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Single Machine Scheduling with Earliness and Tardiness Penalties: When is an Optimal Solution Not Optimal?

by

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ABSTRACT

We consider the problem of finding the optimal schedule for jobs on a single machine when there are penalties for both late and early arrivals. We point out that if attention is paid to how certain parameters are measured, then a change of scale of measurement might lead to the anomalous situation where a schedule is optimal if these parameters are measured in one way, but not if they are measured in a different way that seems equally acceptable. We discuss conditions under which this anomaly is avoided.

1 Introduction

Many practical problems involve the search for an optimal schedule. We consider the problem of scheduling n jobs on a single machine in which each job has a specified due date or completion time and a penalty is applied for a completion time different from the desired one. In many practical problems, a penalty is only applied for tardy completions; while more generally, a penalty is applied to both early and tardy completions, perhaps in a different way. The interest in scheduling problems where penalties are applied to early arrivals as well as late arrivals is closely tied to the concept of "just-in-time" production, the goal being to have "the right amount of materials of the right quality at the right time in the right place to produce the right quantity of items demanded by the next step of the production" (Cheng [1990]). Because penalties can be applied to early arrivals, we shall allow the machine to lie idle. However, we schedule without preemption, i.e., we do not allow a job to be interrupted once it is started. The penalties we shall study involve weighting factors that weight the deviations from desired completion times.

These weights are typically measured using some scale of measurement, and we shall examine the effect on the solution to a scheduling problem if we make admissible changes of scale. We shall show that in some cases such changes can transform an optimal solution into a non-optimal one, and we systematically describe those situations when this anomaly occurs. The main point of this paper is to show that considerations of scale change need to play a role in analysis of scheduling problems and other similar problems of combinatorial optimization. More generally, we wish to make the reader aware of the need to consider sensitivity of conclusions to the change of scales of measurement, not only in the scheduling context, but also in other contexts in operations research.

In the rest of this section, we introduce the basic notation to be used throughout the paper. In Section 2, we introduce relevant concepts from the theory of measurement and give an example to illustrate how an optimal solution could change to a non-optimal one under admissible change of scale. Section 3 gives the systematic description of situations when admissible changes of scale preserve optimality. In Section 4, we discuss the results and present issues for further analysis.

We are interested in n jobs in which the i^{th} job has **processing time** $p_i > 0$ and **desired completion time** $d_i \geq 0$. (We do not assume that p_i and d_i are integers.) At most one job can be carried out at a time and there is no preemption. A **schedule** S (**feasible schedule** S) assigns a **completion time** C_i to each job i, with time starting at 0 and each job i getting processing time p_i. We shall let $C(S) = (C_1, \ldots, C_n)$ and we shall also use the vectors $\boldsymbol{d} = (d_1, \ldots, d_n)$ and $\boldsymbol{p} = (p_1, \ldots, p_n)$, where we use the convention that single boldface letters represent vectors. Since no job may start before time 0, we have $C_i \geq p_i$, for all i. Our terminology follows that of Baker and Scudder [1990].

We shall be interested in two special cases, those with **common due dates**, i.e., where $d_i = d$ for all i, and those with **common processing times**, i.e., where $p_i = p$ for all i. The first case will be abbreviated **cdd**, with **ncdd** standing for the situation where cdd fails. The second case will be abbreviated **cpt**, with **ncpt** standing for the situation where cpt fails.

Typically, we have an **objective function**

$$F(S) = F(d_1, \ldots, d_n, C_1, \ldots, C_n)$$

and we seek a (feasible) schedule S that minimizes $F(S)$. Note that the objective function does not explicitly depend upon the processing times. These play a role in determining whether or not a schedule is feasible. We shall only consider objective functions that depend upon the deviations between desired and actual completion times, and specifically we define the **earliness**

$$E_i(S) = E_i = \max\{0, d_i - C_i\}$$

and the **tardiness**

$$T_i(S) = T_i = \max\{0, C_i - d_i\}.$$

One objective function arises if we simply consider a weighted sum of earliness and tardiness, obtaining

$$F(S) = F_{sumE/T}(S) = \sum_{i=1}^{n} \alpha_i E_i + \sum_{i=1}^{n} \beta_i T_i,$$

where α_i and β_i are positive weights. If only tardiness is penalized, we consider the function

$$F(S) = F_{sumT}(S) = \sum_{i=1}^{n} \beta_i T_i.$$

In objective functions $F_{sumE/T}(S)$ and $F_{sumT}(S)$, we minimize the sum of the deviations. An alternative is to minimize the maximum deviation; we then have the objective functions

$$F(S) = F_{maxE/T} = \max\{\alpha_1 E_1, \ldots, \alpha_n E_n, \beta_1 T_1, \ldots, \beta_n T_n\}$$

and

$$F(S) = F_{maxT} = \max\{\beta_1 T_1, \ldots, \beta_n T_n\}.$$

A variety of other objective functions have been studied in the literature; many are summarized by Baker and Scudder [1990]. We limit ourselves to these four.

In the case where $\alpha_i = \beta_i$ for all i, the objective function has symmetric weights and we say that we are in the case of **symmetry**; we abbreviate this case as **s**. If we are not in case s, we say we are in case **ns**. If $\beta_i = \beta$ for all i and if $\alpha_i = \alpha$ for all i when we are considering earliness penalties, then we say that we have **common weights** and abbreviate this case as **cw**. If we do not have common weights, we say we are in case **ncw**. In Section 3, we shall systematically study the four penalty functions $F_{sumE/T}$, F_{sumT}, $F_{maxE/T}$, and F_{maxT} under all combinations of the four assumptions cdd or ncdd, cpt or ncpt, s or ns, and cw or ncw. In other words, we consider sixteen cases for each objective function. Of course, when we do not consider earliness penalties, considerations of symmetry are irrelevant, so there are only eight cases. We shall make the notational convention that we are in case ns here.

Let us consider the function $F_{sumE/T}(S)$ for example. The case cw, cdd, s (and either cpt or ncpt) gives rise to the objective function

$$F(S) = \alpha \sum_{i=1}^{n} |C_i - d|,$$

where of course $d_i = d$ for all i. For all practical purposes, this is the same as the objective function

$$F(S) = \sum_{i=1}^{n} |C_i - d|.$$

This objective function arises, for example, in the work of Kanet [1981], Sundararaghavan and Admed [1984], Bagchi, Sullivan and Chang [1986], Hall [1986], Emmons [1987], Szwarc [1989], and Hall, Kubiak, and Sethi [1991]. The case cw, cdd, ns (and cpt or ncpt) gives rise to the objective function

$$F(S) = \alpha \sum_{i=1}^{n} E_i + \beta \sum_{i=1}^{n} T_i,$$

and this objective function has been studied by, for example, Panwalker, Smith, and Seideman [1982], Emmons [1987], and Bagchi, Chang, and Sullivan [1987].

The objective function arising in the case ncw, cdd, ns (and cpt or ncpt) has been studied, for example, by Ahmed and Sundararaghavan [1990], Cheng [1987], Emmons [1987], Quaddus [1987], Bector, Gupta and Gupta [1988], Baker and Scudder [1989], Hall and Posner [1991], and Hoogeveen and van de Velde [1991]. The objective function arising in the case ncw, ncdd, ns (and cpt or ncpt) has been studied, for example, by Fry, Armstrong, and Blackstone [1987], Fry and Leong [1987], Fry, Leong, and Rakes [1987], Chand and Schneeberger [1988], Abdul-Razaq and Potts [1988], and Ow and Morton [1988,1989]. The objective function arising in the case cw, ncdd, s (and cpt or ncpt) has been studied, for example, by Cheng [1990].

