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# On Point Covers Of Multiple Intervals And Axis-parallel Rectangles

by

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## ABSTRACT

In certain families of hypergraphs the transversal number is bounded by some function of the packing number. In this paper we study hypergraphs related to multiple intervals and axis-parallel rectangles, respectively. Essential improvements of former established upper bounds are presented here. We explore the close connection between the two problems at issue.

# 1 Introduction

Let $\mathcal{H}$ be a *hypergraph*: a family of nonempty subsets of an underlying set $X$. These subsets are called the *edges* of the hypergraph, the elements of $X$ are its *vertices*. A set $S \subset X$ of vertices is said to *cover* the edges of $\mathcal{H}$ intersecting $S$. The *transversal number* $\tau(\mathcal{H})$ is the minimum size of a set covering all edges of $\mathcal{H}$. The *packing number* $\nu(\mathcal{H})$ is defined as the maximum number of pairwise disjoint edges of $\mathcal{H}$.

The inequality

$$\tau(\mathcal{H}) \geq \nu(\mathcal{H})$$

is trivial. If $\mathcal{H}$ is a finite set of intervals on a line, then $\tau(\mathcal{H}) = \nu(\mathcal{H})$ as it was first observed by Gallai (see Hajnal and Surányi [HaS]). This observation, with its algebraic proof, became an important theorem in the theory of perfect graphs. We note that the result can be extended to arbitrary families of closed intervals on a line (see e.g. [Kar]). On the other hand, there are several interesting types of hypergraphs where $\tau(\mathcal{H})$ may be arbitrarily large though $\nu(\mathcal{H}) = 1$ (see e.g. [DSW] or [GyL2]).

Hypergraphs related to families of multiple intervals were investigated by Gyárfás and Lehel in [GyL1] and [GyL2], and upper bounds were established for the transversal number in terms of the packing number. To state the results we should introduce some terminology.

Let $d$ be a positive integer. Choose and fix $d$ distinct parallel lines $\ell_1, \ldots, \ell_d$. A *$d$-interval* $A$ is the union of $d$ closed intervals one located on each of the lines $\ell_1, \ldots, \ell_d$. We use the superscript notation $A^i$ to denote the component of $A$ on the line $\ell_i$. Here some but definitely not all of the intervals $A^1, \ldots, A^d$ may be empty. Define

$$f(d,k) = \max\{\tau(\mathcal{H}) | \mathcal{H} \text{ is a family of } d\text{-intervals with } \nu(\mathcal{H}) = k\}.$$

We obtain another interesting type of hypergraph if we let the lines $\ell_1, \ldots, \ell_d$ coincide. A *homogeneous $d$-interval* $A$ is the union of $d$ closed intervals on the real line $\mathbb{R}$. We denote these intervals by $A^1, \ldots, A^d$. Analogously to $f(d,k)$ we define

$$f^*(d,k) = \max\{\tau(\mathcal{H}) | \mathcal{H} \text{ is a family of homogeneous } d\text{-intervals with } \nu(\mathcal{H}) = k\}.$$

Clearly $f(d,k) \leq f^*(d,k)$.

The observation of Gallai can be stated as follows.

**Theorem 1.1.** *[HaS]* $f(1,k) = f^*(1,k) = k$ . ■

Gyárfás and Lehel [GyL1] proved that $f(d,k)$ and $f^*(d,k)$ are finite for every $d,k \in \mathbb{N}$. To be more detailed, they proved

**Theorem 1.2.** *[GyL1]* $\quad f(d,k) \leq f(d-1, k((k+1)^{d-1}-1)) + k$ ,

$$f^*(d,k) \leq (f^*(d-1,k))^{d-1} f(d,k) + \sum_{i=1}^{d-1} (f^*(d-1,k))^i \ .$$

■

The recursions in Theorem 1.2 (using Theorem 1.1 as the base case) yield $f(d,k) = O(k^{d!})$ and $f^*(d,k) = O(k^{(d+1)!/2})$ for any fixed $d$. In Section 2 we improve these upper bounds, and also show a lower bound for the function $f(d,k)$. We have to note that a quadratic upper bound for $f^*(2,k)$ was proved by Kostochka [Kos] earlier and later Tardos [Tar] proved a linear upper bound for the same function. Very recently Kaiser [Kai] proved $f^*(d,k) \leq d^2 k$, an upper bound superior to ours.

