[M2] J. B. M. Melissen, Optimal packings of eleven equal circles in an equilateral triangle, Philips Research Manuscript M.S. 17 295 (1992), (submitted).

[MS] J. B. M. Melissen and P. C. Schuur, Packing 16, 17 or 18 circles on an equilateral triangle, *Disc. Math.* (to appear).

[N] D. J. Newman, private communication.

[O] N. Oler, A finite packing problem, *Canad. Math. Bull.* 4 (1961), 153–155.

7 Discussion

While a finite number of patterns for infinite classes have been tentatively identified to date (two one-parameter patterns known or conjectured previously joined by several such patterns in Sec.5 and a two-parameter "matrix" pattern in Sec.6) a countable infinity of such patterns and classes probably exists. Furthermore, each value of n may well be a member of one or more such classes. Thus, the values $n = 18$, 23, 32, and 33, which were not placed into classes in this paper, may well be members of as yet unidentified classes of packings with complex patterns. In fact, a fixed value of n may be on the paths of many, possibly infinitely many, such classes. 12 disks gives an example of this: it is on the path of the class $4\Delta(k)$ and it is also the first term of the classes $\Delta(3k+1)+2$ and $\Delta(2k+3)-3$. As the value of n increases along the path of a class, "hesitations" of the best pattern may occur, wherein several different nonequivalent patterns coexist among the rigid packings and compete for the title of the best. A resolved case of such hesitation occurs for the class $\Delta(2k+1)+1$ where for $k \geq 5$ ($n \geq 67$) two equivalent best patterns finally emerge (at least according to our experiments). We were not able to confirm by experiments the winning pattern for the class $\Delta(k+2)-2$. Will such hesitation always be resolved in favor of one of the competing patterns in a finite initial segment of the path?

References

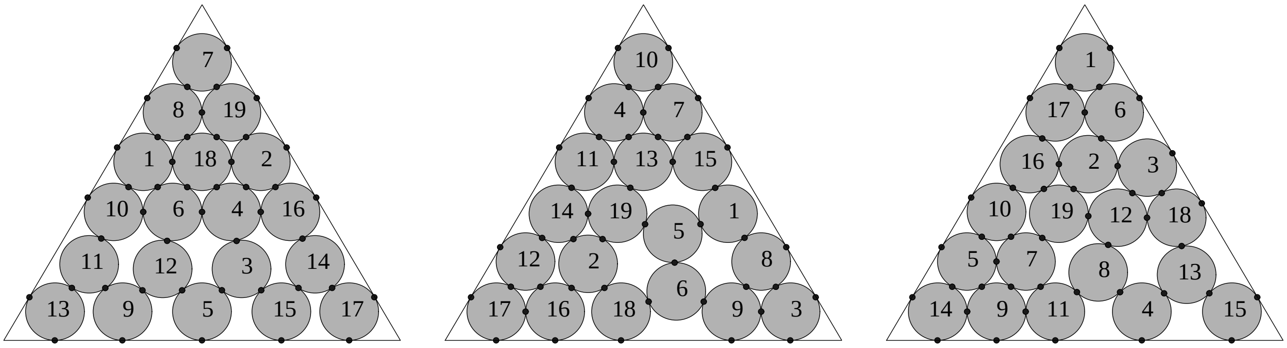
[CFG] H. T. Croft, K. J. Falconer and R. K. Guy, *Unsolved Problems in Geometry*, Springer Verlag, Berlin, 1991, 107–111.

[FG] J. H. Folkman and R. L. Graham, A packing inequality for compact convex subsets of the plane, *Canad. Math. Bull.* **12** (1969), 745–752.

[L] B. D. Lubachevsky, How to simulate billiards and similar systems, *J. Computational Physics* **94** (1991), 255–283.

[LS] B. D. Lubachevsky and F. H. Stillinger, Geometric properties of random disk packings, *J. Statistical Physics* **60** (1990), 561–583.

