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Clique-like Dominating Sets

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ABSTRACT

Continuing work by Bácso and Tuza and Cozzens and Kelleher, we investigate dominating sets which induce subgraphs with small clique covering number, or small independence number. We show that if a graph G is connected and contains no induced subgraph isomorphic to P_6 or H_t (the graph obtained by subdividing each edge of $K_{1,t}, t \geq 3$), then G has a dominating set which induces a connected graph with clique covering number at most $t-1$. We then show that if G has no isolated vertices and contains no induced subgraph isomorphic to mK_2, then G contains a dominating set with independence number at most $2m-2$. For both of these results, the bounds are best possible. Finally, we prove a theorem which implies similar bounds for a variety of classes of graphs, and we show that if G is connected and contains no induced subgraph isomorphic to P_k or H_t, then the domination number of G is bounded above by a constant that depends only on k, t, and the clique number of G.

1 Introduction

By "graph," we mean a finite, undirected graph without loops or multiple edges. Let $G = (V, E)$ be a graph. If $x, y \in V(G)$, then $d(x, y)$ denotes the length of the shortest path in G joining x and y. If $x \in V$, let $N_i[x]$ denote $\{y \in V(G) : d(x, y) \leq i\}$. We may use $N[x]$ to denote $N_1[x]$. For $S \subseteq V$, G_S denotes the subgraph of G induced by S. A *dominating set* for G is a set $D \subseteq V$ such that if $x \in V(G)$, then $N[x] \cap D \neq \emptyset$. Let $\gamma(G)$ denote the size of the smallest dominating set for G. A *clique* is a set of vertices which induce a complete graph; such a set need not be maximal. Let $\theta(G_S)$ denote the clique covering number of G_S, i.e. the smallest integer k such that S can be partitioned into k subsets, each of which induces a clique. Let $\alpha(G_S)$ denote the size of the largest independent set in G_S. If G_1, G_2,..., G_m are finite graphs, $\mathrm{Forb}(G_1, G_2, ...G_m)$ denotes the class of graphs G such that G contains no induced subgraph isomorphic to G_i for $1 \leq i \leq m$. We use P_n (C_n) to denote the path (cycle) on n vertices. Let $R[a, b]$ denote the smallest positive integer n such that every graph on n vertices contains either a clique of size a or an independent set of size b.

Our work is motivated by the following results of Bacsó and Tuza [1], the first of which was proved independently by Cozzens and Kelleher [5]:

Theorem 1 *If $G \in Forb(P_5, C_5)$ is connected, then G has a dominating set D such that G_D is a clique.* □

Theorem 2 *If $G \in Forb(P_5)$ is connected, then G contains either a dominating set D such that G_D is a clique or a P_3.* □

In discussing motivation, Cozzens and Kelleher mention that in a communications network it may be desirable to have a core of nodes such that any two nodes in the core can communicate, and every node in the network communicates with some node of the core. If such a core does not exist, a small number of cliques, which collectively communicate with every node of the network, could serve a similar purpose. For instance, a research community may wish to organize itself around a few groups of researchers who share common interests and who are positioned within the community so that every member of the community can learn the results of at least one group. Thus, we will investigate conditions that guarantee that a graph G will have a dominating set which can be covered with a small number of cliques. We will also discuss dominating sets which are "clique-like" in the sense that they have small independence number. (In a related direction, Bacsó and Tuza have studied dominating sets with small diameter in [2], [3], and [4].)

Let Γ be a class of graphs for which there exists a constant c such that every connected graph $G \in \Gamma$ contains a dominating set D where $\theta(G_D) \leq c$. Then we will say Γ is *θ-dominated*. Similarly, if there exists a constant c such that every connected graph $G \in \Gamma$ contains a dominating set D where $\alpha(G_D) \leq c$, then we will say Γ is *α-dominated.*

We begin with several examples which will illustrate what sort of conditions will be necessary in order for a class Γ to be α-dominated. Note that every θ-dominated class is α-dominated, so these conditions are also necessary for Γ to be θ-dominated. These examples will motivate our results.

First note that every dominating set for P_k contains an independent set of size $\lceil k/4 \rceil$.

