Weighted search in the plane

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Abstract

We present a simple two-dimensional weighted version of Megiddo's multidimensional search technique. This speeds up algorithms for certain convex optimization problems in the plane.
Keywords: Algorithms, combinatorial problems, convex optimization, parametric search.

1 Introduction

We present a two-dimensional weighted version of the multidimensional search technique of Megiddo [4, 7, 9]. The result has applications to a class of two-dimensional convex optimization problems [2, 3, 5, 6, 11]. We need to present some background before stating our result precisely.

Define sign : $\mathbf{R} \rightarrow \{-1, 0, +1\}$ as

$$\operatorname{sign}(x) = \begin{cases} 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Let λ^* be a point in $\mathbf{R}^d$. An affine function $h : \mathbf{R}^d \rightarrow \mathbf{R}$ is *resolved* if $\operatorname{sign}(h(\lambda^*))$ has been computed.

Suppose we have a finite set $\mathcal{H}$ of n d-dimensional affine functions and an oracle $\mathcal{B}$ that can resolve any h. Let S be a finite set on which a weight function $w : S \rightarrow \mathbf{R}^+$ has been defined. For $S' \subseteq S$ we write $w(S')$ to denote $\sum_{s \in S'} w(s)$. The following result is known [2].

Theorem 1 *For each $d \geq 0$, there exist constants $\gamma(d)$ and $\delta(d)$ and an algorithm* WEIGHTED-SEARCH *with the following property. Given a finite set $\mathcal{H}$ of n d-dimensional affine functions and a weight function $w : \mathcal{H} \rightarrow \mathbf{R}^+$,* WEIGHTED-SEARCH *either finds a subset $\mathcal{H}' \subseteq \mathcal{H}$ with $w(\mathcal{H}') \geq \delta(d) \cdot w(\mathcal{H})$ and resolves every $h \in \mathcal{H}'$ by making at most $\gamma(d)$ calls to $\mathcal{B}$ or it returns an affine function h such that $sign(h(\lambda^*)) = 0$. Furthermore, the work done by* WEIGHTED-SEARCH *in addition to the oracle calls is $O(n)$.*

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Agarwala and Fernández-Baca [2] gave two algorithms for this problem, one of which achieved $\gamma(d) = 2^d - 1$ and $\delta(d) = 12/24^{2^{d-1}}$, the other achieved $\gamma(d) = 2(60^{d-1})$ and $\delta(d) = 1/12$. Subsequently, Agarwal, Sharir, and Toledo [1] gave an algorithm that achieved $\gamma(d) = O(d^3 \log d)$ and $\delta(d) = 1/2$. All these algorithms are quite complex and require many oracle calls, even for small d. Since oracle calls tend to be expensive, it is of interest to develop simple algorithms which require fewer oracle calls. Here we give a comparatively simple weighted search algorithm for the plane which always finds a subset $\mathcal{H}' \subseteq \mathcal{H}$ with $w(\mathcal{H}') \geq w(\mathcal{H})/4$ and resolves every $h \in \mathcal{H}'$ by making at most 3 calls to $\mathcal{B}$, thereby achieving $\gamma(2) = 3$ and $\delta(2) = 1/4$ with $O(n)$ additional work. This algorithm uses ideas that were first introduced by Megiddo [10] for a partitioning problem on the plane.

2 The algorithm

Suppose $\mathcal{H} = \{h_1, \ldots, h_n\}$, where $h_i(x,y) = a_i x + b_i y + c_i$. If $a_i, b_i = 0$, $\text{sign}(h_i)$ is simply the sign of c_i, and no oracle calls are needed. Thus, the presence of such h_i's can only help. We shall henceforth assume that $a_i \neq 0$ or $b_i \neq 0$ for $i = 1, \cdots, n$. In this case, each affine function h_i corresponds to a line $H_i \in \mathbf{R}^2$ where $H_i = \{(x,y) : h_i(x,y) = 0\}$. Computing $\text{sign}(h_i)$ is thus equivalent to determining whether H_i contains λ^* and, if not, finding which side of H_i contains λ^*. We shall find it convenient to deal interchangeably with lines and affine functions and to extend the weight function w to these lines by making $w(H_i) = w(h_i)$.

