(i) $A = B_0 <^* B_1 <^* \ldots <^* B_k = C$,

(ii) each B_ℓ is λ–closed,

(iii) $\xi_\ell = \mathbf{w}(B_\ell, B_{\ell+1}, \lambda \restriction (B_{\ell+1} \setminus B_\ell)) > 0$,

(iv) if $C \subseteq B_{\ell+1} \setminus B_\ell$ is non empty λ–closed then $\mathbf{w}(B_\ell, B_\ell \cup C, \lambda \restriction C) > 0$,

(v) if $B_\ell <^* C <^* B_{\ell+1}$ and C is λ–closed *then* $\mathbf{w}_\lambda(C, B_{\ell+1}) < 0$.

[Why $\bar{B}$ exists? As $\{\bar{B}' : \bar{B}'$ satisfies (i)-(iv)$\}$ is not empty (as $\langle A, B \rangle$ belongs to it) and every member has length $< |B \setminus A| + 1$, so there is $\bar{B}$ in the family of maximal length, it is easy to check that $\bar{B}$ satisfies clause (v) too.]

Now, by 4.9(2)(c):

$$\xi =: \mathbf{w}_\lambda(A, B) = \sum_{\ell < k} \mathbf{w}(B_\ell, B_{\ell+1}, \lambda \restriction (B_{\ell+1} \setminus B_\ell)),$$

and let

$$\varepsilon_0 = \min\{\mathbf{w}_\lambda(B_\ell, B_{\ell+1}) : \ell < k\}$$

(it is > 0). For each $(B_\ell, B_{\ell+1})$ we can apply 5.10 (for disjoint to X use clause (b) of 5.10).

More fully let $\varepsilon \in (0, \varepsilon_0)$, let $\mathcal{M}_n$ be random enough and let $f : A \hookrightarrow \mathcal{M}_n$. Let $X \subseteq [n]$, $|X| \leq n^{\varepsilon_0-\varepsilon}$. For $\ell \in \{0, \ldots, k\}$ we let

$$F_\ell = \{g : \ g \text{ is an embedding of } B_\ell \text{ into } \mathcal{M}_n \text{ extending } f \text{ and such that } \mathrm{Rang}(g) \cap X \subseteq \mathrm{Rang}(f)\}.$$

Clearly $|F_0| = 1$. Now prove by induction on ℓ that $|F_\ell| \geq n^{\sum\limits_{i<\ell} \mathbf{w}_\lambda(B_i,B_{i+1})-i\varepsilon}$.
For $\ell = 0$ this is clear; for $\ell+1$ it is enough to prove that for each $f^* \in F_\ell$ the set

$$F_{\ell+1,f^*} = \{g \in F_{\ell+1} : g \restriction B_\ell = f^*\}$$

has $\geq n^{\mathbf{w}_\lambda(B_\ell,B_{\ell+1})-\varepsilon}$ members (noting that $F_{\ell+1}$ is the disjoint union of $\{F_{\ell+1,f^*} : f^* \in F_\ell\}$).
Now clearly

$$F_{\ell+1,f^*} = \{g : \begin{array}{l} g \text{ is an embedding of } B_{\ell+1} \text{ into } \mathcal{M}_n \text{ extending } f^* \\ \text{and such that } \mathrm{Rang}(g) \cap X \subseteq \mathrm{Rang}(f^*)\} \end{array}$$

(as $\mathrm{Rang}(f^*) \cap X \subseteq \mathrm{Rang}(f)$ as $f^* \in F_\ell$). Now the inequality follows by clause (b) of 5.10.
$\blacksquare_{5.12}$

Claim 5.14 *Assume* $A <^*_{pr} B$ *and* $k = |B \setminus A|$, *and we let*

$$\xi = \max\{\mathbf{w}_\lambda(A,B) : \begin{array}{l} (A,B,\lambda) \in \mathcal{T} \textit{and} : \\ \textit{for every } \lambda\textit{-closed non empty } C \subseteq B \setminus A \\ \textit{we have } \mathbf{w}(A, A \cup C, \lambda \restriction C) > 0\}. \end{array}$$

Then *for some* $\varepsilon \in \mathbb{R}^+$ *for every random enough* $\mathcal{M}_n$ *for every* $f : A \hookrightarrow \mathcal{M}_n$ *we have*

(a) *the number of* $g : B \hookrightarrow \mathcal{M}_n$ *extending* f *such that*

$$\mathrm{Rang}(g) \cap \mathrm{cl}^k(\mathrm{Rang}(f), \mathcal{M}_n) \subseteq \mathrm{Rang}(f)$$

is at most $n^{\mathbf{w}_\lambda(A,B)+\varepsilon}$.

PROOF Like the proof of 5.4.

6 The conclusion

Context 6.1 As in §4 and §5, so $p_i = 1/i^\alpha$, for $i > 1$, $p_1 = p_2$ (where $\alpha \in (0,1)_\mathbb{R}$ irrational) and $\mathcal{M}_n = \mathcal{M}^0_n$ (only the graph). (So $\mathcal{K}_\infty = \mathcal{K}$ by 4.4)

The argument here parallels that in [BlSh 528] which is more hidden in [ShSp 304]. The most delicate step is to establish clauses A(δ) and (ε) of Definition 2.11 (almost simply nice). For this, we consider $f : A \hookrightarrow \mathcal{M}_n$ and try to extend f to $g : B \hookrightarrow \mathcal{M}_n$ where $A \leq_s B$ such that $\mathrm{Rang}(g)$ and $\mathrm{cl}^k(A, \mathcal{M}_n)$ are "freely amalgamated" over $\mathrm{Rang}(f)$. The key facts have been established in Section 5. If $\zeta = \mathbf{w}(A,B,\lambda)$ we have shown (Claim 5.12) there are $\geq n^{\zeta-\varepsilon}$ embeddings of B. But we also show (using 5.14) that for each obstruction to free amalgamation there is an $\varepsilon_1 > 0$ such that the number of embeddings satisfying this obstruction is $< n^{\zeta'+\varepsilon_1}$, where $\zeta' = \mathbf{w}(A,B',\lambda)$ (for some B' exemplifying the obstruction) with $\zeta'+\alpha \le \zeta$. So if $\alpha > \varepsilon+\varepsilon_1$ we overcome the obstruction. The details of this computation for various kinds of obstruction are carried out in proving Claim 6.6.

