**DIMACS Technical Report 95-43**
**September 1995**

# Steiner Minimal Trees in Simple Polygons

by

Pawel Winter[1]
Dept. of Computer Science
University of Copenhagen
Universitetsparken 1
DK-2100 Copenhagen Ø
Denmark

[1]This research was completed during a visit at DIMACS and RUTCOR, Rutgers Univ., New Brunswick, NJ, U.S.A. It was partially supported by NSF grant STC 91-9999 and AFORS grant F49620-93-1-0041. It was also partially supported by Danish Research Council Award.

---

DIMACS is a cooperative project of Rutgers University, Princeton University, AT&T Bell Laboratories and Bellcore.

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

## ABSTRACT

An $O(n \log n)$ time and $O(n)$ space algorithm for the Euclidean Steiner tree problem with four terminals in a simple polygon with $n$ vertices is given. Its applicability to the problem of determining good quality solutions for any number of terminals is discussed.

# 1 Introduction

We consider the following variant of the *Euclidean Steiner tree problem* (ESTP):

- **Given:** A simple polygon $P$ and a set $T = \{t_i, t_j, t_k, t_l\}$ of four terminals in $P$.
- **Find:** *Euclidean Steiner minimal tree* (ESMT) spanning $T$ in $P$.

We present an $O(n \log n)$ time and $O(n)$ space algorithm for this problem. Our interest in this special case is due to the fact that it is one of the steps toward a heuristic for the ESTP inside a polygon for *any* number of terminals. More specifically, concatenation of small ESMTs (with up to 4 terminals) proved to yield good quality solutions for the obstacle-free case [9, 1]. Similar approach seems to be applicable when the terminals are inside a simple polygon or amidst polygonal obstacles. The reader interested in a more thorough discussion on issues associated with the concatenation of small ESMTs is referred to [11]. In that paper we also proved that the ESMT for 3 terminals in a simple polygon can be determined in $O(n)$ time and space. The reader is referred to [5] for basic definitions and properties of ESMTs.

The paper is organized as follows. Basic definitions are given in Section 2. The original problem is reduced in Section 3 to the ESTP for up to four *semi-terminals* in a smaller polygon of a very particular shape. The semi-terminals of the reduced problem need not to be identical with the original terminals. In Section 4, the vertices of the smaller polygon are preprocessed in order to determine shortest distances between them and semi-terminals. The algorithm for the ESMT for the semi-terminals in the reduced polygon is given in Section 5. Conclusions and suggestions for further research are collected in Section 6.

# 2 Basic Definitions

A polygon is *simple* if no point of the plane belongs to more than two edges of $P$, and the only points of the plane that belong to precisely two edges are the vertices of $P$. A simple polygon $P$ has a well-defined interior $i(P)$ and exterior $e(P)$. A point $p$ is said to be in $P$ if $p \in i(P) \cup P$. A vertex $v$ on $P$ is *convex* if its interior angle is less than $180^o$. Otherwise, it is *reflex*. A reflex vertex is said to be *wide* if its interior angle is at least $240^o$. Clockwise successor and predecessor vertices of $v$ are denoted by $v^+$ and $v^-$, respectively. A simple polygon is called a *c-kite* iff precisely $c$ of its vertices are convex. A polygon $P$ is *weakly-simple* if any pair of polygonal chains obtained by partitioning $P$ does not properly intersect (but the chains are permitted to touch each other).

A line $L$ is said to be an *interior tangent* of a $c$-kite $P$ at a *touch-point* $v \in P$ iff one of the following cases occurs.

- $v$ is a reflex vertex, and edges $v^-v$ and $vv^+$ are on the same side of $L$.
- $v$ is a convex vertex, and edges $v^-v$ and $vv^+$ are on the opposite sides of $L$.

- $v$ is an interior point of an edge overlapping with $L$.

Two interior tangents of a $c$-kite $P$ are distinct if they have different slopes or different touch-points. We use the notation $e_i || e_j$ if edges $e_i$ and $e_j$ are parallel. A line through two points $p$ and $q$ is denoted by $L(p, q)$.

**Lemma 1** *Every $c$-kite $P$, $c \geq 3$, has exactly $c - 2$ interior tangents for any fixed slope $r \in [-\infty, \infty]$.*

**Proof.** Every triangulation of a simple polygon has $n-2$ triangles. Each triangle contributes to the total sum of interior angles by $\pi$. The sum of interior angles of any simple polygon with $n$ vertices is therefore $(n-2)\pi$. Let $\alpha_i$, $1 \leq i \leq c$, denote interior angles of convex vertices. Let $\gamma_j = \pi + \beta_j$, $1 \leq j \leq n - c$, denote interior angles of reflex vertices. Then

$$\sum_{i=1}^{c} \alpha_i + \sum_{j=1}^{n-c} \beta_j = (n-2)\pi - (n-c)\pi = (c-2)\pi$$

Angles $\alpha_i$ and $\beta_j$ denote maximal rotation of interior tangents at convex and reflex vertices, respectively. Furthermore, the slope interval at a particular vertex does not overlap but has a common point with the slope interval of next vertex on the polygon (Fig. 1). □