[M1] J. B. M. Melissen, Densest packings for congruent circles in an equilateral triangle, *Amer. Math. Monthly* **100** (1993), 916–925.

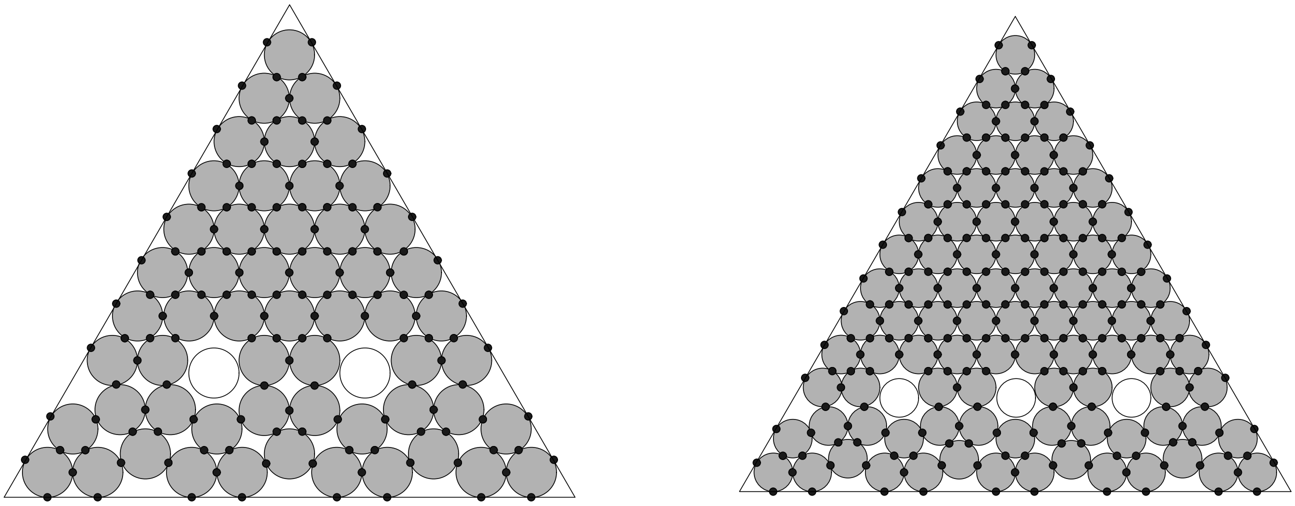


t19a
0.200321458983439 45 bonds

t19b
0.200200894290590 45 bonds

t19c
0.200186574270937 45 bonds

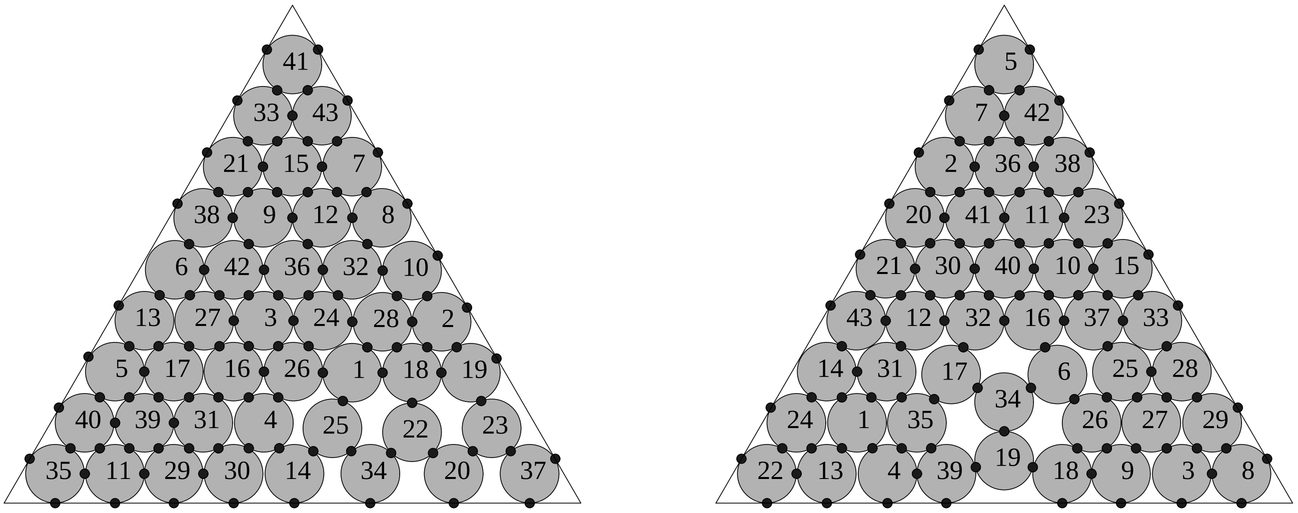
Figure 5.11: The best (t19a), the next best (t19b), and the third best (t19c) packings of 19 disks.