The objective function $F_{maxE/T}$ is considered in the work of such authors as Sidney [1977] and, more recently for example, by Ferris and Vlach [1992]. The objective functions where only tardiness is considered are studied by a wide variety of authors; two survey papers in this literature are that by Koulamas [1994], emphasizing constant-weighted tardiness, and that by Abdul-Razaq, Potts, and van Wassenhove [1990], concerning weighted tardiness.

2 Meaningfulness of the Conclusion of Optimality

In using scales of measurement, we often make somewhat arbitrary choices such as of unit or zero point. One speaks of **admissible transformations** of scale as transformations of scale values that result ¿from changes in these arbitrary choices. For example, in the case of measurement of mass, we can change from pounds to ounces by changing the unit, and this amounts to multiplying each scale value by $\frac{1}{16}$. Similarly, we can change from kilograms to pounds by multiplying each scale value by 2.2. When the admissible transformations of scale correspond exactly to multiplication by a positive constant, as here, we say we have a **ratio scale**. In contrast, if we are measuring temperature, we can change both the unit and the zero point. For example, in going from degrees Centigrade to degrees Fahrenheit, we use the admissible transformation $\frac{9}{5}x + 32$, which changes the unit by 9/5 and the zero point by 32. When the admissible transformations of scale correspond exactly to multiplication by a positive constant and addition of another constant (not necessarily positive) as here, we say we have an **interval scale**. When any (strictly) monotone increasing transformation of scale is admissible, we say we have an **ordinal scale**. An example of an ordinal scale is the Mohs scale of hardness. Every mineral gets a scale value which is an integer between 1 and 10, and the only significance of the numbers is that a mineral with a higher number scratches a mineral with a lower number. Any (strictly) monotone increasing function of scale values would give a scale with similar properties. Preferences sometimes only give rise to ordinal scales. The theory of scale type was introduced into measurement theory by Stevens [1946,1951,1959]. A general introduction to measurement theory can be found in the books by Krantz, et al. [1971], Luce, et al. [1990], Suppes, et al. [1989], and Roberts [1979].

If the truth of a conclusion can depend upon some arbitrary choices involving scales of measurement, as for example about units or zero points, we would probably not want to put much weight behind that conclusion. In measurement theory, we call a statement using scales **meaningful** if its truth or falsity is unchanged after applying admissible transformations to all of the scales in the statement. We shall show that in the case of the single machine scheduling problem with earliness and tardiness penalties, the conclusion that one schedule is optimal can be meaningless. Whether or not it is meaningful depends upon how we measure penalties. For further information about meaningful statements, the reader is referred to Roberts [1985,1994] and Luce, et al [1990]. For other applications of this concept to problems of combinatorial optimization like the scheduling problems discussed here, see Roberts [1990] and Cozzens and Roberts [1991].

To give a simple example of the application of measurement theoretical ideas to the scheduling problem we have considered, let us suppose that the weights in our objective functions are measured on a ratio scale. This is quite naturally the case, since they are often thought of as dollar penalties, and the dollar defines a ratio scale (we can change the unit by changing to cents or Deutschmarks or Yen). Consider the objective function $F_{sumE/T}$. Suppose that schedule S is optimal. This means that

$$F_{sumE/T}(S) \leq F_{sumE/T}(S') \tag{1}$$

for all schedules S'. For conclusion (1) to be meaningful, we would like it to hold if changes in unit are made in the parameters α_i and β_i. Let $F'_{sumE/T}(S')$ denote the value of the objective function for schedule S' if α_i and β_i are transformed by a change of scale into $k\alpha_i$ and $k\beta_i$, respectively, with $k > 0$. We want to show that

$$F'_{sumE/T}(S) \quad \leq \quad F'_{sumE/T}(S')$$

holds for all schedules S'. This is true since

$$F'_{sumE/T}(S') = \sum_{i=1}^{n} k\alpha_i E_i(S') + \sum_{i=1}^{n} k\beta_i T_i(S') = kF_{sumE/T}(S').$$

Hence, if weights are measured on a ratio scale, and if objective function $F_{sumE/T}$ is used, it is meaningful to conclude that a schedule S is optimal. Similar reasoning applies to objective functions $F_{sumT}(S)$, $F_{maxE/T}$, and F_{maxT}.

However, if weights are measured on an interval scale, there can be problems, even in the situation where cases cw, cdd, and cpt all apply. To see why, consider the situation where there are three jobs, there are common due dates with $d = 3$, and there are common processing times with $p = 1$. Suppose that the common weights are given by $\alpha = 1$ and $\beta = 10$. Consider the schedule S defined by the vector of completion times $C(S) = (1, 2, 3)$. For this schedule,

$$F_{sumE/T}(S) = \alpha_1(3-1) + \alpha_2(3-2) + \beta_3(3-3) = 1 \cdot 2 + 1 \cdot 1 + 10 \cdot 0 = 3.$$

This turns out to be an optimal schedule. To see why, consider any other schedule S'. By symmetry, we may assume that in $C(S') = (C'_1, C'_2, C'_3)$, $C'_1 < C'_2 < C'_3$. Since processing times are all 1, we conclude that $C'_1 \geq 1$, $C'_2 \geq 2$, $C'_3 \geq 3$. Since $S' \neq S$, we must have $C'_3 > 3$. If $C'_3 \geq 4$, then

$$F_{sumE/T}(S') \geq \beta_3(4-3) = 10 \cdot 1 = 10 > F_{sumE/T}(S).$$

Suppose that $C'_3 < 4$. Then because $p_3 = 1$, $C'_2 < 3$. Since $p_2 = p_3 = 1$, we have $C'_3 - C'_2 \geq 1$, $C'_3 - C'_1 \geq 2$. Then, since $C'_3 > 3$, we have

$$\begin{aligned} F_{sumE/T}(S') &= \alpha_1(3 - C'_1) + \alpha_2(3 - C'_2) + \beta_3(C'_3 - 3) \\ &= 3 - C'_1 + 3 - C'_2 + 10(C'_3 - 3) \\ &= 8(C'_3 - 3) + (C'_3 - C'_2) + (C'_3 - C'_1) \\ &\geq 8(C'_3 - 3) + 1 + 2 > 3 = F_{sumE/T}(S). \end{aligned}$$

Thus, S is optimal. Now suppose that we change α_i into $k\alpha_i + l$ and β_i into $k\beta_i + l$, where $k = 1$, $l = 100$ give us a change of unit (no change) and a change of zero point, respectively. We get

$$\alpha'_1 = \alpha'_2 = \alpha'_3 = 101, \beta'_1 = \beta'_2 = \beta'_3 = 110.$$

As before, let $F'_{sumE/T}(S)$ denote the value of the objective function for schedule S if α_i and β_i are transformed into α'_i and β'_i, respectively. Then we have

$$F'_{sumE/T}(S) = \alpha'_1(3-1) + \alpha'_2(3-2) + \beta'_3(3-3) = 101 \cdot 2 + 101 \cdot 1 + 110 \cdot 0 = 303.$$

Compare S to the schedule S' with $C(S') = (2, 3, 4)$. We have

$$F'_{sumE/T}(S') = \alpha'_1(3-2) + \beta'_2(3-3) + \beta'_3(4-3) = 101 \cdot 1 + 110 \cdot 0 + 110 \cdot 1 = 211.$$

Thus, schedule S is no longer optimal after a change of unit and zero point in measurement of weights. Hence the conclusion that schedule S is optimal is not meaningful. It does not seem to make sense to assert that a solution to this scheduling problem is optimal if penalties are measured on an interval scale.

