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Balanced Graphs and Network Flows

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ABSTRACT

A graph G is *balanced* if the maximum ratio of edges to vertices, taken over all subgraphs of G, occurs at G itself. This note uses the max-flow/min-cut theorem to prove a good characterization of balanced graphs. This characterization is then applied to some results on how balanced graphs may be combined to form a larger balanced graph. In particular, we show that edge-transitive graphs and complete m-partite graphs are balanced, that a product or lexicographic product of balanced graphs is balanced, and that the normal product of a balanced graph and a regular graph is balanced.

1 Introduction

It is well known that many results in discrete mathematics can be derived from the max-flow/min-cut theorem. Among these results are Menger's theorems, Philip Hall's matching theorem, Dilworth's theorem on chain coverings of partially ordered sets, König's minimax theorem (in any bipartite graph the size of a maximum matching is equal to the size of a minimum line covering), and the Birkhoff-von Neumann theorem (every doubly stochastic matrix is a convex combination of permutation matrices). Details of these theorems and their relationship to the max-flow/min-cut theorem can be found in [2]. The intent of this article is to add another area which can be profitably studied from this point of view: balanced graphs.

Throughout this paper, graphs are assumed to be finite without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $d(G) = \epsilon(G)/\nu(G)$, where $\nu(G)$ and $\epsilon(G)$ denote $|V(G)|$ and $|E(G)|$, respectively. We say that G is *balanced* if for all subgraphs H of G,

$$d(H) \leq d(G), \tag{1}$$

where $V(H)$ is assumed to be non-empty. Clearly, to check whether a given graph G is balanced, it suffices to verify (1) for all non-empty induced subgraphs H. The most commonly cited examples of balanced graphs include trees, complete bipartite graphs, and regular graphs.

In Section 2 we prove a good characterization of balanced graphs. The proof shows that deciding whether a given graph G is balanced is equivalent to finding a minimum cut in a certain network related to G. The main idea of this proof first appeared in [3], which gives an efficient algorithm for the more general problem of finding a subgraph H of a given graph G such that $d(H)$ is maximum. However, our characterization of balanced graphs appears to be new. In Section 3 this characterization is used to give a unified presentation of several facts about balanced graphs. We use the characterization to prove a result on how balanced graphs can be joined together to form a larger balanced graph. As corollaries, we will show that all edge-transitive graphs are balanced, and that if two balanced graphs are joined by placing all possible edges between them, the resulting graph is balanced. We close the paper by showing that a product or lexicographic product of balanced graphs is balanced, and that a normal product of a balanced graph and a regular graph is balanced. These latter results are also consequences of our characterization.

Balanced graphs first arose in the study of random graphs. See [1]. More recently, there have been several papers written on balanced extensions of graphs. Among these is [4], which gives references to several more papers on this topic. In [5], a result is proved which implies that maximal planar graphs and maximal outerplanar graphs are balanced. After deriving *ad hoc* proofs that edge-transitive graphs and complete m-partite graphs are balanced, the present author began to suspect that a common proof was possible. Corollary 1, which is merely a special case of Theorem 2, confirms this suspicion.

2 A Characterization of Balanced Graphs

Let G be a graph. We define the *incidence graph* $I(G)$ to be the bipartite graph with independent sets $A = E(G)$ and $B = V(G)$, and with $e \in A$ adjacent to $v \in B$ if and only if v is an endpoint of e in G. A *normalized edge weight assignment* is an assignment of non-negative real weights to the edges of $I(G)$ such that

1. For every $e \in A$ the sum of the weights of the edges of $I(G)$ incident to e is $\nu(G)$;

2. For every $v \in B$ the sum of the weights of the edges of $I(G)$ incident to v is $\epsilon(G)$.

Theorem 1 *Let G be a graph. Then G is balanced if and only if $I(G)$ has a normalized edge weight assignment.*

Proof: We adopt the network terminology used in [2]. Consider the network D formed by

- orienting all edges of $I(G)$ from A to B and giving these arcs capacity $\nu(G)\epsilon(G)+1$.

- creating a source s with an arc of capacity $\nu(G)$ going from s to each node in A

- creating a sink t with an arc of capacity $\epsilon(G)$ entering t from each node in B.

