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Geometric Probing and Testing - A Survey

by

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ABSTRACT

Geometric probing is the area of computational geometry that studies how to identify, verify, or determine some property of an unknown geometric object using a measuring device known as a probe. It has applications in the areas of robotics, automated manufacturing, computer vision, optical character recognition and tomography. Geometric testing is the subarea of geometric probing that studies the verification problem – given a target object from a class of objects, it looks at how to find a set of probes that distinguishes the target object from all other objects in the class. In this paper we survey results in the field of geometric testing.

1 Introduction

In many application areas such as robotics, automated manufacturing, computer vision, optical character recognition and tomography an object is *probed* using some type of sensing device to learn something about it. For example, a robot equipped with a tactile sensor may touch objects in its environment to learn about them, a parallel-jaw gripper may be used to recognize different parts for an assembly task, x-rays may be studied to reconstruct the image of an organ to diagnose a tumor, or the grey-scale values of pixels of a computer image may be analyzed to determine which character is written.

Geometric probing is the area of computational geometry that examines how to identify, verify, or determine some property of an unknown geometric object using a measuring device known as a *probe*. The area of geometric probing was introduced into the computational geometry field by Cole and Yap [CY87], although probing algorithms had been studied previously in the application areas of robotics [OT77, GLP84a, GLP84b, ERH86, Gri86, Pug86b, Sch86], computer vision [CD86, Hor86, Pug86a] and tomography [Ham63, HR73, Her80, Her83, Vol86]. For a survey of the entire area of geometric probing, the reader may consult [Ski88, Ski89, Lin90, Ski92].

Geometric testing is the subarea of geometric probing that studies the verification task. That is, given a class (i.e. a set) of geometric objects and a specific *target* object in the class, the verification task is to find a set of probes that distinguishes the target object from all other objects in the class. This set of probes is called a *test set* and can be used to test whether an unknown object from the class is equivalent to the target or not. An equivalent object is one that not only has the same shape as the target but also has the same position and orientation. In general a minimum sized set of probes is desired for testing. In this paper we survey results in geometric testing.

Geometric testing is related to the *helpful teacher learning model* [GK91, SDHK91, SM91, ABCST92, JT92, Heg94] in the field of computational learning theory. This field studies learning algorithms, which are algorithms that look for an object to explain a given set of examples. In the helpful teacher learning model, the teacher has knowledge of an object and attempts to teach it using the fewest possible examples. The number of examples that a teacher must present for all students to learn an object is analogous to the notion of the number of test points needed to test an object. Much of the work in helpful teacher learning, however, has focused on learning non-geometric objects such as boolean formulas and threshold functions.

2 Geometric Testing

There are many problems in geometric testing. These problems are determined by different parameters such as the type of probe used for testing, whether the tested object must be exactly equivalent or approximately equivalent to the target, and whether the class of objects is finite or infinite. The three main types of problems that are examined in testing are (1) determining the minimum number of probes needed to test objects from a particular class,

(2) determining the complexity of finding a minimum sized set of probes for testing, and (3) giving an efficient algorithm for generating the probes.

2.1 Types of Probes

There are many different types of probes in the geometric probing literature, each of which models a different physical sensing device. We examine the following types of probes in this paper:

- *Point probe* - This probe returns "positive" if the given input point is inside or on the boundary of the object being tested, or "negative" otherwise. It models point light detectors used in automated manufacturing or pixel data in computer vision.
- *Finger probe* - This probe returns the first point of intersection between a given input ray and the tested object. It models tactile robot sensors.
- *Line probe* - This probe returns the tangent line obtained from moving a line with a given input normal parallel to itself from infinity until it intersects the object. It also models tactile robot sensors.
- *Diameter probe* - This probe returns the distance between two parallel lines, both at angle θ to the x-axis, such that the tested object lies between the two lines and touches both. It models parallel-jaw grippers used in automated manufacturing.
- *X-ray probe* - This probe returns the length of intersection between a line and the tested object. It models the data returned from an x-ray.
- *Histogram probe* - This probe returns a histogram showing the length of intersection between the tested object and a line at a given angle θ as the line is swept through the entire object. It models the data given by tomographic imaging systems.
- *Half-space probe* - This probe returns the area of intersection between a half-space and the tested object. It models probes used in tomography and remote sensing.