* * *

Note that actually the section has two parts of distinct flavours: in 6.2–6.6 we use the probabilistic information from §5 to make the definition of $<_x$ from §1 and of $<^*_x$ from §4. But to actually prove almost niceness, we need more work on the $\le^*_x$ defined in §4; this is done in 6.6, 6.9. Lastly we put everything together.

Claim 6.2 *Assume $A <^* B$. Then the folowing are equivalent:*

(i) *$A <^*_i B$ (i.e. from Definition 4.11(3)),*

(ii) *it is not true that: for some ε, for every random enough $\mathcal{M}_n$ for every $f : A \hookrightarrow \mathcal{M}_n$, the number of $g : B \hookrightarrow \mathcal{M}_n$ extending f is $\ge n^\varepsilon$,*

(iii) *for every $\varepsilon \in \mathbb{R}^+$ for every random enough $\mathcal{M}_n$ for every $f : A \hookrightarrow \mathcal{M}_n$ the number of $g : B \hookrightarrow \mathcal{M}_n$ extending f is $< n^\varepsilon$,*

PROOF We shall use the finite Δ-system lemma: if $f_i : B \to [n]$ is one to one for $i < k$ then for some $w \subseteq \{0,\ldots,k-1\}$, $|w| \ge \mathbf{c}k^{1/2^{|B|}}$, and $A' \subseteq B$ we have: $\bigwedge_{i\in w} f_i \restriction A' = f^*$, $\langle \mathrm{Rang}(f_i \restriction (B \setminus A') : i \in w\rangle$ are pairwise disjoint (so if the f_i's are pairwise distinct then $B \setminus A' \neq \emptyset$).

We use freely Fact 4.2. First clearly (iii) $\Rightarrow$ (ii). Second, if $\neg$(i), i.e. $\neg(A <^*_i B)$ then by 4.14 (equivalence of first and last possibilities) there are A', λ as there, such that:

$A \le A' < B$ and $(A', B, \lambda) \in \mathcal{T}$ and
if $C \subseteq B \setminus A'$ is non empty λ–closed
then $\mathbf{w}(A', A' \cup C, \lambda \restriction C) > 0$ (see 4.14).

So (A', B, λ) satisfies the assumptions of 5.9 which gives $\neg$(ii), i.e. we have proved $\neg$(i) $\Rightarrow$ $\neg$(ii) hence (ii) $\Rightarrow$ (i).

Lastly to prove (i) $\Rightarrow$ (iii) assume $\neg$(iii) (and we shall prove $\neg$(i)). So for some $\varepsilon \in \mathbb{R}^+$:

$(*)_1$ $0 < \limsup_{n\to\infty} \mathrm{Prob}\Big($for some $f : A \hookrightarrow \mathcal{M}_n$, the number of $g : B \hookrightarrow \mathcal{M}_n$ extending f is $\ge n^\varepsilon\Big)$.

But the first paragraph of this proof, shows that from $(*)_1$ we can deduce for some $\zeta \in \mathbb{R}^+$

$(*)_2$ $0 < \limsup_{n\to\infty} \mathrm{Prob}\Big($for some A', $A \le^* A' <^* B$, and $f' : A' \hookrightarrow \mathcal{M}_n$ there are $\ge n^\zeta$ functions $g : B \hookrightarrow \mathcal{M}_n$ which are pairwise disjoint extensions of $f'\Big)$

So for some A', $A \le^* A' <^* B$ and

$(*)_3$ $0 < \limsup_{n\to\infty} \text{Prob}\Big(\text{for some } f' : A' \hookrightarrow \mathcal{M}_n \text{ there are } \geq n^\zeta \text{ functions } g : B \hookrightarrow \mathcal{M}_n \text{ which are pairwise disjoint extensions of } f^*\Big)$.

By 5.4 (and 5.3, 5.2) we have $\neg(A' <^*_c B)$ which (by Definition 4.11(4)) implies $\neg(A <^*_i B)$ so $\neg$(i) as required. $\blacksquare_{6.2}$

Claim 6.3 *For $A <^* B \in \mathcal{K}_\infty$, the following conditions are equivalent:*

(i) $A <^*_s B$,

(ii) *it is not true that: "for every $\varepsilon \in \mathbb{R}^+$, random enough $\mathcal{M}_n$ for every $f : A \hookrightarrow \mathcal{M}_n$, there are not n^ε pairwise disjoint extensions $g : B \hookrightarrow \mathcal{M}_n$ of f",*

(iii) *for some $\varepsilon \in \mathbb{R}^+$ for every random enough $\mathcal{M}_n$ for every $f : A \hookrightarrow \mathcal{M}_n$ there are $\geq n^\varepsilon$ pairwise extensions $g : B \hookrightarrow \mathcal{M}_n$ of f.*

Proof Reflection shows that (iii) $\Rightarrow$ (ii).

If $\neg$(i) i.e. $\neg(A <_s B)$ then by Definition 4.11(4) for some B', $A <_i B' \leq B$, hence by 6.2 easily $\neg$(ii), so (ii) $\Rightarrow$ (i).

Lastly it suffices to prove (i) $\Rightarrow$ (iii). Now by (i) and 4.16 the assumption of 5.9 holds hence the conclusion which give clause (iii). $\blacksquare_{6.3}$

Conclusion 6.4 $<^*_s = <_s$ *and* $<^*_i = <_i$, *and $\mathfrak{K}$ is weakly nice.*

Proof By 6.2, 6.3 and see §1. $\blacksquare_{6.4}$

Conclusion 6.5 *If $A <^*_i B$ then for every $\varepsilon \in \mathbb{R}^+$, for some k for every random enough $\mathcal{M}_n$, for every $f : A \hookrightarrow \mathcal{M}_n$ we have*

$$\{c : \textit{ for some } g : B \hookrightarrow \mathcal{M}_n, \textit{ we have } c \in \text{Rang}(g) \textit{ but } (\forall x \in A)(|c - f(x)| \geq n^\varepsilon)\}$$

has at most k members.