By the definition of D, $I(G)$ has a normalized edge weight assignment if and only if D admits a flow of value $\nu(G)\epsilon(G)$. Since D has a cut of capacity $\nu(G)\epsilon(G)$, by the max-flow/min-cut theorem D has a flow of value $\nu(G)\epsilon(G)$ if and only if it has no cut of capacity strictly less than $\nu(G)\epsilon(G)$. Thus it suffices to show that G is balanced if and only if D has no cut of capacity strictly less than $\nu(G)\epsilon(G)$.

Suppose G is balanced. Let S be an $s-t$ separator in D which determines a cut of minimum capacity. Note that since the cut determined by S has minimum capacity, for every $x \in A$, $x \in S$ if and only if both of x's neighbors in B are in S. Thus, if H is the subgraph of G induced by $S \cap B$, $E(H) = S \cap A$. Therefore, the capacity of the cut determined by S is

$$\nu(G)\epsilon(G) - \nu(G)\epsilon(H) + \nu(H)\epsilon(G),$$

which is less than $\nu(G)\epsilon(G)$ only if $d(G) < d(H)$. Since G is balanced, the capacity of the cut determined by S is at least $\nu(G)\epsilon(G)$.

Conversely, suppose that no cut in D has capacity less than $\nu(G)\epsilon(G)$. Let H be an induced subgraph of G. Define an $s-t$ separator S in D by setting $S = \{s\} \cup E(H) \cup V(H)$. The capacity of the cut determined by S is

$$\nu(G)\epsilon(G) - \nu(G)\epsilon(H) + \nu(H)\epsilon(G).$$

Since no cut has capacity less than $\nu(G)\epsilon(G)$, this implies that $d(H) \leq d(G)$. □

Using the proof of Theorem 1 and any minimum cut algorithm gives a polynomial algorithm for determining whether a given graph G is balanced. In the case where G is not balanced, we may use the minimum cut to find an induced subgraph H such that $d(G) < d(H)$; the vertices of H will be the vertices corresponding to nodes in $S \cap B$. However, there are examples of graphs G where the algorithm produces a subgraph H with $d(H) > d(G)$, but $d(H)$ is not maximum. For a version of this algorithm which does find a subgraph of maximum density in polynomial time, see [3].

3 Applications

In this section we apply Theorem 1 to prove some results on how balanced graphs may be combined to form a larger balanced graph. The advantage in using Theorem 1 is that it gives a structure, the normalized edge weight assignment, which, under the right conditions, can be extended from the smaller graphs to the larger one.

If G is a graph and $U \subseteq V(G)$, let G_U denote the subgraph induced by U. If $x \in V(G)$, let $N(x)$ denote the open neighborhood of x. A *regular partition* of $V(G)$ is a partition of $V(G)$ into sets V_1, V_2,..., V_m such that there exist integers k_1, k_2,..., k_m where $|N(x) \cap V_i| = k_i$ for all $i \in \{1, \ldots, m\}$ and $x \notin V_i$.

Lemma 1 *Let G be a graph. Suppose V_1 and V_2 form a regular partition of $V(G)$ with $d(G_{V_1}) \geq d(G_{V_2})$, and suppose G_{V_1} and G_{V_2} are balanced. Then the following are equivalent:*

1. *G is balanced;*
2. *$d(G_{V_1}) \leq d(G)$;*
3. *$d(G_{V_1}) \leq d(G_{V_2}) + k_1$.*

Proof: By the definition of a balanced graph, Condition 1 implies Condition 2. Condition 2 can be written

$$\frac{\epsilon(G_{V_1})}{\nu(G_{V_1})} \leq \frac{\epsilon(G_{V_1}) + \epsilon(G_{V_2}) + k_1\nu(G_{V_2})}{\nu(G_{V_1}) + \nu(G_{V_2})}.$$

The equivalence of Condition 2 and Condition 3 follows from straightforward algebra. Thus we need only show that Condition 2 implies Condition 1.

Suppose Condition 2 holds. Then $d(G_{V_2}) \leq d(G)$. Since G_{V_1} and G_{V_2} are balanced, we may assume we have fixed normalized edge weight assignments for $I(G_{V_1})$ and $I(G_{V_2})$. Our goal is to produce a normalized edge weight assignment for $I(G)$. Recall that a typical edge of $I(G)$ is of the form (e, v) where $e \in E(G)$, $v \in V(G)$, and v is an endpoint of e. There are 2 cases to consider.

