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Some New Graph Labeling Problems: A Preliminary Report

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ABSTRACT

All "labelings" discussed in this paper are vertex labelings. We call a labeling l of G an N-labeling if there exists a connected induced subgraph H of G with $\sum_{x\in V(H)} l(x) = i$ for every integer i, $1 \leq i \leq N$. Then $\sigma_c(G)$ is the largest integer N such that G has an N-labeling. A labeling l of G is called *irredundant* if, for all connected induced subgraphs H_1 and H_2, $\sum_{x\in V(H_1)} l(x) = \sum_{x\in V(H_2)} l(x)$ if and only if $H_1 = H_2$. Then $\sigma_p(G)$ is the smallest integer N such that there exists an irredundant labeling l of G with $\sum_{x\in V(G)} l(x) = N$. In this report, we study $\sigma_c(G)$ and $\sigma_p(G)$ in the cases where G is a complete graph minus one edge, a star, a path, or a cycle. We obtain some preliminary results, but there are many interesting questions left open.

1 Introduction

Let G be a connected simple graph. We define $\sigma_c(G)$, the *stamp-covering number* of G, and $\sigma_p(G)$, the *stamp-packing number* of G, as follows.

A *vertex labeling* of G is simply a mapping from $V(G)$ to Z^+. Since we will discuss no other kinds of labelings in this paper, we will simply use "labeling" henceforth. Let N be a positive integer. We call a labeling l of G an N-labeling if there exists a connected induced subgraph H of G with $\sum_{x \in V(H)} l(x) = i$ for every integer i, $1 \leq i \leq N$. Then $\sigma_c(G)$ is the largest integer N such that G has an N-labeling.

A labeling l of G is called *irredundant* if, for all connected induced subgraphs H_1 and H_2, $\sum_{x \in V(H_1)} l(x) = \sum_{x \in V(H_2)} l(x)$ if and only if $H_1 = H_2$. Then $\sigma_p(G)$ is the smallest integer N such that there exists an irredundant labeling l of G with $\sum_{x \in V(G)} l(x) = N$.

The quantity $\sum_{x \in V(H)} l(x)$ will sometimes be referred to as the *label sum* of H.

The terms "stamp-covering" and "stamp-packing" are derived from a statement of some special cases of the problem of determining $\sigma_c(G)$ [4]. The problems in this paper are related to the well-studied "postage stamp" problem (see [1]), but only indirectly. For another example of a linkage between the "postage stamp" problem and graph labeling, see [3]. For a survey of results on graph labeling and a good bibliography, see [2].

Note that if $c(G)$ denotes the number of non-empty connected induced subgraphs G, then $\sigma_c(G) \leq c(G) \leq \sigma_p(G)$, and that the first inequality is strict if and only if the second is strict. While graphs G for which $\sigma_c(G) = \sigma_p(G)$ seem to be rare, we do have infinitely many such graphs, since for all positive integers n, $\sigma_c(K_n) = 2^n - 1 = \sigma_p(K_n)$, where K_n denotes the complete graph on n vertices; this is shown by labeling the vertices $1, 2, \ldots, 2^{n-1}$.

In this report, we study $\sigma_c(G)$ and $\sigma_p(G)$ in the cases where G is a complete graph minus one edge, a star, a path, or a cycle. We obtain some preliminary results, but there are many interesting questions left open.

2 Almost Complete Graphs

We begin with a relatively simple case, $K_n - e$, the graph on n vertices obtained by deleting one edge from K_n. The cases $n = 1$ and $n = 2$ are trivial, and it is easy to show that $\sigma_c(K_3 - e) = 6 = \sigma_p(K_3 - e)$.

Theorem 1 *If $n \geq 4$, then $\sigma_c(K_n - e) = 2^n - 3$ and $\sigma_p(K_n - e) = 2^n - 1$.*

Proof: To see that $\sigma_c(K_n - e) \geq 2^n - 3$, use the labels $1, 2, \ldots, 2^{n-2}, 2^{n-1} - 2$, where the vertex labelled 1 is non-adjacent to the vertex labelled $2^{n-1} - 2$, and the other labels are assigned arbitrarily. To see that $\sigma_p(K_n - e) \leq 2^n - 1$, use the labels $1, 2, \ldots, 2^{n-1}$, assigned arbitrarily.

