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Ramsey–Type Results for Geometric Graphs

by

Gyula Károlyi[1,2]
Institute for Advanced Study
Princeton

János Pach[3]
City College, CUNY and
Courant Institute, NYU

Géza Tóth
Courant Institute, NYU

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ABSTRACT

For any 2–coloring of the $\binom{n}{2}$ segments determined by n points in general position in the plane, at least one of the color classes contains a non-selfintersecting spanning tree. Under the same assumptions, we also prove that there exist $\lfloor \frac{n+1}{3} \rfloor$ pairwise disjoint segments of the same color, and this bound is tight. The above theorems were conjectured by Bialostocki and Dierker. Furthermore, improving an earlier result of Larman et al., we construct a family of m segments in the plane, which has no more than $m^{\log 4/\log 27}$ members that are either pairwise disjoint or pairwise crossing. Finally, we discuss some related problems and generalizations.

1 Introduction

A *geometric graph* is a graph drawn in the plane so that every vertex corresponds to a point, and every edge is a closed straight-line segment connecting two vertices but not passing through a third. The $\binom{n}{2}$ segments determined by n points in the plane, no three of which are collinear, form a *complete* geometric graph with n vertices (see [PA95]). In classical Ramsey–theory, we want to find large monochromatic subgraphs in a complete graph whose edges are colored with several colors [B74], [GRS90]. Most questions of this type can be generalized to complete geometric graphs, where the monochromatic subgraphs are required to satisfy certain geometric conditions.

Our first two theorems settle two problems raised by A. Bialostocki and P. Dierker [BD94].

Theorem 1.1 *If the edges of a finite complete geometric graph are colored by two colors, there exists a non-selfintersecting spanning tree, all of whose edges are of the same color.*

Theorem 1.2 *If the edges of a complete geometric graph with $3n-1$ vertices are colored by two colors, there exist n pairwise disjoint edges of the same color.*

For abstract graphs, Theorems 1.1 and 1.2 were noticed by Erdős–Rado (see [EGP91]) and Gerencsér–Gyárfás [GG67], respectively. In fact, Gerencsér and Gyárfás proved the stronger result that for any 2-coloring of the edges of a complete graph with $3n-1$ vertices, there exists a monochromatic path of length $2n-1$. This statement, as well as Theorem 1.2, is best possible, as is shown by the following example. Take the disjoint union of a complete graph of $n-1$ vertices and a complete graph of $2n-1$ vertices, all of whose edges are red and blue, respectively, and color all edges between the two parts red. For many interesting generalizations of these results, consult [EG95], [EGP91], [G89], [G83], [HKS87], [R78].

Theorem 1.2 also has an "asymmetric" version.

Theorem 1.3 *Let k and l be positive integers, $n = k + l + \max\{k, l\} - 1$. If the edges of a complete geometric graph with n vertices are colored by red and blue, one can find either k disjoint red edges or l disjoint blue edges. This result cannot be improved.*

Our Theorem 4.1 (see below) gives a non-trivial upper bound for the smallest positive number $R = R(n)$ such that every complete geometric graph of R vertices whose edges are colored by two colors contains a non-selfintersecting monochromatic path of length n. We have been unable to determine the exact order of magnitude of $R(n)$.

Two segments are said to *cross* each other if they have an interior point in common. It appears to be difficult to obtain any exact results analogous to Theorems 1.2 and 1.3 for pairwise crossing edges. It follows from [AEG94] that if we color the edges of a complete geometric graph of $12n^2$ vertices by two colors, one can always find n pairwise crossing edges of the same color, but the assertion is probably true for much smaller graphs.

In [LMP94], the following question was discussed. What is the smallest positive number $r = r(n)$ such that any family of r closed segments in general position in the plane has n members that are either pairwise disjoint or pairwise crossing? Our next theorem improves the lower bound for $r(n)$, obtained in [LMP94].