In 1965 Wegner [Weg] proposed the following problem. Given a family $\mathcal{H}$ of axis-parallel rectangles in the plane, is it always true that $\tau(\mathcal{H}) < 2\nu(\mathcal{H}) - 1$? See also [GyL2] for a relaxed version of the problem. Though the question is still open, the following approximative result was proved recently by Károlyi [Kar]. Later Fon-Der-Flaass and Kostochka [FFK] gave a simpler proof.

**Theorem 1.3.** *[Kar] If $\mathcal{H}$ is an arbitrary family of axis-parallel rectangles in the plane then*

$$\tau(\mathcal{H}) \leq \nu(\mathcal{H})(\lfloor \log \nu(\mathcal{H}) \rfloor + 2) \ .$$

■

A discrete version of this problem is the following. Let us call a hypergraph $\mathcal{H}$ *rectangular*, if its finite set of vertices $X$ can be placed in the plane so that every edge of $\mathcal{H}$ is of the form $X \cap R$ where $R$ is a suitable rectangle with sides parallel to the coordinate axes. Rectangular hypergraphs were first investigated by Ding, Seymour and Winkler [DSW] in a more general context. As a consequence of their general theorem for arbitrary hypergraphs, they proved the following result.

**Theorem 1.4.** *[DSW] For any rectangular hypergraph $\mathcal{H}$,*

$$\tau(\mathcal{H}) \leq (\nu(\mathcal{H}) + 63)^{127} \ .$$

■

Later Pach and Törőcsik [PaT] observed a connection of these hypergraphs with 4-interval hypergraphs, and proved that $\tau(\mathcal{H}) \leq f(4,1)(\nu(\mathcal{H}))^8$ holds for rectangular hypergraphs $\mathcal{H}$. It is worth noting that the constant $f(4,1)$ here comes from the observation that for rectangular hypergraphs $\mathcal{H}$ with $\nu(\mathcal{H}) = 1$ one has $\tau(\mathcal{H}) \leq f(4,1)$.

In Section 3 we explore the close connection of rectangular hypergraphs with 2-interval hypergraphs which yields the following improvement of Theorem 1.4.

**Theorem 1.5.** *For any rectangular hypergraph $\mathcal{H}$,*

$$\tau(\mathcal{H}) \leq 4\nu(\mathcal{H})(\lfloor \log \nu(\mathcal{H}) \rfloor + 1)^2 \ .$$

This bound of $\tau$ is almost linear in $\nu$. Conjecture 3.5 would even shave off one of the two $\log \nu(\mathcal{H})$ factors. Note that Theorem 1.5 remains valid if we relax the finiteness condition: it is enough to require that the set of vertices $X$ is a closed subset of the plane. In the remaining part of the paper we will refer to rectangular hypergraphs in this more general sense.

## 2 Multiple intervals

The following observation (using Theorem 1.2 to see that $f^*(d,d)$ is finite) yields $f^*(d,k) = O(k^d)$ for any fixed $d$.

**Proposition 2.1.** *For $k \geq d$ we have $f^*(d,k) \leq dk + \binom{k}{d} f^*(d,d)$ .*

**Proof.** Let $\mathcal{H}$ be a system of homogeneous $d$-intervals with $\nu(\mathcal{H}) = k$. Let, in particular, $H_1, H_2, \ldots, H_k$ be pairwise disjoint elements of $\mathcal{H}$. The $dk$ left endpoints of the intervals $H_i^j$ $(1 \leq i \leq k,\ 1 \leq j \leq d)$ will cover the homogeneous $d$-intervals $H_1, H_2, \ldots, H_k$; and also those members $H$ of $\mathcal{H}$ for which some $H^j$ $(1 \leq j \leq d)$ has a common point with at least two different $H_i^l$ $(1 \leq i \leq k,\ 1 \leq l \leq d)$. Therefore it remains to cover those members $H$ of $\mathcal{H}$, for which every $H^j$ $(1 \leq j \leq d)$ meets at most one $H_i$ $(1 \leq i \leq k)$. As these homogeneous $d$-intervals meet at most $d$ of $H_1, \ldots, H_k$ they are contained in one of the sets