Proof By 5.4. $\blacksquare_{6.5}$

Note that we are in a "nice" case (no successor function). Toward proving it we characterize "simply good".

Claim 6.6 *If $A \leq^*_s B$ and $k, m \in \mathbb{N}$* then *for every random enough $\mathcal{M}_n$, for every $f : A \hookrightarrow \mathcal{M}_n$, we can find $g : B \hookrightarrow \mathcal{M}_n$ extending f such that:*

(i) $\text{Rang}(g) \cap \text{cl}^{k,m}(\text{Rang}(f), \mathcal{M}_n) = \text{Rang}(f)$,

(ii) $\mathrm{Rang}(g) \underset{\mathrm{Rang}(f)}{\overset{\mathcal{M}_n}{\Cup}} \mathrm{cl}^{k,m}(\mathrm{Rang}(f), \mathcal{M}_n)$,

(iii) $\mathrm{cl}^{k,\ell}(\mathrm{Rang}(g), \mathcal{M}_n) \subseteq \mathrm{Rang}(g) \cup \mathrm{cl}^{k,\ell}(\mathrm{Rang}(f))$, *for each* $\ell \leq m$.

Remark 6.7 See proof of 7.1.

PROOF We prove this by induction on $|B \setminus A|$, but now, by the character of the desired conclusion, if $A <^*_s B <^*_s C$ to prove it for the pair (A, C) it suffices to prove it for the pairs (A, B) and (B, C). So, without loss of generality, $A <^*_{pr} B$, so 5.14 applies. Let λ be such that $(A, B, \lambda) \in \mathcal{T}$ and

$$\xi =: \mathbf{w}_\lambda(A, B) = \quad \max \Big\{ \mathbf{w}_{\lambda_1}(A, B) : (A, B, \lambda_1) \in \mathcal{T} \text{ satisfies: for every } \lambda_1\text{–closed non-empty } C \subseteq B \setminus A \text{ we have } \quad \mathbf{w}_\lambda(A, A \cup C) > 0 \Big\}$$

Choose $\varepsilon \in \mathbb{R}^+$ small enough, $k(*)$ large enough. The requirement on ε, $k(*)$ will be clear by the end of the argument.

Let $\mathcal{M}_n$ be random enough, and $f : A \hookrightarrow \mathcal{M}_n$. Now by 5.12 we get

$(*)$ $|\mathrm{cl}^{k+|B|,m}(f(A), \mathcal{M}_n)| \leq n^{\varepsilon/k(*)}$,

$(*)_1$ $|G| \geq n^{\xi - \varepsilon/2}$,

where

$$G = \{g : \ g \text{ extends } f \text{ to an embedding of } B \text{ into } \mathcal{M}_n \text{ such that } g(B) \cap \mathrm{cl}^{k+|B|,m+2}(\mathrm{Rang}(f), \mathcal{M}_n) \subseteq \mathrm{Rang}(f)\}.$$

We intend to find $g \in G$ satisfying the requirements in the claim. Now g being an embedding of B into $\mathcal{M}_n$ extending f and clause (i) (i.e. $\mathrm{Rang}(g) \cap \mathrm{cl}^{k,m}(\mathrm{Rang}(f), \mathcal{M}_n) = \mathrm{Rang}(f)$) follows from $g \in G$. So it is enough to prove that $< n^{\xi-\varepsilon}$ members g of G fail clause (ii) and $< n^{\xi-\varepsilon}$ fail clause (iii).

On the failure of (ii):

If $g \in G$ fails clause (ii) in 6.6, then for some

$$x_g \in \mathrm{cl}^{k,m}(\mathrm{Rang}(f), \mathcal{M}_n) \setminus \mathrm{Rang}(f) \quad \text{and} \quad y \in B \setminus A$$

we have: $\{x_g, g(y)\}$ is an edge of $\mathcal{M}_n$. Note $x_g \notin g(B)$ as $g \in G$.

Form a new structure $B^2 = B \cup \{x^*\}$, $(x^* \notin B)$, such that $g \cup \{\langle x^*, x_g\rangle\} : B^2 \hookrightarrow \mathcal{M}_n$ and let $A^2 = B^2 \,^- (A \cup \{x^*\})$. Now up to isomorphism over B there is a finite number (i.e. with a bound not depending on n) of such B^2, say $\langle B^2_j : j < j^* \rangle$.

For $x \in \mathrm{cl}^{k,m}_{\mathcal{M}_n}(\mathrm{Rang}(f))$ and $j < j^*$ let

$$\begin{aligned} G^2_{j,x} &=: \{g : g \text{ is an embedding of } B^2_j \text{ into } \mathcal{M}_n \text{ extending } f \\ &\qquad \text{and satisfying } g(x^*) = x\}, \\ G^2_j &=: \bigcup_{x \in \mathrm{cl}^{k,m}(f(A),\mathcal{M}_n)} G^2_{j,x} \qquad \text{and} \\ G^2 &=: \bigcup_j G^2_j. \end{aligned}$$

So:

$(*)_2$ if $g \in G$ fails clause (ii) of 6.6 then

$$g \in \bigcup_j \Big\{\{g' \restriction B : g' \in G^2_{j,x}\};\ j < j^*_2 \text{ and } x \in \mathrm{cl}^{k,m}(f(A), \mathcal{M}_n)\Big\}.$$

Now, if $\neg(A^2_j <_s B^2_j)$ then as $A <_{pr} B$ easily $A^2_j <_i B^2_j$ so by 6.2 using $(*)$

$(*)_3$ if $\neg(A^2_j <_s B^2_j)$ then $|G^2_j| \le n^{\varepsilon/2}$.

If $A^2_j <_s B^2_j$, then still $A^2_j <_{pr} B^2_j$ and letting

$$\begin{aligned} \xi^2_j =: \max\{\mathbf{w}_\lambda(A^2_j, B^2_j) :\ & (A^2_j, B^2_j, \lambda) \in \mathcal{T} \text{ and} \\ & \text{for every } \lambda\text{-closed non-empty } C \subseteq B^2_j \setminus A^2_j \\ & \text{we have } \mathbf{w}(A^2_j, A^2_j \cup C, \lambda \restriction C) > 0\} \end{aligned}$$

clearly $\xi^2_j < \xi - 2\varepsilon$ (just check). So, by 5.14,

$(*)_4$ if $A^2_j < B^2_j$ then $|G^2_j| \le n^{\xi - 2\varepsilon}$.