t57a
0.104447019187802 143 bonds

t93a
0.07978572868437232 240 bonds

Figure 5.12: The best packings of 57 disks (t57a) and 93 disks (t93a).

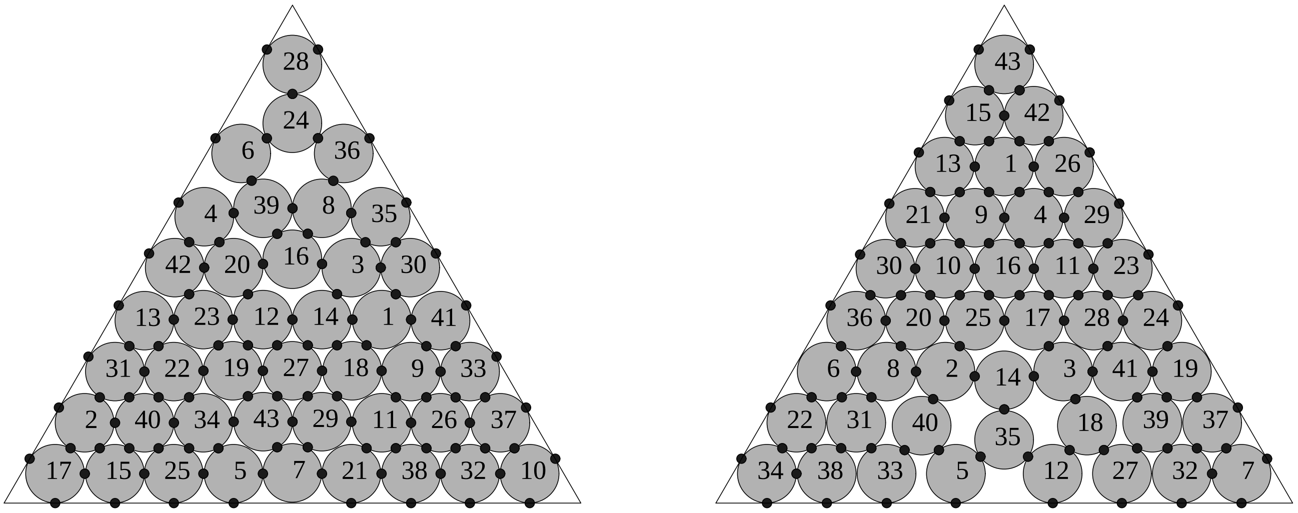


t43a

0.125004280307977 108 bonds

t43b

0.125003232541680 108 bonds



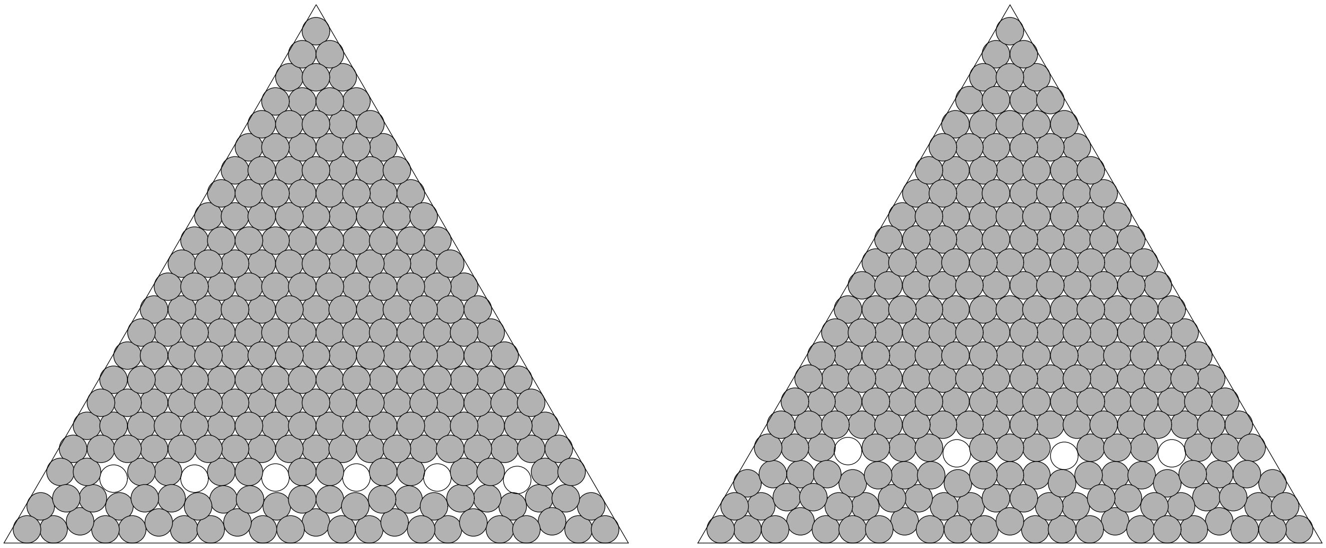
t43c

0.125002529873623 108 bonds

t43d

0.125002234829571 106 bonds

Figure 5.10: The best (t43a), the next-best (t43b), the third-best (t43c) and the fourth-best (t43d) packings of 43 disks.