Let H_t denote the graph obtained by subdividing each edge of $K_{1,t}$. Then every dominating set for H_t contains an independent set of size t.

Let Q_r denote the graph obtained by adding a vertex of degree one to each vertex of K_r, and then subdividing each of these r pendant edges. Then every dominating set for Q_r must contain an independent set of size r.

Thus Γ is not α-dominated if Γ contains P_k for arbitrarily large k, H_t for arbitrarily large t, or Q_r for arbitrarily large r. We would like to show that $\mathrm{Forb}(P_k, H_t, Q_r)$ is α-dominated. In the main results of this paper, we prove three special case.

Our first result shows that $\mathrm{Forb}(P_6, H_t)$ is θ-dominated for all positive integers t. (Note that previous results show this for $t < 3$. Also note that $\mathrm{Forb}(P_6, H_t) = \mathrm{Forb}(P_6, H_t, Q_r)$.)

Theorem 3 *Suppose $t \geq 3$. If $G \in$ Forb(P_6, H_t) is connected, then G contains a dominating set D such that G_D is connected and $\theta(G_D) \leq t-1$.*

Clearly this bound is best possible, since $H_{t-1} \in \mathrm{Forb}(P_6, H_t)$. The result is also sharp in the sense that $\mathrm{Forb}(P_7, 3K_2)$ (and therefore $\mathrm{Forb}(P_7, H_3, Q_3)$) is not θ-dominated, where $3K_2$ denotes the graph consisting of 3 independent edges. Let M_1, M_2, ... be a sequence of graphs such that, for all positive integers i, $\alpha(M_i) = 2$ and $\theta(M_i) = i$. Let G_i be the graph obtained by attaching a vertex of degree 1 to each vertex of M_i. The reader may verify that for all i, $G_i \in \mathrm{Forb}(P_7, 3K_2)$, and that if D is a dominating set for G_i, then the subgraph induced by D has clique covering number $\geq i$.

Since $\mathrm{Forb}(P_7, 3K_2)$ is not θ-dominated, the superclass $\mathrm{Forb}(3K_2)$ is not θ-dominated. However it is α-dominated, as shown by the following result. Let mK_2 denote the graph consisting of m independent edges

Theorem 4 *If $G \in$ Forb(mK_2) has no isolated vertices, then G contains a dominating set D such that $\alpha(G_D) \leq 2m-2$. This bound is best possible.*

Note that this result is even stronger than the statement "$\mathrm{Forb}(mK_2)$ is α-dominated," because it applies even to disconnected graphs in $\mathrm{Forb}(mK_2)$, provided they have no isolated vertices.

Our third main result has several interesting implications. In fact, it was obtained by abstracting the key elements of the proofs of the results which are now stated as corollaries. Let E_m denote the graph on $2m+1$ vertices consisting of a copy of K_m, an independent set on m vertices which is disjoint from the K_m and joined to the K_m by a matching, and a vertex of degree m whose neighborhood is precisely the independent set on m vertices. Let F_n denote the graph on $3n$ vertices formed by joining two disjoint copies of K_n with a matching, and then subdividing each edge of the matching.

Theorem 5 *Forb$(P_k, H_t, Q_r, E_m, F_n)$ is α-dominated for all positive integers k, t, r, m, and n.*

Recall that if a graph G is perfect, then for every $S \subseteq V(G)$, $\theta(G_S) = \alpha(G_S)$, and that comparability graphs and chordal graphs are perfect. Thus Theorem 5, a result on independence numbers, implies the following corollaries on clique covering numbers.