To show that, in fact, equality holds, we shall let $c_n = c(K_n - e) = 2^n - 2$, and show that $K_n - e$ has no c_n-labeling. Suppose that $K_n - e$ has a c_n-labeling, l. This labeling must also be irredundant, so, in particular, no label can be repeated. Let $V(K_n - e) = \{x_1, x_1, \ldots, x_n\}$, and assume that $l(x_1) < l(x_2) < \ldots < l(x_n)$. It is easy to see by induction that $l(x_i) \leq 2^{i-1}$, because otherwise it would be impossible to find an induced subgraph H (connected or not) where $\sum_{x \in V(H)} l(x) = 2^{i-1}$. Since $\sum_{x \in V(K_n - e)} l(x) = 2^n - 2$, there must be a unique vertex x_i such that $l(x_i) < 2^{i-1}$, and, in fact, $l(x_i) = 2^{i-1} - 1$. Clearly, $i = 3$, or else $\sum_{x \in V(H_1)} l(x) = \sum_{x \in V(H_2)} l(x) = 2^{i-1} - 1$, where H_1 is the subgraph induced by $\{x_1, \ldots, x_{i-1}\}$ and H_2 is the subgraph induced by $\{x_i\}$. Also, $x_1 \not\sim x_2$. But then, since $n \geq 4$, we have $\sum_{x \in V(H_1)} l(x) = \sum_{x \in V(H_2)} l(x) = 7$, where H_1 is the subgraph induced by $\{x_1, x_2, x_4\}$ and H_2 is the subgraph induced by $\{x_3, x_4\}$. This contradicts the assumption that l is a c_n-labeling, and is therefore irredundant, so we conclude that no c_n-labeling for $K_n - e$ exists if $n \geq 4$. $\square$

3 Stars

By a star, we mean $K_{1,n}$, the tree on $n+1$ vertices consisting of a vertex of degree n and n vertices of degree 1. Note that $c(K_{1,n}) = 2^n + n$. The main result of this section is that $\sigma_c(K_{1,n}) = 2^n + 2$. We will also briefly discuss $\sigma_p(K_{1,n})$.

Theorem 2 *For all positive integers $n \geq 2$, $\sigma_c(K_{1,n}) = 2^n + 2$.*

Proof: We adopt the convention that the vertex of degree n is x_0, the leaves are $x_1, \ldots, x_n$, and $l(x_1) \leq l(x_2) \leq \ldots \leq l(x_n)$. To see that $2^n + 2 \leq \sigma_c(K_{1,n})$ when $n \geq 2$, consider the labeling $l(x_0) = 3$, and $l(x_i) = 2^{i-1}$ for $1 \leq i \leq n$.

Let $N = 2^n + 3$. We shall prove that there is no N-labeling for $K_{1,n}$. Computer searches reveal that this claim is true if $n \leq 5$, so we shall assume that $n \geq 6$. This assumption will be helpful because it implies that $n - 2 \leq 2^{n-4}$, a useful fact when we argue by contradiction. If l is an N-labeling of $K_{1,n}$, there cannot be disjoint subsets $S, T \subseteq \{x_i : 1 \leq i \leq n\}$ such that $|S| + |T| \leq 4$ and $\sum_{x \in S} l(x) = \sum_{x \in T} l(x)$, because then we would be able to find 2^{n-4} pairs of connected induced subgraphs of $K_{1,n}$ with equal label sums, forcing $N \leq 2^n + n - 2^{n-4} \leq 2^n + n - (n-2) = 2^n + 2$. In particular, if l is an N-labeling and $1 \leq i < j \leq n$, then $l(x_i) < l(x_j)$, a fact we will repeatedly use below without further comment.

Suppose that l is an N-labeling of $K_{1,n}$. We will show that this leads to a contradiction. First note that $l(x_0) < 4$, because otherwise $\{1, 2, 3\} \subset \{l(x_i) : 1 \leq i \leq n\}$, which is impossible by our observation. So we divide the argument into three cases: $l(x_0) = 1$, $l(x_0) = 2$, and $l(x_0) = 3$. We take these cases in reverse order.