In higher dimensions a line probe generalizes to a $(d-1)$-dimensional hyperplane, a diameter probe generalizes to two parallel $(d-1)$-dimensional hyperplanes, and a half-space probe generalizes to return the volume of intersection. All of the other probes remain the same.

2.2 Approximate vs. Exact Testing

For some applications a testing strategy must determine whether or not a tested object is *exactly equivalent* to the target while in other applications it is sufficient to determine that the tested object is *approximately equivalent* to the target. For example, if testing is used

for parts alignment or quality assurance in an automated manufacturing application, then some error tolerance is usually allowed.

An *approximate testing strategy* is given as input not only a target object P, but also an error bound ϵ. It must produce a set of probes that accepts a tested object Q only if the error of Q with respect to P is no greater than ϵ. The *error* of a tested object Q with respect to a target object P in E^d is usually taken to be the d-dimensional volume of the symmetric difference of the two sets. A *two-sided approximate testing strategy* produces a set of probes S such that if S accepts Q then the error of Q with respect to P is no greater than ϵ, and if S rejects Q then the error of Q with respect to P is at least δ, where δ is a function of ϵ and P.

2.3 Model-Based vs. Shape-Based Testing

In model-based testing a finite library of models is given, and a testing strategy must be devised that produces a minimum number of probes to determine which model is present. The strategy can either be designed for *static testing*, producing a set of batch probes that distinguishes between all the models in the library, or for *interactive testing*, producing a decision tree to identify a model. In a decision tree the root corresponds to the set of all models, each internal node corresponds to a probe, and each leaf corresponds to a single model. By following a root-to-leaf path in the tree, using the outcomes of the probes at the internal nodes to direct the path, the model present can be identified. An *optimal* decision tree for interactive testing is a tree of minimum height that identifies all models. Although the work in model-based testing has concentrated on strategies that can identify any model in the library, the results can also be applied to the case where a specific target model is given and a testing strategy must produce probes to determine if that is the model present.

In shape-based testing an infinite class of objects is specified according to the shape of the objects. Some examples of classes are convex polygons in the plane, balls in E^d, or simple polygons with n vertices. For a given class of objects, an exact testing strategy takes as input a specific target object from the class and must determine a minimal set of probes that distinguishes the target object from all other objects in the class. If the strategy performs approximate testing, then it is also given an error bound as input, and it only needs to produce a set of probes that distinguishes objects that differ from the target by more than the error bound.

3 Point Probes

Both shape-based and model-based testing strategies using point probes have been studied.

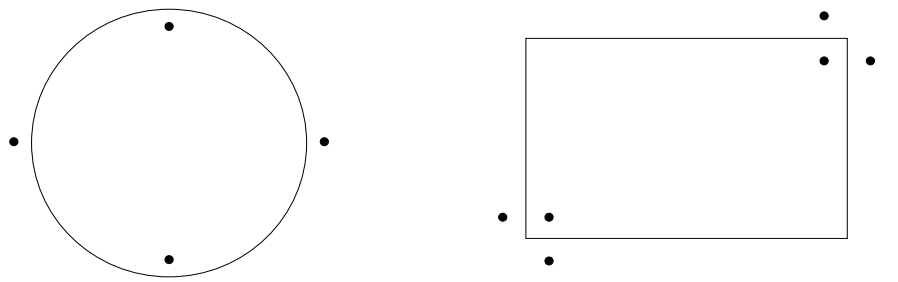
3.1 Shape-based testing with point probes

A testing strategy for an infinite, non-discretized class of objects that uses point probes must necessarily be approximate. This is because a point probe only returns the value "positive"

or "negative" and does not give information about whether the point is on the boundary of the tested object or not. Thus the boundary of a tested object cannot be pinned down exactly with point probes.

A tested object is *consistent* with the target object on a set of point probes if it returns the same value as the target on all the probes in the set. If a tested object is consistent on the probes, then it is accepted, otherwise it is rejected. If the probes produced by an approximate testing strategy accept a tested object, then its error with respect to the target is no greater than the given error bound. If a class of geometric objects has an approximate testing strategy, then the class is *testable*.