Case 1: $e \in E(G_{V_i})$. Give (e, v) the weight

$$\frac{\nu(G)}{\nu(G_{V_i})}w$$

where w is the weight of (e, v) in the normalized edge weight assignment for $I(G_{V_i})$.

Case 2: $v \in V_i$ and e joins v to a vertex in V_j, $i \neq j$. Give (e, v) the edge weight

$$\frac{1}{k_j}\left(\epsilon(G) - \epsilon(G_{V_i})\frac{\nu(G)}{\nu(G_{V_i})}\right).$$

Clearly the edge weights in Case 1 are non-negative, because each is the product of a positive rational and an edge weight from a normalized edge weight assignment. Condition 2 ensures that

$$\epsilon(G) - \epsilon(G_{V_i})\frac{\nu(G)}{\nu(G_{V_i})} \geq 0,$$

and thus the weights are non-negative in Case 2. We leave it to the reader to verify that the remaining conditions of a normalized edge weight assignment are satisfied. □

Lemma 2 *Let a, b, c, d, x be positive real numbers such that $\frac{a}{b} \leq x$ and $\frac{c}{d} \leq x$. Then $\frac{a+c}{b+d} \leq x$.* □

Theorem 2 *Let G be a graph. Suppose V_1, $V_2, \ldots$, V_m, $m \geq 2$, form a regular partition of $V(G)$ with $d(G_{V_1}) \geq d(G_{V_2}) \geq \ldots \geq d(G_{V_m})$, and suppose $G_{V_1}, \ldots, G_{V_m}$ are balanced. If $d(G_{V_i}) \leq d(G_{V_{i+1}}) + k_i$ for $1 \leq i \leq m-1$, then G is balanced.*

Proof: We induct on m. The case $m = 2$ follows from the lemma. Suppose $m > 2$. By the induction hypothesis, $G_{V_1\cup\ldots\cup V_{m-2}}$ and $G_{V_1\cup\ldots\cup V_{m-1}}$ are balanced. Applying Lemma 1, it follows that

$$d(G_{V_1\cup\ldots\cup V_{m-2}}) \leq d(G_{V_{m-1}}) + k_1 + \ldots + k_{m-2},$$

and by the hypothesis of the theorem we have

$$d(G_{V_{m-1}}) \leq d(G_{V_m}) + k_{m-1}.$$

It follows that

$$d(G_{V_1\cup\ldots\cup V_{m-2}}) \leq d(G_{V_m}) + k_1 + \ldots + k_{m-1},$$

and

$$d(G_{V_{m-1}}) + k_1 + \ldots + k_{m-2} \leq d(G_{V_m}) + k_1 + \ldots + k_{m-1}.$$

Applying Lemma 2 gives us

$$d(G_{V_1\cup\ldots\cup V_{m-1}}) \leq d(G_{V_m}) + k_1 + \ldots + k_{m-1}.$$

By Lemma 1, this implies that G is balanced. □

The converse of Theorem 2 is false. Consider a graph G where V_1 and V_2 each induce a complete graph on 7 vertices, V_3 induces an independent set on 7 vertices, and V_1, V_2, and V_3 form a regular partition of $V(G)$ with $k_1 = k_2 = k_3 = 2$. Using Lemma 1, one can show that G is balanced, but $d(G_{V_2}) > d(G_{V_3}) + k_2$.

Before using Theorem 2 as a technical device, we observe that one may use Lemma 1 to paraphrase it as follows. If $G_{V_1}, \ldots, G_{V_m}$ form balanced graphs when they are joined pairwise, then they collectively form a balanced graph. Stated this way, the result seems quite remarkable.

Corollary 1 *Let G be a graph. Suppose that $V_1, \ldots, V_m$ form a regular partition of $V(G)$ with $d(G_{V_1}) = d(G_{V_2}) = \ldots = d(G_{V_m})$, and that G_{V_i} is balanced for $1 \leq i \leq m$. Then G is balanced.* $\Box$

An *automorphism* of a graph G is a bijection from $V(G)$ to itself which preserves edges and non-edges. A graph G is *edge-transitive* if for every pair of edges e_1 and e_2 there is an automorphism which maps e_1 to e_2.