So multiplying by j^*

$(*)_5$ the number of $g \in G$ failing clause (ii) of 6.6 is $\le n^{\xi-\varepsilon}$

On the failure of (iii):

Next assume $g \in G$ fails clause (iii) of 6.6. So for some $\ell < m$ some $k^* \le k^{\ell+1} + k$ there are A^+, B^+, C, g^+ such that

$\otimes$ $A \le_i A^+ \le B^+$, $B \le B^+$, $B^+ \cap A^+ = A$, $|B^+| \le k^*$, $B^+ \setminus A^+ \setminus B \subseteq C \subseteq B^+$, $\mathrm{cl}(C \cap (A^+ \cup B), B^+) \supseteq C$ and $\mathrm{cl}^{k,\ell}(B \cup A^+, B^+) \ne B \cup A^+$, $g \subseteq g^+$, $g^+ : B^+ \hookrightarrow \mathcal{M}_n$, $g^+(A^+) \subseteq \mathrm{cl}^{k,\ell}(f(A), \mathcal{M}_n)$, $g^+(B^+) \cap \mathrm{cl}^{k,\ell+1}(f(A), \mathcal{M}_n) = g^+(A^+)$.

Note: As $g(B) \cap \mathrm{cl}^{k+|B|,m}(f(A), \mathcal{M}_n) = f(A)$ (as $g \in G$), neccessarily $A^+ <_s B^+$. Now if $\lambda \in \Xi(A^+, B^+)$ (see 4.8(2)), as $A^+ \cup B <_i B^+$ easily $\lambda \restriction (B \setminus A) \in \Xi(A, B)$, and $\mathbf{w}(A^+, B^+, \lambda) < \mathbf{w}(A, B, \lambda \restriction (B \setminus A))$.

Let $\langle (A^+_j, B^+_j, \ell_j) : j < j^*_3 \rangle$ list the possible (A^+, B^+, ℓ) up to isomorphism over B as described above. Let for $\bar c \in (\mathrm{cl}^{k,m}(f(A), \mathcal{M}_n))$

$$G^3_{j,\bar c} =: \{g \in G : g \text{ embeds } B^+_j \text{ into } \mathcal{M}^+_n \text{ extending } f\}.$$

So

$(*)_6$ if $g \in G$ fails clause (iii) of 6.6 then

$$g \subseteq \bigcup_{j<j_3^*} \{g' \restriction B : g' \in G_j^3\},$$

$(*)_7$ $|G^3_{j,\bar{c}}| < n^{\xi-2\varepsilon}$.

As $k(*)$ was large enough $|\mathrm{cl}^{k,m}(f(A), \mathcal{M}_n)| \leq n^{\varepsilon/k(*)}$, we can get

$(*)_8$ the number of $g \in G$ failing clause (iii) is $< n^{\xi-\varepsilon}$

(alternatively use 2.16). $\blacksquare_{6.6}$

Conclusion 6.8 *If $A <^*_s B$ and $B_0 \subseteq B$ and $k \in \mathbb{N}$ then (B, A, B_0, k) is simply good (see Definition 2.11(1)).*

PROOF Read 6.6 and Definition 2.11(4), 2.11(1). $\blacksquare_{6.8}$

Toward simple niceness the "only" thing left is the universal part, i.e. 2.11(2)(A).

The following claims 6.9, 6.10 do not use §5; they are the crucial step for proving 2.11(2)(A) (claim 6.9 is a sufficient condition for goodness (by 6.8). The division to two claims is for clarity, though it has a bad effect on the bound.

Claim 6.9 *For every k and m (from $\mathbb{N}$) there are k^* and t such that*

$(*)$ if $m^\otimes + k < m^*$, *$\ell \in \mathbb{N}$ and $M \in \mathcal{K}$, $\bar{a} \in {}^{\ell\geq}M$, $b \in M \setminus \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M)$*

then

$\otimes$ *the set*

$$\mathcal{R} =: \{(c,d) : \begin{array}{l} d \in \mathrm{cl}^k(\bar{a}b, M) \setminus \mathrm{cl}^{k^*,m^\otimes+k}(\bar{a}, M) \ \ \textit{and} \\ c \in \mathrm{cl}^{k^*,m^\otimes}(\bar{a}, M) \ \ \textit{and} \ \ \{c,d\} \ \textit{is an edge of } M\} \end{array}$$

has at most t members. Moreover we can replace k^* by any larger number.

PROOF Choose $\varepsilon \in \mathbb{R}^+$ small enough (such that

$$(C_0, C_1, \lambda) \in \mathcal{T} \ \& \ |C_1| \leq k \quad \Rightarrow \quad \mathbf{w}_\lambda(C_0, C_1) \notin [-\varepsilon, \varepsilon]$$

(even just when $C_0 <_i C_1$)).

Choose $\mathbf{c} \in \mathbb{R}^+$ large enough such that

$$(C_0, C_1, \lambda) \in \mathcal{T}, \ |C_1 \setminus C_0| \leq k \quad \Rightarrow \quad \mathbf{w}_\lambda(C_0, C_1) \leq \mathbf{c}$$

(actually $\mathbf{c} = k$ is enough). Choose $t_1 > 0$ such that $t_1 > \mathbf{c}/\varepsilon$. Choose $t_2 \geq k2^{2^{t_1}}$ (overkill). Choose $k^* \in \mathbb{N}$ large enough (such that $k^* > k$, $\alpha k^* > 1$ and $k^* \geq k \times t_2$).

Suppose the set $\mathcal{R}$ has at least t members. Let $(c_i, d_i) \in \mathcal{R}$ for $i < t$ be pairwise distinct. Choose for each $i < t$ a set C_i such that:

(i) $C_i \le M$ and $\bar{a}b \in C_i$,

(ii) $|C_i| \le k$,

(iii) $d_i \in C_i$,

(iv) $C_i \restriction (C_i \cap (\bar{a}b)) <_i C_i$.