Romanik and Smith [RS94] showed that $d+2$ point probes are necessary and sufficient to test a ball in E^d. Two positive probes are chosen near the ends of one diameter of the target ball, and d evenly spaced negative probes are chosen on the $(d-1)$-dimensional plane that orthogonally bisects this diameter (see Figure 1(a)).



(a) Testing a disc in E^2 (b) Testing a rectangle in E^2

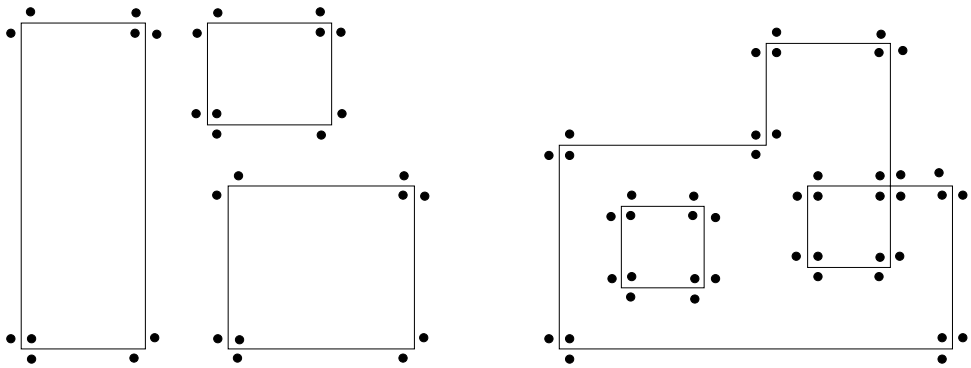
Figure 1: Testing balls and rectangles with point probes

Romanik and Smith [RS94] and Goldman and Kearns [GK91] independently showed that $2d+2$ point probes are necessary and sufficient to test an axis-parallel rectangle in E^d. Two positive probes are chosen near opposite corners of the rectangle, and $2d$ negative probes are chosen by reflecting each positive probe outside the rectangle in each dimension (see Figure 1(b)).

Romanik and Salzberg [RS95] showed that the testing strategy using $2d+2$ point probes for one rectangle can be extended to k rectangles in E^2 yielding a strategy that uses $6k$ point probes, which are also necessary. However, they showed that this strategy does not extend to k rectangles in higher dimensions, so they gave another strategy using $k(4d+2)$ point probes to test k disjoint rectangles in E^d. In this strategy each of the two positive point probes for a rectangle is reflected outside the target object in both directions for each dimension (see Figure 2(a)).

Romanik and Salzberg [RS95] also gave a testing strategy using $3n$ point probes for rectilinear shapes with or without holes in E^2 with n corners (see Figure 2(b)) and a testing strategy using $8n$ point probes for rectilinear shapes with or without holes in E^3 with n corners.

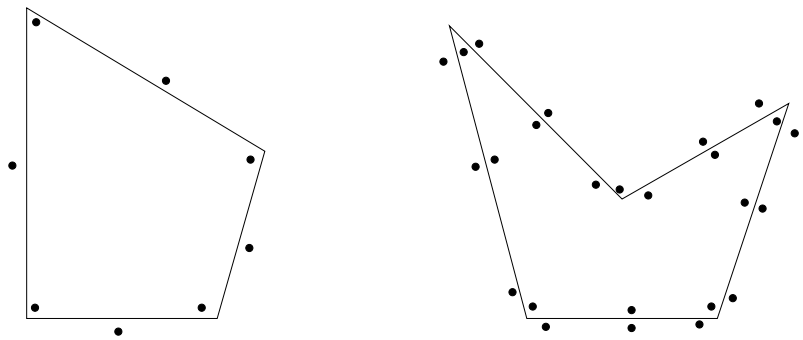
Romanik and Smith [RS94] showed that $2n$ point probes are necessary and sufficient to



(a) Testing k rectangles in E^d (b) Testing rectilinear shapes in E^2 with n corners

Figure 2: Testing rectilinear shapes with point probes

test a convex polygon in E^2, where the target polygon has n vertices. A positive probe is chosen near each vertex and a negative probe is chosen midway along each edge and just outside the target (see Figure 3(a)). This testing strategy does not extend to sets of k convex polygons in E^2. However, by adding $O(n^2)$ negative probes, one on each line segment that connects two positive probes from different polygons, a testing strategy can be obtained for a target object with n vertices.