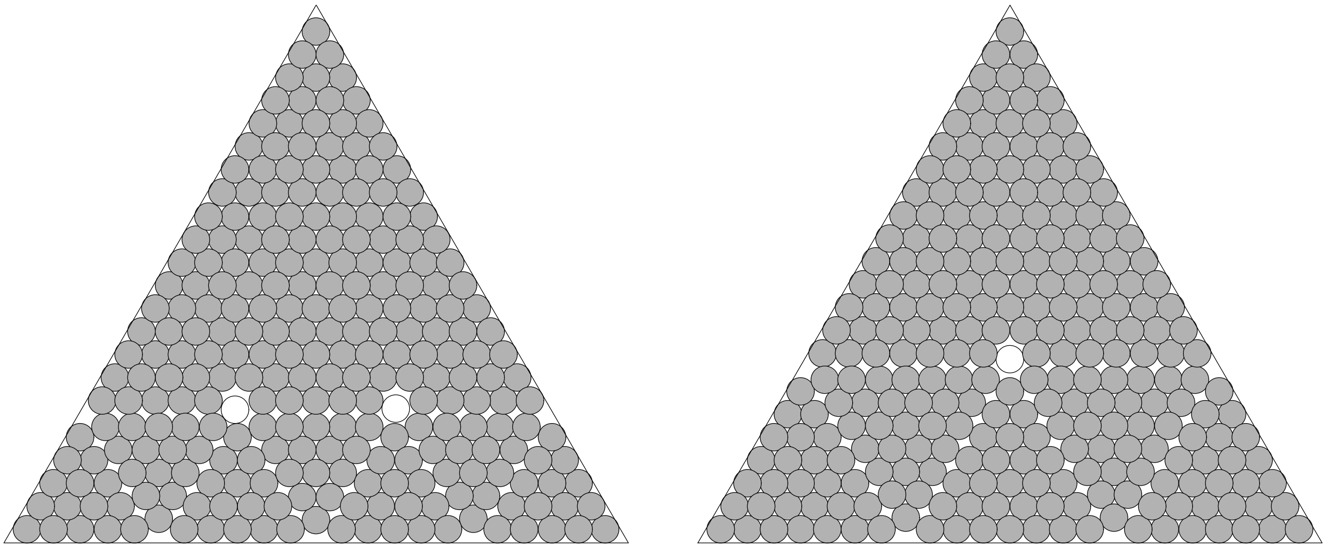


t255a

0.0465999333293566 693 bonds

t256a

0.0465050212813994 703 bonds



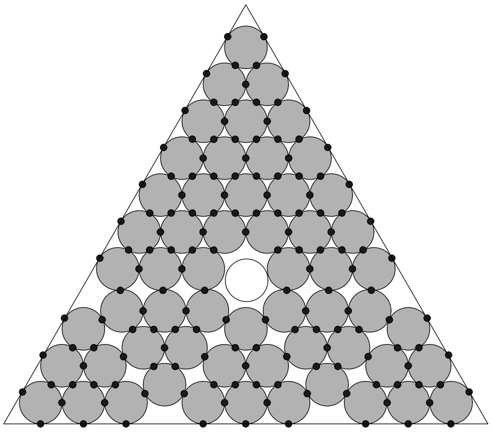
t258a

0.0463515918933559 719 bonds

t260a

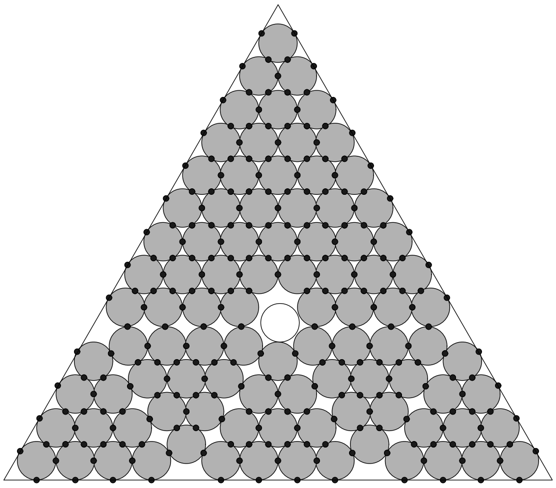
0.0462256883510349 733 bonds

Figure 6.3: The best packings of 255 disks (t255a), 256 disks (t256a), 258 disks (t258a), and 260 disks (t260a).