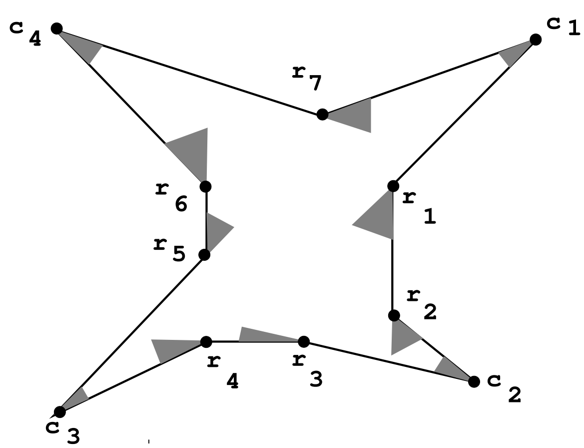


Figure 1: Convex and reflex angles

**Lemma 2** *A $c$-kite $P$, $c = 3, 4$, has at most $c - 2$ wide reflex vertices.*

**Proof.** A wide reflex vertex is a touch-point of interior tangents whose slopes span over at least $240^o$. Existence of 2 such vertices in a 3-kite would imply two parallel interior tangents.

Existence of 3 such vertices in a 4-kite would imply three parallel interior tangents. □

The shortest path between two points $u$ and $v$ in a polygon $P$ will be denoted by $P(u,v)$. $P(u,v)$ is a unique polygonal chain and its interior vertices are reflex vertices of $P$. Linear time algorithms for the determination of $P(u,v)$ are available [2, 6].

# 3 Polygon Reductions

Consider a triangulation of a simple polygon $P$. Remove from it (one at a time) outermost triangles (sharing only one side) if the remaining triangles contain all terminals. Let $P_1$ denote the polygon determined by the boundary of the exterior face of the reduced triangulation. It was shown in [11] that the ESMT for $T$ in $P$ is completely in $P_1$.

Let $P_2$ denote the polygon inside $P_1$ with the minimum length perimeter and containing the set of terminals $T$. In order to determine $P_2$, let $P_2^u$ denote the minimum length perimeter polygon for $T_u = T - \{t_u\}$, $u = i,j,k,l$, inside $P_1$. It was shown in [11] how to determine $P_2^u$ in $O(n)$ time and space. Given $P_2^u$, $u = i,j,k,l$, the polygon $P_2$ can be found as follows.

- If $t_u \in i(P_2^u) \cup P_2^u$ for some $u = i,j,k,l$, then $P_2 = P_2^u$. The inclusion test requires $O(n)$ time and space.

- Otherwise all terminals in $T$ are vertices of $P_2$. For each of six permutations $(t_{\alpha(j)}, t_{\alpha(k)}, t_{\alpha(l)})$ of $(t_j, t_k, t_l)$, consider the closed polygonal chain $P(t_i, t_{\alpha(j)})$, $P(t_{\alpha(j)}, t_{\alpha(k)})$, $P(t_{\alpha(k)}, t_{\alpha(l)})$, $P(t_{\alpha(l)}, t_i)$. Select the shortest one as $P_2$. Again, $O(n)$ time and space is needed.

$P_2$ is a weakly-simple polygon (Fig. 2). Note that $P_2$ can be obtained directly from $P$. However, a triangulation of $P$ is needed anyway in order to determine shortest paths between terminals in $O(n)$ time.

Provan [8] proved that there always exists an obstacle-avoiding ESMT for $T$ completely in $P_2$. If $i(P_2) = \emptyset$, then the ESMT for $T$ in $P_2$ is trivially given. We assume therefore in the following that $i(P_2) \neq \emptyset$.

Let $P_3$ denote the smallest polygon containing $i(P_2)$ (Fig. 2). $P_3$ can be obtained from $P_2$ in $O(n)$ tima and space by a straightforward traversal of $P_2$ (using a stack). Let $q_u, u = i,j,k,l$, denote a vertex on $P_3$ where the path to $t_u$ (overlapping with $P_2$) begins. This path can be of zero-length, i.e., $q_u = t_u$. The vertex $q_u$ will be referred to as a *semi-terminal*. The ESMT for $Q = \{q_i, q_j, q_k, q_l\}$ in $P_3$ together with (trivially given) shortest paths $P_2(t_i, q_i)$, $P_2(t_j, q_j)$, $P_2(t_k, q_k)$, $P_2(t_l, q_l)$ will yield the ESMT for $T$ in $P_2$.

**Lemma 3** *$P_3$ is a c-kite,* $3 \leq c \leq 4$.

**Proof.** All vertices of $P_3$ but semi-terminals are convex. $P_2 \subseteq P_3$ and $i(P_2) \neq \emptyset$ implies that $i(P_3) \neq \emptyset$. Since $i(P_3) \neq \emptyset$, $P_3$ must have at least 3 convex vertices, and therefore $3 \leq c$. Since there are 4 semi-terminals, $c \leq 4$. □

