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Bipartite Steinhaus Graphs

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ABSTRACT

Steinhaus graphs are simple undirected graphs in which the first row of the adjacency matrix $A = (a_{rs})$ (excluding the very first entry which is always 0) is an arbitrary sequence of zeros and ones and the remaining entries in the upper triangular part of A are defined by $a_{rs} = (a_{r-1\,s-1} + a_{r-1\,s}) \bmod 2$ (for $2 \leq r < s \leq n$). Such graphs have already been studied for their various properties. In this paper we characterize *bipartite* Steinhaus graphs, and use this characterization to give an *exact* count as well as linear upper and lower bounds for the number of such graphs on n vertices. These results answer affirmatively some questions posed by W. M. Dymacek (Discrete Mathematics, 59 (1986) pp. 9-20).

1 Introduction.

A Steinhaus graph on n vertices is defined as follows. The graph is simple and undirected. The entries in the adjacency matrix $A = (a_{rs})$ are defined by the following relationship (where $x \oplus y$ denotes addition modulo 2):

$$\begin{array}{ll} a_{rr} = 0 & 1 \le r \le n \\ a_{1s} = 0 \text{ or } 1 & 2 \le s \le n \\ a_{rs} = a_{r-1\,s-1} \oplus a_{r-1\,s} & 2 \le r < s \le n \\ a_{rs} = a_{sr} & 2 \le s < r \le n \end{array}$$

The upper triangular part of A is called a Steinhaus triangle[9]. Such triangles have been studied by Harborth[6] and Chang[1]. Steinhaus graphs are named in honor of Hugo Steinhaus who has asked in [9] whether there is a zero-one sequence of length n (for $n \equiv 0 \text{ or } 3 \pmod 4$) which generates a Steinhaus triangle with the same number of zeros and ones. The question has been answered affirmatively by H. Harborth[6].

Properties of Steinhaus graphs and their complements have been extensively studied [2, 3, 4, 5, 8]. In particular W. Dymacek [5] has studied bipartite Steinhaus graphs and shown that a Steinhaus graph is bipartite if and only if it has no triangles. He has also given a loose upper bound on the number of such graphs on n vertices. In this paper we characterize bipartite Steinhaus graphs, and use this characterization to count the number of such graphs with n vertices. A Steinhaus graph with n vertices has the vertex set $V = \{1, 2, \ldots, n\}$. We count two Steinhaus graphs as distinct if and only if they have different adjacency matrices.

There are exactly 2^{n-1} Steinhaus graphs of size n, because $\langle a_{12} \ldots a_{1n} \rangle$ can be an arbitrary $\{0, 1\}$-sequence. We could consider the sequence $\langle a_{12} \ldots a_{1n} \rangle$ to be a name of a Steinhaus graph and study properties derived from the name. Such an approach would violate a fundamental principle about doing mathematics. If a problem has any kind of symmetries, the mathematical structures defined to handle that problem should reflect these symmetries. In our case, there is just one symmetry. The vertices are ordered $\langle 1, 2, \ldots, n \rangle$, but the reverse order $\langle n, n-1, \ldots, 1 \rangle$ is equally good. There is no reason to distinguish the first vertex from the last vertex of the graph by starting with the first row rather than the last column of the adjacency matrix. We would rather treat rows and columns equally by selecting the sequence $D = \langle a_{12}a_{23}\ldots a_{n-1\,n}\rangle$ in the diagonal $s = r+1$ (just above the main diagonal) as the name of the graph.

Correspondingly, we better reformulate the definition of Steinhaus graphs in a symmetric way.

$$\begin{array}{ll} a_{rr} = 0 & 1 \le r \le n \\ a_{r\,r+1} = 0 \ \text{ or } \ 1 & 1 \le r \le n-1 \\ a_{r\,s} = a_{r\,s-1} \oplus a_{r+1\,s} & 1 < r+1 \le s-1 < n \\ a_{r\,s} = a_{sr} & 1 \le s < r \le n \end{array}$$

We will see that the new name $D = \langle a_{12}a_{23}\ldots a_{n-1\,n}\rangle$ reflects structural properties of its graph. In particular, bipartite Steinhaus graphs will be characterized by the non-occurrence of some simple patterns in their names. On the other hand, the first rows of bipartite Steinhaus graphs are not known to have such nice properties.

In Section 2, we characterize the bipartite Steinhaus graphs by the pattern of D. In Section 3, we count their number as a function of n and give linear upper and lower bounds on $b(n)$.

2 A Characterization of bipartite Steinhaus graphs.

The previous discussion has shown that a Steinhaus graph is uniquely determined by the sequence of zero and one elements $D = \langle d_1 d_2 \ldots d_{n-1}\rangle$ just above the main diagonal, namely the elements $a_{r\,r+1}$ (for $1 \le r \le n-1$), since the remaining entries of the adjacency matrix A can be computed from them.

Notation *We use D to denote the sequence of zeros and ones above the main diagonal and n to denote the number of vertices of the Steinhaus graph throughout the remaining part of this paper.*

The main purpose of this paper is to prove the following characterization of bipartite

Steinhaus graphs in terms of forbidden substrings in the sequence D.