Theorem 1.4 *For infinitely many n, there exists a family of $n^{\log 27/\log 4}$ segments in general position in the plane, which has at most n members that are pairwise disjoint and at most n members that are pairwise crossing.*

2 Proofs of Theorems 1.1–1.3

Proof of Theorem 1.1: Let $P = \{p_1, \ldots, p_n\}$ denote the vertex set of a complete geometric graph K_n whose edges are colored with red and blue. Suppose without loss of generality that no two vertices have the same x-coordinate and that the vertices are listed in increasing order of their x-coordinates. The assertion is trivial for $n \leq 2$. Thus, we can assume that $n \geq 3$ and the theorem has already been proved for all complete geometric graphs having fewer than n vertices.

We can also assume that all edges along the boundary of the convex hull of P are of the same color (say, red). Indeed, if two consecutive edges of the convex hull have different colors, then remove their common endpoint from K_n. By the induction hypothesis, the remaining graph has a monochromatic non-selfintersecting spanning tree. Depending on its color, this spanning tree can be completed to a monochromatic non-selfintersecting spanning tree of K_n, by putting back one of the two previously deleted edges of the convex hull of P.

For every i, $1 < i < n$, let K_i^l and K_i^r denote the subgraphs of K_n induced by the points $\{p_1, \ldots, p_i\}$ and $\{p_i, \ldots, p_n\}$, respectively. By the induction hypothesis, both K_i^l and K_i^r

have a monochromatic non-selfintersecting spanning tree, T_i^l resp. T_i^r. We can assume that these two trees have different colors, otherwise their union will meet the requirements of the theorem. We can also assume that T_2^l is red and T_2^r is blue. Otherwise, T_2^r would be red, and it could be completed to a non-selfintersecting red spanning tree of K_n by the addition of any edge of the convex hull of P incident to p_1. Similarly, we can suppose that T_{n-1}^l is blue and T_{n-1}^r is red. Hence, there exists an i, $1 < i < n-1$ such that
(a) T_i^l is red and T_i^r is blue,
(b) T_{i+1}^l is blue and T_{i+1}^r is red.

Connecting T_i^l and T_{i+1}^r by any edge of the convex hull of P which intersects a vertical line separating p_i and p_{i+1}, we obtain a non-selfintersecting red spanning tree of K_n, as required. $\square$

Proof of Theorem 1.2: Let P denote the vertex set of a complete geometric graph K_{3n-1}, whose edges are colored with red and blue. Suppose for contradiction that K_{3n-1} does not contain n pairwise disjoint edges of the same color. Since the theorem is trivial for $n = 1$, we can assume that $n \geq 2$ and that the statement has already been proved for every complete geometric graph with $3k-1$ vertices, where $1 \leq k < n$.

An i-element subset of P is called an *i-set* if it can be obtained by intersecting P with an open half-plane. It is easy to see that all i-sets can be generated by the following procedure [ELS73]: Take an oriented line ℓ passing through precisely one point $p \in P$ and having i elements of P on its left side. Rotate ℓ around p in the clockwise direction until it hits another point $q \in P$, and then continue the rotation around q, etc. Whenever ℓ passes through only one element of P, the points lying on its left side form an i-set.

By the induction hypothesis, the subgraph of K_{3n-1} induced by any $(3k-1)$-set contains k disjoint edges of the same color $(1 \leq k < n)$. If these edges are red (blue), we say that the *type* of the corresponding $(3k-1)$-set is *red (blue)*. Note that a set may have both types. Just like in the previous proof, we can suppose that all edges along the boundary of the convex hull of P are of the same color (say, red). In other words, for $k = 1$, the type of every $(3k-1)$-set is supposed to be red.

