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Defective Coloring of Toroidal Graphs and Graphs of Bounded Genus

by

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ABSTRACT

A graph is (k, d)-colorable if one can color the vertices with k colors such that no vertex is adjacent to more than d vertices of the same color. In this paper we investigate the existence of such colorings in surfaces. It is shown that a toroidal graph is $(3, 2)$- and $(5, 1)$- colorable, and that a graph of genus γ is $(\chi_\gamma/(d+1)+4, d)$-colorable, where χ_γ is the maximum chromatic number of a graph embeddable on the surface of genus γ.

1 Introduction

We define a (k, d)-coloring of a graph as a coloring of the vertices with k colors such that each vertex has at most d neighbors of its same color. For a graph G we define $\chi_d(G)$ as the minimum k such that there is a (k, d)-coloring of G. So a $(k, 0)$-coloring is a proper coloring, and $\chi_0(G)$ is the usual vertex chromatic number of the graph.

The parameter $\chi_d(G)$, also called the *defective chromatic number* of G, has been well-studied for planar graphs (see, for example, [5, 8, 10, 14, 16]). In this paper, we extend the study of $\chi_d(G)$ to graphs embeddable on the torus, and surfaces of higher genera.

1.1 Previous Work

Defective coloring was introduced almost simultaneously by Burr and Jacobson (see [1]), Harary and Jones [11] and Cowen, Cowen and Woodall [5], and has continued to be an active area of research. Surveys of this and related colorings are given in [8] and [16]. Cowen, Cowen and Woodall [5] focussed on graphs embedded on surfaces and gave a complete characterization of all k and d such that every outerplanar graph is (k, d)-colorable, and a complete characterization of all k and d such that every planar graph is (k, d)-colorable. Namely, every outerplanar graph is 3-colorable, every outerplanar graph is $(2, 2)$-colorable, and there exist outerplanar graphs which are not $(2, 1)$-colorable. There does not exist a d such that every planar graph is $(1, d)$- or $(2, d)$-colorable; there exist planar graphs which are not $(3, 1)$-colorable, but every planar graph is $(3, 2)$-colorable. Together with the $(4, 0)$-coloring implied by the 4-Color Theorem, this solves defective chromatic number for the plane. Recently, Poh [14] and Goddard [10] showed that any planar graph has a special $(3, 2)$-coloring in which each color class is a linear forest (no cycles), though this can in fact be read out of a more general result of Woodall [16, Theorem 2.2].

For general surfaces, it was shown in [5] that for each genus $g \geq 0$, there exists a $k = k(g)$ such that every graph on the surface of genus g is $(4, k)$-colorable. This was improved to $(3, k)$-colorable by Archdeacon [2].

For general graphs, a result of Lovász from the 1960s [13], which has been rediscovered many times since (cf. [3, 4, 12, 16]), provides an upper bound on the defective chromatic number of a graph.

Theorem 1.1 (Lovász) *For any k, any graph of maximum degree Δ can be $(k, \lfloor \Delta/k \rfloor)$-colored.*

Proof. Take a k-coloring that minimizes the number of monochromatic edges. This must be the desired coloring, because if not, consider some vertex v with more than $\lfloor \Delta/k \rfloor$ of its neighbors self-colored. Since in any k-coloring of the vertices of G, there is always one color class with at most $\lfloor \Delta/k \rfloor$ members in the neighbor set of v, we can flip v's color to this color, thereby decreasing the number of monochromatic edges and contradicting the minimality of the original coloring. ∎

Corollary 1.2

$$\chi_d(G) \leq \left\lceil \frac{\Delta + 1}{d + 1} \right\rceil = \left\lfloor \frac{\Delta}{d + 1} \right\rfloor + 1. \qquad ∎$$

The papers [1, 8, 9] provide other bounds on the defective chromatic number in terms of other parameters and in terms of other defective chromatic numbers.

The complexity of constructing defective colorings is less well-studied. However, R. Cowen [7] showed that $(2,1)$-coloring is NP-Complete for general graphs. Recently, [6] extended this, showing that determining if a graph is $(2,1)$-colorable is NP-complete for planar graphs, and this generalizes to $(2,d)$-coloring for $d \geq 1$, and also showed that determining if a planar graph is $(3,1)$-colorable is also NP-Complete. In general graph (k,d)-coloring is NP-Complete for all $k \geq 3$, and all $d \geq 0$, as expected, and [6] go on to generalize approximate coloring algorithms to defects.

1.2 This paper

In this paper, we first extend the results on the plane to the torus. First, we show that any graph embeddable on the torus is $(3,2)$-colorable. Then we show that any such graph is also $(5,1)$-colorable. The question of whether or not every toroidal graph is $(4,1)$-colorable remains an open question.