Note that it is reasonable in some applications to think of weights as being measured on a scale different from a ratio scale. In particular, if we are not given weights, but only know that some jobs are more important than others, we might only have an ordinal scale. Even if weights are thought of as monetary penalties, the scale they define is like an interval scale if we have a fixed fee for currency transactions, e.g., from dollars to Deutschmarks. (This isn't quite an interval scale if there is no fee for transactions changing dollars to cents.) If the weights measure some value or utility (or disutility) associated with a given early or tardy completion, or loss of goodwill due to completion at a time different than the requested time, then it is possible that we would measure these with a scale where change of zero point as well as change of unit is allowed, as is commonly done in utility theory. In this case, we have an interval scale. Quaddus [1987] calls α_i and β_i "costs", which might suggest measuring them in dollar amounts and therefore possibly on a ratio scale. However, he points out that techniques from preference and value theory, such as those developed by Keeney and Raiffa [1976], can be used to estimate these costs; this suggests that they are measures of utility or value. Hall and Posner [1991] speak of the α_i and β_i as weights or "values", which suggests again that they could be measured on scales other than ratio scales.

As the previous discussion shows, the conclusion that S is an optimal schedule may not be meaningful if weights are measured on interval scales. In the next section, we systematically study the four objective functions defined above if weights are measured on interval scales and determine the meaningfulness of the conclusion of optimality under the sixteen cases defined by the presence or absence of the conditions cw, cdd, cpt, and s. A similar analysis for ordinal scales is briefly presented in the discussion section.

3 Meaningfulness of Optimality Under Interval Scales

Our results are summarized in Table 1. They show that with common and symmetric weights, the conclusion of optimality is meaningful for all the objective functions we have considered, but with common and non-symmetric weights, it can be meaningless for $F_{sumE/T}$ and $F_{maxE/T}$. With non-common weights, the conclusion of optimality can be meaningless in all cases except the ncw, cdd, cpt, s case for $F_{sumE/T}$ and the corresponding ncw, cdd, cpt, ns case for F_{sumT}. The detailed conclusions are not as important as the principle that the conclusion of optimality may indeed be meaningless and that it is necessary to take considerations of meaningfulness into account in analysis of scheduling problems and, more generally, of problems of operations research.

The entries in Table 1 are justified by a series of examples, theorems, and principles that give us a way of constructing new examples from old ones. Some of the details are left to the appendix at the end. The reader who is most interested in the general principles developed in this paper or who wishes to use the table for reference without going through the detailed justifications may wish to skip the appendix. Since one example suffices to show that a conclusion can be meaningless, the reader might naturally ask if the examples we present are simply isolated cases. The answer is that they are not, and that it is not hard to produce many other examples.

Throughout this section, we adopt the convention that when scale values are required to be positive, by admissible transformations of interval scales we mean the transformations x into $kx + l$, $k > 0$, that leave all weights positive; in other words, $k > 0$ and l must be chosen so that all $k\alpha_i + l$ and $k\beta_i + l$ are positive. We still call the corresponding scale an interval scale although technically not all transformations of the form x into $kx + l$ are allowed.

Example 1 *If the objective function is $F_{sumE/T}$, then in the cw, cdd, cpt, ns case, the conclusion of optimality can be meaningless.*

Proof. The example here is the example given in the previous section, and we simply give it a number here so we may refer to it in Table 1. To summarize, we have $n = 3$, $d = 3$, $p = 1$, $\alpha = 1$, $\beta = 10$. The schedule S defined by $C(S) = (1, 2, 3)$ is optimal, but if $k = 1$, $l = 100$, then with $C(S') = (2, 3, 4)$, we have after transformation of weights that $F'_{sumE/T}(S') < F'_{sumE/T}(S)$. So S is no longer optimal. ∎

Remark. We say in Example 1 (and Table 1) that the condition of optimality **can** be meaningless. However, this is simply to say that there are examples satisfying the case cw, cdd, cpt, ns under the objective function $F_{sumE/T}$ for which this conclusion is meaningless. It is not to say that all situations satisfying these conditions under this objective function will make the conclusion meaningless. For instance, let us take $n = 2$, $d = 2$, $p = 1$, $\alpha = 1$, $\beta = 10$, and let $C(S) = (1, 2)$. It can be seen that S is optimal and that it remains optimal after a transformation x into $kx + l$ with $k > 0$ and $\alpha' = k\alpha + l > 0$ and $\beta' = k\beta + l > 0$. Actually, it suffices to prove the latter because a special case has $k = 1$, $l = 0$, which gives us back α and β.

Example 2 *If the objective function is $F_{sumE/T}$, then in the cw, cdd, ncpt, ns case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 3$, $d = 3.1$, $\boldsymbol{p} = (1, 1, 1.1)$, $\alpha = 1$, $\beta = 10$. Then $C(S) = (2.1, 3.1, 1.1)$ is optimal (see appendix). Now, let $k = 1$, $l = 100$. Then after change of scale, we get $\alpha' = 101$, $\beta' = 110$. We have

$$F'_{sumE/T}(S) = 101(3.1 - 2.1) + 110(3.1 - 3.1) + 101(3.1 - 1.1) = 303.$$

However, if $C(S') = (2, 3, 4.1)$, we have

$$F'_{sumE/T}(S') = 101(3.1 - 2) + 101(3.1 - 3) + 110(4.1 - 3.1) = 231.2,$$

and so S is no longer optimal. ∎

Remark. Example 2 can be generalized to give one example that covers all cases with cw, ns. We take $n = 3$, $\boldsymbol{d} = (3, 3, 3+q)$, $\boldsymbol{p} = (1, 1, 1+r)$, $\alpha = 1$, $\beta = 10$, with $q, r \geq 0$, $|q-r| < 1$. If $C(S) = (1+x, 2+x, 3+q)$, where $x = q - r$, then S turns out to be optimal. However, if $k = 1$, $l = 100$, then after the corresponding change of scale, S' with $C(S') = (2, 3, 4 + r)$ has a lower objective function value than S. Details are a bit messy and are left to the reader.

Example 3 *If the objective function is $F_{sumE/T}$, then in the ncw, cdd, cpt, ns and ncw, cdd, ncpt, ns cases, the conclusion of optimality can be meaningless.*

Proof. Let $n = 3$, $d = 2$, $\boldsymbol{p} = (1, 1, 1 + q)$ with $q = 0$ or $.01$, $\boldsymbol{\alpha} = (\frac{1}{2}, \frac{1}{3}, 1)$, $\boldsymbol{\beta} = (\frac{1}{16}, \frac{1}{81}, 1)$, and take $C(S) = (3, 4, 2)$. The case $q = 0$ of course will give us cpt, while the case $q = .01$ will give us ncpt. Then S is optimal (see appendix). Let $k = 1$, $l = 1/2$, giving us $\boldsymbol{\alpha}' = (1, \frac{5}{6}, \frac{3}{2})$, $\boldsymbol{\beta}' = (\frac{9}{16}, \frac{83}{162}, \frac{3}{2})$. Now we have

$$F'_{sumE/T}(S) = \frac{9}{16}(3 - 2) + \frac{83}{162}(4 - 2) + \frac{3}{2}(2 - 2) > 1.58.$$