Suppose $l(x_0) = 3$. We shall argue that $l(x_i) \leq 2^{i-1}$ for $1 \leq i \leq n$. It follows that $\sum_{x \in V(K_{1,n})} l(x) < N$, so l is not an N-labeling, a contradiction. Clearly, we must have $l(x_1) = 1$ and $l(x_2) = 2$. Suppose that $l(x_i) \leq 2^{i-1}$ for $1 \leq i < j$, but $l(x_j) > 2^{j-1}$. Since l is an N-labeling we must be able to find a connected induced subgraph H such that $\sum_{x \in V(H)} l(x) = 2^{j-1} + 3$. Clearly this can only occur if $l(x_m) = 2^{j-1} + 3$ for some m, but then $2^{j-1} + 1$ and $2^{j-1} + 2$ cannot appear as labels by our earlier observation, so $l(x_j) = 2^{j-1} + 3$. But then $2^{j-1} + 4$ cannot appear as a label, and it is then impossible to find a connected induced subgraph H such that $\sum_{x\in V(H)} l(x) = 2^{j-1}+4$, contradicting the assumption that l is an N-labeling.

The argument for the case where $l(x_0) = 2$ is similar, and is left to the reader.

Now suppose $l(x_0) = 1$. Note that either $l(x_1) = 1$, or $l(x_1) = 2$. We shall first consider when $l(x_1) = 2$, and we will break this into the two subcases, $l(x_2) = 3$ and $l(x_2) = 4$.

Suppose $l(x_0) = 1$, $l(x_1) = 2$, and $l(x_2) = 3$. In order to have a connected induced subgraph with label-sum 5, we must have $l(x_3) \le 5$, but by our observation $l(x_3) \ne 5$, so $l(x_3) = 4$. We claim that, in fact, $l(x_i) \le 2^{i-1}$ for $3 \le i \le n$; it follows that $\sum_{x\in V(K_{1,n})} l(x) < N$, so l is not an N-labeling, a contradiction. We have alreday shown the case $i = 3$. Suppose that $l(x_i) \le 2^{i-1}$ for $3 \le i < j$, but $l(x_j) > 2^{j-1}$. Clearly, no subset of $\{x_0, x_1, \ldots, x_{j-1}\}$ induces a connected subgraph H with $\sum_{x\in V(H)} l(x) = 2^{j-1}+1$, so we must have $l(x_j) = 2^{j-1}+1$. Note that this forces $l(x_{j+1}) \ne 2^{j-1}+2$ (or else $l(x_2)+l(x_j) = l(x_1)+l(x_{j+1})$), and $l(x_{j+1}) \ne 2^{j-1}+3$ (or else $l(x_1)+l(x_j) = l(x_{j+1})$). But then there is no connected induced subgraph H such that $\sum_{x\in V(H)} l(x) = 2^{j-1}+3$, contradicting the assumption that l is an N-labeling.

Now suppose $l(x_0) = 1$, $l(x_1) = 2$, and $l(x_2) = 4$. In order to have a connected induced subgraph with label-sum 6, we must have $l(x_3) \le 6$, but by our observation $l(x_3) \ne 6$, so $l(x_3) = 5$. We claim that $l(x_i) \le 2^{i-1}$ for $4 \le i \le n$; it follows that $\sum_{x\in V(K_{1,n})} l(x) \le N+1$. Note that there is no connected induced subgraph H such that $\sum_{x\in V(H)} l(x) = (\sum_{x\in V(K_{1,n})} l(x)) - 1$, so l is not an N-labeling, a contradiction. We prove our claim by induction, beginning with the base $i = 4$. In order to have a connected induced subgraph with label-sum 9, we must have $l(x_4) \le 9$, but by our observation $l(x_4) \ne 9$, so $l(x_4) \le 8$. Suppose that $l(x_i) \le 2^{i-1}$ for $4 \le i < j$, but $l(x_j) > 2^{j-1}$. Then there exists a positve integer $m \le 2j-1+1$ such that no subset of $\{x_0, x_1, \ldots, x_{j-1}\}$ induces a connected subgraph H with $\sum_{x\in V(H)} l(x) = m$. Thus, we must have $l(x_j) = 2^{j-1}+1$. Note that this forces $l(x_{j+1}) \ne 2^{j-1}+2$ (or else $l(x_3)+l(x_j) = l(x_2)+l(x_{j+1})$), and $l(x_{j+1}) \ne 2^{j-1}+3$ (or else $l(x_1)+l(x_j) = l(x_{j+1})$). But then there is no connected induced subgraph H such that $\sum_{x\in V(H)} l(x) = 2^{j-1}+3$, contradicting the assumption that l is an N-labeling. The completes the argument for the case $l(x_1) = 2$.