Second, we consider defective colorings of graphs on arbitrary surfaces. For genus γ, let $\chi_d(\gamma)$ be the maximum d-defect coloring number of all graphs embeddable on the surface S_γ. We show that $\chi_d(\gamma) \leq \chi_0(\gamma)/(d+1)+4$. Also, Archdeacon [2] showed that every graph embeddable on the surface S_γ is $(3, 3\gamma+O(1))$-colorable. We show that this is improveable to $(3, c\sqrt{\gamma})$-colorable, which shows that the maximum defect needed for 3-colorability is within a constant factor of that needed for the maximum clique on that surface.

2 Defective coloring on the torus

Since every planar graph embeds on the torus there does not exist a d such that every toroidal graph is $(2,d)$-colorable. For 3 colors we need a result that is slightly stronger than, but whose proof is strongly similar to the proofs of, Theorem 5 in [5] and Theorem 1 in [10].

Theorem 2.1 *Every planar graph can be $(3,2)$-colored such that any two specified vertices v_1 and v_2 receive specified colors and such that for $i = 1, 2$ v_i has no neighbor with the same color (except for possibly v_{3-i}).*

Proof. We prove this by induction on the number of vertices. First case: v_1 and v_2 are adjacent, and are required to be the same color. Then we contract them to a single vertex, choose a second vertex arbitrarily, and then use the induction hypothesis to $(3,2)$-color the resultant graph so that the vertex v_1v_2 is given the specified color. When we uncontract, v_1 and v_2 are properly colored in the resulting $(3,2)$-coloring except for the edge v_1v_2, as required.

Second case: v_1 and v_2 are not adjacent. We may assume G is a maximal planar graph. So there must be a cycle that separates v_1 and v_2 in G: let W be such a cycle of minimum length (so W is chord-free). Let G_1 (G_2) consist of v_1 (v_2) and all the other vertices and edges inside (outside) W. Let G_1' (G_2') be obtained from G_1 (G_2) by contracting W into a single new vertex w. These are both planar graphs with fewer vertices than G. Now by induction, color G_1' (G_2') with the requisite specified color for v_1 (v_2) and w specified to get a color distinct from either the color specified for v_1 or v_2 (possible, since there are three colors), and the vertices v_1, v_2 and w each without defect. We now transfer these colors back to G, giving all the vertices of W the color assigned to w.

Third case: v_1 and v_2 are adjacent and required to be different colors. Then we insert a new vertex on the edge between them, and without violating planarity add edges if necessary, to make G again a maximal planar graph. We then proceed as in the second case. ▮

Theorem 2.2 *Every toroidal graph can be* $(3,2)$*-colored.*

Proof. Without loss of generality we may assume that G is a maximal toroidal graph. Let C be a minimal noncontractible cycle of G. Then cut down the middle of C: split every vertex and every edge of C into two parts yielding G' with two copies C_1 and C_2 of the cycle. For each edge linking a vertex v in C to a vertex w outside C that edge remains linking w and one of the copies of v as indicated. See Figure 1. At the same time this cut turns the torus into a sphere with the graph G' embedded on the sphere such that C_1 and C_2 are the boundaries of regions.

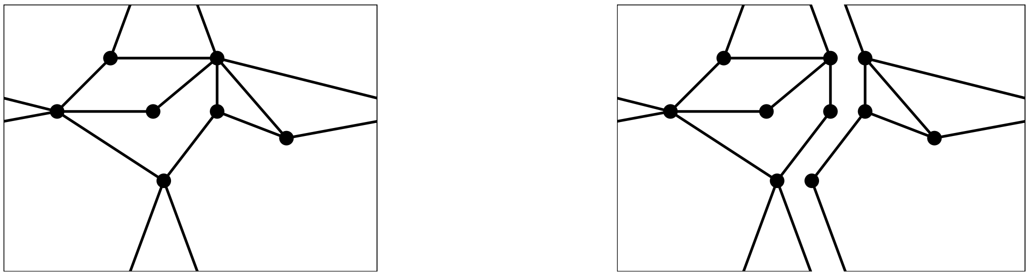

Figure 1: Cutting along minimum noncontractible cycle C making C_1 and C_2 yields a planar graph; C_1 and C_2 are then contracted to single vertices.

Form graph G'' from G' by contracting C_1 and C_2 each to a single vertex v_1 and v_2. Since G'' is planar, by the above theorem we can 3-color the vertices of G'' such that each color class has maximum degree two, and v_1 and v_2 both receive color 1 while none of their neighbors have color 1.

This yields immediately a coloring of G where all the vertices of C recieve color 1. This is the desired $(3,2)$-coloring. ▮

We turn next to colorings with defect 1.