Main Theorem *A Steinhaus graph is bipartite if and only if D does not contain a substring of the form $\langle 0^{2^v} 1 0^{(2u+1)2^v-1} 1\rangle$ or $\langle 1 0^{(2u+1)2^v-1} 1 0^{2^v}\rangle$.*

We need to prove a few results before we provide the proof of the Main Theorem. To facilitate further discussions, we introduce the concept of (doubly) *infinite* Steinhaus graphs as Steinhaus graphs with the vertex set $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The usefulness of infinite Steinhaus graphs will be apparent later when we will classify (finite) bipartite Steinhaus graphs depending on whether or not they are subgraphs of some infinite bipartite Steinhaus graph.

Usually, we describe graphs by their (possibly infinite) adjacency matrices. As is customary with finite matrices, we enumerate the rows of adjacency matrices from top to bottom and their columns from left to right (even when they are indexed by $\mathbf{Z}$ rather than $\{1, \ldots, n\}$).

Since Steinhaus graphs are undirected, they are uniquely defined by their upper triangular part of the adjacency matrix (excluding the main diagonal). So, from now on, when displaying entries in an adjacency matrix, we will show the entries in the upper triangular part only.

Before we focus on adjacency matrices, we study another infinite matrix $P = (p_{rs})$ whose rows r and columns s are nonnegative integers. P is not an adjacency matrix, but we will see P, as well as several superposed copies of P, appear in interesting adjacency matrices. While the columns of P are enumerated as usual from left to right, the rows are enumerated bottom up from 0 to infinity, because this is the orientation of copies of P appearing in adjacency matrices. The entries in P are defined as follows:

$$p_{rs} = \begin{cases} 1 & \text{if } r = 0 \text{ or } s = 0 \\ p_{r\,s-1} \oplus p_{r-1\,s} & \text{otherwise} \end{cases}$$

Hence, it is trivial to prove by induction on $r + s$ that

$$p_{rs} \equiv \binom{r+s}{s} \pmod 2 \tag{1}$$

This implies that the matrix P is *symmetric*, i.e., $p_{rs} = p_{sr}$.

It is apparent from its definition that P forms a rotated Pascal's triangle modulo 2. Therefore, we use the name Pascal matrix for P.

$$P = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & \dots \end{pmatrix}$$

We are interested in knowing how many zeros follow the first 1 in any row of P before the second 1 appears. For this purpose, let us define, for each row numbered r, the parameter α_r as follows:

$$p_{rs} = \begin{cases} 0 & \text{if } 0 < s < \alpha_r \\ 1 & \text{if } s = \alpha_r \end{cases}$$

Naturally, Pascal's triangle has been studied for a long time. One of its most important properties is expressed in the following well known theorem (for example, see [7, page 68]).

Theorem 2.1 (E. Lucas, 1877) *If* $t = \sum_k t_k 2^k$ *and* $s = \sum_k s_k 2^k$ *are two nonnegative integers, with* $t_k, s_k \in \{0, 1\}$*, then*

$$\binom{t}{s} \equiv \prod_k \binom{t_k}{s_k} \pmod 2$$

∎

The following lemma states the value of α_r for any r (note that any positive integer can be uniquely expressed in the form $(2u+1)2^v$ for some integers $u, v \geq 0$).

Lemma 2.1 *For any two integers* $u, v \geq 0$*, let* $r = (2u+1)2^v - 1$ *and* $0 < s < 2^v$*. Then we have*

(a) $p_{rs} = p_{sr} = 0$

(b) $p_{r\,2^v} = p_{2^v\,r} = 1$

(c) $\alpha(r) = 2^v$

Proof. Let

$$(r+s) = \sum_k t_k 2^k \quad \text{and} \quad s = \sum_k s_k 2^k$$

where $s_k, t_k \in \{0,1\}$. Applying Theorem 2.1 to the characterization of P mod 2 by congruence (1), we get

$$p_{rs} \equiv 0 \pmod 2 \quad \text{if and only if} \quad \exists k\,(t_k = 0) \wedge (s_k = 1) \tag{2}$$

From this characterization, it is not hard to see that p_{rs} is 0, if and only if the addition of the numbers r and s in binary produces at least one carry anywhere.

Now, let us write r in the following form

$$r = \sum_k r_k 2^k = (2u+1)2^v - 1$$

i.e., $r_0 = r_1 = \ldots = r_{v-1} = 1$ and $r_v = 0$.

To prove part (a), we first assume $0 < s < 2^v$. Then there exists an index $k < v$ with $s_k = 1$. Let k be the smallest index with this property. Observing that $t_k = r_k \oplus s_k = 0$, we have found an index k with $t_k = 0$ while $s_k = 1$ implying $\binom{t_k}{s_k} = 0$ and therefore

$$p_{rs} \equiv \binom{r+s}{s} \equiv 0 \pmod 2 \quad (\text{for } 0 < s < 2^v)$$

Together with the symmetry of the $\{0,1\}$-matrix P, this proves claim (a).