Suppose $b_i = 0$ for some $i \in \{1, \cdots, n\}$. Pick a β such that $\beta \neq (-b_i/a_i)$ for $i = 1, \cdots, n$. By using a change of variables $x = x' + \beta \cdot y$ and $b_i = b_i' - \beta \cdot a_i$, we get a set of lines in which $b_i' \neq 0$ for $i = 1, \cdots, n$. This change of variables simplifies the exposition and needs to be reversed before making an oracle call. For convenience, we now drop the primes on x' and b_i'. Henceforth, $b_i \neq 0$ for $i = 1, \cdots, n$.

Since for every scalar $l \neq 0$, $\text{sign}(h(x,y)) = \text{sign}(l) \cdot \text{sign}(h(x,y)/l)$, we can rewrite the equations of these lines so that $a_i \geq 0$. Let the *slope* α_i of H_i be the same as that of $a_i x + b_i y + c_i = 0$ with respect to $y = 0$; i.e., let $\alpha_i = (-a_i/b_i)$. Since $b_i \neq 0$ for $i = 1, \cdots, n$, the slopes are well defined. Let α^* be the weighted median of the set $\{\alpha_i\}$ where the weight of α_i is $w(H_i)$. Now we make the slopes of roughly weighted half of the lines nonnegative and weighted half nonpositive by using the change of variables $y = y' + \alpha^* \cdot x$ and $a_i = a_i' - \alpha^* \cdot b_i$. As before, this change of variables is only done to simplify the exposition and needs to be reversed before making an oracle call. We drop the primes on y' and a_i'. Recalculate the slopes of the lines after making this change in variables. All lines that originally had a slope of α^* will have 0 slope. Let $\mathcal{H}_0 = \{H_i : \alpha_i = 0\}$, $\mathcal{H}_- = \{H_i : \alpha_i < 0\}$, and $\mathcal{H}_+ = \{H_i : \alpha_i > 0\}$.

Let $W = w(\mathcal{H})$. Since 0 is our new weighted median slope, $w(\mathcal{H}_-) \leq W/2$ and $w(\mathcal{H}_-) + w(\mathcal{H}_0) \geq W/2$. Therefore, $w(\mathcal{H}_-) \geq W/2 - w(\mathcal{H}_0)$. Similarly, $W/2 \geq w(\mathcal{H}_+) \geq W/2 - w(\mathcal{H}_0)$. Assume without loss of generality that $\mathcal{H}_+ = \{H_1, \cdots, H_k\}$, $\mathcal{H}_- = \{H_{k+1}, \cdots, H_m\}$, and $\mathcal{H}_0 = \{H_{m+1}, \cdots, H_n\}$. Let $W_P(c)$ $[W_N(c)]$ be the weighted median of the set $\{p_1, \cdots, p_k\}$ $[\{p_{k+1}, \cdots, p_m\}]$, where p_i is the point of intersection of lines H_i and $y = c$ (i.e., $p_i = (-(c \cdot b_i + c_i)/a_i, c)$) and $w(p_i) = w(H_i)$. Let W_P $[W_N]$ be the curve that we get if we plot $W_P(c)$ $[W_N(c)]$ from $c = -\infty$ to $c = \infty$ (See Figure 1).