Corollary 1 *For all positive integers t and r, the class of perfect graphs which are contained in Forb(P_7, H_t, Q_r) is θ-dominated.*

Proof. The class of perfect graphs which are contained in Forb(P_7, H_t, Q_r) is a subclass of Forb$(P_7, H_t, Q_r, E_2, F_3)$, since $E_2 = C_5$ is not perfect and F_3 contains an induced P_7. □

Corollary 2 *For all positive integers k and t, the class of comparability graphs which are contained in Forb(P_k, H_t) is θ-dominated.*

Proof. The class of comparability graphs which are contained in Forb(P_k, H_t) is a subclass of Forb$(P_k, H_t, Q_3, E_2, F_3)$, because Q_3, E_2, and F_3 are not comparability graphs. □

Corollary 3 *For all positive integers k, t, and r, the class of chordal graphs which are contained in Forb(P_k, H_t, Q_r) is θ-dominated.*

Proof. The class of chordal graphs which are contained in Forb(P_k, H_t, Q_r) is a subclass of Forb$(P_k, H_t, Q_r, E_2, F_2)$, because $E_2 = C_5$ and $F_2 = C_6$ are not chordal. □

Note that a similar result for cochordal graphs follows immediately from Theorem 2. A similar result for cocomparability graphs follows from the fact that every connected cocomparability graph has a dominating set which induces a chordless path (see [6]).

Corollary 4 *For all positive integers k and t there exists a function $f_{k,t}$ such that if $G \in$ Forb(P_k, H_t) is connected, then $\gamma(G) \leq f_{k,t}(\omega(G))$, where $\omega(G)$ denotes the size of the largest clique in G.*

Proof. Let $c(k, t, r, m, n)$ denote the constant that witnesses that Forb$(P_k, H_t, Q_r, E_m, F_n)$ is α-dominated. If $G \in$ Forb(P_k, H_t) is connected and $\omega(G) = w$, then by Theorem 5, G has a dominating set D such that $\alpha(G_D) \leq c(k, t, w + 1, w + 1, w + 1)$. By Ramsey's theorem, $|D| < R[w + 1, c(k, t, w + 1, w + 1, w + 1) + 1]$. The result follows. □

Observe that P_k and H_t are connected and triangle-free, yet $\gamma(P_k) = \lceil k/3 \rceil$ for all k and $\gamma(H_t) = t$ for all t. Thus, no analogous result is possible for a class of graphs which contains P_k for arbitrarily large k or H_t for arbitrarily large t.

2 Proofs

We begin this section by proving Theorem 3. The structure of the proof is modeled after the proof of Theorem 2 in [1]. One of the most powerful tools from that proof is the following lemma. If G is a graph and S is a cutset in $V(G)$ such that $S - \{x\} \subseteq N(x)$ for some $x \in S$, then S is a *star-cutset*.

Lemma 1 *Let* $\mathbf{D}$ *be a class of connected graphs. Let* G *be a minimal connected graph such that if* D *is a dominating set for* G, $G_D \notin \mathbf{D}$. *Then either* G *has a cutvertex or* G *has no star-cutset.* □

Taking $\mathbf{D}$ to be the class of connected graphs H with $\theta(H) \leq t-1$, we need to prove Theorem 3 only for the two cases mentioned in Lemma 1. Lemma 2 will help us with these two cases.

If $G \in \text{Forb}(P_6, H_t)$, we say that $v \in V(G)$ is *critical* if either v is the endpoint of a chordless P_5 in G, or $v \in S \subseteq V(G)$ such that $S - \{v\}$ induces H_{t-1} and v's only neighbor in S is the central vertex of the induced H_{t-1}. In other words, v is critical if v belongs to an induced subgraph of G in which attaching a leaf to v gives a graph isomorphic to either P_6 or H_t. The motivation for this definition is the fact that a cutvertex of $G \in \text{Forb}(P_6, H_t)$ is not critical.

Lemma 2 *Suppose* $t \geq 2$. *Let* $G \in$ *Forb*(P_6, H_t) *be connected. Suppose there is a vertex* $v \in V(G)$ *such that* v *is not critical. Then* G *has a dominating set* D *such that* $v \in D$, G_D *is connected,* G_D *contains no chordless* P_4*'s and* $\theta(G_D) \leq t-1$.

Proof. We apply induction on the number of vertices of G. The result is trivial for a graph with one vertex. Suppose that $G \in \text{Forb}(P_6, H_t)$ is connected and has n vertices, that $v \in V(G)$ is not critical, and that the result holds for all connected graphs in $\text{Forb}(P_6, H_t)$ with $n-1$ vertices. If v has a neighbor x which is not a cutvertex of G, then by the induction hypothesis $G - \{x\}$ has a dominating set D with the desired properties. Since $v \in D$, D is a dominating set for G as well.