If $C(S') = (1, 3 + q, 2 + q)$, then

$$F'_{sumE/T}(S') = 1 \cdot (2 - 1) + \frac{83}{162}(3 + q - 2) + \frac{3}{2}(2 + q - 2) = 1 + \frac{83}{162} + q(\frac{83}{162} + \frac{3}{2}) < 1.55,$$

since $q \leq .01$. We conclude that S is no longer optimal. ∎

Example 4 *If the objective function is $F_{sumE/T}$, then in the ncw, cdd, ncpt, s case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 3$, $d = 0$, $\boldsymbol{p} = (1, 1, 2)$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = (2, 2, 5)$. Let $C(S) = (3, 4, 2)$. Then S is optimal. To see why, note that since $d_i = 0$ for all i, an optimal schedule will start at time 0 and have no idle times. Without loss of generality job 1 completes before job 2. Hence, the only other candidates for optimal schedule are defined by $C(S') = (1, 4, 3)$ and $C(S'') = (1, 2, 4)$. A simple calculation shows that $F_{sumE/T}(S) = 24$, while $F_{sumE/T}(S') = 25$ and $F_{sumE/T}(S'') = 26$, so S is indeed optimal. Now take $k = 1$, $l = 3$, giving $\boldsymbol{\alpha}' = \boldsymbol{\beta}' = (5, 5, 8)$. We have $F'_{sumE/T}(S) = 51$, but $F'_{sumE/T}(S') = 49$. So S is no longer optimal. ∎

Remark. We have used $d = 0$ in the previous example. If that makes the reader uncomfortable, we could use an arbitrary d, add a fourth job with $p_4 = d$, $\alpha_4 = \beta_4 = 10^{10}$. Then it is clear that the fourth job is always first and the rest of the argument is as in Example 4.

Example 5 *If the objective function is $F_{sumE/T}$, then in the ncw, ncdd, cpt, s case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 4$, $\boldsymbol{d} = (2, 2, 2, 1)$, $p = 1$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = (9, 9, 9, 1)$. Then $C(S) = (2, 1, 3, 4)$ is optimal (see appendix). Now, let $k = 1/8$, $l = 7/8$. Then $\boldsymbol{\alpha}' = \boldsymbol{\beta}' = (2, 2, 2, 1)$ and

$$F'_{sumE/T}(S) = 2(2-2) + 2(2-1) + 2(3-2) + 1 \cdot (4-1) = 7.$$

However, if $C(S') = (2, 3, 4, 1)$, then

$$F'_{sumE/T}(S') = 2(2-2) + 2(3-2) + 2(4-2) + 1 \cdot (1-1) = 6,$$

so S is no longer optimal. ∎

Example 6 *If the objective function is F_{sumT}, then in the ncw, cdd, ncpt, ns case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 2$, $d = 1$, $\boldsymbol{p} = (1, 2)$, $\boldsymbol{\beta} = (1, 3)$. Let $C(S) = (3, 2)$. Since only tardiness matters and $p_i \geq 1 = d$, an optimal schedule will start at time 0 and have no idle time. Thus, we need only consider the alternative schedule S' given by $C(S') = (1, 3)$. We have $F_{sumT}(S) = 5$, while $F_{sumT}(S') = 6$, so S is optimal. However, if we take $k = 1$, $l = 2$, we get $\boldsymbol{\beta}' = (3, 5)$ and now $F'_{sumT}(S) = 11$ while $F'_{sumT}(S') = 10$. So S is no longer optimal. ∎

Example 7 *If the objective function is F_{sumT}, then in the ncw, ncdd, cpt, ns case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 4$, $\boldsymbol{d} = (2, 2, 2, 1)$, $p = 1$, $\boldsymbol{\beta} = (9, 9, 9, 1)$. Then $C(S) = (1, 2, 3, 4)$ is optimal (see appendix). Now let $k = 5/8$, $l = 99/8$. Then $\boldsymbol{\beta}' = (18, 18, 18, 13)$ and

$$F'_{sumT}(S) = 18 \cdot 0 + 18 \cdot 1 \;\; + 13 \cdot 3 = 57,$$

while if $C(S') = (2, 3, 4, 1)$,

$$F'_{sumT}(S') = 18 \cdot 0 + 18 \cdot 1 + 18 \cdot 2 + 13 \cdot 0 = 54.$$

Hence, S is no longer optimal. ∎

Example 8 *If the objective function is $F_{maxE/T}$, then in the cw, cdd, cpt, ns case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 3$, $d = 3$, $p = 1$, $\alpha = 1$, $\beta = 10$. Then $C(S) = (\frac{13}{11}, \frac{24}{11}, \frac{35}{11})$ is optimal (see appendix). Now take $k = 1$, $l = 100$, giving us $\alpha' = 101$, $\beta' = 110$. It follows that

$$F'_{maxE/T}(S) \geq 101(\frac{20}{11}) > 180.$$

However, if $C(S') = (2, 3, 4)$, then

$$F'_{maxE/T}(S') = \max\{101(3-2), 110(3-3), 110(4-3)\} = 110 < 180,$$

and we conclude that S is no longer optimal. ∎

Example 9 *If the objective function is $F_{maxE/T}$, then in the cw, cdd, ncpt, ns case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 3$, $d = 3$, $\boldsymbol{p} = (1, 1, 1.1)$, $\alpha = 1$, $\beta = 10$. Then $C(S) = (\frac{24}{11}, \frac{35}{11}, \frac{13}{11})$ is optimal (see appendix). Now, let $k = 1$, $l = 100$, giving us $\alpha' = 101$, $\beta' = 110$. It follows that

$$F'_{maxE/T}(S) \geq 101(3 - \frac{13}{11}) > 180.$$

However, if $C(S') = (2, 3, 4.1)$, then

$$F'_{maxE/T}(S') = \max\{101(3-2), 110(3-3), 110(4.1-3)\} = 121.$$

Hence, S is no longer optimal. ∎

Example 10 *If the objective function is $F_{maxE/T}$, then in the ncw, cdd, cpt, s case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 4$, $d = 1$, $p = 1$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = (10, 9, 3, 2)$. Then $C(S) = (1, 2, 4, 3)$ is optimal (see appendix). Let $k = 1$ and $l = 1$, giving us $\boldsymbol{\alpha}' = \boldsymbol{\beta}' = (11, 10, 4, 3)$. Then

$$F'_{maxE/T}(S) = \max\{11(1-1), 10(2-1), 4(4-1), 3(3-1)\} = 12,$$

while if $C(S') = (1, 2, 3, 4)$, we have

$$F'_{maxE/T}(S') = \max\{11(1-1), 10(2-1), 4(3-1), 3(4-1)\} = 10.$$

We conclude that S is no longer optimal. ∎

Example 11 *If the objective function is $F_{maxE/T}$, then in the ncw, cdd, ncpt, s case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 3$, $d = 3$, $\boldsymbol{p} = (2, 1, 1)$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = (1, 2, 2)$. Then $C(S) = (5, 2, 3)$ (see appendix). Let $k = 1$, $l = 1$, giving us $\boldsymbol{\alpha}' = \boldsymbol{\beta}' = (2, 3, 3)$. Then

$$F'_{maxE/T}(S) = \max\{2(5-3), 3(3-2), 3(3-3)\} = 4,$$

while if $C(S') = (2, 3, 4)$, then

$$F'_{maxE/T}(S') = \max\{2(3-2), 3(3-3), 3(4-3)\} = 3.$$

We conclude that S is no longer optimal. ∎

Example 12 *If the objective function is F_{maxT}, then in the ncw, cdd, cpt, ns case, the conclusion of optimality can be meaningless.*