On the other hand, if $s = 2^v$, then $s_v = 1$ and $s_k = 0$ for all $k \neq v$. Therefore,

$$t_v = r_v \oplus s_v = 1$$

and there is no k with $(t_k = 0) \wedge (s_k = 1)$. This implies

$$p_{rs} \equiv \binom{r+s}{s} \equiv 1 \pmod 2 \quad (\text{for } s = 2^v)$$

Together with the symmetry of the $\{0,1\}$-matrix P, this proves claim (b).

Claim (c) is equivalent to (a) and (b) by the definition of α_r. ∎

Lemma 2.2 *If r and s are positive integers whose sum is a power of 2, then $p_{rs} = 0$.*

Proof. With $t = r + s$, we know that $p_{rs} = \binom{t}{s} \bmod 2$. Let $t = 2^v$. Then there is an index $k < v$ with $s_k = 1$, while $t_k = 0$ for all $k < v$. Thus the lemma follows from characterization (2) above. ∎

Lemma 2.3 *Let $D = \langle \ldots d_{-2}\, d_{-1}\, d_0\, d_1\, d_2 \ldots \rangle$ be an infinite sequence of zeros and ones that does not contain a substring of the form $\langle 0^{2^v} 1 0^{(2u+1)2^v-1} 1 \rangle$ or $\langle 1 0^{(2u+1)2^v-1} 1 0^{2^v} \rangle$. Then D has one of the following forms:*

(a) *D does not contain any 1.*

(b) *D contains exactly one 1.*

(c) *D is an infinite repetition of the substring $\langle 1 0^{2^k-1} \rangle$ for some integer $k \geq 0$.*

Proof. If D contains at most one 1, then obviously D cannot contain any forbidden substring. If D contains at least two 1's, let $\ell = (2u+1)2^k - 1$ be the smallest nonnegative integer such that D contains the substring $\langle 1 0^\ell 1 \rangle$. As both substrings $\langle 1 0^\ell 1 0^{2^k} \rangle$ and $\langle 0^{2^k} 1 0^\ell 1 \rangle$ are forbidden, every occurrence of a substring $\langle 1 0^\ell 1 \rangle$ is contained in a substring $\langle 1 0^{\ell'} 1 0^\ell 1 0^{\ell''} 1 \rangle$ of D. The minimality of ℓ implies $\ell \leq \ell'$ and the forbidden substring $\langle 0^{2^k} 1 0^\ell 1 \rangle$ implies $\ell' < 2^k$. Hence, $u = 0$ and $\ell' = \ell = 2^k - 1$. A symmetric argument shows $\ell'' = \ell$. Therefore, D is an infinite repetition of the sequence $\langle 1 0^\ell \rangle$ with $\ell = 2^k - 1$. ∎

Lemma 2.4 *Let $D = \langle \ldots d_{-2}\, d_{-1}\, d_0\, d_1\, d_2 \ldots \rangle$ be the infinite sequence of zeros and ones just above the main diagonal for an infinite Steinhaus graph $G = (\mathbf{Z}, E)$. If D does not contain a substring of the form $\langle 0^{2^v} 1 0^{(2u+1)2^v-1} 1 \rangle$ or $\langle 1 0^{(2u+1)2^v-1} 1 0^{2^v} \rangle$, then G is bipartite.*

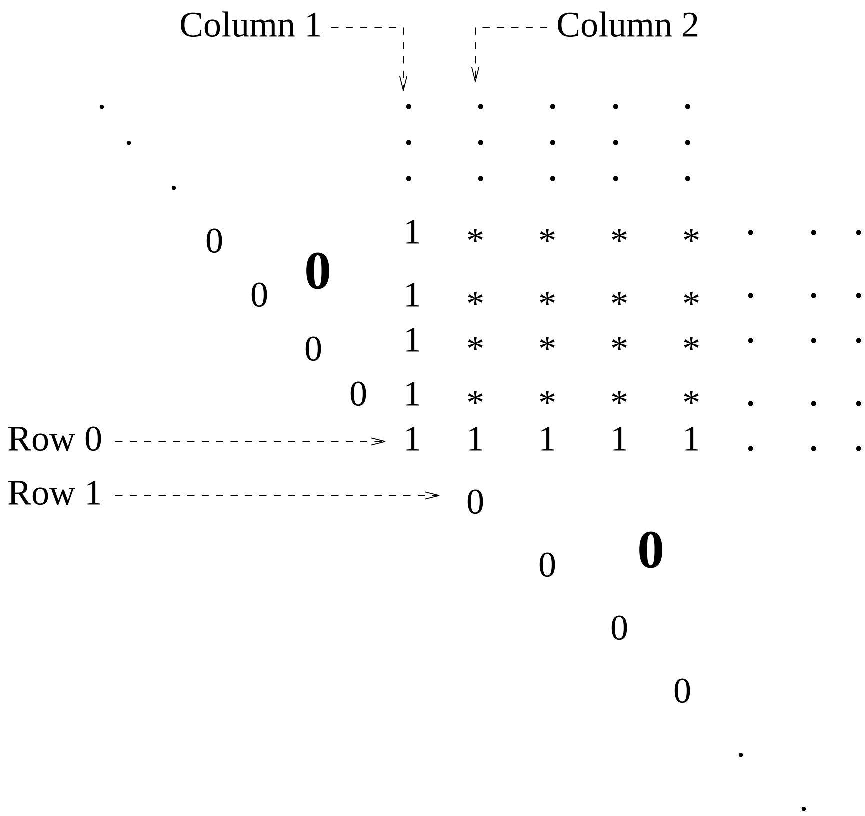

Figure 1: *D contains exactly one 1.*

Proof. By Lemma 2.3, we distinguish the following three cases:

Case 1: D does not contain any 1's. Then the graph has no edges and is obviously bipartite.