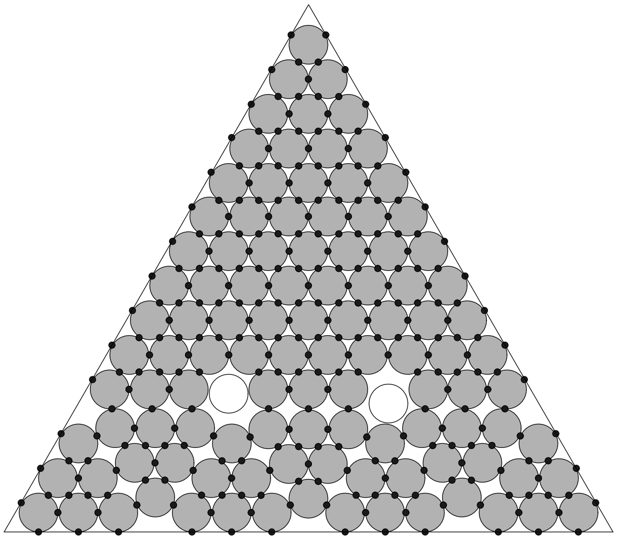
t58a

0.103809894100183 151 bonds



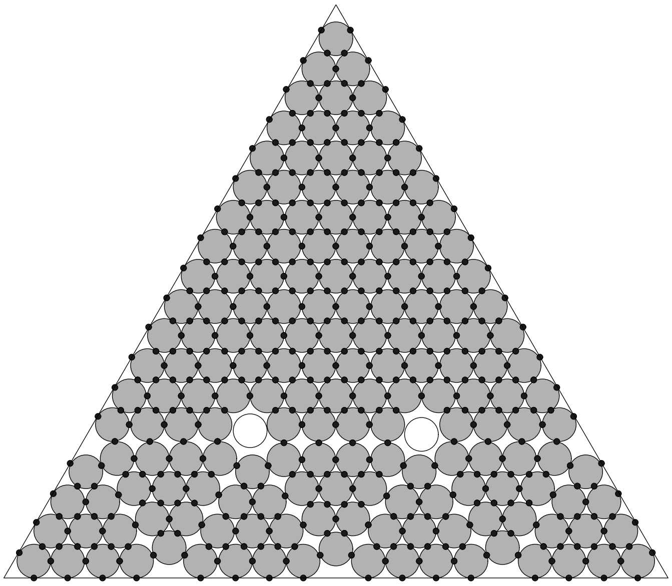
t95a

0.0791578042660103 256 bonds



t108a

0.0736689924964931 287 bonds



t175a

0.0569014905693285 479 bonds

Figure 6.2: The best packings of 58 disks (t58a), 95 disks (t95a), 108 disks (t108a), and 175 disks (t175a).

small and bonds are not seen. The pictures with a larger scale (omitted here) show that all the bonds exist in right places.