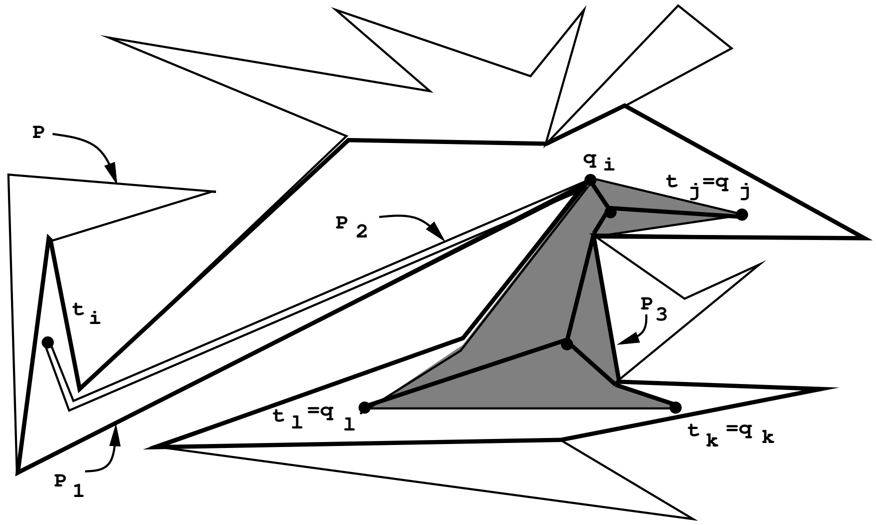


Figure 2: Nested polygons with the ESMT ($P_3$ is shaded)

# 4 Preprocessing of $c$-Kites

We need to associate shortest path distances $d_i(v), d_j(v), d_k(v), d_l(v)$ between every vertex $v$ of $P_3$ and semi-terminals of $Q$. We will distinguish between the case when $P_3$ is a 3-kite and a 4-kite.

## 4.1 Preprocessing of 3-Kites

If $P_3$ is a 3-kite, one of the semi-terminals is certainly not a convex vertex of $P_3$. Assume that $q_l$ is such a semi-terminal. Hence, $q_l = t_l$, and $Q_l = Q - \{q_l\}$ is the set of convex vertices of $P_3$ (Fig. 3). It was shown in [11] how to determine $d_i(v), d_j(v), d_k(v)$ for all vertices $v$ of $P_3$ in $O(n)$ time and space. We therefore only need to show that also $d_l(v)$ can be determined in $O(n)$ time and space for all vertices $v$ on $P_3(q_i, q_j)$. Values of $d_l(v)$ for vertices on $P_3(q_j, q_k)$ and $P_3(q_k, q_i)$ are determined in analogous way.

Suppose that $q_l \in i(P_3) \cup P_3(q_j, q_k) \cup P_3(q_k, q_i)$. Consider the polygon bounded by the shortest paths $P_3(q_i, q_j)$, $P_3(q_j, q_l)$ and $P_3(q_l, q_i)$. Traverse $P_3(q_i, q_j)$ beginning at $q_i$. Let $v$ denote currently traversed vertex. As long as $v$ also belongs to $P_3(q_l, q_i)$, let $d_l(v) = |P_3(q_l, q_i)| - d_i(v)$. Let $q_i'$ denote the last vertex belonging to both $P_3(q_i, q_j)$ and $P_3(q_l, q_i)$. Possibly $q_i' = q_i$. Similarly, traverse $P_3(q_i, q_j)$ in the other direction, beginning at $q_j$. As long as the currently traversed vertex also belongs to $P_3(q_j, q_l)$, let $d_l(v) = |P_3(q_j, q_l)| - d_j(v)$. Let $q_j'$ denote the last vertex belonging to both $P_3(q_i, q_j)$ and $P_3(q_j, q_l)$. Possibly $q_j' = q_j$. Consider the 3-kite bounded by $P_3(q_i', q_j')$, $P_3(q_j', q_l)$ and $P_3(q_l, q_i')$. Values of $d_l(v)$ in the 3-kite are equal to the corresponding values in $P_3$. Hence, they can be determined in linear time.

If $q_l \in P_3(q_i, q_j)$, the values $d_l(v)$ can be determined by traversing $P_3(q_i, q_j)$ in both

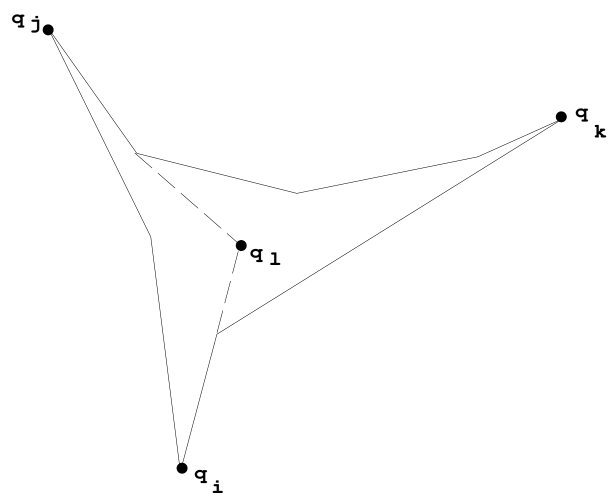


Figure 3: 3-kite

directions, beginning at $q_l$, and accumulating the lengths of traversed edges.

## 4.2 Preprocessing of 4-Kites

The value of $d_i(v)$ for every vertex $v$ invisible from $q_i$ will be determined in linear time (as an intermediate step) of computing shortest paths from $q_i$ to the other three semi-terminals.

For every vertex $v$ visible from $q_i$, $d_i(v) = |q_i v|$. We need to be able to identify in $O(n)$ time all vertices of $P_3$ visible from $q_i$. This can be achieved in the following way. Consider the rays $R_l$ and $R_r$ both rooted at $q_i$ and through $q_i^+$ and $q_i^-$, respectively. The semi-terminal $q_j$ is to the left or on $R_l$ while the semi-terminal $q_l$ is to the right or on $R_r$. We consider three cases depending on the location of the semi-terminal $q_k$. It can be to the left or on $R_l$, between $R_l$ and $R_r$, and to the right or on $R_r$.