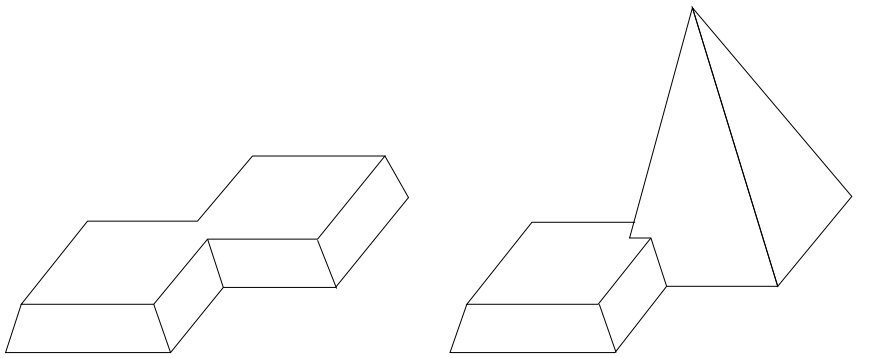


(a) Testing a convex polygon in E^2 (b) Testing a simple n-gon in E^2

Figure 3: Testing polygonal shapes with point probes

Arkin, Belleville, Mitchell, Mount, Romanik, Salzberg and Souvaine [ABM+93] proved that $5n$ point probes are sufficient to test a simple n-vertex polygon in E^2 with no three collinear vertices. The testing strategy places three collinear point probes near each vertex of the target and an additional pair of probes somewhere in the middle of each edge (see Figure 3(b)). If an additional assumption is made that no vertex is within distance δ of a nonadjacent edge, then the same $5n$ testing strategy works when the tested polygon is allowed to have n or $n+1$ vertices. However, this strategy does not generalize to an $8f$ testing strategy for simple polyhedra in E^3 with f facets. By carefully distributing 4 pairs of probes on each facet of the target polyhedron, a testing strategy can assure that any tested

polyhedron will have a facet close to it, but these facets may connect to yield a very different polyhedron (see Figure 4). Finally, if the polygons can have collinear edges, then a simple n-vertex polygon in E^2 can be tested with $6n + 6l$ point probes, where l is the number of pairs of consecutive, aligned edges of the target polygon.



(a) A 10-sided target polyhedron (b) A consistent 10-sided test polyhedron

Figure 4: Testing strategy does not work for polyhedra in E^3

Notice that all of the testing strategies illustrated in Figures 1, 2 and 3 are two-sided. If the positive probes had instead been chosen on the boundaries of the target objects, then the strategies would not be two-sided.

Romanik [Rom92] studied general properties that make a class of geometric objects testable or untestable. Characterizing properties that make a class of objects testable is difficult since adding a finite set of points to each object in a class can make it either testable or untestable. For example, the class $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ in E^1 is obviously testable since it only contains two objects. However the class $\{[0, \frac{1}{2}] \triangle p \mid p$ is any finite set of points in $[0, 1]\} \cup \{[\frac{1}{2}, 1] \triangle p \mid p$ is any finite set of points in $[0, 1]\}$, where $\triangle$ denotes the symmetric difference of the two sets, is not testable for any error bound less than 1. This is because any set of point probes used to verify an object that differs from $[0, \frac{1}{2}]$ by a finite set of points can be "fooled" by an object that differs from $[\frac{1}{2}, 1]$ by a finite set of points. However, if the class of objects is changed so that the set p of points used to define an object always contains the point $\frac{3}{4}$, then the new class is again testable, using only the point $\frac{3}{4}$ as a probe.

3.2 Model-based testing with point probes

In model-based testing, the infinite number of possible locations for a point probe can be reduced to a finite number by considering the set of regions formed by overlaying all of the models. Since two probes in the same region yield the same values for all models, a testing strategy only needs to consider at most one probe in each region. It is easy to see that the number of probes needed to distinguish among a library of m models is at least $\lceil \log m \rceil$ and no more than $m - 1$ [JE94b].