Lemma 2.1 **(i)** *For any given $k < n$, all $(3k-1)$-sets are of the same type.*
(ii) *For any $k, l > 1$ for which $k + l = n$, the $(3k-1)$-sets and the $(3l-1)$-sets have opposite types.*
(iii) *For any $k, l > 1$ for which $k + l = n - 1$, either all $(3k-1)$-sets or all $(3l-1)$-sets are of blue type.*

To establish (i) and (ii), consider an oriented line ℓ passing through precisely one point $p \in P$ and dividing $P - \{p\}$ into a $(3k-1)$-set $P^-(\ell)$ and a $(3l-1)$-set $P^+(\ell)$, where $k+l=n$. If $P^-(\ell)$ and $P^+(\ell)$ had the same type, then P would contain $k+l=n$ edges of the same color, contradicting our assumption. Now rotate ℓ around p in the clockwise direction until it hits another point $q \in P$, and let ℓ' denote a line obtained by slightly continuing the rotation around q. Notice that either $P^-(\ell) = P^-(\ell')$ or $P^+(\ell) = P^+(\ell')$. Since $P^-(\ell')$ and $P^+(\ell')$ have opposite types, we can conclude that $P^-(\ell)$ and $P^-(\ell')$ are of the same type. Thus, (i) follows from the fact that any $(3k-1)$-set can be reached from $P^-(\ell)$ by repeating the above step a finite number of times.

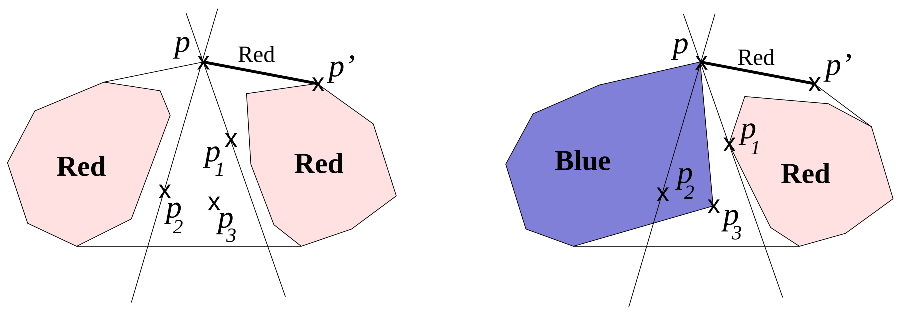

Fig. 1.

To show (iii), fix a vertex p of the convex hull of P, and let $p' \in P$ be the next vertex of the convex hull immediately after p in the clockwise order. Let p_1 and p_2 denote those elements of P, for which there are exactly $3k-1$ points to the left of the oriented line pp_1 and exactly $3l-1$ points to the right of pp_2, where $k+l=n-1$. That is, we have $|P^-(pp_1)| = 3k-1, |P^+(pp_2)| = 3l-1$, and there is just one point $p_3 \in P$ in the angular region p_1pp_2; see Figure 1. Assume now, for contradiction, that all $(3k-1)$-sets and all $(3l-1)$-sets are of red type. In particular, the type of $P^-(pp_1)$ is red, which implies that the type of $P^+(pp_2) \cup \{p, p_2, p_3\}$ must be blue; otherwise we could find $k+(l+1)=n$ disjoint red edges. This, in turn, yields that $P^-(pp_1) \cup \{p_1\} - \{p'\}$ is a $(3k-1)$-element set that cannot contain k disjoint blue edges. Thus, $P^-(pp_1) \cup \{p_1\} - \{p'\}$ has k disjoint red edges, $P^+(pp_2)$ has l disjoint red edges, and by our assumption that every edge of the convex hull of P is red (including pp'), we would obtain $k+l+1=n$ pairwise disjoint red edges. This contradiction proves the lemma.

Now we are in a position to complete the proof of Theorem 1.2. We distinguish between two cases.

Case 1: n is even. Consider a line ℓ passing through precisely one point of P and dividing the remaining points into two equal classes. Applying Lemma 2.1 (i) with $k = n/2$, we obtain that these classes are of the same type (say, red). Thus, there are $n/2$ pairwise disjoint red edges on both sides of ℓ, contradicting our assumption that there are no n disjoint edges of the same color in K_{3n-1}.