Proof. Use Example 10 except leave $\boldsymbol{\alpha}$ undefined. Then $F_{maxT}(S')$ in this problem is the same as $F_{maxE/T}(S')$ in Example 10, and similarly $F'_{maxT}(S')$ is the same as $F'_{maxE/T}(S')$ in Example 10. This is because any schedule has all $C_i \geq 1$. Thus, the analysis of Example 10 applies. ∎

Example 13 *If the objective function is F_{maxT}, then in the ncw, cdd, ncpt, ns case, the conclusion of optimality can be meaningless.*

Proof. Let $n = 3$, $d = 2$, $\boldsymbol{p} = (3, 2, 2)$, $\boldsymbol{\beta} = (10, 2, 1)$, and let $C(S) = (3, 7, 5)$. Then

$$F_{maxT}(S) = \max\{10(3-2), 2(7-2), 1 \cdot (5-2)\} = 10.$$

For any S' with $C(S') = (C_1', C_2', C_3')$, we have $C_1' \geq 3$ since $p_1 = 3$, and therefore

$$F_{maxT}(S') \geq 10(3-2) = 10 = F_{maxT}(S).$$

Hence, S is optimal

Now let $k = 1$, $l = 5$, giving us $\boldsymbol{\beta}' = (15, 7, 6)$. Then

$$F'_{maxT}(S) \geq 7(7-2) = 35$$

while, if $C(S') = (3, 5, 7)$,

$$F'_{maxT}(S') = \max\{15(3-2), 7(5-2), 6(7-2)\} = 30.$$

It follows that S is no longer optimal. ∎

We now present some positive results.

Theorem 1 *If the objective function is $F_{sumE/T}$, then in all cases in which cw, s hold, the conclusion of optimality is meaningful.*

Proof. We have

$$F_{sumE/T}(S') = \alpha \sum E_i(S') + \alpha \sum T_i(S').$$

Thus,

$$F'_{sumE/T}(S') = (k\alpha + l) \sum E_i(S') + (k\alpha + l) \sum T_i(S') = (k + \frac{l}{\alpha}) F_{sumE/T}(S').$$

By our convention, $k\alpha + l > 0$. Thus, since $\alpha > 0$,

$$F_{sumE/T}(S) \leq F_{sumE/T}(S') \quad \text{if and only if}$$

$$(k + \frac{l}{\alpha}) F_{sumE/T}(S) \leq (k + \frac{l}{\alpha}) F_{sumE/T}(S') \quad \text{if and only if}$$

$$F'_{sumE/T}(S) \leq F'_{sumE/T}(S').$$

∎

The next three theorems can also be proved similarly.

Theorem 2 *If the objective function is F_{sumT}, then in all cases in which cw holds, the conclusion of optimality is meaningful.*

Theorem 3 *If the objective function is $F_{maxE/T}$, then in all cases in which cw, s hold, the conclusion of optimality is meaningful.*

Theorem 4 *If the objective function is F_{maxT}, then in all cases in which cw holds, the conclusion of optimality is meaningful.*

Theorem 5 *If the objective function is $F_{sumE/T}$, then in all cases in which cdd, cpt, s hold, the conclusion of optimality is meaningful.*

Proof. Suppose without loss of generality that

$$\alpha_1 \geq \cdots \geq \alpha_n.$$

We obtain a **greedy schedule** as follows: Let j be the largest integer such that $d - jp \geq p$. Then one greedy solution is schedule T defined by

$$C(T) = (d, d-p, d+p, d-2p, d+2p, \ldots, d-jp, d+jp, d+(j+1)p, d+(j+2)p, \ldots).$$

Note that we cannot use $d-(j+1)p$ because $C_i \geq p$. All other greedy schedules are obtained from T by switching C_{2i} and C_{2i+1} for possibly some i with $C_{2i} \leq d - jp$, and by switching C_i and C_j for possibly some pairs (i,j) with $\alpha_i = \alpha_j$.

We say that (i,j) is a **reversing pair** for S if $\alpha_i > \alpha_j$ and $|C_i - d| > |C_j - d|$. It is easy to see that S with reversing pairs cannot be optimal: simply switch C_i and C_j to obtain a schedule S' with a smaller $F_{sumE/T}(S')$. Also, since no two C_i can be equal, at most two $|C_i - d|$ can be equal. It follows that S is optimal if and only if it is greedy.

We complete the proof by noting that any transformation x into $kx + l$, $k > 0$, doesn't change the greedy solutions since the ordering of weights is unchanged. Thus, the same solutions remain optimal. ∎

Theorem 6 *If the objective function is F_{sumT}, then in all cases in which cdd, cpt hold, the conclusion of optimality is meaningful.*

Proof. The proof is similar to that of Theorem 5. We suppose without loss of generality that

$$\beta_1 \geq \cdots \geq \beta_n.$$

Then the **greedy solutions** are those obtained from

$$C(T) = (0, p, 2p, \ldots)$$

by switching some C_i and C_j that are less than are equal to d (since we do not penalize earliness) and switching C_i and C_j for possibly some pairs (i,j) with $\beta_i = \beta_j$. It is easy to see that S is optimal if and only if it is greedy and that the transformation x into $kx + l$, $k > 0$, does not change the greedy solution because it does not change the order of weights. ∎

We conclude the verification of Table 1 by introducing two principles for generating new examples from old examples.

Principle 1 *For $F = F_{sumE/T}$, F_{sumT}, $F_{maxE/T}$, or F_{maxT}, suppose E is an example in a case with cdd for which the conclusion of optimality is meaningless. Then we can find an example E' in which the conclusion of optimality is meaningless and ncdd holds. Moreover, if ncpt or s or ns holds in E, it also holds in E'. However, if cw holds in E, we can arrange the example so that cw also holds in E' and we can arrange it so that ncw holds in E'. Also, if cpt holds in E, we can arrange the example so that cpt also holds in E' and we can arrange it so that ncpt holds in E'.*

Proof. Given example E, define E' by adding a new $(n+1)$-st job with a very large due date d_{n+1}, a processing time p_{n+1} equal to the minimum p_i from E, and weights $\alpha_{n+1} = \min\ \alpha_i$ and $\beta_{n+1} = \min\ \beta_i$ from E. To be specific, we could take

$$d_{n+1} = \max_{i \leq n} d_i + \sum_{i=1}^{n} p_i + \max_{i \leq n} p_i$$

If S was optimal for E, then taking $C_{n+1} = d_{n+1}$ certainly gives an optimal schedule $\overline{S}$ for E'. If in E, S' had a lower objective function value than S after a transformation of weights, then in E', taking $C_{n+1} = d_{n+1}$ gives a schedule $\overline{S'}$ with a lower objective function value than $\overline{S}$ after the same transformation of weights.