For each $i < t$, as $C_i \cap \mathrm{cl}^{k^*, m^{\otimes}+k}(\bar{a})$ is a proper subset of C_i (witnessed by d_i and also b) clearly it has $< k$ elements and hence for some $k[i] < k$ we have

(v) $C_i \cap \mathrm{cl}^{k^*, m^{\otimes}+k[i]+1}(\bar{a}, M) \subseteq \mathrm{cl}^{k^*, m^{\otimes}+k[i]}(\bar{a}, M)$.

So without loss of generality

(vi) $i < t/k^2 \Rightarrow k[i] = k[0]$ & $|C_i| = |C_0| = k'$,

remember $t_2 < t/k^2$. As $k^* \ge t_2 \times k$ (by the choice of k^*), neccessarily $D' <_s D$, where $D = \bigcup\limits_{i<t_2} C_i \cup \mathrm{cl}^{k^*, m^{\otimes}}(\bar{a}, M)$, $D' = D \cap \mathrm{cl}^{k^*, m^{\otimes}+k[0]}(\bar{a}, M)$. So we can choose $\lambda \in \Xi(D', D)$. Let $C_i = \{d_{i,s} : s < k'\}$, with $d_{i,0} = d_i$ and $b \ne d_i \;\Rightarrow\; b = d_{i,1}$ and no repetitions.

By the finite Δ-system lemma for some $S_0, S_1, S_2 \subseteq \{0, \ldots, k_1 - 1\}$ and $u \subseteq \{0, \ldots, t_2 - 1\}$ with $\ge t_1$ elements we have:

$\oplus_1$ **(a)** $\lambda' =: \{(s_1, s_2) : d_{i,s_1} \lambda d_{i,s_2}\}$ is the same for all $i \in u$,

(b) for each $j < \ell g(\bar{a}) + 1$, and $s < k'$, the truth value of $c_{i,s} = (\bar{a}b)_j$ is the same for all $i \in u$,

(c) $d_{i_1,s_1} = d_{i_2,s_2} \;\Rightarrow\; s_1 = s_2$ for $i_1, i_2 \in u$,

(d) $d_{i_1,s} = d_{i_2,s} \;\Leftrightarrow\; s \in S_1$ for $i_1 \ne i_2 \in u$,

(e) $d_{i_1,s_1} \lambda d_{i_2,s_2} \;\Rightarrow\; d_{i_1,s_1} \lambda d_{i_1,s_2}$ & $d_{i_1,s_2} \lambda d_{i_2,s_2}$,

(f) $d_{i_1,s} \lambda d_{i_2,s} \;\Leftrightarrow\; s \in S_2$ for $i_1 \ne i_2 \in u$,

(g) $b = d_{i,0}$ has the same truth value for all $i \in u$.

$\oplus_2$ $0 \notin S_2$ (i.e. $\lambda \restriction \{d_i : i \in u\}$ is equality).

[Why? Otherwise for $i \in u$ letting $X = d_i/\lambda$ the triple $(D', D' \cup X, \lambda \restriction X) \in \mathcal{T}$ has weight

$$\begin{aligned} & \mathbf{w}(D', D' \cup X, \lambda \restriction X) = \\ = \;& \mathbf{v}(D', D' \cup X, \lambda \restriction X) - \alpha \mathbf{e}(D', D' \cup X, \lambda \restriction X) \\ = \;& 1 - \alpha \times |\{e : e \text{ an edge of } M \text{ with one end in } D' \text{ and the other in } X\}| \\ & (\text{as } c_i \in \mathrm{cl}^{k^*, m^{\otimes}}(\bar{a}, M) \text{ and } \{c_i, d_i\} \in \mathrm{edge}(M) \text{ distinct for different } i) \\ \le \;& 1 - \alpha \times |\{d_i : i \in u\}| = 1 - \alpha \times |u| = 1 - \alpha \times k^* < 0; \end{aligned}$$

contradiction to $\lambda \in \Xi(D', D)$.]

Let $D^0 = \{d_{i,s} : s \in S_2 \text{ and } i \in u\}$.

$\oplus_3$ $b = d_{i,1}$ and $1 \in S_2$.

[Why? As $b \in C_i$, $b \in \{d_{i,0}, d_{i,1}\}$ and $\oplus_2$ and (g) of $\oplus_1$.]

$\oplus_4$ For each $i \in C$ we have $\mathbf{w}_\lambda(C_i \cap D_0, C_i) < 0$.

[Why? As $\bar{a}b \subseteq C_i \cap D_0$ and $C_i{}^-\, (\bar{a}b) <_i C_i$.]

Hence

$\oplus_5$ $\mathbf{w}_\lambda(C_i \cap D_0, C_i) \leq -\varepsilon$ for $i \in C$.

So:

$$\begin{aligned}
&\mathbf{w}_\lambda(D', \bigcup\{x/\lambda : x \in \bigcup_{i\in u} C_i \setminus D'\} \cup D') \\
&= \mathbf{w}_\lambda(D', D' \cup D^0, \lambda^- D^0) + \mathbf{w}_\lambda(D' \cup D^0, D' \cup \{d_{i,s}\colon i \in u, s < k', s \notin S_2\}) \\
&\leq \mathbf{w}_\lambda(D', D' \cup D^0, \lambda^- D^0) + \sum_{i\in u} \mathbf{w}_\lambda(D' \cup D^0, D' \cup D^0 \cup \{d_{i,s}\colon s < k', s \notin S_2\}) \\
&\leq \mathbf{w}_\lambda(D' \cap D^0, D^0, \lambda^-(D^0 \setminus D')) + \sum_{i\in u} \mathbf{w}_\lambda(C_i \cap D^0, C_i) \\
&\leq \mathbf{c} + |u| \times (-\varepsilon) = \mathbf{c} - t_1\varepsilon < 0,
\end{aligned}$$

contradicting the choice of λ. $\blacksquare_{6.9}$

Claim 6.10 *For every k, m and ℓ from $\mathbb{N}$ from some k^*, m^* we have*

$(*)$ *if $M \in \mathcal{K}$, $\bar{a} \in {}^{\ell\geq}M$, $b \in M \setminus \mathrm{cl}^{k^*,m^*}(\bar{a}, M)$*

then *for some $m^\otimes \leq m^* - m$ we have*

$$\mathrm{cl}^k(\bar{a}b, M) \cap \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M) \subseteq \mathrm{cl}^{k^*,m^\otimes}(\bar{a}, M).$$

Proof Let t, k^* be as in the previous claim 6.9. Choose m^* such that e.g. $m^*/m \to (t)^{2k}_{2^{k!}}$ (we could get more reasonable bounds but no reason as for now). By 6.9 we can increase k^* so without loss of generality $k^* \geq km^*$.