Assume first that $q_k$ is to the left or on $R_l$. Consider the unique interior tangent $L$ for a vertex on $P_3(q_j, q_k)$ through $q_i$. There are three possibilities:

- $L$ is to the left or overlapping with $R_l$ (Fig. 4a). The only vertices visible from $q_i$ are those on $P_3(q_k, q_l)$ between $R_l$ and $R_r$.

- $L$ is to the right of $R_l$ and to the left of $R_r$ (Fig. 4b). The vertices visible from $q_i$ are those on $P_3(q_j, q_k)$ between $R_l$ and $L$ that can be reached from $q_j$ without passing the touch-point of $L$, and those on $P_3(q_k, q_l)$ between $L$ and $R_r$.

- $L$ is to the right or overlapping with $R_r$ (Fig. 4c). The vertices visible from $q_i$ are those on $P_3(q_j, q_k)$ between $R_l$ and $R_r$ that can be reached from $q_j$ before reaching $R_r$.

The case when $q_k$ is to the right or on $R_r$ is analogous and will not be covered here. Assume that $q_k$ is between $R_l$ and $R_r$. If $q_k$ is visible to $q_i$ (Fig. 4d), then the vertices visible from $q_i$ are those on $P_3(q_j, q_k)$ between $R_l$ and $q_k$ and those on $P_3(q_k, q_l)$ between $q_k$ and $R_r$. If $q_k$ is invisible to $q_i$, this visibility can be obstructed by either $P_3(q_j, q_k)$ or $P_3(q_k, q_l)$. Since they are analogous, we only examine the former case. Consider once again the unique interior tangent $L$ for a vertex on $P_3(q_j, q_k)$ through $q_i$. There are two possibilities shown in Fig. 4e and Fig. 4f. Note that these two cases are analogous to cases shown in Fig. 4b and Fig. 4c.

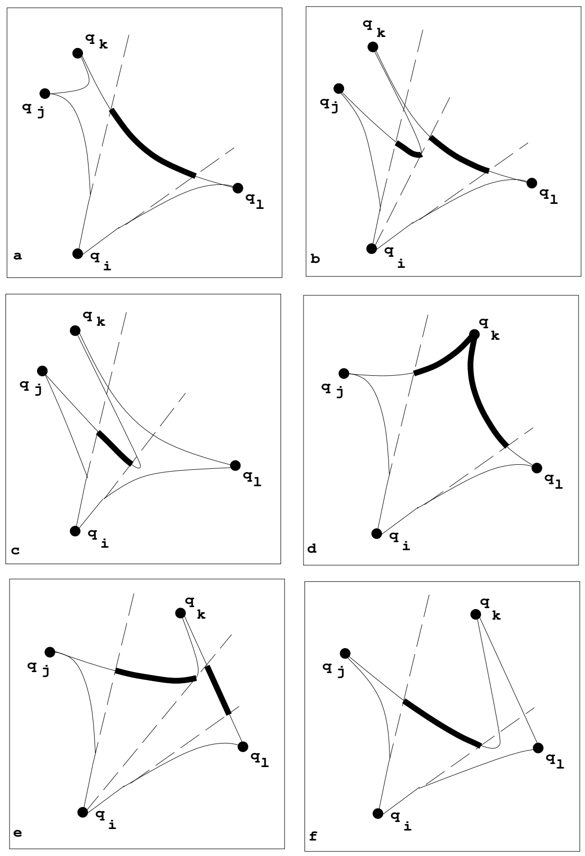


Figure 4: Preprocessing of a 4-kite

# 5 ESMT for Semi-Terminals

When determining the ESMT for $Q$ in $P_3$, $i(P_3) \neq \emptyset$, we need to distinguish between the following three cases. If $i(P_3)$ is not connected, then $i(P_3)$ consists of two regions, each with two semi-terminals. The problem breaks down into two smaller subproblems, each with fewer terminals. If $i(P_3)$ is connected and $P_3$ is a 4-kite, then the ESMT for $Q$ in $P_3$ is the shortest of the following trees spanning $Q$ (classified by the number of Steiner points).

- No Steiner points. Select three shortest paths spanning all semi-terminals.

- One Steiner point. Given an ESMT for $Q_u$, $u = i, j, k, l$, a tree spanning $Q$ is obtained by connecting $q_u$ to the closest of the semi-terminals in $Q_u$. Retain the shortest of these four trees.

  The Steiner point could also be degenerated in the sense that it could overlap with a wide reflex vertex (its incident edges do not need to make $120^o$ with each other). There are at most two such wide reflex vertices. Select four shortest paths spanning $Q$ and a wide reflex vertex. Retain the shortest of these two trees.

- Two Steiner points. The case when Steiner points are visible to each other in $i(P_3)$ is covered in Subsection 5.1. The case when they are invisible to each other is covered in Subsection 5.2.

  If one Steiner point is degenerate, consider each of six possible pairs of semi-terminals. Determine an ESMT for such a pair and each wide reflex vertex. Connect the other two terminals to the wide reflex vertex by shortest paths. Among up to twelve trees generated in this way, retain the shortest one.

  If both Steiner points are degenerate, select five shortest paths spanning semi-terminals and wide reflex vertices.

Finally, if $i(P_3)$ is connected and $P_3$ is a 3-kite, the following lemma reduces the problem to four problems with fewer terminals.