Note that E' has property ncdd. If E had ncpt or ns or s, so does E'. If E had cw, then E' as constructed has cw. However, we can modify the construction of E' by changing α_{n+1} and β_{n+1} to be some very small numbers, getting an example E' with ncw. If E had cpt, then E' as constructed has cpt. However, we can modify E' by changing p_{n+1} to be some very small amount and get an example E' with ncpt. ∎

Principle 2 *For $F = F_{maxE/T}$, suppose E is an example in the case cw, cdd, cpt, ns for which the conclusion of optimality is meaningless. Suppose that schedule S is optimal for $F_{maxE/T}$ but not optimal after a transformation of weights using the transformation x into $kx + l$,* **with** *$l > 0$ (rather than any real l), because $F'_{maxE/T}(S) > F'_{maxE/T}(S')$. Then we can find an example E' in which the conclusion of optimality is meaningless and ncw, cdd, and ns hold. Moreover, we can arrange the example so that cpt also holds in E' and we can arrange it so that ncpt holds in E'.*

Proof. Given E, define E' by adding an $(n+1)^{st}$ job with $d_{n+1} = d$, and $p_{n+1} = q$, where we use $q = p$ if we want cpt and arbitrary $q \neq p$ otherwise. Pick $\gamma > 0$ so that

$$\gamma|(\max_{i \leq n} C_i) + q - d| < F_{maxE/T}(S) \tag{2}$$

and

$$\gamma|(\max_{i \leq n} C'_i) + q - d| < F'_{maxE/T}(S'). \tag{3}$$

Let $\alpha_{n+1} = \beta_{n+1} = \gamma$. Define $\overline{S}$ from S by letting $C_{n+1} = (\max_{i \leq n} C_i) + q$ and $\overline{S'}$ ¿from S' by letting $C'_{n+1} = (\max_{i \leq n} C'_i) + q$. Then by Equation 2,

$$F_{maxE/T}(\overline{S}) = F_{maxE/T}(S).$$

For every schedule $\overline{U}$ for the $(n+1)$ jobs-problem, let U be $\overline{U}$ restricted to the first n jobs. It follows that

$$F_{maxE/T}(\overline{U}) \geq F_{maxE/T}(U) \geq F_{maxE/T}(S) = F_{maxE/T}(\overline{S}),$$

and so $\overline{S}$ is optimal. Note that since $l > 0$, the transformation x into $kx + l$ is admissible since it takes α_{n+1} and β_{n+1} into positive weights. By Equation 3, we have

$$F'_{maxE/T}(\overline{S'}) = F'_{maxE/T}(S') < F'_{maxE/T}(S) \leq F'_{maxE/T}(\overline{S}).$$

Hence, $\overline{S}$ is no longer optimal after the transformation. ∎

4 Discussion

We have shown that there are situations in scheduling in which an admissible change of scale can change an optimal solution to one which is not optimal. The idea that one needs to consider such a possibility is the most important message in this paper, more important than the specific examples we give.

If weights are measured on a ratio scale, then conclusions about optimality of a schedule are meaningful under the four objective functions we have considered. However, this is not always true for interval scales. The results of Table 1 can be summarized as follows: in the case of common weights, if weights are symmetric or if we only apply penalties for tardiness, then conclusions about optimality are meaningful for all of the objective functions we have considered. In other cases, these conclusions can be meaningless, except in the cases ncw, cdd, cpt, s for objective function $F_{sumE/T}$ and ncw, cdd, cpt, ns for objective function F_{sumT}.

This paper leaves room for considerably more analysis. For one thing, we have considered only single machines, rather than multiple machines.

Second, we have limited discussion to four specific objective functions. A variety of other objective functions are studied in the literature (many are surveyed by Baker and Scudder [1990]). It would be interesting to extend the analysis in this paper to some of these other objective functions as well.

Third, we have not considered other types of scales, in particular ordinal scales. It turns out that the results in Table 1 depend only on ordinal scales. For instance, it is easy to see that Theorems 1-4 hold for ordinal scales as well as interval scales. To see this, note that in the proof of Theorem 1, if $\phi(x)$ is a monotone increasing function so that $\phi(\alpha) > 0$, then after this transformation,

$$F'_{sumE/T}(S') = \phi(\alpha) \sum E_i(S') + \phi(\alpha) \sum T_i(S') = \frac{\phi(\alpha)}{\alpha} F_{sumE/T}(S').$$

Meaningfulness now follows for ordinal scales as it does for interval scales. (Actually, the proof does not use the hypothesis that we have an ordinal scale, but only the hypothesis that $\phi(\alpha) > 0$.) A similar conclusion holds for Theorems 2-4. What is more surprising is that Theorems 5 and 6 hold for ordinal scales as well as interval scales. The heart of the proofs of these theorems is to show that a greedy solution depends only on the ordering of the weights, and so is unchanged not only by transformations x into $kx + l$, $k > 0$, but by (strictly) monotone increasing transformations as well. These observations show that for the four objective functions we have studied, the conclusions of Table 1 hold for ordinal scales, since every example we give which shows that a conclusion is meaningless under interval scales is also an example to show that the conclusion is meaningless under ordinal scales. However, the situation for ordinal scales should be different from that for interval scales if one considers other objective functions, and so it will be useful to investigate such a variety of objective functions for ordinal scales as well as for interval scales.

We have not discussed measurement of parameters in our problem other than weights. Specifically, we have not discussed the measurement of desired and scheduled completion times and of processing times. Time is measured on a ratio scale when we talk about durations (we can change from minutes to hours or days or years). A change of time scale would, therefore, not affect the truth or falsity of the conclusion of optimality under the four objective functions discussed here. Even if time were measured on an interval scale, as it would be if we talked about 'the year 1999' as opposed to 'four years from now (1995),' the conclusion of optimality would still be meaningful under these four objective functions. However, this might no longer be true if we were to use 'nonlinear' objective functions. Hence for such objective functions, one should analyze meaningfulness not just under admissible transformations of weights, but also under admissible transformations of times, even if both define ratio scales.

Table 1
Meaningfulness of the Conclusion of Optimality under Interval Scales.

	Objective	Functions		
Case	$F_{sumE/T}$	F_{sumT}	$F_{maxE/T}$	F_{maxT}
cw cdd cpt s	Yes: T1	ND	Yes: T3	ND
cw cdd cpt ns	No: E1	Yes: T2	No: E8	Yes: T4
cw cdd ncpt s	Yes: T1	ND	Yes: T3	ND
cw cdd ncpt ns	No: E2	Yes: T2	No: E9	Yes: T4
cw ncdd cpt s	Yes: T1	ND	Yes: T3	ND
cw ncdd cpt ns	No: E1+P1	Yes: T2	No: E8+P1	Yes: T4
cw ncdd ncpt s	Yes: T1	ND	Yes: T3	ND
cw ncdd ncpt ns	No: E1+P1	Yes: T2	No: E8+P1	Yes: T4
ncw cdd cpt s	Yes: T5	ND	No: E10	ND
ncw cdd cpt ns	No: E3	Yes: T6	No: E8+P2	No: E12
ncw cdd ncpt s	No: E4	ND	No: E11	ND
ncw cdd ncpt ns	No: E3	No: E6	No: E8+P2	No: E13
ncw ncdd cpt s	No: E5	ND	No: E10+P1	ND
ncw ncdd cpt ns	No: E1+P1	No: E7	No: E8+P1	No: E12+P1
ncw ncdd ncpt s	No: E4+P1	ND	No: E10+P1	ND
ncw ncdd ncpt ns	No: E1+P1	No: E6+P1	No: E8+P1	No: E12+P1

Yes = meaningful, No = can be meaningless, ND = case not defined,
Ei = justified by Example i,
Ti = justified by Theorem i,
Ei+Pj = justified by example constructed from Example i by Principle j,
cw = common weights, ncw = non-common weights,
cdd = common due dates, ncdd = non-common due dates,
cpt = common processing times, ncpt = non-common processing times,
s = symmetric, ns = non-symmetric (by notational convention, ns applies to F_{sumT} and F_{maxT}).