For static testing, both Belleville and Shermer [BS93] and Jia and Erdmann [JE94b] showed that it is NP-Complete to find a minimum cardinality set of point probes that can be used to distinguish among a library of m polygonal models, even if all the polygons are convex. Jia and Erdmann also gave an $O(m^2n^2)$ time approximation algorithm that produces no more than $2 \log m$ times the optimal number of probes, where n is the total number of vertices of all the models. Their algorithm works for models that are simple polygons.

Arkin, Meijer, Mitchell, Rappaport and Skiena [AMM+93] considered the problem of computing interactive testing strategies using point probes. They gave an $O(n \log m)$ time algorithm to construct an optimal $\lceil \log m \rceil$ height decision tree to distinguish among m polygonal models with a total of n vertices, where all models contain a common point and the set of models is *non-degenerate*, which means that no two models share an edge. They also proved that constructing a minimum height decision tree to distinguish among m possibly degenerate models is NP-Complete, even if all the models are convex polygons, and the problem remains NP-Complete if the models are non-degenerate but share no common point. They went on to give an $O(hn^2)$ time algorithm for constructing a decision tree of height h using a greedy heuristic. This decision tree has height no more than $\lceil \log m \rceil$ times the height of an optimal decision tree.

4 Finger Probes

Both shape-based and model-based testing strategies using finger probes have been studied. The finger probes used by most strategies originate from infinity and hit the object from the outside.

4.1 Shape-based testing with finger probes

Cole and Yap [CY87] showed that $2n$ finger probes are both necessary and sufficient to test a convex polygon where the target polygon has n vertices. The probes must contact each vertex and the interior of each edge once (see Figure 5). This result can be generalized to $n + f$ for convex polytopes in E^d, where f is the number of $(d-1)$-dimensional faces of the target.

Skiena [Ski88] extended Cole and Yap's result to two disjoint convex polygons, but showed that for $m > 2$ there exist collections of m disjoint convex polygons that cannot be verified with finger probes. For example, in Figure 6(b) edge (v_i, v_j) cannot be verified since no finger probe originating from infinity can reach it. Skiena also showed that star-shaped polygons, even with vertices in general position, cannot be verified with finger probes. A polygon P is *star-shaped* if there exists a point $z \in P$ such that for all other points $p \in P$, the line segment $\overline{zp}$ is contained in P. The star-shaped polygon in Figure 6(a) cannot be verified because the edge (v_i, v_j) cannot be verified. The dashed lines in the figure indicate another star-shaped polygon that is consistent with the probes. Finally, Skiena showed that if all convex polygons in a class have n vertices, then only $3\lceil n/2 \rceil$ probes are needed for testing.

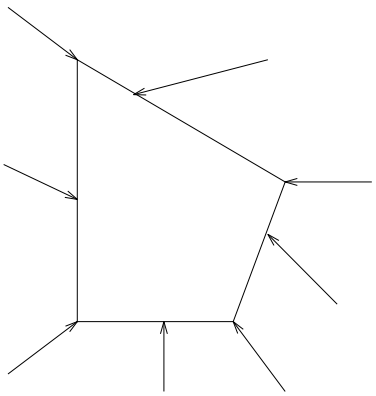
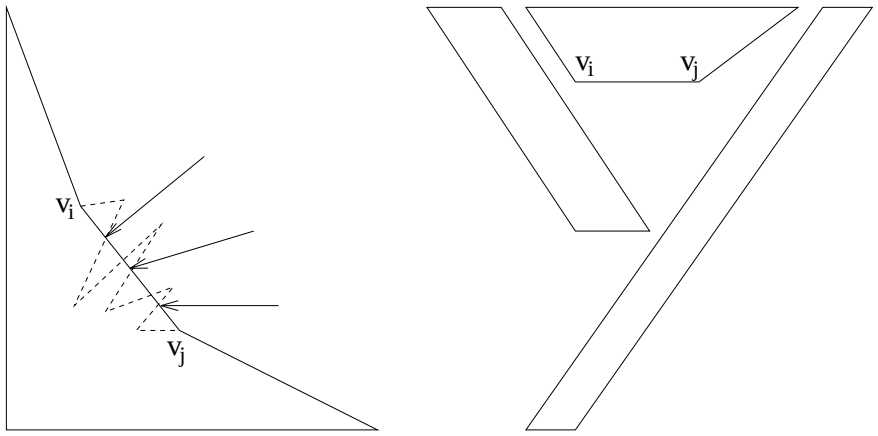