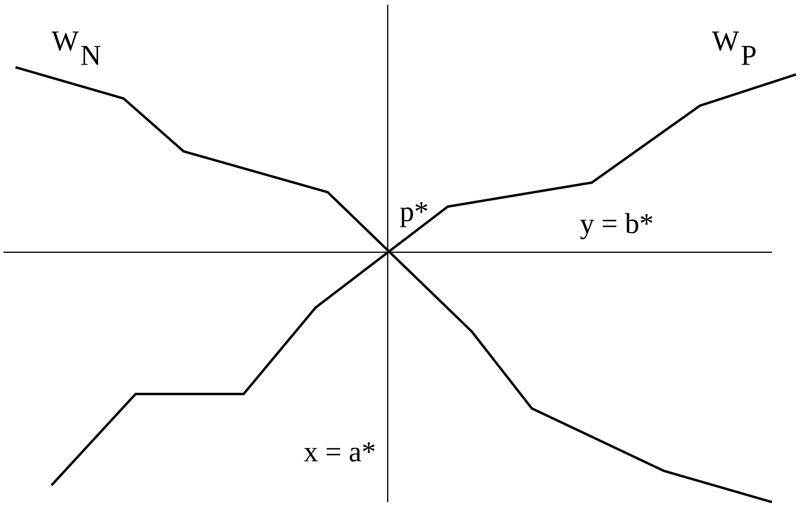

Figure 1: W_P and W_N.

Observation 1 *W_P [W_N] is a monotonically increasing [decreasing] function.*

Let $p^* = (a^*, b^*)$ be the point where W_P intersects W_N. Let $H_P^* \in \mathcal{H}_+$ and $H_N^* \in \mathcal{H}_-$ be the lines whose intersection point is p^*. Suppose that, using two oracle calls, we resolve the lines $y = b^*$ and $x = a^*$. We have the following cases:

Case 1: λ^* lies on the line $x = a^*$ and on the line $y = b^*$. Then $\lambda^* = p^*$ and we have resolved all the lines in $\mathcal{H}$.

Case 2: λ^* lies on $y = b^*$. In this case we have resolved a set of lines $\mathcal{H}_1 \subseteq \mathcal{H}_+$ and $\mathcal{H}_2 \subseteq \mathcal{H}_-$ such that $w(\mathcal{H}_1) \geq w(\mathcal{H}_+)/2$ and $w(\mathcal{H}_2) \geq w(\mathcal{H}_-)/2$. (For example, if λ^* lies on the line described by $y = b^*$ and $x > a^*$, then we resolve $\{h_i \in \mathcal{H}_+ \cup \mathcal{H}_- : -(b^* \cdot b_i + c_i)/a_i \leq a^*\}$.) Furthermore, every $H \in \mathcal{H}_0$ is also resolved.

Case 3: λ^* lies on $x = a^*$. As in the previous case, we resolve $\mathcal{H}_1 \subseteq \mathcal{H}_+$ and $\mathcal{H}_2 \subseteq \mathcal{H}_-$ such that $w(\mathcal{H}_1) \geq w(\mathcal{H}_+)/2$ and $w(\mathcal{H}_2) \geq w(\mathcal{H}_-)/2$.

Case 4: λ^* lies in one of the following four quadrants: $y < b^*$ and $x > a^*$, $y < b^*$ and $x < a^*$, $y > b^*$ and $x > a^*$, $y > b^*$ and $x < a^*$ In this case either we resolve $\mathcal{H}' \subseteq \mathcal{H}_+$ such that $w(\mathcal{H}') \geq w(\mathcal{H}_+)/2$ or we resolve $\mathcal{H}' \subseteq \mathcal{H}_-$ such that $w(\mathcal{H}') \geq w(\mathcal{H}_-)/2$. (For example, if λ^* lies in the quadrant described by $y < b^*$ and $x > a^*$, then we resolve $\{h_i \in \mathcal{H}_+ : -(b^* \cdot b_i + c_i)/a_i \leq a^*\}$; see [9] for details.)

In cases 1 and 2, we have resolved a subset of $\mathcal{H}$ with weight at least $W/2$. Thus, we have achieved a fraction $\delta(2) = 1/2$ with $\gamma(2) = 2$ oracle calls. In cases 3 and 4, we have resolved a subset of $\mathcal{H}_+ \cup \mathcal{H}_-$ of weight at least $(1/2)(W/2 - w(\mathcal{H}_0))$. Let c^* is the weighted median of the set $\{p_i : p_i = -c_i/b_i, h_i \in \mathcal{H}_0\}$, where $w(p_i) = w(h_i)$. By resolving the line $y = c^*$, we will be resolving a subset of $\mathcal{H}_0$ having total weight at least $w(\mathcal{H}_0)/2$. The total weight of the lines resolved is thus at least $(1/2)(W/2 - w(\mathcal{H}_0)) + w(\mathcal{H}_0)/2 = W/4$, giving us a fraction $\delta(2) = 1/4$ with $\gamma(2) = 3$ oracle calls.