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Appendix

In this appendix we shall provide the missing proofs that S is optimal in some of the examples.
Example 2. Here $n = 3$, $d = 3.1$, $\boldsymbol{p} = (1, 1, 1.1)$, $\alpha = 1$, $\beta = 10$ and $C(S) = (2.1, 3.1, 1.1)$. Then

$$F_{sumE/T}(S) = 1 \cdot (3.1 - 2.1) + 10(3.1 - 3.1) + 1 \cdot (3.1 - 1.1) = 3.$$

We show that S is optimal. Suppose $C(S') = (C_1', C_2', C_3')$. If any $C_i' \geq 3.4$, then

$$F_{sumE/T}(S') \geq 10(3.4 - 3.1) = 3 = F_{sumE/T}(S).$$

So, we assume that $C_i' < 3.4$ for all i. Without loss of generality, we may assume that $C_1' < C_2'$. We now consider three cases.
Case 1: $C_3' > C_2'$. Then since $C_1' + C_2' + C_3' \geq p_1 + p_2 + p_3$, we have $C_3' \geq 3.1$. Suppose it is equal to $3.1 + \delta$, for $\delta \geq 0$. Since $C_3' < 3.4$, we have $\delta < .3$. Now $C_2' \leq 2 + \delta$ and $C_1' \leq 1 + \delta$. Hence,

$$F_{sumE/T}(S') \geq 1 \cdot (3.1 - (1 + \delta)) + 1 \cdot (3.1 - (2 + \delta)) + 10((3.1 + \delta) - 3.1)$$

$$= 3.2 + 8\delta \geq 3.2 > F_{sumE/T}(S).$$

Case 2: $C_1' < C_3' < C_2'$. Then $C_2' \geq 3.1$; say it is $3.1 + \delta$. As in Case 1, we have $0 < \delta < .3$. Now $C_3' \leq 2.1 + \delta$ and $C_1' \leq 1 + \delta$. Hence,

$$F_{sumE/T}(S') \geq 1 \cdot (3.1 - (1 + \delta)) + 10((3.1 + \delta) - 3.1) + 1 \cdot (3.1 - (2.1 + \delta))$$

$$= 3.1 + 8\delta \geq 3.1 > F_{sumE/T}(S).$$

Case 3: $C_3' < C_1' < C_2'$. In this case, as in the other cases, we have $C_2' = 3.1 + \delta$, $0 < \delta < .3$, and we have $C_1' \leq 2.1 + \delta$, $C_3' \leq 1.1 + \delta$. Then

$$F_{sumE/T}(S') \geq 1 \cdot (3.1 - (2.1 + \delta)) + 10((3.1 + \delta) - 3.1) + 1 \cdot (3.1 - (1.1 + \delta))$$

$$= 3 + 8\delta \geq 3 \geq F_{sumE/T}(S).$$

We conclude that S is optimal. ∎

Example 3. Here $n = 3$, $d = 2$, $\boldsymbol{p} = (1, 1, 1 + q)$ with $q = 0$ or $.01$, $\boldsymbol{\alpha} = (\frac{1}{2}, \frac{1}{3}, 1)$, $\boldsymbol{\beta} = (\frac{1}{16}, \frac{1}{81}, 1)$, and $C(S) = (3, 4, 2)$. The case $q = 0$ of course will give us cpt, while the case $q = .01$ will give us ncpt. Then

$$F_{sumE/T}(S) = \frac{1}{16}(3 - 2) + \frac{1}{81}(4 - 2) + 1 \cdot (2 - 2) = \frac{1}{16} + \frac{2}{81} = \lambda.$$

Let $C(S') = (C_1', C_2', C_3')$. If $C_3' \geq 2 + \lambda$ or $C_3' \leq 2 - \lambda$, then

$$F_{sumE/T}(S') \geq 1 \cdot ((2 + \lambda) - 2) = \lambda = F_{sumE/T}(S),$$

or

$$F_{sumE/T}(S') \geq 1 \cdot (2 - (2 - \lambda)) = \lambda = F_{sumE/T}(S).$$

Suppose $2 - \lambda < C_3' < 2 + \lambda$. Observe that $\lambda < 1/4$. If C_1' (C_2') is less than C_3', then C_1' (C_2') is at most $1 + \lambda - q$. Hence we have

$$F_{sumE/T}(S') \geq \frac{1}{2}(2 - (1 + \lambda - q)) \geq \frac{1}{2}(1 - \lambda) > \lambda = F_{sumE/T}(S),$$

or

$$F_{sumE/T}(S') \geq \frac{1}{3}(2 - (1 + \lambda - q)) \geq \frac{1}{3}(1 - \lambda) > \lambda = F_{sumE/T}(S).$$

Thus, we may assume that C'_1, $C'_2 > C'_3$. If $C'_3 \geq 2$, then $C'_1 \geq 3$ and $C'_2 \geq 4$, or $C'_1 \geq 4$ and $C'_2 \geq 3$. It follows that

$$F_{sumE/T}(S') \geq \frac{1}{16}(3-2) + \frac{1}{81}(4-2) = \lambda = F_{sumE/T}(S).$$

Thus, we may suppose that $C'_3 < 2$. Then $C'_3 = 2 - \delta$, $\delta > 0$. We know $\delta < \lambda$ since $C'_3 > 2 - \lambda$. It follows that $C'_1 \geq 3 - \delta$ and $C'_2 \geq 4 - \delta$ or $C'_1 \geq 4 - \delta$ and $C'_2 \geq 3 - \delta$. Since $\delta < \lambda < 1/4$, both jobs 1 and 2 are tardy. Hence,

$$\begin{aligned} F_{sumE/T}(S') &\geq \frac{1}{16}(3-\delta-2) + \frac{1}{81}(4-\delta-2) + 1 \cdot (2-(2-\delta)) \\ &= \frac{1}{16} + \frac{2}{81} + \delta(1 - \frac{1}{16} - \frac{1}{81}) \\ &= \lambda + \delta(1 - \frac{1}{16} - \frac{1}{81}) \\ &> \lambda = F_{sumE/T}(S). \end{aligned}$$

It follows that S is optimal. ▮

Example 5. Here $n = 4$, $\boldsymbol{d} = (2,2,2,1)$, $p = 1$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = (9,9,9,1)$ and $C(S) = (2,1,3,4)$. Then

$$F_{sumE/T}(S) = 9(2-2) + 9(2-1) + 9(3-2) + 1 \cdot (4-1) = 21.$$

Consider S' with $C(S') = (C'_1, C'_2, C'_3, C'_4)$. By symmetry, we may assume that $C'_1 < C'_2 < C'_3$. If $C'_4 < C'_2$, then $C'_2 \geq 3$ and $C'_3 \geq 4$, so

$$F_{sumE/T}(S') \geq 9(3-2) + 9(4-2) = 27 > F_{sumE/T}(S).$$

Thus, suppose that $C'_2 < C'_4$. If $C'_1 \geq 2$, then $C'_2 \geq 3$, $C'_3 \geq 4$ and again we obtain

$$F_{sumE/T}(S') \geq 9(3-2) + 9(4-2) = 27 > F_{sumE/T}(S).$$

Thus, we may assume that $C'_1 = 2 - \lambda$, with $\lambda > 0$. Note that since $p_1 = 1$, $\lambda \leq 1$. Also, we have $C'_2 \geq 3 - \lambda$. Moreover, either
Case 1: $C'_3 \geq 4 - \lambda$ and $C'_4 \geq 5 - \lambda$ or
Case 2: $C'_3 \geq 5 - \lambda$ and $C'_4 \geq 4 - \lambda$.
In Case 1,