If the conclusion fails, then the set

$$Z = \{j \leq m^* - m : \mathrm{cl}^k(\bar{a}b, M) \cap \mathrm{cl}^{k^*,j+1}(\bar{a}, M) \nsubseteq \mathrm{cl}^{k^*,j}(\bar{a}, M)\}$$

satisfies:

$$j \leq m^* - m \quad \Rightarrow \quad Z \cap [j, j+m) \neq \emptyset.$$

Hence $|Z| \geq m^*/m$. For $j \in Z$ there is $C_j \leq^* M$ such that

$|C_j| \leq k$ and $(C_j \cap (\bar{a}, b)) <_i C_j$, and
$d_j \in C_j \cap \mathrm{cl}^{k^*,j+1}(\bar{a}, M) \setminus \mathrm{cl}^{k^*,j}(\bar{a}, M)$.

Let $w(j) = \{i : C_j \cap \mathrm{cl}^{k^*,i+1}(\bar{a}, M) \nsubseteq \mathrm{cl}^{k^*,i}(\bar{a}, w)\}$.
Let $C_j = \{d_{j,s} : s < s_j \leq k\}$ with $d_{j,0} = d_j$, no repetitions. Let $D = \bigcup\{C_i : i \in Z'\}$, $D' = D \cap \mathrm{cl}^{k^*,m^*-k'}(\bar{a}, M)$, and choose $\lambda \in \Xi(D', D)$. We can find $u \subseteq Z$, $|u| = t+5$ and $S_0, S_1 \subseteq \{0, \ldots, k'-1\}$ such that

(a) $i \in u \;\Rightarrow\; s_j = s^*$,

(b) for each $j < \ell g(\bar a) + 1$ the truth value of $c_{i,s} = (\bar a b)_j$ is the same for all $i \in u$,

(c) if $i \neq j$ are from u then $|i + j| > k$,

(d) the truth value of "$\{d_{i,s_1}, d_{i,s_2}\}$ is an edge" is the same for all $i \in u$,

(e) for all $i_0 < i_1$ from u:
$$d_{i_0,\ell} \in \mathrm{cl}^{k^*,i_1}(\bar a, M) \quad\Leftrightarrow\quad \ell \in S_0,$$

(f) for all $i_0 < i_1$ from u:
$$d_{i_1,\ell} \in \mathrm{cl}^{k^*,i_0}(\bar a, M) \quad\Leftrightarrow\quad \ell \in S_1.$$

Now let $i(*)$ be e.g. the third element of the set u and
$$B_1 \stackrel{\rm def}{=} C_{i(*)} \cap \mathrm{cl}^{k^*,\min(u)}(\bar a, M), \quad \text{and} \quad B_2 \stackrel{\rm def}{=} C_{i(*)} \cap \mathrm{cl}^{k^*,\max(u)}(\bar a, M).$$

So $B_1 <^* B_2 \leq^* C_{i(*)}$ (note: $d_i \in B_2 \setminus B_1$) and $(\bar a b) \cap B_2 \subseteq B_1$ (by the Δ-system requrements) and there is no edge in $(C_i \setminus B_2) \times (B_2 \setminus B_1)$. [Why? Let the edge be $\{d_{i(*),\ell_1}, d_{i(*),\ell_2}\}$ with
$$d_{i(*),\ell_1} \in C_i \setminus B_2 \quad \text{and} \quad d_{i(*),\ell_2} \in B_2 \setminus B_1;$$
necessarily this holds for every i, and also necessarily $\langle d_{i,\ell_2} : i \in u\rangle$ is with no repetitions. So the set of edges $\{\{d_{i,\ell_1}, d_{i,\ell_2}\} : i \in u \setminus \{\max u\} \setminus \{\min u\}\}$ contradicts 6.9 (and our choice of parameters).]
As $C_{i(*)} \restriction (\bar a b) <_i C_{i(*)}$ and $B_2 \cap (\bar a b) \subseteq B_1$, clearly $\bar a b \subseteq C_{i(*)} \setminus (B_2 \setminus B_1) \subseteq C_i$. Hence $(C_{i(*)} \setminus (B_2 \setminus B_1)) <_i C_{i(*)}$. But, as stated above, $C_{i(*)} \setminus (B_2 \setminus B_1) \biguplus_{B_1} B_2$, hence by the previous sentence (and smoothness, see 4.1) we get $B_1 <_i B_2$; also $|B_2| \leq |C_{i(*)}| \leq k \leq k^*$. By their definitions, $B_1 \subseteq \mathrm{cl}^{k^*,\max(u)}(\bar a, M)$, but $B_1 \leq_i B_2$, $|B_2| \leq k$ and hence $B_2 \subseteq \mathrm{cl}^{k^*,2^{\rm nd}\text{ member of } u}(\bar a, M)$. Contradiction to the choice of $d_{i(*)}$. $\blacksquare_{6.10}$

Conclusion 6.11 *For every k, m and ℓ (from $\mathbb{N}$), for some m^*, k^* and t:*

$(*)$ *if $M \in \mathcal{K}$, $\bar a \in {}^{\ell\geq}M$ and $b \in M \setminus \mathrm{cl}^{k^*,m^*}(\bar a, M)$ then for some $m^\otimes \leq m^* - m$ and B we have*

- **(i)** $|B| \leq t$,
- **(ii)** $B \subseteq \mathrm{cl}^k(B, M) \subseteq \mathrm{cl}^{k^*,m^\otimes}(\bar a, M)$,
- **(iii)** $\mathrm{cl}^{k^*,m^\otimes+m}(\bar a, M)$, $\mathrm{cl}^k(\bar a b, M)$ *are free over B inside M,*
- **(iv)** $B \leq^*_s B^* = M \restriction ((\mathrm{cl}^k(\bar a b, M) \setminus \mathrm{cl}^k(B, M)) \cup B)$.