**Lemma 4** *If $P_3$ is a 3-kite with connected and non-empty $i(P_3)$, then the ESMT for $Q$ has at most one non-degenerate Steiner point.*

**Proof.** Suppose first that the ESMT for $Q$ has two Steiner points $s_{jk}$ and $s_{li}$ *visible* to each other (Fig. 5a). Since edges incident with Steiner points make $120^o$, $v_k s_{jk} || v_i s_{li}$ as well as $v_j s_{jk} || v_l s_{li}$. End-points of at least one edge-pair ($v_k s_{jk}$ and $v_i s_{li}$, say) are vertices of $P_3$. Furthermore, these two edges must overlap with interior tangents. But $P_3$ has exactly one interior tangent with any given slope. Consequently, $v_k, s_{jk}, v_i, s_{li}$ are co-linear. This implies that $|s_{jk} s_{li}| = 0$, a contradiction.

Suppose next that the ESMT for $Q$ has 2 Steiner points *invisible* to each other (Fig. 5b). Three of the vertices $v_i, v_j, v_k, v_l$ must be on $P_3$. Their incident edges must overlap with interior tangents at these vertices. Suppose that $v_i, v_j, v_k$ are such vertices on $P_3$. The ESMT for $Q$ partitions the interior of $P_3$ into 3 regions. Consider the region containing $v_l$. Whatever shape $P_3(q_k, q_l)$ has, it must contain a point $v$ with interior tangent parallel to the interior tangent at $v_j$. □

## 5.1 Visible Steiner Points

In this subsection we discuss the problem of determining a shortest tree spanning four semi-terminals of a 4-kite $P_3$ with two Steiner points $s_{jk}$ and $s_{li}$ *visible* to each other (Fig. 6).

There is an obvious algorithm. Consider all $O(n^4)$ subsets of four vertices one by one. For each subset, its Steiner trees (at most two) are determined. If none exists, next subset

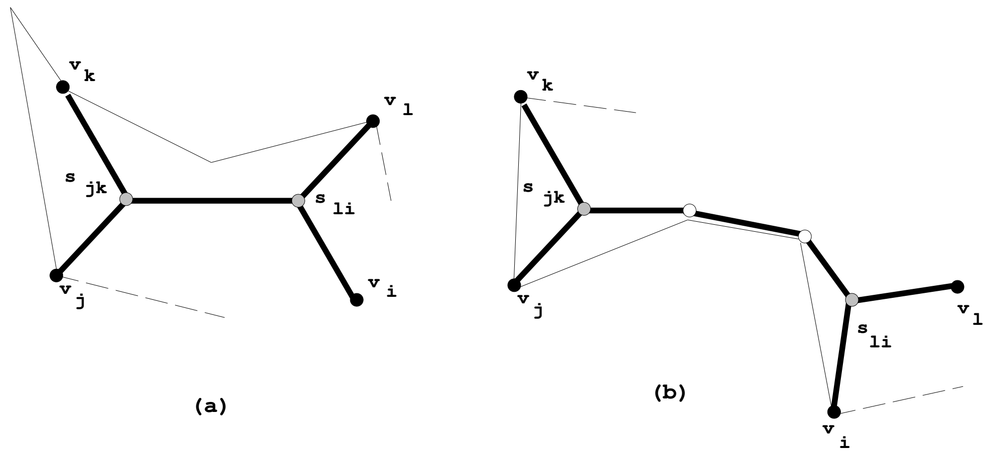


Figure 5: Invisible Steiner points in a 3-kite

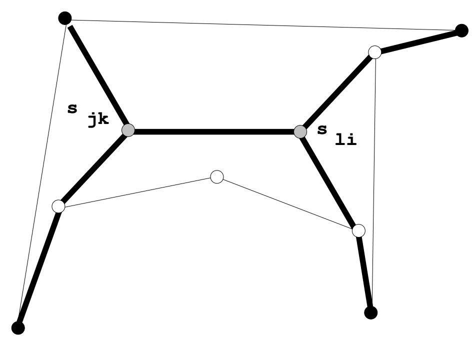


Figure 6: A tree with 2 visible Steiner points

of 4 vertices is considered. If both exist, the longer one is discarded. If the retained full Steiner tree (if any), together with the shortest paths from semi-terminals, is shorter than any tree found so far, it is checked if the edges incident with Steiner points overlap with interior tangents. It is also checked if the line segment connecting the two Steiner points is in $i(P_3)$. As will be seen in Lemma 5, this operation requires $O(\log n)$ time. This dominates all other $O(1)$ time operations. Consequently, the trivial algorithm requires $O(n^4 \log n)$ time and $O(n)$ space.

In the remainder of this section we give a more elaborate $O(n \log n)$ time and $O(n)$ space algorithm. This considerable improvement is due to the fact that only an $O(n)$ subsets of four vertices of $P_3$ are examined.

- **Initialization:** Let $v_j$ denote the vertex of $P_3$ which is a touch-point of a vertical interior tangent $L_j$. If $L_j$ overlaps with an edge of $P_3$, let $v_j$ be its first end-point (when $P_3$ is traversed clockwise). Traverse the vertices of $P_3$ clockwise, beginning at $v_j$, until reaching a vertex $v_k$ with an interior tangent $L_k$ making $60^o$ with $L_j$. Continue the clockwise traversal until reaching a vertex $v_l$ which admits a vertical tangent $L_l$

other than $L_j$. Continue the clockwise traversal until reaching a vertex $v_i$ which admits an interior tangent $L_i$ parallel to $L_k$.