Case 2: D contains exactly one 1. Without loss of generality, assume that $d_0 = a_{01} = 1$. Then the upper triangular part of the adjacency matrix A is formed by a Pascal matrix P with its origin shifted to a_{01}. Outside this shifted Pascal matrix, all entries are 0. Figure 1 shows a finite portion of the matrix A. Entries marked by '$*$' are not studied here. More precisely,

$$a_{s\ -r} = a_{-r\ s} = \begin{cases} p_{r\ s-1} & \text{for } r \geq 0, s \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

The 0's outside P imply that $G = (\mathbf{Z}, E)$ is bipartite with parts $V_1 = \{\ldots, -2, -1, 0\}$ and $V_2 = \{1, 2, 3, \ldots\}$.

Case 3: D contains two or more 1's. By Lemma 2.3, D is an infinite repetition of a substring of the form $\langle 10^{2^v - 1}\rangle$ for some integer $v \geq 0$. Without loss of generality, let

$$d_k = \begin{cases} 1 & \text{for } k \equiv 0 \pmod{2^v} \\ 0 & \text{otherwise} \end{cases}$$

Then the upper triangular part of the adjacency matrix A is an infinite superposition (by addition modulo 2) of copies of the Pascal matrix P with its origin shifted to $a_{2^v\ 2^v+1}$ ($v \in \mathbf{Z}$). Each location is covered by only finitely many copies of P. More precisely, A is characterized by

$$a_{rs} \equiv \sum_k p_{k\cdot 2^v - r\ \ s - k\cdot 2^v - 1} \pmod 2 \tag{3}$$

where k is restricted to integers with $r \leq k \cdot 2^v < s$ (such that both indices of P are nonnegative).

Claim: All entries a_{rs} on the diagonal $s = r + 2^v + 1$ (parallel to the main diagonal) are 0.

We distinguish two cases. If r is a multiple of 2^v, then the sum (3) has two terms: $p_{0\ 2^v}$ and $p_{2^v\ 0}$, which are both 1 by definition of P. Otherwise the sum (3) has only one term $p_{r'\!s'}$