In the next section we describe how to find point p^* in $O(m)$ time by adapting a technique due to Megiddo [10]. Hence, we have a search algorithm for Euclidean plane with $\gamma(2) = 3$ and $\delta(2) = 1/4$, which does $O(n)$ work, in addition to the time spent on oracle calls.

3 Finding p^*

We rely on an idea introduced by Megiddo [10] who gave an algorithm for the case where W_N and W_P correspond to unweighted medians. The technique relies on a line query algorithm that finds the position of p^* with respect to any straight line in $O(m)$ time. We solve our problem by invoking the line query algorithm repeatedly; each step either finds p^* or it discards at least $1/8$ of the lines as candidates for H_P^* and H_N^*. Point p^* is just the intersection point of H_P^* and H_N^*.

An adaptation of Megiddo's line query algorithm to handle the weighted case is presented in Section 3.1. A summary of the steps required to find H_P^* and H_N^* is presented in Section 3.2.

3.1 The line query

Suppose L is a vertical line given by $x = c$. The analysis for the case where L is horizontal is analogous. L intersects W_P and W_N exactly once. Let $P = (c, b_P)$ be the point where W_P intersects L and $N = (c, b_N)$ be the point where W_N intersects L. N [P] is the weighted median of the y values of the intersection points of lines in $\mathcal{H}_-$ [$\mathcal{H}_+$] with L. It is easy to see that if $b_N < b_P$ [$b_N > b_P$] then $a^* < c$ [$a^* > c$]. Furthermore, if $b_N = b_P$ we have $p^* = (c, b_N)$.

Now consider the case where L has positive slope; the analysis for the case where L has negative slope is analogous. Line L will intersect with W_N exactly once, at a point $N = (a, b)$ that is the weighted median of the x values of the intersection points of lines in $\mathcal{H}_-$ with L. Thus N can be found in $O(m - k)$ time. Let P be the weighted median of the y values of the intersection points of lines in $\mathcal{H}_+$ and the line $x = a$. This takes $O(k)$ time. As before, the relative position of N and P along with the fact that N lies on W_N and P lies on W_P is sufficient to determine the relative position of L and p^*.

3.2 Finding H_P^* and H_N^*

Let R be the set of candidate lines for H_P^* and H_N^* at the beginning of the current iteration. At the start of the algorithm $R = \mathcal{H}_+ \cup \mathcal{H}_-$. Each iteration either finds p^* or discards $1/8$ of the candidate lines as follows:

Step 1. Find the median slope s of the lines in R.

Step 2. Partition R into sets G and S such that the lines in G [S] have slopes greater [smaller] than s.

Step 3. Form disjoint pairs of lines so that each pair has one line from G and another from S.

Step 4. Compute the x values of the intersection points of the paired lines. Let c be the median of these values.

Step 5. Do a line query on line $x = c$. This either finds p^*, in which case we are done, or it determines that $a^* < c$, or $a^* > c$. Suppose $a^* < c$. The case for $a^* > c$ is analogous.

Step 5.1 Let L be the line with slope s and y intercept equal to the median of the set $\{y - sx\}$, where (x, y) runs over the intersection points with x values greater than or equal to c. Note that there are at least $|R|/4$ such points and that L divides them into two equal-size sets.

Step 5.2 Do a line query on L. Lines L and $x = c$ identify a quadrant Q containing p^*. The quadrant opposite to Q has $|R|/8$ intersection points; with respect to each such point, there is a line that does not cross Q and thus can not be H_P^* or H_N^*. These lines are discarded from R and we are left with a similar problem with $7|R|/8$ lines.

Acknowledgements

We thank Naoki Katoh for drawing our attention to [10]. The referee noticed an error in an earlier version of this note and gave suggestions for improving the presentation.

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