$$F_{sumE/T}(S') \geq 9(2-(2-\lambda)) + 9((3-\lambda)-2) + 9((4-\lambda)-2) + 1 \cdot ((5-\lambda)-1)$$

$$= 9(3-\lambda) + 4 - \lambda \geq 9 \cdot 2 + 3 = 21 = F_{sumE/T}(S).$$

In Case 2,

$$F_{sumE/T}(S') \geq 9(2-(2-\lambda)) + 9((3-\lambda)-2) + 9((5-\lambda)-2) + 1 \cdot ((4-\lambda)-1)$$

$$= 9(4-\lambda) + 3 - \lambda \geq 9 \cdot 3 + 2 = 29 > F_{sumE/T}(S).$$

Hence, S is optimal. ▮

Example 7. Here $n = 4$, $\boldsymbol{d} = (2,2,2,1)$, $p = 1$, $\boldsymbol{\beta} = (9,9,9,1)$ and $C(S) = (1,2,3,4)$. Then

$$F_{sumT}(S) = 9 \cdot 0 + 9 \cdot 1 + 1 \cdot 3 = 12.$$

Consider $C(S') = (C'_1, C'_2, C'_3, C'_4)$. We may assume without loss of generality that $C'_1 < C'_2 < C'_3$. If $C'_3 \geq 4$, then

$$F_{sumT}(S') \geq \beta_3(C'_3 - 2) \geq \beta_3(4-2) = 9 \cdot 2 = 18 > F_{sumT}(S).$$

Thus, we may assume that $C_3' < 4$ and hence, since all $p_i = 1$, we have $C_4' \geq 4$. Also, again since all $p_i = 1$, we have $C_1' \geq 1$, $C_2' \geq 2$, $C_3' \geq 3$. Hence, if $C_4' = 4$, then $S' = S$. If $C_4' > 4$, then

$$F_{sumT}(S') > \beta_3(3-2) + \beta_4(4-1) = 9(3-2) + 1 \cdot (4-1) = 12 = F_{sumT}(S).$$

Thus, S is optimal. ▮

Example 8. Here $n = 3$, $d = 3$, $p = 1$, $\alpha = 1$, $\beta = 10$ and $C(S) = (\frac{13}{11}, \frac{24}{11}, \frac{35}{11})$. We have

$$F_{maxE/T}(S) = \max\{1 \cdot (3 - \frac{13}{11}), 1 \cdot (3 - \frac{24}{11}), 10(\frac{35}{11} - 3)\} = \frac{20}{11}.$$

To see that S is optimal, consider $C(S') = (C_1', C_2', C_3')$. By symmetry, we may assume that $C_1' < C_2' < C_3'$. If $C_1' \leq \frac{13}{11}$, then

$$F_{maxE/T}(S') \geq 1 \cdot (3 - \frac{13}{11}) = \frac{20}{11} = F_{maxE/T}(S).$$

Similarly, if $C_1' \geq \frac{13}{11}$, then $C_3' \geq \frac{13}{11} + 2 = \frac{35}{11}$, and so

$$F_{maxE/T}(S') \geq 10 \cdot (\frac{35}{11} - 3) = \frac{20}{11} = F_{maxE/T}(S).$$

Hence S is optimal. ▮

Example 9. Here $n = 3$, $d = 3$, $\boldsymbol{p} = (1, 1, 1.1)$, $\alpha = 1$, $\beta = 10$ and $C(S) = (\frac{24}{11}, \frac{35}{11}, \frac{13}{11})$. Then

$$F_{maxE/T}(S) = \max\{1 \cdot (3 - \frac{24}{11}), 10(\frac{35}{11} - 3), 1 \cdot (3 - \frac{13}{11})\} = \frac{20}{11}.$$

To see that S is optimal, consider S' with $C(S') = (C_1', C_2', C_2')$. If for $i = 1$, 2, or 3, $C_i' \leq \frac{13}{11}$, then

$$F_{maxE/T}(S') \geq 1 \cdot (3 - \frac{13}{11}) = \frac{20}{11} = F_{maxE/T}(S).$$

Similarly, if $C_i' \geq \frac{35}{11}$, then

$$F_{maxE/T}(S') \geq 10(\frac{35}{11} - 3) = \frac{20}{11} = F_{maxE/T}(S).$$

Hence, we may assume that

$$\frac{13}{11} < C_i' < \frac{35}{11}, \qquad i = 1, 2, 3. \tag{4}$$

Without loss of generality, $C_1' < C_2'$. If $C_2' < C_3'$, then

$$C_3' \geq C_1' + p_2 + p_3 > \frac{13}{11} + 2.1 = 3 + \frac{31}{110},$$

so

$$F_{maxE/T}(S') > 10((3 + \frac{31}{110}) - 3) = \frac{31}{11} > F_{maxE/T}(S).$$

If $C_1' < C_3' < C_2'$, then as above,

$$C_2' \geq C_1' + p_2 + p_3 > 3 + \frac{31}{110},$$

and again

$$F_{maxE/T}(S') > \frac{31}{11} > F_{maxE/T}(S).$$

If $C_3' < C_1' < C_2'$, then

$$C_2' \geq C_3' + p_1 + p_2 > \frac{13}{11} + 2 = \frac{35}{11},$$

which contradicts Equation 4. We conclude that S is optimal. ∎

Example 10. Here $n = 4$, $d = 1$, $p = 1$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = (10, 9, 3, 2)$ and $C(S) = (1, 2, 4, 3)$. Note that

$$F_{maxE/T}(S) = \max\{10(1-1), 9(2-1), 3(4-1), 2(3-1)\} = 9.$$

Given any other S' with $C(S') = (C_1', C_2', C_3', C_4')$, note that $C_1' \geq 2$ or $C_2' \geq 2$, since $p = 1$. In the former case,

$$F_{maxE/T}(S') \geq 10(2-1) = 10 > F_{maxE/T}(S),$$

while in the latter case,

$$F_{maxE/T}(S') \geq 9(2-1) = 9 = F_{maxE/T}(S).$$

Thus, S is optimal. ∎

Example 11. Here $n = 3$, $d = 3$, $\boldsymbol{p} = (2, 1, 1)$, $\boldsymbol{\alpha} = \boldsymbol{\beta} = (1, 2, 2)$ and $C(S) = (5, 2, 3)$. Then

$$F_{maxE/T}(S) = \max\{1 \cdot (5-3), 2(3-2), 2(3-3)\} = 2.$$

We note that S is optimal. For suppose $C(S') = (C_1', C_2', C_3')$. One of the C_i' is at least $p_1 + p_2 + p_3 = 4$. If C_2' or C_3' is at least 4, then

$$F_{maxE/T}(S') \geq 2(4-3) = 2 = F_{maxE/T}(S).$$

Suppose $C_1' \geq 4$ and C_2', $C_3' \leq 4$. If $C_1' \geq 5$, then

$$F_{maxE/T}(S') \geq 1 \cdot (5-3) = 2 = F_{maxE/T}(S).$$

If $4 \leq C_1' < 5$, then since C_2', $C_3' \leq 4$ and $p_2 = p_3 = 1$ while $p_1 = 2$, we have $C_2' < 2$ or $C_3' < 2$, and therefore

$$F_{maxE/T}(S') > 2(3-2) = 2 = F_{maxE/T}(S).$$

We conclude that S is optimal. ∎