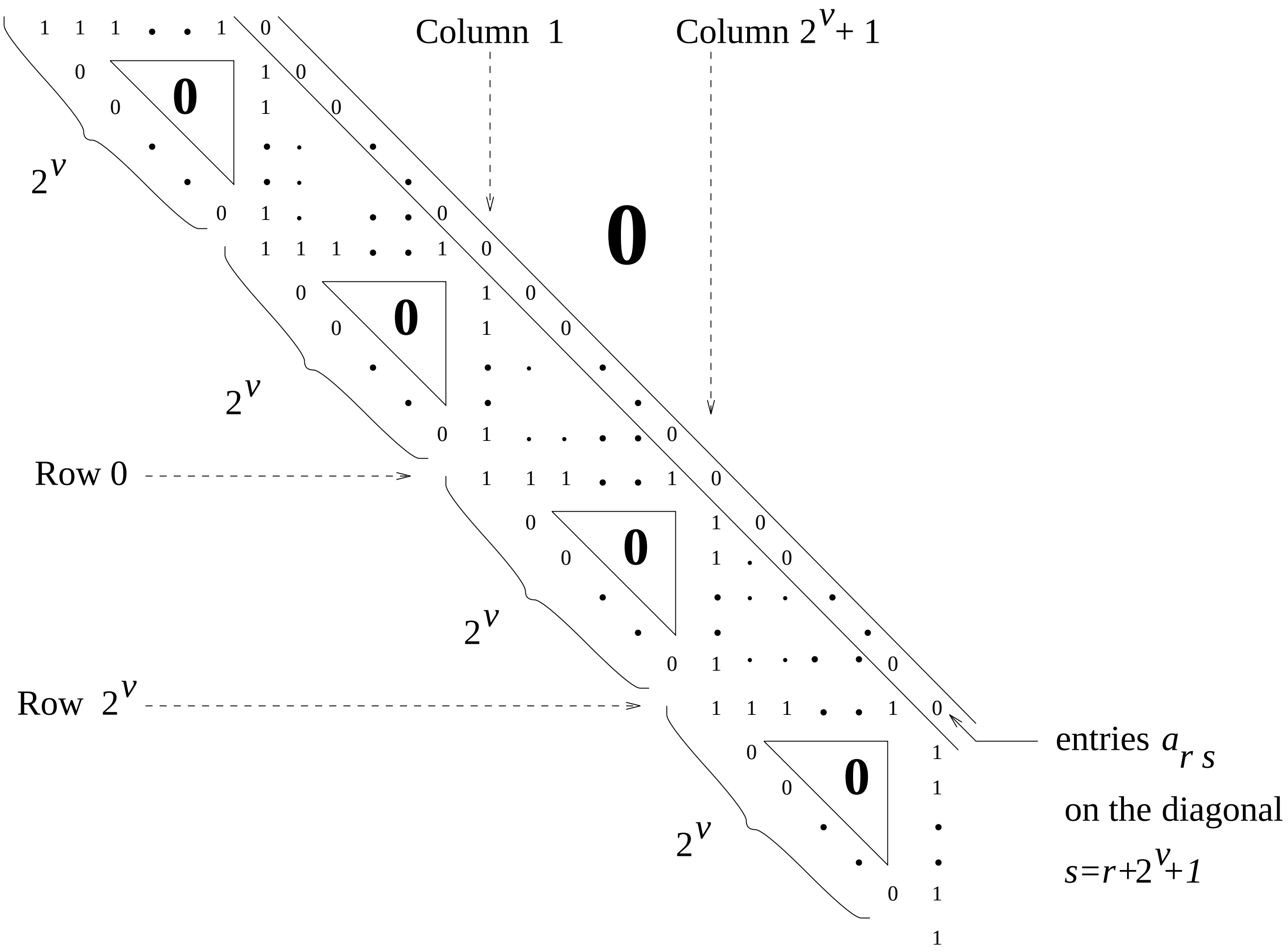

Figure 2: *D is periodic.*

with $r' = (-r) \bmod 2^v$ and $s' = (r) \bmod 2^v$. In this case, $r' + s' = 2^v$ and $a_{rs} = p_{r's'} = 0$ by Lemma 2.2. This proves the Claim.

Once the entries a_{rs} of the matrix A are 0 on the whole diagonal $s = r + 2^v + 1$, its entries are also 0 everywhere above this diagonal (see Figure 2).

$$a_{rs} = 0 \qquad \text{for} \qquad s \geq r + 2^v + 1 \tag{4}$$

Furthermore,

$$a_{rs} = 0 \qquad \text{for} \qquad \left\lfloor \frac{r-1}{2^v} \right\rfloor = \left\lfloor \frac{s-1}{2^v} \right\rfloor \tag{5}$$

because the sum (3) is 0, as it does not contain any terms.

We define

$$V_1 = \{r \ : \ \left\lfloor \frac{r-1}{2^v} \right\rfloor \equiv 0 \pmod 2\}$$

and

$$V_2 = \{r \ : \ \left\lfloor \frac{r-1}{2^v} \right\rfloor \equiv 1 \pmod 2\}$$

Now we show that all edges are between adjacent blocks of 2^v vertices, i.e., if $\{r, r'\} \in E$ then the block distance

$$\Delta = \left| \left\lfloor \frac{r-1}{2^v} \right\rfloor - \left\lfloor \frac{r'-1}{2^v} \right\rfloor \right|$$

is 1. If $\Delta = 0$, there is no edge between r and r' because of (5). If $\Delta \geq 2$, there is no edge between r and r' because of (4). Therefore $G = (V, E)$ is bipartite with partition V_1, V_2. ∎

We now want to study the finite bipartite Steinhaus graphs G (those with $u > 0$ below) that are not extendable to infinite bipartite Steinhaus graphs. This means that the sequence D of G cannot be extended to any one of the infinite sequences described in Lemma 2.3. In other words, G (with vertex set $V = \{1, \ldots, n\}$) is not a subgraph of any infinite bipartite Steinhaus graph (with vertex set $\mathbf{Z}$). Note that we do not study whether or not G might be isomorphic to a subgraph of an infinite bipartite Steinhaus graph.

Lemma 2.5 *Let $\ell = 2^v$ and $m = (2u+1)2^v$ for some nonnegative integers u and v. If the sequence D of a (finite) Steinhaus graph G is a substring of $\langle 0^{\ell-1}10^{m-1}10^{\ell-1} \rangle$ then G is bipartite.*