Corollary 2 *If G is an edge-transitive graph, then G is balanced.*

Proof: If G is regular, then clearly it is balanced. If it is not regular, then there exist integers r and s, $r \neq s$, such that every edge of G joins a vertex of degree r to a vertex of degree s. The vertices of degree r and the vertices of degree s form a pair of independent sets, so we may apply Corollary 1 by letting V_1 be the set of vertices of degree r and letting V_2 be the set of vertices of degree s. $\Box$

Corollary 3 *Let G be a graph. Suppose $V_1, \ldots, V_m$ form a partition of $V(G)$ such that G_{V_i} is balanced for $1 \leq i \leq m$, and $x \sim y$ whenever $x \in V_i$, $y \in V_j$ and $i \neq j$. Then G is balanced.*

Proof: Without loss of generality, $d(G_{V_1}) \geq d(G_{V_2}) \geq \ldots \geq d(G_{V_m})$. Apply Theorem 2, with $k_i = |V_i|$, noting that $d(H) \leq \nu(H)$ for all graphs H. $\Box$

A *complete m-partite graph* is a graph G such that $V(G)$ can be partitioned into m independent sets $V_1, \ldots, V_m$ such that $x \sim y$ whenever $x \in V_i$, $y \in V_j$, and $i \neq j$. Note that Corollary 1 and Corollary 3 each imply the following.

Corollary 4 *If G is a complete m-partite graph, then G is balanced.* $\Box$

Finally, we prove a result which tells us how the balanced property behaves under three kinds of products of graphs. Let G and H be graphs. The *product* $G \times H$ is the graph on the vertex set $V(G) \times V(H)$ with $(x, y) \sim (x', y')$ if and only if $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y = y'$. The *lexicographic product* $G[H]$ is the graph on $V(G) \times V(H)$ with $(x, y) \sim (x', y')$ if and only if $x \sim x'$, or $x = x'$ and $y \sim y'$. The *normal product* $G \cdot H$ is the graph on the vertex set $V(G) \times V(H)$ with $(x, y) \sim (x', y')$ if and only if $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y = y'$, or $x \sim x'$ and $y \sim y'$.

Theorem 3 *Let G and H be balanced graphs with $V(G) = \{1, 2, ..., m\}$. Suppose P is a graph and k is a positive integer such that $V(P)$ can be partitioned into sets $V_1, V_2, \ldots V_m$ with P_{V_i} isomorphic to H for $1 \le i \le m$, $|N(x) \cap V_j| = k$ whenever $x \in V_i$ and $i \sim j$ in G, and $|N(x) \cap V_j| = 0$ whenever $x \in V_i$ and $i \not\sim j$ in G. Then P is balanced.*

Proof: By Theorem 1 we may assume we have fixed normalized edge weight assignments for $I(G)$ and $I(H)$, and thus for $I(P_{V_i})$, $1 \le i \le m$. We must produce a normalized edge weight assignment for $I(P)$. A typical edge of $I(P)$ is of the form (e, v), where $e \in E(P)$, $v \in V(P)$, and v is an endpoint of e. There are two cases to consider.

Case 1: $e \in E(P_{V_i})$ for some i. Assign (e, v) the weight $\nu(G)w$, where w is the weight of (e, v) in the normalized edge weight assignment for $I(P_{V_i})$

Case 2: $v \in V_i$ and e joins v to a vertex in V_j, $j \ne i$. Assign (e, v) the weight $\nu(H)w$, where w is the weight of $(\{i, j\}, i)$ in the normalized edge weight assignment for $I(G)$.

We leave it to the reader to verify that we have defined a normalized edge weight assignment for $I(P)$. □

Corollary 5 *If G and H are balanced graphs, then $G \times H$ is balanced.*

Proof: Apply Theorem 3 with $k = 1$. □

Corollary 6 *If G and H are balanced graphs, then $G[H]$ is balanced.*

Proof: Apply Theorem 3 with $k = \nu(H)$. □

Corollary 7 *If G is a balanced graph and H is a regular graph of degree d, then $G \cdot H$ is balanced.*

Proof: Apply Theorem 3 with $k = d + 1$. □

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