Case 2: n is odd. Applying part (iii) of Lemma 2.1 with $k = (n-1)/2$, we obtain that all $3(n-1)/2 - 1$-sets are of blue type. By part (ii), this implies that the type of all $3(n+1)/2 + 1$-sets is red. Applying (iii) again, we find that all $3(n-3)/2 - 1$-sets are of blue type. Proceeding like this, we conclude that the type of every 2-set is blue. In other words, every edge of the convex hull of P is blue, contradicting our assumption. □

Proof of Theorem 1.3: Let $R(k,l)$ denote smallest number R with the property that in any complete geometric graph of R vertices, whose edges are colored with red and blue, one can find either k disjoint red edges or l disjoint blue edges. It is enough to show that $R(k,l) = 2k + l - 1$, for every $l, 1 \leq l \leq k$. This is trivial for $l = 1$, and, according to Theorem 1.2, it also holds for $l = k$. To complete the proof, it is sufficient to verify that $R(k,l) < R(k,l+1)$, for every $l < k$. Indeed, adding a new point p to any complete geometric graph which contains neither k disjoint red edges nor l disjoint blue edges, and connecting p to every other point by a blue edge, does not change the maximum number of disjoint red edges, and the maximum number of disjoint blue edges can only increase by one. □

3 A construction

The aim of this section is to prove Theorem 1.4 by a construction. A family of segments is in *general position* if no three of their endpoints are collinear. Let $\mathcal{S}_1$ denote the family of 27 segments depicted in Figure 2. Clearly, $\mathcal{S}_1$ is in general position, and it is easy to check that it has no 5 pairwise crossing and no 5 pairwise disjoint members.

Let $\mathcal{S} = \{s_1, \ldots, s_n\}$ be a family of segments in general position in the plane. We say that $\mathcal{S}$ can be *flattened* if for every $\varepsilon > 0$ there are two discs of radius ε at unit distance from each other, and there is another family of segments $\mathcal{S}' = \{s'_1, \ldots, s'_n\}$ in general position such that s'_i and s'_j are disjoint if and only if s_i and s_j are disjoint, and every s_i connects two points belonging to different discs.

Lemma 3.1 *Any system $\mathcal{S}$ of segments in general position, whose endpoints form the vertex set of a convex polygon, can be flattened.*

Proof: Let $p_1, \ldots, p_{2n}$ denote the endpoints of the segments in counterclockwise order. Notice that moving the endpoints to any convex curve does not effect the crossing pattern of $\mathcal{S}$, provided that the order of the endpoints remains unchanged. Let $p'_i = (\varepsilon/4^{i-1}, \sqrt{\varepsilon}/2^{i-1}), 1 \leq i \leq 2n$. Since all of these points are on the parabola $y = \sqrt{x}$, connecting the corresponding pairs by segments, we obtain a family $\mathcal{S}'$, which has the same crossing pattern as $\mathcal{S}$.

It can be shown by easy calculation that if we have two disjoint segments $p'_i p'_j, p'_k p'_l \in \mathcal{S}'$ for some $i < k < l < j$,, then the slope of $p'_i p'_j$ is smaller than the slope of $p'_k p'_l$. Thus, extending all segments of $\mathcal{S}'$ to the right until they hit the line $x = 1 + \varepsilon$, does not change the crossing pattern of the family. The lemma follows by applying an affine transformation $(x, y) \to (x, \delta y)$ for some $\delta > 0$, and moving the points into general position. $\square$

Consider now the family $\mathcal{S}_1$ depicted in Figure 2, and let ε denote the minimum distance between the endpoints. Replace every segment $s \in \mathcal{S}_1$ by a suitably flattened copy of $\mathcal{S}_1$, consisting of 27 segments whose endpoints are closer to the endpoints of s than $\varepsilon/2$. Replacing every member of the resulting family $\mathcal{S}_2$ by a (very) flattened copy of $\mathcal{S}_1$, we obtain $\mathcal{S}_3$, etc. In this manner, for every k, we construct a family $\mathcal{S}_k$ of 27^k segments in general position, which has at most 4^k pairwise disjoint members and at most 4^k pairwise crossing members. This completes the proof of Theorem 1.4.