Figure 5: Testing a convex polygon in E^2 with finger probes



(a) Star-shaped polygons (b) Sets of k disjoint convex polygons

Figure 6: Objects that are untestable with finger probes

Alevizos, Boissonnat and Yvinec [ABY89] considered the problem of identifying a simple polygon using a stronger type of finger probe that is allowed to be a polygonal path instead of a straight ray and that returns not only a contact point but also the normal to the edge on which the contact point is located. They assumed that a probe collinear with an edge returns no contact point on the edge, and a probe that hits a vertex returns the normals to the two incident edges. Using this model a simple n-vertex polygon can be verified with n finger probes, one hitting each vertex.

4.2 Model-based testing with finger probes

Belleville and Shermer [BS93] showed that it is NP-Complete to find a minimum cardinality set of finger probes that can be used in static testing to distinguish among a library of m convex unimodal polygons in standard position. A convex polygon P is *unimodal* if the

distance function from a vertex of P to all other vertices in clockwise order around the boundary of P has exactly one local maximum. Given a line L and a point $p \in L$, a convex polygon P is in *standard position* with respect to L and p if p is a vertex of P and the edge of P clockwise from p is contained in L. Belleville and Shermer's result holds even if all probes must be aimed at p through at least one other vertex of some polygon in the library.

Lyons and Rappaport [LR94] showed that $m-1$ batch finger probes are both necessary and sufficient to distinguish among m n-sided convex polygons in standard position with respect to a line L and point p. Necessity in this case means that it is possible to construct m convex polygons such that any probing strategy must use at least $m-1$ probes to distinguish among the polygons. To show sufficiency they gave an $O(mn + m^2)$ time algorithm that uses $O(mn)$ space and produces a set of no more than $m-1$ probes to distinguish among the polygons. All of the probes generated by their algorithm are aimed at the point p.

For interactive testing, Bernstein [Ber86] showed that $2n+2$ finger probes are sufficient to identify one of m n-sided convex polygons. This result was improved to $n+3$ by Joseph and Skiena [JS92], who also showed that $n-1$ probes are necessary.

Natarajan [Nat87] looked at the problem of how to determine the orientation of an n-vertex object with n possible orientations using a weaker form of finger probe that only detects whether or not it intersects the object. He showed that in two dimensions $n/2$ batch probes are necessary and n are sufficient to determine the orientation, and in three dimensions $n/4$ probes are necessary and $6n$ are sufficient. Skiena [Ski88] improved his results and showed that in two dimensions $n-3$ probes are necessary and n are sufficient, and in three dimensions $n-4$ probes are necessary and $6n-12$ are sufficient. Skiena's result also holds for nonconvex objects that are distinguishable using this kind of probe.

For nonconvex polygons Skiena [Ski88] proved that $O(mn)$ finger probes are both necessary and sufficient using an interactive strategey to identify one of m n-sided star-shaped polygonal models when the orientation of the model being probed is unknown.

5 Line Probes

There is a duality between finger probes and line probes (or hyperplane probes in higher dimensions) [Gre85, DEY86, Ski88], so the $2n$ result for testing convex polygons with finger probes [CY87] also applies to line probes.

Several people have looked at the problem of approximating non-polygonal convex objects using line probes. Lindenbaum and Bruckstein [LB94] considered approximating planar convex sets where the error measure ϵ is the Hausdorff metric between the probed object and the approximating polygon. The Hausdorff metric between two planar sets A and B is defined by

$$\delta_H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\}$$

where $|x-y|$ is the Euclidean distance between the two points. They gave a lower bound of $2\sqrt{\frac{L}{2\pi\epsilon}}$ probes and an upper bound of $\sqrt{\frac{\pi}{8}}\sqrt{\frac{L}{\epsilon}}\log_2(\frac{L}{\epsilon})$ probes for approximate reconstruction

of an object with perimeter L, which implies similar bounds for verification. The upper bound for reconstruction was improved to $3\sqrt{\pi(\frac{L}{\epsilon}+2\pi)}$ probes by Richardson [Ric95].