$$\mathcal{H}_S = \{H \in \mathcal{H} | H \cap U = \emptyset \text{ for any } U \in S\}$$

where $S \subset \{H_1, \ldots, H_k\}$ and $|S| = k - d$. By definition all elements of $S$ are disjoint from any element of $\mathcal{H}_S$ and therefore

$$\nu(\mathcal{H}_S) \leq k - |S| = d \ .$$

Thus the homogeneous $d$-intervals in $\mathcal{H}_S$ can be covered by at most $f^*(d,d)$ points. The statement of the theorem follows by taking the $dk$ left endpoints and then covering each $\mathcal{H}_S$ separately. ■

The same proof yields the analogous statement for $d$-intervals:

**Proposition 2.2.** *For $k \geq d$ we have $f(d,k) \leq dk + \binom{k}{d} f(d,d)$.* ■

The best upper bound we can show to $f(d,k)$ comes from the recursion in the following lemma. Although it yields stronger bounds, the statement and the proof is similar to Theorem 1.2.

**Lemma 2.3.** *For positive integers $k$ and $d$ we have $f(d+1,k) \leq d^2k + 1 + f(d+1,k-1) + f(d,kd)$.*

**Proof.** Let $\mathcal{H}$ be a family of $(d+1)$-intervals with $\nu(\mathcal{H}) = k$. We must cover $\mathcal{H}$ with at most $d^2k + 1 + f(d+1,k-1) + f(d,kd)$ points. For compactness reasons it is enough to do this for finite families $\mathcal{H}$.

Let us identify the first line $\ell_1$ with the real line $\mathbb{R}$. Take a subset $I \subset \ell_1$ and consider the set

$$\mathcal{H}_I = \{H \in \mathcal{H} | H^1 \subset I\}.$$

Let $\mathcal{H}'_I$ consist of the $d$-intervals obtained from the $(d+1)$-intervals in $\mathcal{H}_I$ by removing their first component. Let us take

$$x_0 = \sup\{x | \nu(\mathcal{H}'_{(-\infty,x)}) \leq dk\}.$$

As $\nu(\mathcal{H}'_{(-\infty,x_0)}) \leq dk$ we can cover $\mathcal{H}'_{(-\infty,x_0)}$ and thus $\mathcal{H}_{(-\infty,x_0)}$ by $f(d,kd)$ points. In case $x_0$ is infinity this finishes the proof. Suppose therefore that $x_0$ is finite.

By the finiteness of $\mathcal{H}$ we have $\nu(\mathcal{H}'_{(-\infty,x_0]}) > dk$. We can take therefore a $dk+1$ element set $S \subset \mathcal{H}_{(\infty,x_0]}$ of $(d+1)$-intervals that are pairwise disjoint except for their first components. For $i = 2, \ldots, d+1$ we can take $dk$ points on $\ell_i$ that separates the $dk+1$ disjoint $i$th components of $S$. Let $T$ be the set of all these $d^2k$ points.

Let us take $\mathcal{H}^*$ to be the set of $(d+1)$-intervals of $\mathcal{H}_{(x_0,\infty)}$ not covered by $T$. We claim that $\nu(\mathcal{H}^*) < k$. We prove this by contradiction. Suppose $S^* \subset \mathcal{H}^*$ consists of $k$ pairwise disjoint $(d+1)$-intervals. For any element $H$ of $S^*$ the first component $H^1$ is disjoint from the elements of $S$ and any other component $H^i$ $(i = 2, \ldots d+1)$ can only intersect at most one of the elements in $H$ since $T$ does not cover $H$. Thus at least one of the elements in $S$ is disjoint from each element of $S^*$. So $S^*$ plus this single element of $S$ are $k+1$ pairwise disjoint elements of $\mathcal{H}$, a contradiction.