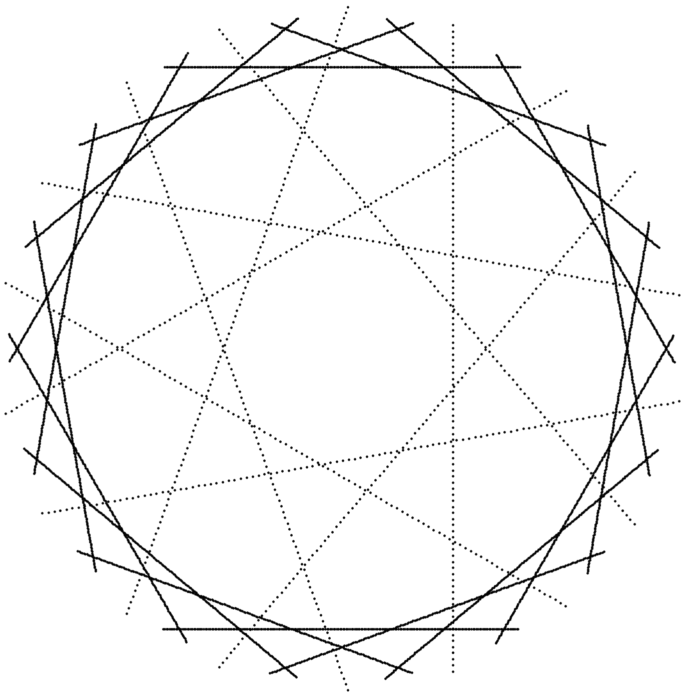

Fig. 2: The family of segments $\mathcal{S}_1$

4 Related Problems and Generalizations

Geometric Ramsey Numbers. Let $\mathcal{G}_1, \ldots, \mathcal{G}_k$ be not necessarily different classes of geometric graphs. Let $R(\mathcal{G}_1, \ldots, \mathcal{G}_k)$ denote the smallest positive number R with the property that any complete geometric graph of R vertices whose edges are colored with k colors ($1, \ldots, k$, say) contains, for some i, an i-colored subgraph belonging to $\mathcal{G}_i$. If $\mathcal{G}_1 = \ldots = \mathcal{G}_k = \mathcal{G}$, we write $R(\mathcal{G}; k)$ instead of $R(\mathcal{G}_1, \ldots, \mathcal{G}_k)$. If $k = 2$, for the sake of simplicity, let $R(\mathcal{G})$ stand for $R(\mathcal{G}; 2)$.

In Theorems 1.1 and 1.2, we determined $R(\mathcal{G})$ in the special case when $\mathcal{G}$ is the class of all non-selfintersecting trees of n vertices and the class of all geometric graphs having n disjoint edges, respectively. Theorem 1.3 gives the exact value of $R(\mathcal{G}_1, \mathcal{G}_2)$, when $\mathcal{G}_1$ and $\mathcal{G}_2$ denote the classes of all geometric graphs consisting of k disjoint edges and l disjoint edges, respectively.

The proof of Theorem 1.2 can be easily generalized to give an upper bound for $R(\mathcal{H})$, when $\mathcal{H}$ is e.g. the class of all geometric graphs consisting of n pairwise disjoint triangles. More generally, for any class of geometric graphs $\mathcal{G}$ and for any positive positive integer n, let $n\mathcal{G}$ denote the class of all geometric graphs that can be obtained by taking the union of n pairwise disjoint members of $\mathcal{G}$, any two of which can be separated by a straight line.

Theorem 4.1 *Let $\mathcal{G}$ be any class of geometric graphs, and let n be a power of 2. Then*

$$R(n\mathcal{G}) \leq (R(\mathcal{G}) + 1)n - 1.$$

In particular, if $\mathcal{G}$ is the class of triangles, we have $R(\mathcal{G}) = 6$. Moreover, in this case Theorem 4.1 cannot be improved. We will return to these questions in a forthcoming paper.

Non-Selfintersecting Paths. The *length* of a path is the number of its edges. Let $\mathcal{P}_n$ denote the class of all non-selfintersecting paths of length n.