- **Iteration:** Determine a full Steiner tree for $v_i, v_j, v_k, v_l$, with $v_j, v_k$ having a common Steiner point $s_{jk}$, and $v_l, v_i$ having a common Steiner point $s_{li}$ (or decide that such a tree does not exist). If the edge $s_{jk}s_{li}$ is completely in $i(P_3)$, and $L(s_{jk}, v_j)$, $L(s_{jk}, v_k)$, $L(s_{li}, v_i)$, $L(s_{li}, v_l)$ are interior tangents, then connect each semi-terminal of $Q$ to the closest of vertices $v_i, v_j, v_k, v_l$. Save this tree if no other tree has been found so far or if its length is less than the length of the best solution found so far.

- **Sweep:** Rotate the interior tangents (counterclockwise) around their touch-points $v_i, v_j, v_k, v_l$ until one of them overlaps with an edge of $P_3$. Such rotation amounts to finding the smallest angle between interior tangents $L_i, L_j, L_k, L_l$ and edges $v_iv_i^+$, $v_jv_j^+$, $v_kv_k^+$, $v_lv_l^+$. Suppose that $L_i$ and $v_iv_i^+$ meet at smallest angle. Replace $v_i$ by $v_i^+$.

- **Termination:** Stop if the interior tangents have been rotated at least $180^o$; otherwise perform next **Iteration**.

**Lemma 5** *Shortest tree spanning four semi-terminals of the 4-kite $P_3$ with the Steiner points visible to each other can be determined in $O(n \log n)$ time and $O(n)$ space.*

**Proof.** The initialization requires at most two scans of vertices of $P_3$. During the first scan, the initial vertex $v_i$ is determined. The determination of initial vertices $v_j, v_k, v_l$ requires one more scan. The entire initialization requires $O(n)$ time. Given $v_i, v_j, v_k, v_l$, the existence of the full Steiner tree with a given full topology (where $v_j$ and $v_k$ are adjacent to one Steiner point, and $v_l$ and $v_i$ are adjacent to the other Steiner point) can be verified in $O(1)$ time. Furthermore, the locations of Steiner points can be determined in constant time. Also constant time is needed to verify if the edges incident with $v_i, v_j, v_k, v_l$ in fact overlap with interior tangents of $P_3$ at these vertices. It remains to establish the complexity of determining whether $s_{jk}s_{li} \in i(P_3)$.

Consider the convex hull $C$ of the vertices of $P_3$. $C$ can have four or three semi-terminals as its only extreme vertices. Suppose first that $C$ has 4 extreme points (Fig. 7a). It can be verified in constant time whether $s_{jk}$ or $s_{li}$ is outside of $C$. If this is the case, then $s_{jk}s_{li} \notin P_3$. Let $B_{ij}$ denote the *bay region* bounded by $P_3(q_i, q_j)$ and $q_iq_j$. Since $B_{ij}$ is convex, $O(\log n)$ time is needed to check if $s_{jk} \in B_{ij}$ or $s_{li} \in B_{ij}$. Assume that $s_{jk}, s_{li} \notin B_{ij}$. The line segment $s_{jk}s_{li}$ can still penetrate $B_{ij}$. Represent $P_3(q_i, q_j)$ as a balanced binary search tree where the middle edge is associated with the root. Left and right subtrees (defined recursively) contain edges of $P_3(q_i, q_j)$ preceding and following the middle edge, respectively. This tree can be constructed in $O(n)$ time and space. Traverse the tree beginning at the root. Let $e = vv^+$ denote the edge associated with currently visited node. Distinguish between the following cases.

- Line segment $s_{jk}s_{li}$ intersects $e$ (Fig. 8a). Stop, $s_{jk}s_{li}$ intersects $P_3$.