We shall assume henceforth that $l(x_1) = 1$. Clearly, either $l(x_2) = 2$, or $l(x_2) = 3$. In the latter case, the argument subdivides further into the cases $l(x_3) = 4$ and $l(x_3) = 5$. If $l(x_3) = 4$, then by our initial observation, $l(x_4) \notin \{5, 6, 7, 8\}$. But then we cannot have a connected subgraph H such that $\sum_{x\in V(H)} l(x) = 7$. On the other hand, if $l(x_3) = 5$, $l(x_4) \notin \{6, 7, 8, 9\}$, which implies that we cannot have a connected subgraph H such that $\sum_{x\in V(H)} l(x) = 8$. In either of these subcases, we have that l is not an N-labeling, a contradiction.

So now we assume that $l(x_0) = 1$, $l(x_1) = 1$ and $l(x_2) = 2$. If $l(x_i) \le 2^{i-1}$ for $1 \le i \le n$, then clearly l is not an N-labeling. So suppose there exists a positive integer j such that $l(x_i) \le 2^{i-1}$ for $1 \le i < j$, but $l(x_j) > 2^{j-1}$. We shall show that this leads to a contradiction.

Since $\sum_{i=0}^{j-1} l(x_i) \le 2^j$, $l(x_j) = 2^{j-1}+1$. If $j = n$, then $\sum_{i=0}^{n} l(x_i) < N$, so l is not an N-labeling, a contradiction. If $j = n-1$, then $l(x_n) \le 2^j+2$. If $l(x_n) < 2^j+2$, then $\sum_{i=0}^{n} l(x_i) < N$, so we must have $l(x_n) = 2^j+2$, and $\sum_{i=0}^{n} l(x_i) = N$. But, then there is no connected induced subgraph H such that $\sum_{x\in V(H)} l(x) = 2^{j-1} + 2^j + 3$, contradicting the assumption that l is an N-labeling.

So assume that $j < n-1$. We claim that $l(x_{j+1}) \leq 2l(x_j)$ and that for $j+2 \leq m \leq n$, $l(x_m) \leq \sum_{i=j}^{m-1} l(x_i)$. These claims imply that $\sum_{i=0}^{n} l(x_i) < N$, as the reader may easily verify. This means l is not an N-labeling, a contradiction. Thus, the theorem will have been proved when these two claims are shown.

The fact that $l(x_{j+1}) \leq 2l(x_j)$ follows easily from the fact that $\sum_{i=0}^{j} l(x_i) < 2^j+2 = 2l(x_j)$. To see that $l(x_m) \leq \sum_{i=j}^{m-1} l(x_i)$ for $j+2 \leq m \leq n$, suppose the inequality does not hold for some m. Then there is no connected induced subgraph H such that $\sum_{x\in V(H)} l(x) = (\sum_{i=0}^{m-1} l(x_i)) - 2^j$, so l is not an N-labeling, a contradiction. □

The problem of determining even an asymptotic formula for $\sigma_p(K_{1,n})$ appears to be more difficult. Labeling the central vertex $2^{n-1}+1$ and the leaves $1, 2, \ldots, 2^{n-1}$ shows that $\sigma_p(K_{1,n}) \leq 2^n + 2^{n-1}$. However, this is known to be best possible only for $n \leq 4$. A computer search shows that $\sigma_p(K_{1,5}) = 46$; this is realized by labeling the central vertex 11, and the leaves 1,2,4,9, and 19.

4 Paths

In this section, we discuss $\sigma_c(P_n)$ and $\sigma_p(P_n)$, where P_n denotes the path on n vertices. We show that $\sigma_c(P_n)$ is bounded above and below by a quadratic in n, and that $\sigma_p(P_n)$ is, at most, cubic.

Theorem 3 *For positive integers $n \geq 4$*

$$\frac{n^2+6n-4}{4} \leq \sigma_c(P_n) \leq \binom{n+1}{2} - 1.$$

Proof: We shall adopt the convention that $V(P_n) = \{x_i : 1 \leq i \leq n\}$, with the subscripts reading from left to right along the path.