By the observation above $\mathcal{H}^*$ can be covered by $f(d+1,k-1)$ points. Thus $\mathcal{H}_{(x_0,\infty)}$ can be covered by $f(d+1,k-1) + |T| = f(d+1,k-1) + d^2k$ points. We finish the proof by recalling that $\mathcal{H}_{(-\infty,x_0)}$ can be covered by $f(d,dk)$ points and observing that the rest of $\mathcal{H}$ is covered by the single pont $x_0$. ■

Using $f(1,k) = k$ and $f(d,0) = 0$ as the base case for the recursion in Lemma 2.3 one obtains $f(d,k) = O(k^d)$ for any fixed $d$ again. We remark that this proof proves the existence of a covering set of this size which has only $k$ points on the first line. (The number $k$ is minimal here.)

We get better upper bounds for $f(d,k)$ if we use the following theorem from [Tar] as the base case when applying Lemma 2.3.

**Theorem 2.4.** *[Tar]* $f(2,k) = 2k$. ■

**Corollary 2.5.** *For any fixed* $d \geq 2$ *we have* $f(d,k) = O(k^{d-1})$.

**Proof.** Lemma 2.3 and Theorem 2.4 yield the proof. ■

Let us remark here that this exponential upper bound is inferior to the recent $f(d,k)) \leq d^2k - d$ bound in [Kai]. Even this better upper bound on $f(d,k)$ is not known to be tight. There are however special types of families of multiple intervals for which the dependence between $\tau$ and $\nu$ can be computed exactly.

A map $l$ defined on closed intervals is called *admissible* if $l(I)$ is a point of $I$ for every non-empty closed interval $I$ and $l(I)$ is undefined if $I = \emptyset$. We say that a family $\mathcal{H}$ of $d$-intervals is *ordered*, if there exists an admissible map $l$ such that $l(A^i) < l(B^i)$ implies $l(A^j) \leq l(B^j)$ for every pair $A, B \in \mathcal{H}$ and superscripts $1 \leq i, j \leq d$, whenever $l(A^i)$, $l(B^i)$, $l(A^j)$ and $l(B^j)$ are defined. Introduce

$$g(d,k) = \max\{\tau(\mathcal{H}) | \mathcal{H} \text{ is an ordered family of } d\text{-intervals with } \nu(\mathcal{H}) = k\}$$

**Theorem 2.6.** $g(d,k) = dk$ .

**Proof.** The upper bound $g(d,k) \leq dk$ is a consequence of Theorem 1.1. Indeed, let $\mathcal{H}$ be a family of $d$-intervals with $\nu(\mathcal{H}) = k$, and suppose that $\mathcal{H}$ is ordered. We may assume that $l(A^i) \neq l(B^i)$ for every $A, B \in \mathcal{H}$ $(A \neq B)$ and $1 \leq i \leq d$. Therefore we may choose homeomorphisms

$$g_i : \ell^i \longrightarrow \mathbb{R} \ (i = 1, 2, \ldots, d)$$

with the following property: for every $H \in \mathcal{H}$ and every $1 \leq i, j \leq d$ we have $g_i(l(H^i)) = g_j(l(H^j))$ if $H^i$ and $H^j$ are nonempty. Then $g(H) = \cup_{j=1}^{d} g_j(H^j)$ is an interval for every $H \in \mathcal{H}$. If the intervals $g(H_1), \ldots, g(H_l)$ are pairwise disjoint for some $H_1, \ldots, H_l \in \mathcal{H}$, then the $d$-intervals $H_1, \ldots, H_l$ are also pairwise disjoint, thus $l \leq k$. Therefore, by Theorem 1.1, the family of intervals $\{g(H) \mid H \in \mathcal{H}\}$ can be covered by at most $k$ points, $x_1, \ldots, x_k$. Obviously, the inverse images $g_j^{-1}(x_i)$ $(1 \leq i \leq k,\ 1 \leq j \leq d)$ cover all the elements of $\mathcal{H}$, proving the assertion.