A variation on line probes uses a half-line that pivots around a given input point until it hits the object. Two of these half-line probes pivoting in opposite directions can be combined to make a *cone probe* that touches the object on either side (see Figure 7). When an object is contained in a cone probe in this way, it is said to be *inscribed* in the cone.

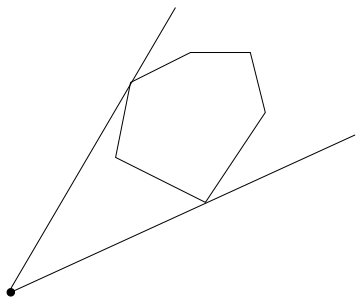


Figure 7: Probing a convex polygon with a cone probe

Jia and Erdmann [JE94a] considered the problem of determining the *pose* (i.e. the orientation and position) of a known convex polygon using cone probes. They proved that there exist at most $6n$ possible poses for a convex n-gon inscribed in two cones with distinct vertices, and they gave an $O(mn)$ algorithm to compute all possible poses for a convex n-gon inscribed in m cones.

6 Diameter Probes

For diameter probes the goal is to recognize an object using a small number of diameter measurements. In a more general setting, one can consider the problem of recognizing an object using the entire *diameter function*. This is a function $d : [0, 2\pi) \rightarrow \mathbb{R}$ that returns, for any angle, the diameter of the object at that angle. However, for every non-trivial valid diameter function d there exist infinitely many convex polygons having diameter function d [RG94] (see Figure 8). This implies that shape-based testing is not possible for convex polygons using diameter probes. Therefore, model-based testing strategies have been studied.

In model-based testing a set of polygonal parts with a total of n vertices is given, and the shortest interactive testing strategy for disambiguating the parts is sought. Each internal node of the decision tree for the testing strategy gives an angle for the next diameter probe. A particular orientation for each part is not assumed, but the only diameter measurements allowed are those in which the part is in a *stable grasp* by the gripper (see [RG94] for a definition of a stable grasp by a parallel-jaw gripper). Therefore, when a diameter measurement is made (i.e. when the part is gripped) it is assumed that the part moves until it is in a stable grasp. There are at most n stable grasps on an n-sided convex polygon [RG94].

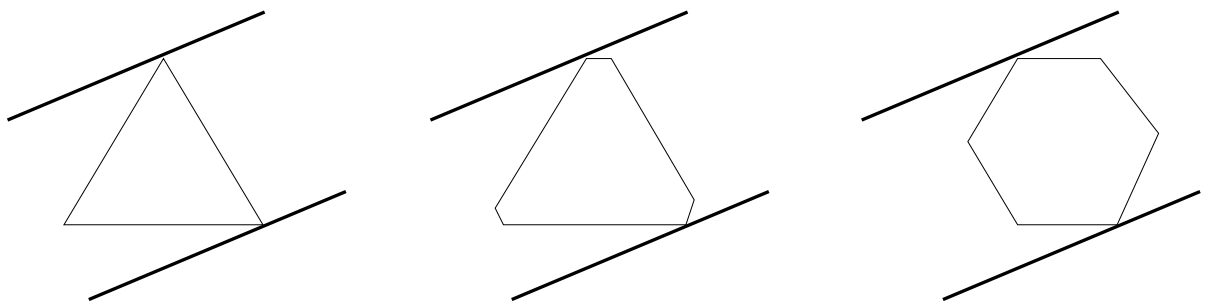


Figure 8: Triangle and hexagons with the same diameter function

Arkin, Held, Mitchell and Skiena [AHMS94] showed that it is NP-Complete to find an optimal decision tree to identify one of a library of distinguishable convex polygonal models using diameter probes. Rao and Goldberg [RG94] gave an $O(n^4 2^n)$ algorithm to construct an optimal decision tree and an $O(n^2 \log n)$ algorithm to construct a suboptimal decision tree that uses no more than n diameter probes to identify any model. Their algorithms also work for nonconvex models if the convex hulls of the models are distinguishable with diameter probes.

7 X-Ray Probes

No work has been done on model-based testing with x-ray probes although the techniques used by Natarajan [Nat87] and Skiena [Ski88] to determine the orientation of an object using a weaker form of finger probe that only detects whether it intersects the object can also be used with x-ray probes since they are a stronger form of these probes.