Assume henceforth that every neighbor of v is a cutvertex.

Since v is not critical, the set $\{x \in V(G) : d(x,v) \leq 2\}$ is a dominating set for G which induces a connected subgraph in G. Let N_2 be a minimal subset of $\{x \in V(G) : d(x,v) = 2\}$ such that $D = \{v\} \cup N(v) \cup N_2$ is a dominating set for G which induces a connected subgraph of G. We claim that D is the desired dominating set. Clearly, $v \in D$, and G_D is connected. We first show that $\alpha(G_D) \leq t-1$. Then we will argue that G_D does not contain a chordless P_4. The result will follow since P_4-free graphs are perfect.

First we observe that if $\{a_1, \ldots, a_m\} \subseteq N_2$ is independent, then there exists $x \in N(V)$ and $\{b_1, \ldots, b_m\} \subseteq V(G) - D$ such that $\{x\} \cup \{a_i : 1 \leq i \leq m\} \cup \{b_i : 1 \leq i \leq m\}$ induces H_m, where $x \sim a_i$ and $a_i \sim b_i$ for $1 \leq i \leq m$. By the minimality of N_2, for each a_i, $1 \leq i \leq m$, there exists a vertex $b_i \in V(G) - D$ such that $N(b_i) \cap D = \{a_i\}$. Then $\{b_i : 1 \leq i \leq m\}$ is an independent set; if, for example, $b_1 \sim b_2$, then $\{b_2, b_1, a_1, x, v\}$, where $x \in N(v)$ is a neighbor of a_1, is a chordless P_5 which has v for an endpoint, which is a contradiction. Moreover, if $1 \le i < j \le m$, either $N(a_i) \cap N(v) \subseteq N(a_j) \cap N(v)$ or $N(a_j) \cap N(v) \subseteq N(a_i) \cap N(v)$. If there exists $c_i \in (N(a_i) \cap N(v)) - (N(a_j) \cap N(v))$ and $c_j \in (N(a_j) \cap N(v)) - (N(a_i) \cap N(v))$, then either $c_i \sim c_j$ and $\{b_i, a_i, c_i, c_j, a_j, b_j\}$ induces P_6, or $c_i \not\sim c_j$ and $\{b_i, a_i, c_i, v, c_j, a_j, b_j\}$ induces P_7; in either case, we find a chordless P_6. Thus, there exists $x \in N(v)$ such that $x \sim a_i$ for $1 \le i \le m$. This justifies our observation.

Suppose $I \subseteq D$ is an independent set, and $|I| = t$. Since $t \ge 2$, if $v \in I$, then $\emptyset \ne I - \{v\} \subseteq N_2$. By the previous observation, this implies that v is critical, a contradiction, so assume $v \notin I$.

First consider the case where $I \cap N_2 \ne \emptyset$; say $I = \{a_1, a_2, \ldots, a_t\}$. By the previous observation, $I \not\subseteq N_2$. By our assumption that $I \cap N_2 \ne \emptyset$, there exists, without loss of generality, a positive integer $m < t$ such that $a_i \in N_2$ if and only if $i \le m$. By our previous work, there exists $x \in N(v)$ and $\{b_1, \ldots, b_m\} \subseteq V(G) - D$ such that $\{x\} \cup \{a_i : 1 \le i \le m\} \cup \{b_i : 1 \le i \le m\}$ induces H_m, where $x \sim a_i$ and $a_i \sim b_i$ for $1 \le i \le m$. For $m+1 \le i \le t$, $a_i \in N(v)$, so a_i is a cutvertex of G. Therefore, we may find vertices b_i, $m+1 \le i \le t$, such that $b_i \notin \{v, x\} \cup \{a_i : 1 \le i \le t\} \cup \{b_i : 1 \le i \le m\}$ and $N(b_i) \cap (\{v, x\} \cup \{a_i : 1 \le i \le t\} \cup \{b_i : 1 \le i \le m\}) = \{a_i\}$; simply choose a neighbor of a_i which is not in the same component of $G - \{a_i\}$ as v. Note that, if chosen in this manner, the set $\{b_i : m+1 \le i \le t\}$ is independent. Also note that, if $m+1 \le i \le t$, $a_i \sim x$, or else $\{b_i, a_i, v, x, a_1, b_1\}$ induces P_6. But then $\{x\} \cup \{a_i : 1 \le i \le t\} \cup \{b_i : 1 \le i \le t\}$ induces H_t, a contradiction.