Proof By the last two claims 6.9, 6.10. $\blacksquare_{6.11}$

Conclusion 6.12 *Requirements (A) of 2.11(2) and (B) of 2.11(4) hold.*

PROOF Requirement (B) of 2.11(4) holds by 6.8. Requirement (A) of 2.11(2) holds by 6.11 (and the previous sentence). $\blacksquare_{6.12}$

Conclusion 6.13 *1. $\mathfrak{K}$ is smooth and transitive and local and transparent.*

2. $\mathfrak{K}$ is simply nice (hence almost nice).

3. $\mathfrak{K}$ satisfies the 0–1 law.

PROOF 1) Check the definition.
2) By 6.12 $\mathfrak{K}$ is simply nice.
3) By 4.2 we know that for each k, for every random enough $\mathcal{M}_n$, $\mathrm{cl}^k(\emptyset, \mathcal{M}_n)$ is empty. Hence by 2.19(1) we get the desired conclusion. $\blacksquare_{6.13}$

7 Random graphs with the successor function

Theorem 7.1 *Let $\mathfrak{K}$ be as follows:*

$\alpha \in (0,1)_{\mathbb{R}}$ *be irrational,*
$p_\ell = \frac{1}{\ell^\alpha}$ *for* $\ell > 1$, $p_1 = \frac{1}{2}$ *(can omit this) and* $\mathcal{M}_n = \mathcal{M}_n^1$

(so $\mathfrak{K}$ is defined). Then $\mathfrak{K}$ satisfies the convergence law.

PROOF This is proved later.

Theorem 7.2 *Let $\alpha \in (0,1)_{\mathbb{R}}$ be irrational, $p_i = 1/i^\alpha$ for $i > 1$, $p_1 = \frac{1}{2}$. Let $\mathcal{M}_n = \mathcal{M}_n^{0.5}$ be $([n], R, S_n)$ where R is a graph on $[n]$ chosen randomly:*
$\{i,j\} \in R$ has probability p_k where $k \in \{1, \ldots, n-1\}$ is minimal such that n divides $i-j-k$, the choices independent for distinct edges, lastly

$$S_n \stackrel{\text{def}}{=} \{(x,y) : x \in [n],\ y \in [n] \ \ \textit{and}\ x+1 = y \bmod n\}.$$

So a 0–1 context $\mathfrak{K}$ is defined.
Then $\mathfrak{K}$ is simply almost nice and for every k for every random enough $\mathcal{M}_n$ we have: $\mathrm{cl}^k(\emptyset, \mathcal{M}_n) = \emptyset$ hence $\langle \mathcal{M}_n \restriction \mathrm{cl}^k(\emptyset, \mathcal{M}_n) : n < \omega \rangle$ satisfies the 0–1 law. Consequently $\mathfrak{K}$ satisfies the 0–1 law.

PROOF We repeat the proof for $\mathcal{M}_n$ with small changes. Let $\mathcal{K}$ be the class of finite models (X, R, S) such that: R is a symmetric irreflexive two–place relation on A, S is an irreflexive antisymmetric two–place relation satisfying

$$(\forall x_0, y_0)(\forall x_1, y_1)(S(x_0, y_0) \ \wedge\ S(x_1, y_1) \quad \Rightarrow \quad (x_0 = x_1 \equiv y_0 = y_1)).$$

Let $\mathcal{K}'$ be the set of $(X, R, S) \in \mathcal{K}$ with no S–cycle. $\mathcal{K}_n$ is the set of possible values of $\mathcal{M}_n$.
Easily $\mathcal{K}_n \subseteq \mathcal{K}$ (by the proof of 4.2)

$(*)$ for each $A \in \mathcal{K}'$ for some $\mathbf{c} \in \mathbb{R}^+$ for every random enough $\mathcal{M}_n$ there are $\geq \mathbf{c} \times n$ pairwise disjoint embeddings of A into $\mathcal{M}_n$.

Hence $\mathcal{K}' = \mathcal{K}_\infty \subseteq \mathcal{K}$, and hence (the parallel of) 4.3, 4.4 hold. If $A \subseteq B$ we let

$$\begin{aligned}\mathrm{scl}(A,B) = \{x \in B : \ & x \in A \quad \text{or } x \text{ is } S^B\text{–connected to a member of } A \\ & \text{or } x \text{ is } S^B\text{–connected to some } y \in B \text{ satisfying} \\ & P_f(y) \vee P_f(x)\}.\end{aligned}$$

Let $A \leq B$ mean $A \subseteq B$ (submodel) and let

$A \leq^{**} B$ if $A \leq B$ and for no $y \in \mathrm{scl}(A,B) \setminus A$ and $x \in A$ is $\{x,y\} \in \mathbb{R}^B$,
$A \leq^* B$ if $\mathrm{scl}(A,B) = A$.

We say that B, C are *freely amalgamated over A inside N* if:

$A \leq^* B \leq N$, $A \leq^* C \leq N$ and
for no $x \in B \setminus A$, $y \in C \setminus A$ does $\{x,y\} \in R^N$.

We say that B, C are *almost freely amalgamated over A inside N* if:

$A \leq^{**} B \leq N$, $A \leq^* C \leq N$ and
for no $x \in B \setminus A$, $y \in C \setminus A$ is $\{x,y\} \in R^N$.

Now we define $\mathcal{T}$ (see definition 4.6):

$$\begin{aligned}\mathcal{T} = \ & \{(A,B,\lambda) : A <^* B \in \mathcal{K}_\infty,\ \lambda \text{ an equivalence relation on } B \setminus A \text{ and:} \\ & x \in B \setminus A \ \&\ y \in B \ \&\ (S(x,y) \vee S(y,x)) \quad \Rightarrow \quad y \in (x/\lambda) \subseteq B \setminus A \\ & \text{so } A \leq^* B\}.\end{aligned}$$

Now $\mathbf{v}$, $\mathbf{e}$, $\mathbf{w}$ are defined as in 4.7, 4.8 and $A <^*_x B$ are defined as in 4.11, and the parallel to 4.14–4.17 hold.