To show that the upper bound is tight we construct an ordered family $\mathcal{H} = \mathcal{H}_{d,k}$ of $d$-intervals with $\nu(\mathcal{H}) = k$ and $\tau(\mathcal{H}) \geq kd$. In fact it is enough to construct $\mathcal{H}_d = \mathcal{H}_{d,1}$ with the desired properties, then $\mathcal{H}_{d,k}$ can be obtained as the union of $k$ pairwise disjoint translates of $\mathcal{H}_d$.

In the construction we identify the lines $\ell_j$ with the real line $\mathbb{R}$ for $j = 1, \ldots, d$. Let us take $\mathcal{H}_d$ as the set of the $d$-intervals $H_i$ for $1 \leq i \leq d(d-1)+1$ where we define

$$H_i^j = [i, d(d-1)+2] \quad \text{if} \quad j = \lceil i/(d-1) \rceil \qquad \text{and}$$

$$H_i^j = [i, i+1/2] \quad \text{otherwise}$$

for all $1 \leq i \leq d(d-1)+1$ and $1 \leq j \leq d$.

Now it is easy to check that

1) $\mathcal{H}_d$ is ordered (an appropriate map $l$ maps the intervals to their left endpoints);
2) for $1 \leq i < i' \leq d(d-1)+1$ the $d$-intervals $H_i$ and $H_{i'}$ both contain the point $i'$ on the line $\ell_j$ where $j = \lceil i/(d-1) \rceil$, and therefore $\nu(\mathcal{H}_d) = 1$;
3) the intervals $H_i^j$ $(1 \leq i \leq d(d-1)+1\ ,\ \lceil i/(d-1) \rceil \neq j)$ are pairwise disjoint for each $1 \leq j \leq d$, and therefore each point covers at most $d$ distinct elements of $\mathcal{H}_d$.

Since $|\mathcal{H}_d| > d(d-1)$, the last assertion implies that $d-1$ points are not enough to cover all the elements of $\mathcal{H}_d$. Thus we have $\nu(\mathcal{H}_d) = 1$ and $\tau(\mathcal{H}_d) \geq d$, as claimed. ■

Let us remark that the trivial lower bound $f(d,k) \geq g(d,k) = dk$ is not tight in general as [GyL1] proves $f(3,1) = 4$. Proving better lower bounds seems to be hard, even for $f(d,1)$ we are unable to improve the trivial lower bound by more than a constant.

## 3 Axis-parallel rectangles

To bound the transversal number of rectangular hypergraphs (Theorem 1.5) we need to deal first with the following special type of rectangular hypergraphs.

A hypergraph $\mathcal{H}$ is called *pointed rectangular* if its vertex set $X$ can be identified with a closed set in the plane so that for every $H \in \mathcal{H}$ there exist an axis-parallel rectangle $R_H$ such that $H = X \cap R_H$ and $\cap_{H \in \mathcal{H}} R_H \neq \emptyset$.

Bounding the transversal number of pointed rectangular hypergraphs $\mathcal{H}$ shows close connection to bounding the transversal number of families of multiple intervals. A straightforward generalization of Lemma 1 in [PaT] yields that $\tau(\mathcal{H}) \leq f(4, \nu(\mathcal{H}))$. The following statement is an improvement upon this result. In the view of the recent $f(d,k) \leq d^2k - d$ result in [Kai] this improvement is only by a constant factor.

**Lemma 3.1.** *For any pointed rectangular hypergraph* $\mathcal{H}$

$$\tau(\mathcal{H}) \leq 2f(2, \nu(\mathcal{H}))\ .$$

**Proof.** Let us identify the vertex set $X$ of $\mathcal{H}$ with a closed set in the plane according to the definition. For an edge $H \in \mathcal{H}$ let $R_H$ be the axis parallel rectangle with $H = X \cap R_H$, such that these rectangles $R_H$ contain a common point. We may assume that the common point is the origin $(0,0)$.