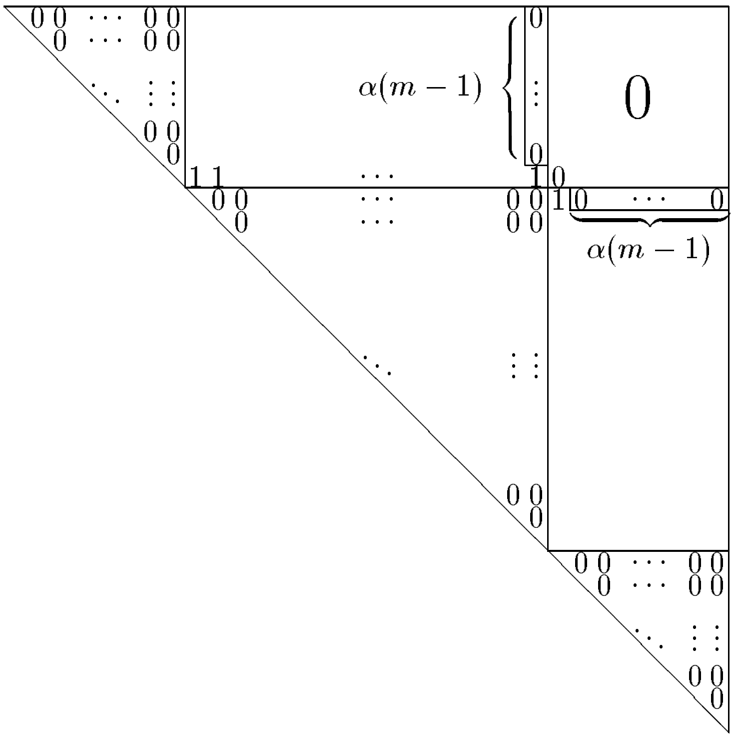

Figure 3: D contains two 1's

It is sufficient to show the lemma for $D = \langle 0^{\ell-1}10^{m-1}10^{\ell-1}\rangle$. Figure 3 shows the important parts of the adjacency matrix of G. The 1's of D are the entries $a_{\ell\,\ell+1}$ and $a_{\ell+m\,\ell+m+1}$. The graph G is bipartite with the parts $V_1 = V_1' \cup V_1''$ and $V_2 = \{\ell+1, \ldots, \ell+m\}$ where $V_1' = \{1, \ldots, \ell\}$ and $V_1'' = \{\ell+m+1, \ldots, \ell+m+\ell\}$.

Clearly, there are no edges within each of V_1', V_1'' and V_2. To show that there are no edges between V_1' and V_1'', we notice that the entry $a_{\ell+1\,\ell+m+1} = 1$ is followed by a block of $\alpha(m-1) - 1 = 2^v - 1$ zeros

$$a_{\ell+1\,\ell+m+2} = \ldots = a_{\ell+1\,\ell+m+\ell} = 0$$

Symmetrically, $a_{\ell\,\ell+m} = 1$ is located below a block of $\alpha(m-1)-1 = 2^v - 1$ zeros

$$a_{\ell-1\,\ell+m} = \ldots = a_{1\,\ell+m} = 0$$

Hence, the top right square ($1 \leq r \leq \ell$, $\ell+m+1 \leq s \leq \ell+m+\ell$) has a zero entry $a_{\ell\,\ell+m+1} = 0$ in its lower left corner and is otherwise completely flanked by 0's. Therefore $a_{rs} = 0$ in this whole square.

Theorem 2.2 *Let the sequence D of a Steinhaus graph not contain any forbidden substring $\langle 0^{2^v}10^{(2u+1)2^v-1}1\rangle$ or $\langle 10^{(2u+1)2^v-1}10^{2^v}\rangle$. Then G is bipartite.*

Proof. If the number of 1's in the sequence D is two, then the result follows from the preceding Lemma 2.5. Otherwise D can be Extended to a doubly infinite sequence $\langle \ldots 10^{2^v-1}10^{2^v-1}\ldots 10^{2^v-1}\ldots\rangle$ with period 2^v for some v, and the result follows from Lemma 2.4. ∎

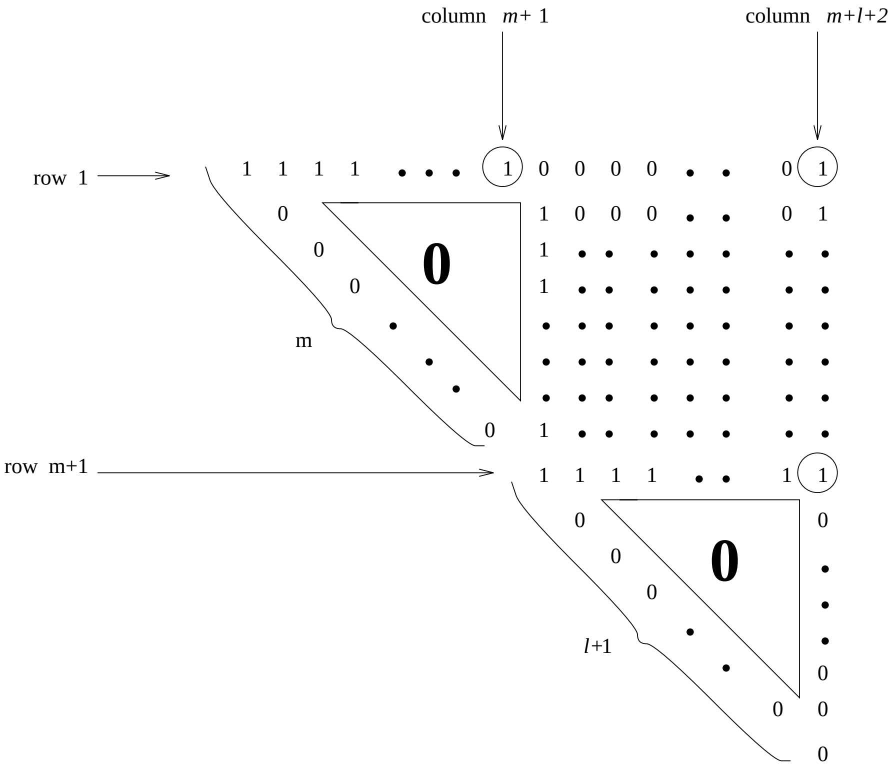

Figure 4: *The forbidden sequence $D = \langle 10^{m-1}10^{\ell}\rangle$*

Theorem 2.3 *If the sequence D of a Steinhaus graph G contains a forbidden substring* $\langle 0^{2^v}10^{(2u+1)2^v-1}1\rangle$ *or* $\langle 10^{(2u+1)2^v-1}10^{2^v}\rangle$*, then G contains a triangle.*

Proof. As the two cases are symmetric, we may assume that D contains the sequence $\langle 10^{m-1}10^{\ell}\rangle$ with $m=(2u+1)2^v$ and $\ell=2^v$. In fact, we can assume that D is this sequence. Figure 4 shows part of the adjacency matrix.