To give a non-trivial upper bound on $R(\mathcal{P}_k, \mathcal{P}_l)$, we recall the following wellknown (and very easy) lemma of Dilworth [D50].

Lemma 4.2 *Any partially ordered set of size $kl + 1$ either has a totally ordered subset of size $k + 1$ or contains $l + 1$ pairwise incomparable elements.*

Theorem 4.3 *If the edges of a complete geometric graph of $kl + 1$ vertices are colored by red and blue, one can find either a non-selfintersecting red path of length k or a non-selfintersecting blue path of length l.*

Proof: Let $p_i (0 \le i \le kl)$ denote the vertices of a complete geometric graph. Suppose that they are listed in increasing order of their x-coordinates, which are all distinct. Define a partial ordering of the vertices, as follows. Let $p_i < p_j$ if $i < j$ and there is an x-monotone red path connecting p_i to p_j. By Lemma 4.2, one can find either $k+1$ elements that form a totally ordered subset $Q \subset P$, or $l+1$ elements that are pairwise incomparable. In the first case, there is an x-monotone red path visiting every vertex of Q. In the second case, there is an x-monotone blue path of length l, because any two incomparable elements are connected by a blue edge. Since an x-monotone path cannot intersect itself, the proof is complete. $\square$

Using the notation introduced above, Theorem 4.3 implies that $R(\mathcal{P}_n) = O(n^2)$, but the best lower bound we are aware of is linear in n.

Constructive Vertex and Edge Ramsey Numbers. Given a class of geometric graphs $\mathcal{G}$, let $R_v(\mathcal{G})$ denote the smallest number R such that *there exists* a (complete) geometric graph of R vertices with the property that for any 2-coloring of its edges, it has a monochromatic subgraph belonging to $\mathcal{G}$. Similarly, let $R_e(\mathcal{G})$ denote the minimum number of *edges* of a geometric graph with this property. $R_v(\mathcal{G})$ and $R_e(\mathcal{G})$ are called the *vertex* and *edge Ramsey number* of $\mathcal{G}$, respectively. Clearly, we have

$$R_v(\mathcal{G}) \le R(\mathcal{G}),$$

$$R_e(\mathcal{G}) \le \binom{R(\mathcal{G})}{2}.$$

(For abstract graphs, a similar notion is dicussed in [EFR78] and [B83].)

It follows from the previous subsection that for $\mathcal{P}_n$, the class of non-selfintersecting paths of length n, $R_v(\mathcal{P}_n) = O(n^2)$ and $R_e(\mathcal{P}_n) = O(n^4)$. It is not difficult to improve the latter bound, as follows.

Proposition 4.4 $R_e(\mathcal{P}_n) = O(n^2)$.

Proof: Construct a geometric graph G on the vertex set $P = \{(i,j) | 0 \le i,j \le n\}$ by connecting every (i,j) to the points $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$ (provided that they belong to P). For any coloring of the edges of G with red and blue, color every triangle of G red (blue) if at least two of its sides are red (blue). Notice that any two vertices of a red (blue) triangle can be joined in G by a red (blue) path of length at most two. Thus, any two vertices belonging to the same connected component of the closure of the union of the red (blue) triangles can be joined by a red (blue) path in G. The result now follows from the fact that one can always find a pair of vertices lying on opposite sides of the square $\{(x, y) \in \mathbf{R}^2 | 0 \leq x, y \leq n\}$, which belong to the same connected component of the closure of the union of the red triangles or the closure of the union of the blue triangles. $\square$

Covering with Non-Selfintersecting Monochromatic Paths. Is it true that for every k there exists an integer $C(k)$ such that the vertex set of every complete geometric graph whose edges are colored by k colors can be covered by $C(k)$ non-selfintersecting monochromatic paths? We cannot even decide the following weaker question for $k = 2$: Does there exist a positive ε such that every complete geometric graph G whose edges are colored by k colors contains a non-selfintersecting monochromatic path of length $\varepsilon|V(G)|$?

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