First we prove the lower bound. If n is even, let $l(x_i) = 1$ for $1 \leq i \leq \frac{n}{2}$, $l(x_i) = \frac{n}{2}+2$ for $\frac{n}{2} < i < n$, and $l(x_n) = \frac{n}{2}+1$. It is easily verified that l is an $\frac{n^2+6n-4}{4}$-labeling. If n is odd, let $l(x_i) = 1$ for $1 \leq i \leq \frac{n-1}{2}$, $l(x_i) = \frac{n-1}{2}+2$ for $\frac{n-1}{2} < i < n$, and $l(x_n) = \frac{n-1}{2}+1$. It is easily verified that l is an $\frac{n^2+6n-3}{4}$-labeling. Thus $\frac{n^2+6n-4}{4} \leq \sigma_c(P_n)$

We now prove the upper bound for $n \geq 4$. Note that $c(P_n) = \binom{n+1}{2}$. Thus, to prove the upper bound, we will show that P_n cannot have an $\binom{n+1}{2}$-labeling.

Suppose there is an $\binom{n+1}{2}$-labeling l for P_n. Note that l must be irredundant, and, in particular, one-to-one. Moreover, it is easily seen that

$$\sum_{1\leq i\leq n} l(x_i) = \binom{n+1}{2}.$$

It follows that $\{l(x_i) : 1 \le i \le n\} = \{1, 2, \ldots, n\}$. Since $\binom{n+1}{2} - 1$ and $\binom{n+1}{2} - 2$ must be label sums for some connected induced subgraphs, we may assume without loss of generality that $l(x_1) = 1$ and $l(x_n) = 2$. We then have $l(x_2) = n$, or else $l(x_1) + l(x_2) = l(x_i)$ for some i. Since $n \ge 4$, $x_{n-1} \ne x_2$, so $l(x_{n-1}) + l(x_n) \le n+1$, which implies that l is not irredundant, a contradiction. □

In fact, an upper bound of $\binom{n+1}{2} - 2$ can be proven for $n \ge 5$, but we have chosen to omit that proof. It was proved in [5] that $\sigma_c(P_5) = 13$, and computer searches have shown that $\sigma_c(P_6) = 18$, $\sigma_c(P_7) = 24$, and $\sigma_c(P_8) = 29$. Even if an exact formula for $\sigma_c(P_n)$ is not found, it would be interesting to find a constant k such that $\sigma_c(P_n)$ is asymptotic to kn^2.

It would be of interest to determine whether there is an upper bound on $\sigma_p(P_n)$ which is quadratic in n. At present, we have the folowing cubic bound.

Theorem 4 *For all positive integers n, $\sigma_p(P_n) \le \frac{n^3-n^2+2n}{2}$.*

Proof: Let $V(P_n) = \{x_1, \ldots, x_n\}$. We claim that the labeling $l(x_i) = 1 + (i-1)n$ is irredundant. If $\sum_{x\in V(H_1)} l(x) = \sum_{x\in V(H_2)} l(x)$ for connected induced subgraphs H_1 and H_2, then obviously $\sum_{x\in V(H_1)} l(x) \equiv \sum_{x\in V(H_2)} l(x) \bmod n$. By our labeling, this implies $|V(H_1)| = |V(H_2)|$. But then it is easy to see that $H_1 = H_2$. Moreover, $\sum_{x\in V(P_n)} l(x) = \frac{n^3-n^2+2n}{2}$. □

In fact, this bound can be improved, because, for example one may use the labeling $l(x_i) = 1 + (i-1)n$, $1 \le i \le 2n$ for P_{2n} to obtain an irredundant labeling. However, since this still results in a cubic bound, we shall omit the details.

We suspect a quadratic upper bound for $\sigma_p(P_n)$ is possible. Computer searches have revealed that $\sigma_p(P_5) = 17$, $\sigma_p(P_6) = 25$, $\sigma_p(P_7) = 34$, $\sigma_p(P_8) = 44$, and $\sigma_p(P_9) \le 55$. Also, we investigated the "greedy" method for finding an irredundant labeling for P_n: let $l(x_1) = 1$, and, if $l(x_i)$, $1 \le i < j$, have been chosen, let $l(x_j)$ be the smallest positive integer such that l restricted to $\{x_i : 1 \le i \le j\}$ is irredundant. This method yields the following sequence, which may be growing slowly enough to yield a quadratic upper bound on $\sigma_p(P_n)$: $1, 2, 4, 5, 8, 10, 14, 21, 15, 16, 26, 25, 34, 22, 48, 38, 71, 40, 74, 90, 28, 69$. Note that this sequence shows the bound from Theorem 4 is far from tight for $n \le 22$.