For shape-based testing Skiena [Ski88] showed that n x-ray probes are necessary and $(3n/2) + 6$ are sufficient to test a convex polygon when the target polygon has n vertices. He conjectured that $(3n/2) + k$ probes, for some constant k, are also necessary.

8 Histogram Probes

Giering [Gie62] proved that three histogram probes are sufficient to verify any convex set, and Gardner [Gar83] showed that three are necessary.

Meijer and Skiena [MS93] considered the problems of determining and verifying simple polygons in E^2 using histogram probes. The loss of convexity leads to complications since there may exist invisible vertices in the histogram (see Figure 9). If the number of vertices of the tested polygon is unknown, then no finite number of probes suffices. However, if all polygons in the class have no more than n vertices, then they proved that any two polygons in the class that are consistent with the results of any $n(n-1)/2 + 1$ histogram probes are identical. Moreover, they gave an interactive probing strategy for determining a polygon

from this class using $2n + 2$ histogram probes, which implies an upper bound of $2n + 2$ for verification.

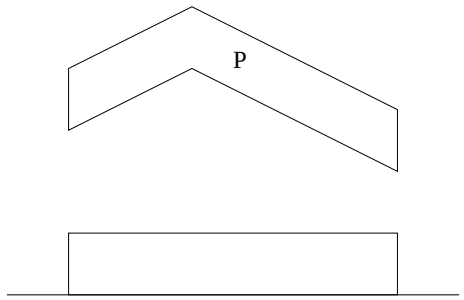


Figure 9: Invisible vertices in the histogram of a simple polygon P

9 Half-Space Probes

Not much work has been done on testing with half-space probes. Skiena [Ski88] showed that $2n/3$ half-space probes are necessary and $n + 1$ are sufficient to test a convex polygon when the target polygon has n vertices. The upper bound is achieved by probing in both directions of the defining line of one of the edges of the target polygon and then probing once along the defining line for each of the remaining edges of the target. This strategy can be generalized to convex polytopes in d dimensions yielding an upper bound of $f + 1$ probes, where f is the number of $(d - 1)$-dimensional faces of the target.

10 Conclusions

We have surveyed results in the area of geometric testing, which studies how to devise testing strategies that produce a set of probes to either distinguish between a finite set of models or to verify an unknown object from an infinite class with respect to a target object. Strategies for both batch and interactive testing can be devised, as well as for approximate and exact testing. There are still many interesting open problems in this area.

References

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Object Class	**Probe**	**Lower Bound**	**Upper Bound**
Balls in E^d	Point	$d+2$	$d+2$
Axis-parallel rectangles in E^d	Point	$2d+2$	$2d+2$
Sets of k disjoint axis-parallel rectangles in E^2	Point	$6k$	$6k$
Sets of k disjoint axis-parallel rectangles in E^d	Point	$k(2d+2)$	$k(4d+2)$
Rectilinear shapes with n corners in E^2	Point	n	$3n$
Rectilinear shapes with n corners in E^3 (f = # of faces)	Point	f	$8n$
Convex polygons in E^2 (n = # of vertices)	Point	$2n$	$2n$
Simple n-gons in E^2 with vertices in general position	Point	$2n$	$5n$
Sets of k disjoint convex polygons in E^2 (n = # of vertices)	Point	$2n$	$O(n^2)$
Convex polygons in E^2 (n = # of vertices)	Finger	$2n$	$2n$
Convex n-gons in E^2	Finger	n	$3\lceil n/2 \rceil$
Convex polytopes in E^d (n = # of vertices) (f = # of $(d-1)$-dimensional faces)	Finger	$n+f$	$n+f$
Convex polygons in E^2 (n = # of vertices)	X-ray	n	$(3n/2)+6$
Convex sets in E^2	Histogram	3	3
Simple polygons in E^2 with at most n vertices	Histogram	3	$2n+2$
Convex polygons in E^2 (n = # of vertices)	Half-space	$2n/3$	$n+1$
Convex polytopes in E^d (f = # of $(d-1)$-dimensional faces)	Half-space	2	$f+1$

Figure 10: Summary of bounds for shape-based testing

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