The idea behind the proof is the following. With a method similar to that of described in [PaT] we may replace $\mathcal{H}$ with a 4-interval hypergraph $\mathcal{H}_1$ that has the same packing and transversal number as $\mathcal{H}$. The crucial observation is that restricting $\mathcal{H}_1$ to two lines which represent two neighboring quadrants one obtains an ordered 2-interval hypergraph. Hence, similar to the way we proved Theorem 2.6, we may construct a 2-interval hypergraph $\mathcal{H}_2$ with $\nu(\mathcal{H}_2) \leq \nu(\mathcal{H})$ and $2\tau(\mathcal{H}_2) \geq \tau(\mathcal{H})$.

Let us define $X_1 = \{(x,y) \in X | x \geq 0\}$ and $X_2 = \{(x,y) \in X | x \leq 0\}$. Let $p_1$ be the projection to the $x$ axis and $p_2$ the projection to a different line parallel to the $x$ axis. For an edge $H \in \mathcal{H}$ we define the following 2-interval $I_H$. The first component $I_H^1$ is the convex hull of $p_1(H \cap X_1)$ while $I_H^2$ is the convex hull of $p_2(H \cap X_2)$. Let $\mathcal{H}' = \{I_H | H \in \mathcal{H}\}$ be the family of 2-intervals so obtained.

Let $H_1$ and $H_2$ be two intersecting edges of $\mathcal{H}$. If their common vertex is $x \in X_i$ ($i = 1$ or 2) then $p_i(x)$ is a common point of $I_{H_1}$ and $I_{H_2}$. Thus $\nu(\mathcal{H}') \leq \nu(\mathcal{H})$.

Let $P$ be a point covering some of the 2-intervals in $\mathcal{H}'$. We claim that the corresponding edges of $\mathcal{H}$ can be covered by two points. By symmetry we may suppose $P$ is on the first line, the $x$ axis, so $P = (x_0, 0)$. Consider the set $\{(x,y) \in X_1 | x \leq x_0 \text{ and } y \geq 0\}$ and let $P_1 = (x_1, y_1)$ be an element of the set with minimal $y$-coordinate. Similarly let $P_2 = (x_2, y_2)$ be an element of the set $\{(x,y) \in X_1 | x \leq x_0 \text{ and } y \leq 0\}$ with maximal $y$-coordinate. If the corresponding sets are empty then $P_1$ or $P_2$ or both are undefined. Take an edge $H \in \mathcal{H}$ such that $P \in I_H$. By definition there are points $P_3 = (x_3, y_3)$ and $P_4 = (x_4, y_4)$ in $H \cap X_1$ such that $x_3 \leq x_0 \leq x_4$. We claim that if $y_3 \geq 0$ then $P_1$ covers $H$ and if $y_3 \leq 0$ then $P_2$ covers $H$. By symmetry it is enough to prove the first assertion. Since $P_3 \in \{(x,y) \in X_1 | x \leq x_0 \text{ and } y \geq 0\}$, $P_1$ is defined and $0 \leq y_1 \leq y_3$. Therefore any axis-parallel rectangle containing $P_3$, $P_4$, and the origin also contains $P_1$. Thus $P_1 \in H$ as claimed.

The last paragraph implies $\tau(\mathcal{H}) \leq 2\tau(\mathcal{H}')$. As $\mathcal{H}'$ is a system of 2-intervals with $\nu(\mathcal{H}') \leq \nu(\mathcal{H})$ we have $\tau(\mathcal{H}') \leq f(2, \nu(\mathcal{H}))$. The statement of the lemma follows. ■

Let us remark that an upper bound on the transversal number of rectangular hypergraphs follows from Theorem 1.3, Theorem 2.4, and Lemma 3.1. Let $\mathcal{H}$ be a rectangular hypergraph. The packing number of the system of the corresponding rectangles is at most