The only remaining case is $I = \{a_1, a_2, \ldots, a_t\} \subseteq N(v)$. Arguing as in the previous paragraph, we may use the assumption that each $a_i \in I$ is a cutvertex of G to find a set of vertices $\{b_i : 1 \le i \le t\}$ such that $\{v\} \cup \{a_i : 1 \le i \le t\} \cup \{b_i : 1 \le i \le t\}$ induces H_t. This is a contradiction, so $\alpha(G_D) \le t - 1$.

Now we claim that G_D does not contain a chordless P_4. Suppose $P = \{w, x, y, z\} \subseteq D$ induces a chordless P_4 with endpoints w and z. If $\{w, z\} \subseteq N(v)$, then w and z are cutvertices of G, and we may find vertices p and q such that $\{p, w, x, y, z, q\}$ induces P_6, a contradiction. If one endpoint of P is in $N(v)$, the other endpoint cannot be v. Thus we may assume without loss of generality that $z \in N_2$. By the minimality of N_2 we may find a vertex $q \in V(G) - D$ such that $N(q) \cap D = \{z\}$, and thus $\{w, x, y, z, q\}$ induces P_5. Since v is not critical, $w \ne v$. If $w \in N(v)$, we may use the assumption that w is a cutvertex of G to find a vertex p such that $\{p, w, x, y, z, q\}$ induces P_6, a contradiction. If $w \in N_2$, we may find $p \in V(G) - D$ such that $N(p) \cap D = \{w\}$. Note that $p \not\sim q$, or else $\{q, p, w, x, v\}$, where $x \in N(v)$ and $x \sim w$, induces a chordless P_5 with endpoint v, a contradiction. But then $\{p, w, x, y, z, q\}$ induces P_6, a contradiction. Thus no subset of D induces a chordless P_4.

Since graphs which lack chordless P_4's are known to be perfect, $\theta(G_D) = \alpha(G_D) \le t - 1$, and the lemma is proved. □

Proof of Theorem 3. Let **D** be the class of connected graphs H with $\theta(H) \le t - 1$. By Lemma 1 it suffices to prove the theorem for connected graphs in $\mathrm{Forb}(P_6, H_t)$ which either have a cutvertex or do not have a star-cutset.

Suppose $G \in \mathrm{Forb}(P_6, H_t)$ is connected and has a cutvertex v. Note that v is not critical. By Lemma 2, G has a dominating set D such that G_D is connected and $\theta(G_D) \leq t-1$.

Suppose $G \in \mathrm{Forb}(P_6, H_t)$ is connected and has no star-cutset. Choose any $v \in V(G)$, and any $u \in N(v)$. Let $Y = \{v\} \cup (V(G) - N[u])$. By supposition, $N[u] - \{v\}$ is not a cutset, so G_Y is connected. Moreover, v is not critical in G_Y, or else we could attach u to v to form either P_6 or H_t. Thus, the hypotheses of Lemma 2 are satisfied, and we may find a dominating set D_Y for G_Y which has the properties given in the Lemma. If D_Y is a dominating set for G, then we are done, so assume there exists a vertex $w \in N(u)$ such that $N(w) \cap D_Y = \emptyset$. We claim that $D' = \{u, v, w\} \cup (N(v) \cap D_Y)$ is a dominating set for G. Suppose $x \in Y$ has no neighbor in D'. Then there exists a vertex $y \in (D_Y - D') \cap N(x)$. Since G_{D_Y} is connected and $y \in (D_Y - D')$, there exists a chordless path from y to v in G_{D_Y} of length at least 2. But then there exists a chordless path in G from x to w of length at least 5, a contradiction since this implies the existence of a chordless P_6 in G.