5 Cycles

Perhaps the most intriguing case we have investigated to date is C_n, the cycle on n vertices. It is interesting to see that $\sigma_c(C_n) = \sigma_p(C_n)$ for some, but not all, small values of n.

Theorem 5 *If $n \in \{3, 4, 5, 6, 8, 9\}$, $\sigma_c(C_n) = n(n-1) + 1 = \sigma_p(C_n)$.*

Proof: Let $V(C_n) = \{x_1, \ldots x_n\}$, and let $N = n(n-1) + 1$. We shall give an explicit N-labeling, l, for each of the values of n given in the statement of the theorem.

For $n = 3$, let $l(x_1) = 1$, $l(x_2) = 2$, $l(x_3) = 4$.

For $n=4$, let $l(x_1)=1$, $l(x_2)=2$, $l(x_3)=6$, $l(x_4)=4$.

For $n=5$, let $l(x_1)=1$, $l(x_2)=3$, $l(x_3)=10$, $l(x_4)=2$, $l(x_5)=5$.

For $n=6$, let $l(x_1)=1$, $l(x_2)=2$, $l(x_3)=5$, $l(x_4)=4$, $l(x_5)=6$, $l(x_6)=13$.

For $n=8$, let $l(x_1)=1$, $l(x_2)=2$, $l(x_3)=10$, $l(x_4)=19$, $l(x_5)=4$, $l(x_6)=7$, $l(x_7)=9$, $l(x_8)=5$.

For $n=9$, let $l(x_1)=1$, $l(x_2)=2$, $l(x_3)=4$, $l(x_4)=8$, $l(x_5)=16$, $l(x_6)=5$, $l(x_7)=18$, $l(x_8)=9$, $l(x_9)=10$. □

Remarkably, $\sigma_c(C_7) < \sigma_p(C_7)$; according to our computer searches, $\sigma_c(C_7) = 39$ and $\sigma_p(C_7) = 48$. We do not know the exact values of our two parameters for $n \geq 10$. It would be of great interest to know whether there exist infinitely many values of n for which $\sigma_c(C_n) = \sigma_p(C_n)$.

Our only general results on $\sigma_c(C_n)$ and $\sigma_p(C_n)$ are derived from the constructions we used to obtain bounds on $\sigma_c(P_n)$ and $\sigma_p(P_n)$. Better bounds, whose existence is suggested by our results for small values of n, would be very desirable.

Theorem 6 *For all positive integers n, $\frac{n^2+6n-4}{2} \leq \sigma_c(C_n)$ and $\sigma_p(C_n) \leq n^3 - 4n^2 + 7n - 3$.*

Proof: To see the lower bound on $\sigma_c(C_n)$, use the labeling of P_{n-1} from the proof of the lower bound in Theorem 3 to label $x_1, \ldots, x_{n-1}$, and label x_n $\frac{n^2+6n-4}{4} + 1$.

To see the upper bound on $\sigma_p(C_n)$, use the labeling of P_{n-1} from the proof of Theorem 4 to label $x_1, \ldots, x_{n-1}$, and label x_n $\frac{n^3-n^2+2n}{2} + 1$. □

Note that we have actually shown that $\sigma_p(C_n) \leq 2\sigma_c(P_{n-1}) + 1$.

6 Open Problems

We have already indicated several interesting questions, all of which involve computing (or estimating) σ_c and σ_p for some special cases. We now propose three general questions.

Question 1 *Is there a positive constant k_0 such that $k_0 c(G) \leq \sigma_c(G)$ for all connected graphs G?*

Question 2 *Is there a constant k_1 such that $\sigma_p(G) \leq k_1 c(G)$ for all connected graphs G?*

Question 3 *Is there an integer N such that if G is a connected graph on $n \geq N$ vertices with $\sigma_c(G) = \sigma_p(G)$, then $G = K_n$?*

Finally, we pose a question that was suggested by one of our early attempts to establish a lower bound on $\sigma_c(C_n)$. It has been verified that the answer is affirmative for $n \leq 22$.

Question 4 *Let $N = \frac{n^2+n}{2}$. Is it true that for every integer $n \geq 3$ there exists a one-to-one N-labeling l of C_n such that $l(x) \leq n$ for all $x \in V(C_n)$?*

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