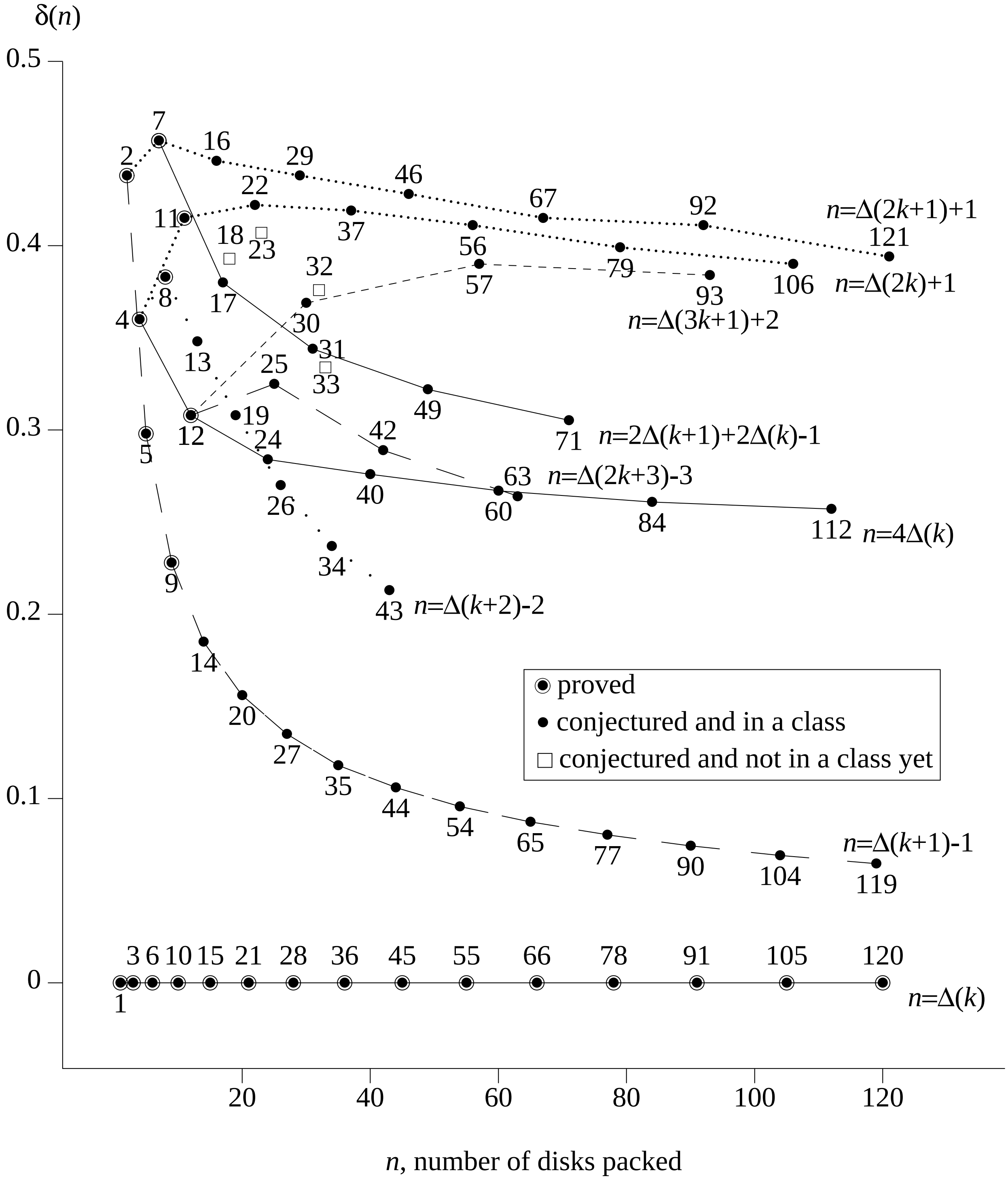


Figure 6.1: Discrepancy between side length of the triangle and its lower bound for different n.

row or a column of this matrix we obtain a distinct infinite class of packings. Our conjecture is that each row class is tight and each column class is loose.

This matrix contains three infinite classes conjectured in Section 5: the row at $p = 1$ is the class $4\Delta(k)$, the column at $k = 1$ is the class $\Delta(2p) + 1$, and the column at $k = 2$ is the class $\Delta(3p + 1) + 2 = \Delta(3p - 1) + (2p + 1)\Delta(2)$.

The conjecture about the full matrix is also reinforced by the fact that k and p in (3) are unique for each given value of $n = n_p(k)$. This can be easily seen using the first expression for $n_p(k)$ in (3). To further test our matrix conjecture, we generated the list of all n of the form $n = n_p(k)$ for $n \leq 300$. These are: 4, 11, 12, 22, 24, 30, 37, 40, 56, 57, 58, 60, 79, 84, 93, 95, 106, 108, 112, 137, 138, 141, 144, 172, 174, 175, 180, 192, 196, 211, 220, 254, 255, 256, 258, 260, 264, and 280. Some increments in this increasing sequence are small. Specifically, in each following subsequence increments do not exceed 2: (11, 12), (22, 24), (56, 57, 58, 60), (93, 95), (106, 108), (137, 138), (172, 174, 175), and (254, 255, 256, 258, 260).

Now, take for example, $255 = n_7(2)$ and $256 = n_5(3)$. These are two "almost" equal numbers of disks. However according to the matrix conjecture they should produce different patterns of densest packings: the pattern for 255 should consist of one large and 15 small triangles with 6 rattlers and the pattern for 256 of one large and 11 small triangles and 4 rattlers. Similarly, the matrix conjecture prescribes specific patterns for the densest packing of the other numbers of disks n of this sequence, e.g., $254 = n_{11}(1)$, $258 = n_3(5)$, and $260 = n_2(7)$.

One might think it would be a stress test for both the matrix conjecture and our packing procedure to try to pack these numbers of disks. Note that many of the packings for smaller values in the sequences above have been generated (as discussed above) and they all conform to the matrix conjecture. Thus, we experimented with packing $n = 254$, 255, 256, 258, and 260 disks. Recall that the procedure of packing has no idea, so to say, of the desirable packing. Starting with random initial conditions the disks perform chaotic movements, they collide with each other and with the boundaries millions of times and each collision evaluation is subject to roundoff error.