Let N_1 be a minimal subset of $N(v) \cap D_Y$ such that $D = \{u, v, w\} \cup N_1$ is a dominating set for G. Clearly, G_D is connected. We will show that $\theta(G_D) \leq t-1$.

We claim that $\alpha(G_{N_1}) \leq t-2$. Suppose $\{a_1, \ldots, a_{t-1}\} \subseteq N_1$ is an independent set. By the minimality of N_1, for each i, $1 \leq i \leq t-1$ there exists a vertex $b_i \in V(G) - D$ such that $N(b_i) \cap D = \{a_i\}$. Note that $\{b_i : 1 \leq i \leq t-1\}$ is independent; if, for example, $b_1 \sim b_2$, then $\{b_2, b_1, a_1, v, u, w\}$ induces P_6. But then $\{u, v, w\} \cup \{a_i : 1 \leq i \leq t-1\} \cup \{b_i : 1 \leq i \leq t-1\}$ induces H_t, a contradiction.

By Lemma 2, G_{N_1} contains no chordless P_4, so $\theta(G_{N_1}) \leq t-2$. If $N_1 = \emptyset$, then the result is obvious because $\theta(G_{\{u,v,w\}}) = 2 \leq t-1$, since $t \geq 3$. Otherwise, we may add v to any one of the cliques in a minimum clique covering for G_{N_1} to show that $\theta(G_{N_1 \cup \{v\}}) \leq t-2$. Since $\{u, w\}$ induces a clique, $\theta(G_D) \leq t-1$, and the Theorem is proved. □

Proof of Theorem 4. Suppose $G \in \mathrm{Forb}(mK_2)$ has no isolated vertices. Let D_1 be a maximal subset of $V(G)$ such that G_{D_1} is isomorphic to jK_2 for some $j < m$. Let S be the set of vertices of G which have no neighbors in D_1. By the maximality of D_1, S is an independent set. Since G has no isolated vertices, every vertex in S has a neighbor in $N = \{x : x \sim y \in D_1\}$. Let D_2 be a minimal subset of N such that $D = D_1 \cup D_2$ is a dominating set for G. We claim that $\alpha(G_D) \leq 2m-2$. Clearly $\alpha(G_{D_1}) \leq m-1$, so it suffices to show that $\alpha(G_{D_2}) \leq m-1$. Suppose D_2 contains an independent set $\{x_1, x_2, \ldots, x_m\}$ on m vertices. By the minimality of D_2, for $1 \leq i \leq m$ there exists $y_i \in S$ such that $N(y_i) \cap D_2 = \{x_i\}$. But then $\{x_i : 1 \leq i \leq m\} \cup \{y_i : 1 \leq i \leq m\}$ induces mK_2, a contradiction.

Note that the graph consisting of $m-1$ disjoint copies of C_5 realizes this bound, so the proof is complete. □

Theorem 5 follows from a technical lemma which gives a sufficient condition for finding an induced H_t or Q_r. Consider the following function of three variables defined by induction on the first variable. Let $f(1, t, r) = t$ for all positive integers t and r. Let $f(i+1, t, r) = (2t-1)R[r, f(i, t, r)]$ for all positive integers i, t, and r.

Lemma 3 *Suppose G^* is a graph and $x \in V(G^*)$. Let $m = f(i,t,r)$. Suppose there exist independent sets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ in $V(G^*)$ such that for $1 \le j \le m$, $d(x, a_j) = i$ and $N[b_j] \cap (\{a_1, \ldots, a_m\} \cup N_{i-1}[x]) = \{a_j\}$. Then G^* contains an induced subgraph isomorphic to either H_t or Q_r.*