$\nu(\mathcal{H})$. Thus by Theorem 1.3 $\nu(\mathcal{H})(\lfloor \log \nu(\mathcal{H}) \rfloor + 2)$ points of the plain is enough to cover these rectangles. For any covering point $P$ the edges of $\mathcal{H}$ for which the corresponding rectangle is covered by $P$ form a pointed rectangular hypergraph, with packing number at most $\nu(\mathcal{H})$. Thus it can be covered by $2f(2, \nu(\mathcal{H})) = 4\nu(\mathcal{H})$ points by Lemma 3.1 and Theorem 2.4. Covering these subsystem of $\mathcal{H}$ separately proves

$$\tau(\mathcal{H}) \leq 4(\nu(\mathcal{H})^2)(\lfloor \log \nu(\mathcal{H}) \rfloor + 2).$$

To prove the much stronger Theorem 1.5 we replace Theorem 1.3 with Corollary 3.4 in the above argument. We need the following definition that we use only for families of rectangles.

We call a hypergraph $\mathcal{H}$ *disintegrated* if there is a partition $\mathcal{H} = \cup_{i \in I} \mathcal{H}_i$ such that $\cap_{H \in \mathcal{H}_i} H \neq \emptyset$ for each $i \in I$ and the elements of $\mathcal{H}_{i_1}$ are disjoint from the elements of $\mathcal{H}_{i_2}$ for each distinct pair of indices $i_1, i_2 \in I$.

**Lemma 3.2.** *Let $\mathcal{H}'$ be a disintegrated family of axis-parallel rectangles and $X$ a closed set in the plane. The rectangular hypergraph $\mathcal{H} = \{R \cap X | R \in \mathcal{H}'\}$ satisfies $\tau(\mathcal{H}) \leq 4\nu(\mathcal{H})$.*

**Proof.** Let $\mathcal{H}' = \cup_{i \in I} \mathcal{H}'_i$ be the partition showing that $\mathcal{H}'$ is disintegrated. Let $\mathcal{H}_i = \{R \cap X | R \in \mathcal{H}'_i\}$, these sets form a partition of $\mathcal{H}$ into pointed rectangular hypergraphs. Thus by Lemma 3.1 and Theorem 2.4 we have $\tau(\mathcal{H}_i) \leq 2f(2, \nu(\mathcal{H}_i)) = 4\nu(\mathcal{H}_i)$. By the disjointness property of $\mathcal{H}'_i$ we have $\nu(\mathcal{H}) = \sum_{i \in I} \nu(\mathcal{H}_i)$ and $\tau(\mathcal{H}) = \sum_{i \in I} \tau(\mathcal{H}_i)$. The lemma follows. ■

We call a line *horizontal* if it is parallel to the $x$ axis, and we call it *vertical* if it is parallel to the $y$ axis.

**Lemma 3.3.** *Let $\mathcal{H}$ be a system of axis-parallel rectangles in the plane and let $i$ and $j$ be positive integers. Suppose there are $2^i - 1$ horizontal lines such that each rectangle in $\mathcal{H}$ intersects at least one of these lines. Suppose there are $2^j - 1$ vertical lines with the same property. Then $\mathcal{H}$ can be partitioned into $ij$ disintegrated hypergraphs.*

**Proof.** The proof is by induction on $i$ and $j$. If $i = j = 1$ then all rectangles in $\mathcal{H}$ contain the intersection of the only horizontal and the only vertical line. Thus $\mathcal{H}$ is disintegrated as claimed.

Suppose one of $i$ and $j$ is not one. By symmetry we may suppose $i > 1$. Let us call the central of the $2^i - 1$ horizontal lines $\ell$. We partition $\mathcal{H}$ with respect to $\ell$: let $\mathcal{H}_0$ consist of the rectangles in $\mathcal{H}$ intersecting $\ell$, $\mathcal{H}_1$ consist of the rectangles on the one side of $\ell$, while $\mathcal{H}_2$ consist of the rectangles in $\mathcal{H}$ on the other side of $\ell$. By induction $\mathcal{H}_0$ can be partitioned into $j$ disintegrated hypergraphs, while $\mathcal{H}_1$ and $\mathcal{H}_2$ can be partitioned into $(i-1)j$ disintegrated hypergraphs each. As the elements of $\mathcal{H}_1$ are separated from the elements of $\mathcal{H}_2$ by $\ell$ we can take the union of a disintegrated subset of $\mathcal{H}_1$ and a disintegrated subset of $\mathcal{H}_2$ and still get a disintegrated hypergraph. Thus matching the parts of $\mathcal{H}_1$ to the parts of $\mathcal{H}_2$ we get a partition of $\mathcal{H}$ into $(i-1)j + j = ij$ disintegrated hypergraphs as claimed. ■