In the proof of 5.4 we should be careful to preserve $\pm S$ in the relevant cases, so in stage C, choose m^* large enough and in the type of g fix $g(b) + m^*\mathbb{Z}$. So we have $m^\otimes \in \{0, \ldots, m^* - 1\}$ such that for no b is $g(b) = m^\otimes \mathrm{mod}\ m^*$, and we move blocks

$$\{m^* \cdot i + m^\otimes + 1, m^* \cdot i + m^\otimes + 2, \ldots, m^* \cdot (i+1) + m^\otimes\}$$

together. The other cases are even less serious.

In 5.9, on each λ–equivalence class we should preserve S. In §6 we have to change somewhat 6.6:

Claim 7.3 *For every $k, m \in \mathbb{N}$ there is $r(k,m) \in \mathbb{N}$ such that:*
if *$A <^*_s B$ and $B_0 \subseteq B$ satisfy*

$(*)^2_{B_0,B}$ *if $x \in B_0$, let its S–component in B be $\{x_0, \ldots, x_j\}$ with $S^B(x_\ell, x_{\ell+1})$, and $x = x_{i(*)}$, then $P_f(x_0) \ \vee\ (i(*) > r)$ and $P_f(x_j) \ \vee\ (|j - i(*)| > r)$*

then *for every random enough* $f : A \hookrightarrow \mathcal{M}_n$ *we can find* $g : B \hookrightarrow \mathcal{M}_n$ *extending* f *and such that*

(i) $\mathrm{Rang}(g) \cap \mathrm{cl}^{k,m}(\mathrm{Rang}(f), \mathcal{M}_n) = \mathrm{Rang}(f)$,

(ii) $\mathrm{Rang}(g) \underset{\mathrm{Rang}(f)}{\overset{\mathcal{M}_n}{\Cup}} \mathrm{cl}^{k,m}(\mathrm{Rang}(f), \mathcal{M}_n)$

(iii) $\mathrm{cl}^{k,\ell}(g(B_0), \mathcal{M}_n) \subseteq g(B) \cup \mathrm{cl}^{k,\ell}(g(A))$, *for* $\ell \leq m$.

Observation: It is enough to prove this for $m = 1$.
[Why? Choose k' such that for every $A \subseteq N_1 \in \mathcal{K}$ we have $\mathrm{cl}^{k,m}(A, N) \subseteq \mathrm{cl}^{k'}(A, N)$. Let $\mathcal{M}_n$ be random enough and $f : A \hookrightarrow \mathcal{M}_n$. Find $g : B \hookrightarrow \mathcal{M}_n$ as required for $(k', 1)$ instead (k, m). Now

$$\mathrm{cl}^{k,\ell}(g(B_0), \mathcal{M}_n) \subseteq \mathrm{cl}^{k'}(g(B_0), \mathcal{M}_n) \subseteq g(B) \cup \mathrm{cl}^{k'}(g(A), \mathcal{M}_n).$$

By smoothness we can prove by induction on $\ell \leq m$ that clause (iii) holds.]

$\blacksquare_{7.2}$

Proof of Theorem 7.1:
Like 7.2, but in the vocabulary we have also two unary predicates P_f, P_ℓ, and

$$\begin{aligned}\mathcal{K} = \ & \big\{(X, R, S, P_f, P_\ell) : (X, R, S) \text{ as in 7.2}, |P_f| \leq 1, |P_\ell| \leq 1, \text{ and} \\ & \quad S(x, y) \ \Rightarrow \ \neg P_\ell(x) \wedge \neg P_f(y) \quad \text{and} \\ & \quad (P_f(x) \wedge P_\ell(x) \wedge (x, y) \text{ are } S\text{–connected}) \ \Rightarrow \ X \text{ is } S\text{–connected}\big\}.\end{aligned}$$

We can prove, as in 7.2, that

$\otimes$ $\mathfrak{K}$ is simply almost nice.

The difference in the proof with 7.2 is that here for positive $k \in \mathbb{N}$, for random enough $\mathcal{M}_n$, $\mathrm{cl}^{k,m}(\emptyset, \mathcal{M}_n)$ is not empty, so we get only convergence in 4.3 (in fact e.g.

$$(\exists x, y, z)(S(x, y) \wedge S(x, y) \wedge P_f(x) \wedge R(x, z))$$

has probability p_2). Still applying 2.19 we need

$\otimes_1$ for every f.o. φ the sequence

$$\langle \mathrm{Prob}(\mathcal{M}_n \,^{-}\, \mathrm{cl}^k(\emptyset, \mathcal{M}_n)) : n \in \mathbb{N} \rangle$$

converges.

For this it suffices to prove

$\otimes_2$ **(a)** for every $\varepsilon \in \mathbb{R}^+$ and $k \in \mathbb{N}$, for some $1 - \varepsilon \in \mathrm{Prob}(E^n_k)$, where E^n_k is the event

$$\mathrm{cl}^k(\emptyset, \mathcal{M}_n) \subseteq \{1, \dots, m\} \cup \{n - m + 1, \dots, n - 1, n\},$$

(b) for every first order φ, E^n_k occurs, $n > 2k + 1$, the satisfaction of $\mathcal{M}_n \models \varphi$ depends only on the isomorphism type of

$\mathcal{M}_n \restriction (\{1, \dots, m\} \cup \{n - m + 1, \dots, n\})$,

(c) for all $n > 2k$, for every $N \in \mathcal{K}$ with $2k$ elements, the probability $\mathrm{Prob}(\mathcal{M}_n \restriction (\{1, \dots, m\} \cup \{n - m + 1, \dots, n\})$ does not depend on n.

Now in $\otimes_2$, clauses (b) and (c) are immediate. For proving (a) show by induction on ℓ that

$\otimes_3$ for every ε for some m, for every n and $w \subseteq [n]$ with $k - \ell$ elements

$$1 - \varepsilon \text{ ??? } \mathrm{Prob}\begin{pmatrix} \text{if } \mathcal{M}_n \restriction w <_i A \leq \mathcal{M}_n, \ |A| \leq k \\ \text{then } (\forall x \in A \setminus w)(\exists y \in w)[|x - y| \leq m] \end{pmatrix}.$$

If you have read 5.4 this should be clear. $\blacksquare_{7.1}$

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