Proof. We induct on i. The case $i = 1$ is obvious. Suppose the result is true for $i = h$. Let $m = f(h+1, t, r)$, and suppose $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are independent sets in $V(G^*)$ such that for $1 \le j \le m$, $d(x, a_j) = h+1$ and $N[b_j] \cap (\{a_1, \ldots, a_m\} \cup N_h[x]) = \{a_j\}$. There exist vertices $c_1, \ldots, c_m$ such that $d(x, c_j) = h$ and $a_j \sim c_j$ for $1 \le j \le m$. (Note that the c_j's are not necessarily distinct.) Let A be the directed graph on the vertex set $\{1, \ldots, m\}$ with an arc directed from p to q if and only if $c_p \sim a_q$. If A has a node of outdegree t, then G^* contains an induced H_t, and we are done. Therefore, assume that each node of A has outdegree at most $t-1$. It follows that A can be properly $(2t-1)$-colored, so it must contain an independent set of size $s = R[r, f(h,t,r)]$. Without loss of generality, this independent set in A is on the vertex set $\{1, \ldots, s\}$. In other words, $N[a_j] \cap \{c_1, \ldots, c_s\} = \{c_j\}$ for $1 \le j \le s$. By Ramsey's Theorem, the graph induced by $\{c_1, \ldots, c_s\}$ contains either a clique on r vertices, in which case G^* contains an induced Q_r, or an independent set on $f(h,t,r)$ vertices, in which case we find H_t or Q_r by the induction hypothesis. □

Proof of Theorem 5. Suppose $G \in \mathrm{Forb}(P_k, H_t, Q_r, E_m, F_n)$ is connected. Choose any $x \in V(G)$. Note that $d(v, x) \le k-2$ for all $v \in V(G)$. Then $N_{k-3}[x]$ is a connected dominating set for G. Choose sets S_{k-3}, S_{k-4}, ... , S_1 inductively (and in this order) as follows. Choose S_{k-3} to be a minimal subset of $N_{k-3}[x] - N_{k-4}[x]$ such that

$$N_{k-4}[x] \bigcup S_{k-3}$$

is a connected dominating set for G. If $j \ge 1$ and S_{k-3}, S_{k-4},..., S_{j+1} have been chosen, choose S_j to be a minimal subset of $N_j[x] - N_{j-1}[x]$ such that

$$N_{j-1}[x] \bigcup \left(\bigcup_{j \le i \le k-3} S_i \right)$$

is a connected dominating set for G. Let $S_0 = \{x\}$. For $1 \le p \le k-3$, let

$$D_p = \bigcup_{0 \le i \le p} S_i.$$

Let $D = D_{k-3}$. Note that if $v \in S_i$, then $d(x, v) = i$ in G_D.

We claim that

$$\alpha(G_D) < 1 + \sum_{1 \le i \le k-3} R\left[(2m-1)R[n,r], f(i,t,r)\right],$$

because G_{S_i} cannot contain an independent set on $R\left[(2m-1)R[n,r], f(i,t,r)\right]$ vertices for $1 \le i \le k-3$.

Let $s = R[(2m-1)R[n,r], f(i,t,r)]$, and suppose $\{a_1, \ldots, a_s\} \subseteq S_i$ is an independent set. By our method of constructing D, for $1 \leq j \leq s$ there exists a vertex $b_j \in N_{i+1}[x] - N_i[x]$ such that $N[b_j] \cap D = \{a_j\}$. If $\{b_1, \ldots, b_s\}$ contains an independent set of size $f(i,t,r)$, then by Lemma 1 (where G^* is the graph induced by $D \cup \{b_1, \ldots, b_s\}$), we may find an induced H_t or Q_r. Thus we may assume $\{b_1, \ldots, b_p\}$ induces a clique, where $p = (2m-1)R[n,r]$.

For $1 \leq j \leq p$, there exists a vertex $c_j \in D_{i-1}$ such that $a_j \sim c_j$. An argument using a directed graph, as in the proof of Lemma 1, shows that either there is vertex in $\{c_1, \ldots, c_p\}$ which is adjacent to m vertices in $\{a_1, \ldots, a_p\}$, or else we may assume $N[a_j] \cap \{c_1, \ldots, c_q\} = \{c_j\}$ for $1 \leq j \leq q$, where $q = R[n,r]$. Since the former would imply the existence of an induced E_m, we assume the latter. But then either $\{c_1, \ldots, c_q\}$ contains a clique on n vertices, in which case G contains an induced F_n, or an independent set on r vertices, in which case G contains an induced Q_r. Either case is a contradiction, so G_{S_i} does not contain an independent set of size $s = R[(2m-1)R[n,r], f(i,t,r)]$. $\square$

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