Lemma 2.3 *If H is planar and u and v are vertices of H, then H can be* $(5,1)$*-colored such that the induced graph* $\langle N(u) \cup N(v) - \{u,v\} \rangle$ *is* 3*-colored.*

Proof. Find a maximal collection $\mathcal{P}$ of internally disjoint (u,v)-paths of length 3. Construct H' by, for each path ua_ib_iv in $\mathcal{P}$, contracting the edge a_ib_i. The resultant graph H' is planar.

So by the 4-color theorem, we can 4-color the graph H'. If we uncontract to get H we have a $(4,1)$-coloring of H, since all vertices are properly colored, save those pairs of vertices that were contracted, which are adjacent with the same color and so have defect 1. If u and v receive the same color in this coloring, we are done.

Otherwise, suppose u receives color 1 and v receives color 2. Then any common neighbor is colored 3 or 4. In particular, all the internal vertices of the paths in $\mathcal{P}$ receive color 3 or 4, and color classes 1 and 2 are both independent sets. Now re-color every vertex in $\langle N(u) \cup N(v) - \{u, v\}\rangle$ that has color 1 or 2 with a new color, color 5. Trivially $\langle N(u) \cup N(v) - \{u, v\}\rangle$ is 3-colored.

To prove the theorem we need to show that the new coloring is a $(5,1)$-coloring. For this it is sufficient to show that the vertices with color 5 form an independent set. But suppose that there are vertices a and b which are adjacent and colored 5. Then in the original coloring of H they must have had different colors. Say vertex a had color 2 and vertex b had color 1. Then vertex a cannot be adjacent to vertex v and so must be adjacent to vertex u. Similarly, vertex b must be adjacent to vertex v. This yields a path $uabv$ of length 3 in H. Since neither a nor b received color 3 or 4, this is a path internally disjoint from the ones in $\mathcal{P}$—a contradiction. ∎

Theorem 2.4 *One can $(5,1)$-color any graph embedded in the torus.*

Proof. Let G be embedded in the torus. Then there exists a minimal noncontractible cycle C that is an induced cycle. Construct a planar graph H by cutting along the edges of C, to form two copies of C, and contracting the two cycles to two single vertices u and v. By the above lemma we can $(5,1)$-color H so that the neighbors of u and v are 3-colored. This translates to a $(5,1)$-coloring of $G - C$ where the neighbors of C are 3-colored. Since there are two colors which we may use for C, we obtain the desired conclusion. ∎

Actually, we obtain the conclusion with at least one of the color classes being an independent set. But we are unable to resolve the following question.

Question 2.5 *Is every graph on the torus $(4,1)$-colorable?*

3 General genera

To prove upper bounds for graphs on general surfaces it helps to have the following upper bound on the defective chromatic number of dense graphs.

Theorem 3.1 *For all d and for all graphs with q edges it holds that $\chi_d < \sqrt{2q}/(d+1) + 2$.*

Proof. Suppose that $\chi_d = k$. Label the vertices of G $v_1, v_2, \ldots, v_n$ such that v_i has the maximum degree Δ_i in the graph $G_i = G - \{v_1, v_2, \ldots, v_{i-1}\}$. (Note that $\Delta_1 \geq \Delta_2 \geq \ldots \geq \Delta_n$.) Then by Lovász (Theorem 1.1) $\Delta_1 \geq (k-1)(d+1)$. Similarly $\Delta_{d+2} \geq (k-2)(d+1)$, else one can color $v_1, v_2, \ldots, v_{d+1}$ with one color and G_{d+2} with $k-2$ colors. In general, $\Delta_{j(d+1)+1} \geq (k-j-1)(d+1)$ for $0 \leq j \leq k-2$. It follows that

$$2q = \sum_{i=1}^{n} \Delta_i \geq (k-1)(d+1) + \sum_{j=1}^{k-2} (k-j-1)(d+1)^2 = (d+1)(k-1)(dk+k-2d)/2. \quad (1)$$

In particular, $2q > (d+1)^2(k-2)^2$, whence the result. ∎

One can extract a sharper bound from Inequality 1 that is best possible for $d = 0$ and $d = 1$. If $d = 0$ it follows that $q \geq k(k-1)/2$, while the clique K_k has the fewest edges for a graph with $\chi_0 = k$. If $d = 1$ it follows that $q \geq 2(k-1)^2$. This is best possible because the graph constructed by taking K_{2k-1} and removing the edges of a maximum matching has $2(k-1)^2$ edges but is not $(k-1,1)$-colorable. (It cannot have three vertices of the same color.)