**Corollary 3.4.** *Let $\mathcal{H}$ be a family of axis-parallel rectangles in the plane with finite packing number. Then $\mathcal{H}$ can be partitioned into $(\lfloor \log \nu(\mathcal{H}) \rfloor + 1)^2$ disintegrated hypergraphs.*

**Proof.** Let $\mathcal{H}'$ be the set of projections to the $x$ axis of the rectangles in $\mathcal{H}$. Using Theorem 1.1 one finds $\nu(\mathcal{H}') \leq \nu(\mathcal{H})$ vertical lines such that each rectangle in $\mathcal{H}$ intersects one of them. Similarly one can find at most $\nu(\mathcal{H})$ horizontal lines with the same property. An application of Lemma 3.3 finishes the proof. ■

**Proof of Theorem 1.5.** Let $X$ be the set of vertices of $\mathcal{H}$ and let $\mathcal{H}'$ be a collection of axis-parallel rectangles such that $\mathcal{H} = \{R \cap X | R \in \mathcal{H}'\}$. Apply Corollary 3.4 to partition $\mathcal{H}'$ into disintegrated hypergraphs $\mathcal{H}'_i$ $(i \in I)$. As $\nu(\mathcal{H}') \leq \nu(\mathcal{H})$ the number of the parts is at most $|I| \leq (\lfloor \log \nu(\mathcal{H}) \rfloor + 1)^2$. For $i \in I$ let $\mathcal{H}_i = \{R \cap X | R \in \mathcal{H}'_i\}$, these sets form a partition of $\mathcal{H}$. Lemma 3.2 yields $\tau(\mathcal{H}_i) \leq 4\nu(\mathcal{H}_i)$. Using the trivial $\nu(\mathcal{H}_i) \leq \nu(\mathcal{H})$ and $\tau(\mathcal{H}) \leq \sum_{i \in I} \tau(\mathcal{H}_i)$ bounds this last observation proves the theorem. ■

One way to improve on the bound in Theorem 1.5 would be to improve Corollary 3.4. Unfortunately it is optimal except for a constant factor. Let $\mathcal{H}$ be the family of all rectangles with integer coordinates inside a $k$ by $k$ square. It is easy to show that $\nu(\mathcal{H}) = \Theta(k^2)$ and the minimum number of disintegrated hypergraphs $\mathcal{H}$ can be partitioned to is $\Theta(\log^2 k)$. One can gain though by restricting attention to *Sperner systems*, i. e. hypergraphs with no edge containing another edge. As the packing and transversal numbers of a finite hypergraph does not change by removing all non-minimal edges we can assume without loss of generality that the hypergraph is a Sperner system. In the case of rectangular hypergraphs the corresponding family of rectangles is also a Sperner system then. So the next conjecture would be enough to prove

$$\tau(\mathcal{H}) = O(\nu(\mathcal{H}) \log \nu(\mathcal{H}))$$

for any finite rectangular hypergraph. Although removing the non-minimal edges from an infinite hypergraph can change $\tau$ and $\nu$ (we can remove all edges) a simple reasoning shows that the conjecture implies this inequality for *all* rectangular hypergraphs. It is worth noting that this bound matches the best known bound for families of rectangles.

**Conjecture 3.5.** *Any Sperner system $\mathcal{H}$ of axis-parallel rectangles can be partitioned into $O(\log \nu(\mathcal{H}))$ disintegrated hypergraphs.*

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