As in Figure 3, we have an important block

$$a_{2\,m+3}=a_{2\,m+4}=\ldots a_{2\,m+\ell+1}=0$$

of $\ell-1$ zeros. Now this block is followed by an entry $a_{2\,m+\ell+2}=1$, implying also $a_{1\,m+\ell+2}=1$. Combining this fact with the simple observation that $a_{1\,m+1}=a_{m+1\,m+\ell+2}=1$, we have found a triangle with vertices $\{1,m+1,m+\ell+2\}$. ∎

The Theorems 2.2 and 2.3 imply the Main Theorem and provide a new proof of the following result.

Corollary 2.4 (Dymacek [5]) *A Steinhaus graph is bipartite if and only if it does not contain a triangle.* ∎

3 The number of bipartite Steinhaus graphs.

Let $b(n)$ denote the number of bipartite Steinhaus graphs on n vertices. We want to study this function. We will present an explicit expression allowing to compute $b(n)$ in time $O(\log n)$. Furthermore, we show linear upper and lower bounds for $b(n)$.

Previous results by Dymacek [5] showed that $n+1\leq b(n)$ and $b(n)\leq 50\cdot 2^{n-24}$ for $n\geq 24$. The first inequality corresponds to the bipartite Steinhaus graphs with zero, one or $n-1$ 1's in the sequence D. The second inequality has been obtained from the recurrence relation $b(n+1)\leq 2b(n)$ and the computation of $b(n)$ for $n\leq 24$ by exhaustive search.

Table 1 [5] shows $b(n)$ for a few more values of n. It is interesting to note that before our present investigation, it would have been computationally infeasible to compute even this

n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	11	21	21	43	31	69	41	87
2	2	12	23	22	44	32	73	42	87
3	4	13	27	23	47	33	79	43	89
4	6	14	28	24	50	34	77	44	91
5	9	15	31	25	55	35	77	45	95
6	10	16	34	26	55	36	77	46	97
7	13	17	39	27	57	37	79	47	101
8	15	18	38	28	59	38	79	48	105
9	19	19	39	29	63	39	81	49	111
10	19	20	40	30	65	40	83	50	110

Table 1: *The number of bipartite Steinhaus graphs for small n*

small table. This contrasts sharply with the fact that we can now compute $b(n)$ for large values of n, and even enumerate all bipartite Steinhaus graphs for every reasonable value of n with minimal computational effort.

Looking at this table of initial values of $b(n)$, one might come up with the following conjecture: $b(n)$ seems to be $2(n-1)$ plus some nonnegative noise which follows no obvious pattern. One does not expect the existence of a simple expression for $b(n)$ or even a fast algorithm to compute it. As a consequence of the Main Theorem, we can show that all such conjectures are wrong.

Let $n' = n - 1$ be the length of the sequences D, and let k be defined by $2^k \leq n' < 2^{k+1}$. We differentiate between two types of bipartite Steinhaus graphs: those which are subgraphs of the infinite bipartite Steinhaus graphs and those which are not.

We can count the number of bipartite Steinhaus graphs of n vertices which are subgraphs of infinite bipartite Steinhaus graphs as follows:

There is one Steinhaus graph without any ones in the sequence D.

For every $v \leq k$ there are 2^v bipartite Steinhaus graphs of n vertices which are subgraphs of infinite bipartite Steinhaus graphs of periodicity 2^v (i.e., there are $2^v - 1$ zeros between two successive 1s). And in these cases, v is uniquely determined by D.

Finally, there are $2(n' - 2^k)$ cases where the sequence D contains exactly one 1 so close to either end that a periodicity 2^k is impossible.

Alternatively, we could count as follows:

1	without 1's
n'	with exactly one 1
$n' - 2^k$	with two 1's at distance 2^k
2^v	with 1's at a distance 2^v for $0 \leq v < k$ for each k with $2^k \leq n' < 2^{k+1}$
	giving in all $2^k - 1$ cases.
$2n'$	total number of graphs.

Finally, we have those bipartite Steinhaus graphs which are not subgraphs of infinite bipartite Steinhaus graphs. Each such graph has a sequence D of the form $\langle 0^r 1 0^s 1 0^t \rangle$ for some non-negative integers r, s, t, u, v with $s = (2u+1)2^v - 1$, $u \geq 1$ and $r, t < 2^v$. (Those graphs with $u = 0$ are already counted, because they are subgraphs of infinite Steinhaus graphs.) We count the number of such sequences D for given values of u and v. As before, let $n' = n-1$ be the length of D. Then $n' = (2u+1)2^v+1+r+t$ and $r+t = (n'-1-2^v) \bmod 2^{v+1}$. For a given sum $w = r + t$, we distinguish two cases.