Theorem 3.2 *For genus γ, let $\chi_d(\gamma)$ be the maximum d-defect chromatic number of all graphs embeddable on the surface S_γ. Then:*

$$\chi_d(\gamma) < \chi_0(\gamma)/(d+1) + 4.$$

Proof. Recall that $\chi_0 = \sqrt{12\gamma} + 7/2 + o(1)$, and the extremal graph is the complete graph (see [15]). We show that $\chi_d < \sqrt{12\gamma} + 7$.

Suppose first that $n \leq (d+1)\sqrt{12\gamma}$. Recall that in a graph of genus γ it holds that $q \leq 3n + 6\gamma$. So $2q \leq 6(d+1)\sqrt{12\gamma} + 12\gamma$. Thus by Theorem 3.1

$$\chi_d(G) < \sqrt{2q}/(d+1) + 2 < (\sqrt{12\gamma} + 3(d+1))/(d+1) + 2 = \sqrt{12\gamma}/(d+1) + 5.$$

For graphs with $n > (d+1)\sqrt{12\gamma}$, the proof is by induction on n. Such a graph has a vertex v of degree at most $(6n + 12\gamma)/n < 6 + \sqrt{12\gamma}/(d+1)$. So one can remove vertex v, color the graph $G - v$ by the induction hypothesis, and then re-insert vertex v and properly color it. ∎

The above theorem shows that the number of colors needed is only a few more than those needed for the maximum clique on that surface. For fixed number of colors, namely 3, Archdeacon [2] showed that a graph is approximately $(3, 3\gamma)$-colorable. The next theorem shows that this is improveable to $(3, c\sqrt{\gamma})$-colorable, which shows that the maximum defect needed for 3-colorability is within a constant factor of that needed for the maximum clique on that surface. We use virtually the same approach that Archdeacon used.

Lemma 3.3 *Let $t > 12$, and suppose G is a graph with minimum degree at least 3, the vertices of G of degree less than t form an independent set, and G has a 2-cell embedding on the surface of genus γ. Then the number of vertices of degree at least t is at most $24(\gamma - 1)/(t - 12)$.*

Proof. Let S denote the set of vertices of degree less than t and T the vertices of degree at least t. Now, in each region that is not a triangle add edges between vertices of T. One way to do this is, if $v_1, \ldots, v_4$ are four consecutive vertices on the boundary of the region with $v_1 \in T$, then if $v_3 \in T$ then join v_1 and v_3 by an edge inside the region, otherwise join v_2 and v_4, and repeat as necessary. The result is a triangulation of a multigraph H which has minimum degree at least 3, and in which the vertices S of degree less than t still form an independent set. (Multiple edges can arise between two vertices x and y with several common neighbors, when edges are added between them in each region.)

Let α denote the number of edges between S and T, and β denote the number of edges between vertices of T. Let v_i denote the number of vertices of degree i. Since S is an independent set, it follows that $\alpha = \sum_{i<t} iv_i$ and $\alpha + 2\beta = \sum_{i \geq t} iv_i$. Since the embedding is

a triangulation, it also follows that $\alpha \leq 2\beta$. Hence

$$\begin{aligned}
(t/2-6)|T| &= (t/2-6)\sum_{i\geq t} v_i \\
&\leq \sum_{3\leq i<t}(2i-6)v_i + \sum_{i\geq t}(i/2-6)v_i \\
&= \sum_{i<t}(i-6)v_i + \alpha + \sum_{i\geq t}(i/2-6)v_i \\
&\leq \sum_{i<t}(i-6)v_i + (\alpha/2+\beta) + \sum_{i\geq t}(i/2-6)v_i \\
&= \sum_{i}(i-6)v_i \\
&= 12\gamma - 12,
\end{aligned}$$

where the last equality is Euler's fomula for triangulations. ∎

Theorem 3.4 *A graph of genus γ is $(3, \sqrt{12\gamma}+6)$-colorable.*

Proof. Let $t = \sqrt{12\gamma}+7$. By induction on the number of edges. If G contains a vertex v of degree at most 2 then we easily obtain a suitable coloring of G from a coloring of $G-v$. Furthermore, if there is an edge e that joins two vertices of degree less than t then remove the edge, induct and conclude. Therefore we may assume that the minimum degree is at least 3 and that the vertices of degree less than t form an independent set S.

By the above lemma it follows that $T = V(G) - S$ has at most $24(\gamma-1)/(t-12)$ members. We form a 3-coloring by making all the members of S the first color, and then half the members of T receive the second color and half the third color. This gives the desired bound. ∎

4 Open problems

The two most immediate open problem is questions 2.5 that asks whether every toroidal graph is $(4,1)$-colorable. This would complete the characterization of defective colorings on the torus. It is also open to compute defective chromatic numbers exactly for surfaces of genus 2 or higher.

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