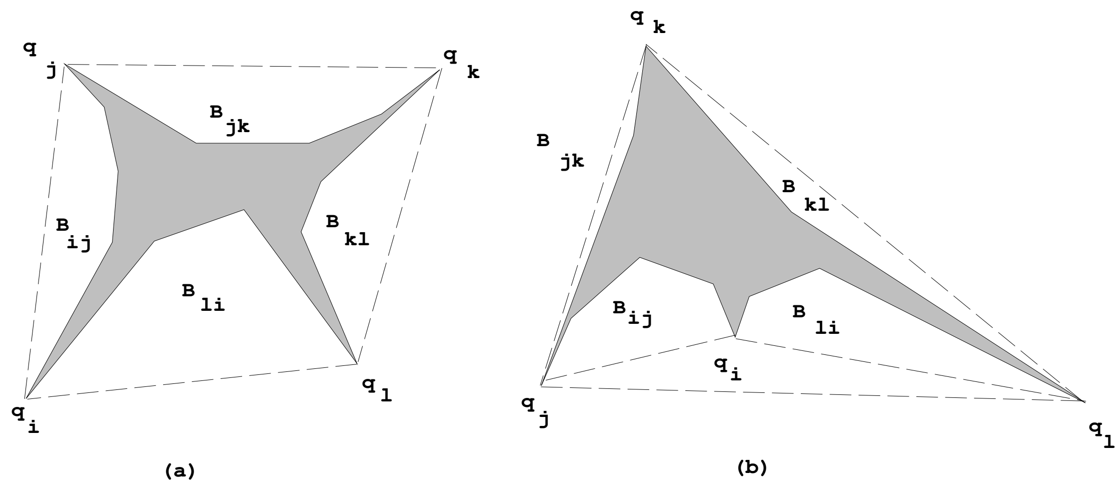


Figure 7: 4-kites with 4 and 3 extreme vertices on their convex hull

- $L(s_{jk}, s_{li})$ separates $e$ from $q_i$ and $q_j$ (Fig. 8b). Since $s_{jk}$ and $s_{li}$ are both in $P_3$, $s_{jk}s_{li}$ intersects $P_3$.
- $L(s_{jk}, s_{li})$ is parallel to the line through $e$ (Fig. 8c). There is no intersection with $P_3(q_i, q_j)$.
- $L(s_{jk}, s_{li})$ intersects the line through $e$ closer to $v$ (Fig. 8d). Search through the left subtree.
- $L(s_{jk}, s_{li})$ intersects the line through $e$ closer to $v^+$ (Fig. 8e). Search through the right subtree.

Constant time is needed to recognized which case applies. Since the balanced binary tree has $O(n)$ nodes, $O(\log n)$ time is needed. If no intersection has been identified, similar treatment is applied to the bays $B_{jk}, B_{kl}, B_{li}$ bounded by $P_3(q_j, q_k)$, $P_3(q_k, q_l)$, $P_3(q_l, q_i)$, respectively.

The case where $C$ has 3 extreme vertices (Fig. 7b) is treated similarly. It is first checked if both Steiner points are inside $C$. If so, it is checked if they are not in the bays $B_{ij}, B_{jk}, B_{kl}, B_{li}$. Furthermore, they must be outside of the triangle $\triangle q_k q_l q_i$. If both $s_{jk}$ and $s_{li}$ are in $P_3$, it is checked if $s_{jk}s_{li}$ intersects any of the bays $B_{ij}, B_{jk}, B_{kl}, B_{li}$ (as already described). Finally, the bay $B_{lj}$ bounded by $P_3(q_l, q_j)$ is considered. If both $s_{jk}$ and $s_{li}$ are outside $B_{lj}$, it is checked for the intersection of $s_{jk}s_{li}$ with $B_{lj}$.

It follows that verifying if $s_{jk}s_{li} \subseteq i(P_3)$ requires $O(\log n)$ time for each subset of four vertices on $P_3$. Since only $O(n)$ subsets is examined, lemma follows. □

## 5.2 Invisible Steiner Points

In this subsection, we discuss the problem of determining shortest tree spanning four semi-terminals of a 4-kite $P_3$ with two Steiner points $s_{jk}$ and $s_{li}$ *invisible* to each other (Fig. 9).

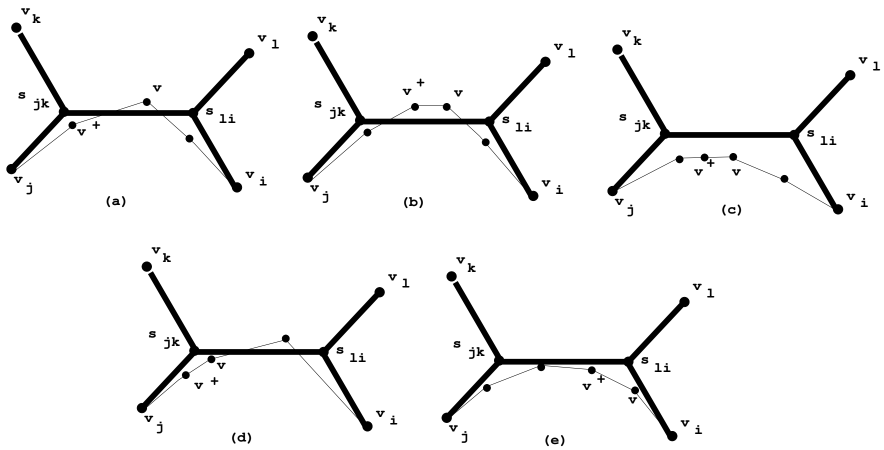


Figure 8: Searching for intersection between $s_{jk}s_{li}$ and $P_3(q_i, q_j)$

The polygonal chain connecting $s_{jk}$ and $s_{li}$ touches $P_3(q_i, q_j)$ in at least one reflex vertex.

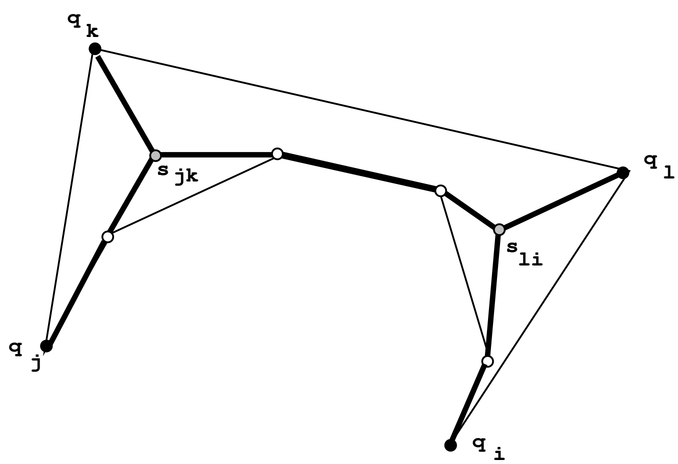


Figure 9: A tree with 2 invisible Steiner points

Consider the polygon $P_3^l$ bounded by the shortest paths $P_3(q_i, q_j)$, $P_3(q_j, q_k)$ and $P_3(q_k, q_i)$. Let $d_i(v)$, $d_j(v)$, $d_k(v)$ denote shortest distances between a vertex $v$ on $P_3^l$ and semi-terminals $q_i, q_j, q_k$, respectively. It was argued in [11] that these distances can be determined in linear time and space for *all* vertices.