If $0 \leq w < 2^v$ then we have $w + 1$ sequences D with $0 \leq r \leq w$ and $0 \leq t = w - r \leq w$. If $2^v \leq w < 2^{v+1}$ then we have $2^{v+1} - 1 - w$ sequences D with $w - 2^v + 1 \leq r \leq 2^v - 1$ and $w - 2^v + 1 \leq t = w - r \leq 2^v - 1$.

Notation Let $b_\infty(n)$ be the number of bipartite Steinhaus graphs extendable to infinite bipartite Steinhaus graphs, and $b_{\text{fin}}(n)$ be the number of bipartite Steinhaus graphs not so extendable. In order to find a simple expression for $b_{\text{fin}}(n)$, we introduce the functions

$\text{tooth}_v : \mathbf{R} \to \mathbf{R}^+$ defined by

$$\text{tooth}_v(x) = \begin{cases} 2^v & \text{for} \quad x \equiv 0 \pmod{2^{v+1}} \\ 0 & \text{for} \quad x \equiv 2^v \pmod{2^{v+1}} \end{cases}$$

and by linear interpolation between these values. We can also write it as

$$\text{tooth}_v(x) = |x \bmod 2^{v+1} - 2^v|$$

The preceding investigation has shown that there are $\text{tooth}_v(n')$ sequences D of length n' for a given value of v with $0 \le v \le \log_2 \lfloor \frac{n'}{3} \rfloor$. We summarize these results as follows.

Theorem 3.1 **(a)** $b(n) = b_\infty(n) + b_{\mathit{fin}}(n)$

(b) $b_\infty(n) = 2(n-1)$

(c) $b_{\mathit{fin}} = \sum_{v=0}^{\lfloor \log_2 \frac{n-1}{3} \rfloor} \mathit{tooth}_v(n-1)$

∎

Remark 3.1 *The periodic functions tooth_v can easily be expressed as a Fourier series*

$$\mathit{tooth}_v(x) = \frac{2^{v+2}}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi x / 2^{v+1})}{(2k+1)^2} + 2^{v-1}$$

thus giving an explicit expression for $b(n)$:

$$b(n) = 2n - 2 + \sum_{v=0}^{\lfloor \log_2 \frac{n-1}{3} \rfloor} \left(\frac{2^{v+2}}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi(n-1)/2^{v+1})}{(2k+1)^2} + 2^{v-1}\right)$$

The next lemma shows the linear lower and upper bounds for $b(n)$.

Lemma 3.1 $2n - 2 < b(n) \le \frac{1}{2}(5n-7)$

Proof. By Theorem 3.1, it is sufficient to show that $b_{\text{fin}}(n) \leq \frac{1}{2}(n-3)$. If $n = 2^k + 1$, then $\text{tooth}_v(n-1) = 2^v$ for all $v \leq k-2 = \lfloor \log_2 \frac{n-1}{3} \rfloor$. Hence,

$$b_{\text{fin}}(n) = \sum_{v=0}^{k-2} \text{tooth}_v(2^k) = 2^{k-1} - 1 = \frac{1}{2}(n-1) - 1 = \frac{1}{2}(n-3)$$

To show that b_{fin} is smaller otherwise, we distinguish two cases. From the start of a new largest tooth until the linear upper bound is reached, we bound all other tooths by their maxima.

Case (a): Let $3 \cdot 2^{k-2} \leq n-1 < 2^k$.

Then $\lfloor \log_2 \frac{n-1}{3} \rfloor = k-2$ and the contribution of the largest tooth $\text{tooth}_{k-2}(n-1)$ is increasing.

$$\begin{aligned}
b_{\text{fin}}(n) &= \sum_{v=0}^{k-2} \text{tooth}_v(n-1) \\
&\leq \sum_{v=0}^{k-3} 2^v + \text{tooth}_{k-2}(n-1) \\
&= 2^{k-2} - 1 + |(n-1) \bmod 2^{k-1} - 2^{k-2}| \\
&= 2^{k-2} - 1 + n - 1 - 2^{k-1} - 2^{k-2} \\
&= n - 2 - 2^{k-1} \\
&< \frac{1}{2}(n-1) - 1 \\
&= \frac{1}{2}(n-3)
\end{aligned}$$

Case (b): Let $2^k < n-1 < 3 \cdot 2^{k-1}$.

Then $k-2 = \lfloor \log_2 \frac{n-1}{3} \rfloor$ is still the index of the largest tooth.

$$\begin{aligned}
b_{\text{fin}}(n) &= \sum_{v=0}^{k-2} \text{tooth}_v(n-1) \\
&\leq \sum_{v=0}^{k-2} 2^v \\
&= 2^{k-1} - 1 \\
&< \frac{1}{2}(n-1) - 1 \\
&= \frac{1}{2}(n-3)
\end{aligned}$$

∎

Remark 3.2 *It is also possible to show that $b_{\mathit{fin}}(n) \geq \lceil \frac{n}{8} \rceil$ and derive some nice recurrences fulfilled by $b_{\mathit{fin}}(n)$ (and hence $b(n)$). We omit the details of these derivations from this paper.*

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