We next show how to determine obstacle-avoiding full Steiner trees for $q_j, q_k$ and $v$ for *all* reflex vertices $v$ on $P_3(q_i, q_j)$. Let $z_{jk}(v)$ denote the length of the shortest such trees. Initially, $z_{jk}(v) = \infty$. Circular sweep described in [11] requires linear time and space to determine *all* full Steiner trees for triples of vertices of the 3-kite $P_3^l$. For each triple $v_i$, $v_j$ and $v_k$ admitting a full Steiner tree, let $w$ denote its length. Assume that $v_i$ is a reflex vertex of $P_3(q_i, q_j)$ closest to $q_i$. If such $v_i$ exists and

$$z_{jk}(v_i) > w + \min\{d_j(v_j), d_j(v_k)\} + \min\{d_k(v_j), d_k(v_k)\}$$

then reduce $z_{jk}(v_i)$ to the right-hand side of the above inequality.

When all full Steiner trees have been generated in this way, reflex vertices of $P_3(q_i, q_j)$ are scanned in the direction from $q_j$ to $q_i$. Suppose that $v$ is the first reflex vertex with $z_{jk}(v) \neq \infty$. Let

$$z_{jk}(v^-) = \min\{z_{jk}(v) + |vv^-|, z_{jk}(v^-)\}$$

When the scan is completed, $z_{jk}(v)$ is the length of shortest tree spanning $q_j, q_k, v$ with a Steiner point in $i(P_3^l)$.

Similar procedure can be applied to determine lengths $z_{li}(v)$ of shortest trees spanning $q_i, q_l, v$ for all reflex vertices $v$ of $P_3(q_i, q_j)$.

Once the values $z_{jk}(v)$ and $z_{li}(v)$ are given for all reflex vertices of $P_3(q_i, q_j)$, the shortest tree with two Steiner points $s_{jk}$ and $s_{li}$ invisible to each other and touching $P_3(q_i, q_j)$ can be determined in linear time by scanning reflex vertices of $P_3(q_i, q_j)$ one more time and determining $z_{jk}(v) + z_{li}(v)$.

Similar procedure can be used to determined shortest trees spanning four semi-terminals in a 4-kite such that the chain connecting its two Steiner points touches respectively $P_3(q_j, q_k)$, $P_3(q_k, q_l)$ and $P_3(q_l, q_i)$. We obtain the following lemma.

**Lemma 6** *Shortest tree spanning four vertices of a 4-kite with its two Steiner points connected by a chain touching the 4-kite can be determined in $O(n)$ time and space.*

# 6 Conclusions

We presented an $O(n \log n)$ time and $O(n)$ space algorithm for the ESTP for four terminals inside a simple polygon. There is a number of interesting issues that remain open. One is how to preprocess the polygon with any number of terminals in order to identify reasonable clusters of up to four terminals. This would lead to a general heuristic for the ESTP for any number of terminals inside a polygon. Such a heuristic would be based on the concatenation of small ESMT. Determination of ESMTs for small subsets of terminals in presence of several (convex) obstacles is also of interest. In this context, Steiner visibility graph introduced in [10] could prove useful. Finally, we mention the problem of preprocessing a simple polygon so that three and/or four terminals queries for ESMTs can be answered efficiently.

# References

[1] J.E. Beasley, A heuristic for the Euclidean and rectilinear Steiner tree problem, Working paper, Management School, Imperial College, London (1989).

[2] B.M. Chazelle, A theorem on polygon cutting with applications, *Proc. of the 23-th IEEE Symp. on Foundations of Computer Science* (1982) 339-349.

[3] B.M. Chazelle, Triangulating a simple polygon in linear time, *Disc. Comp. Geom.* **6** (1991) 485-524.

[4] L.J. Guibas and J. Stolfi, Primitives for the manipulation of general subdivisions and the computation of Voronoi diagrams, *ACM Trans. on Graphics* **4** (1985) 74-123.

[5] F.K. Hwang, D.S. Richards and P. Winter, *The Steiner Tree Problem*, North-Holland (1992).

[6] D.T. Lee and F.P. Preparata, Euclidean shortest paths in the presence of rectilinear barriers, *Networks* **14** (1984) 393-410.

[7] F.P. Preparata and M.I. Shamos, *Computational Geometry: An Introduction*, Springer, Berlin (1985).

[8] J.S. Provan, An approximation scheme for finding Steiner trees with obstacles, *SIAM J. Comput.* **17** (1988) 920-934.

[9] J. MacGregor Smith, D.T. Lee and J.S. Liebman, An $O(n \log n)$ heuristic for the Steiner minimal tree problem on the Euclidean metric, *Networks* **11** (1981) 23-29.

[10] P. Winter, Euclidean Steiner minimal trees with obstacles and Steiner visibility graphs, *Discrete Applied Mathematics* **47** (1993) 187-206.

[11] P. Winter, Euclidean Steiner minimum trees for 3 terminals in a simple polygon, Manuscript submitted for publication (1995).

[12] P. Winter and J. MacGregor Smith, Steiner minimal tree for three points with one convex polygonal obstacle, *Ann. Oper. Res.* **33** (1991) 577-599.