The experiments turned out to be not so difficult. (Case of 53 disks proved to be harder.) As expected, the best packing of 254 disks has the pattern of its class $\Delta(2k)+1$ with 10 rattlers ($d(254) = 0.0467170396481042$, 679 bonds; we omit the diagram). Fig.6.3 shows the patterns of packings t255a, t256a, t258, and t260a. These too are consistent with formula (3). Note that because of the large number of disks in the packings the scale of drawing in Fig.6.3 is the form

$$(3) \qquad n_p(k) = \Delta((k+1)(p+1)-2) + k \ = \ \Delta((k+1)p-1) + (2p+1)\Delta(k)\ .$$

Two equal expressions for $n_p(k)$ are given in (3). The second expression suggests $r = 2(p+1)$ triangles. If we take, for example, $k = p = 2$, we get $n_2(2) = 30$ disks and $r = 6$. The conjectured t30a (Fig 3.7) indeed consists of 6 densely packed triangles, if we attach rattler 8 to the top triangle $\Delta(5)$. Take now $58 = n_2(3)$. The pattern of our experimental packing t58a (Fig 6.2) looks like t30a (Fig 3.7) with 6 triangles again (with the rattler attached to the top triangle).

Thus, just as t30a is a member of the class $\Delta(3k-1) + (2k+1)\Delta(2)$ for $k = 2$, t58a is (perhaps) a member of the class $\Delta(4k-1) + (2k+1)\Delta(3)$ for $k = 2$. The pattern of the kth packing of this class is composed of $2(k+1)$ densely packed triangles. Is this class tight or loose? We believe it is loose because $\Delta(4k-1) + (2k+1)\Delta(3) = \Delta(3k+1) + 2$, and a class of the form "$\Delta(.) + const$" is always loose (we think). Incidentally, the number of the triangles in the class is unbounded with k.

However, if (as we believe) the sequence of densest packings $n_p(k), k = 1, 2, 3, ...$ for fixed $p = 2$ can be continued with all packings having the same pattern of six triangles, then t58a might also be a member of the class $\Delta(2k+1) + 5\Delta(k)$ for $k = 3$. The next term in the latter class would be the densest packing of $n = n_2(4) = \Delta(9) + 5\Delta(4) = 95$ disks. Our experiments with 95 disks, indeed, produced the desired pattern of six triangles in the densest packing t95a (Fig. 6.2). This reinforces our suspicion that the class $n_2(k)$, $k = 1, 2, ...$, exists in which each densest packing consists of six densely packed triangles. The class $n_2(k)$, $k = 1, 2, ...$, should be tight because each densest packing in it consists of a fixed number of triangles.

By increasing p we are moving into a different class, which is again tight if the conjecture above is correct. Thus, for $p = 3$ we have the sequence $n_3(1) = 22, n_3(2) = 57, n_3(3) = 108, n_3(4) = 175, ...$ Packings t22a (Fig 3.1), t57a (Fig 5.12) indeed each consist of $2(p+1) = 8$ densely packed triangles. Our experiments with 108 and 175 disks yield the same pattern in the best packings (Fig 6.2) so the class $n_3(k)$ probably exists too. In the same way, the class $n_p(k), k = 1, 2, ...$ exists for any fixed index p and has a distinct pattern with $r = 2(p+1)$ triangles and $p-1$ rattlers.

If this is correct, then Figure 6.2 can be seen as the 2×2 submatrix for $2 \leq p \leq 3$ and $3 \leq k \leq 4$ of the matrix of dense packings of $n_p(k)$ disks where $1 \leq k, p \leq \infty$. By traversing a the value of $\delta(\Delta(k)+c)$ is bounded away from zero. In other words, packings of $\Delta(k)$ disks are so tight that any attempt to accommodate even one additional disk noticeably worsens the packing in that $L(n)$ increases by at least some positive amount independent of n. In this sense the packings for the class $\Delta(k)$ are "tight".

On the other hand, we believe that after any fixed positive number of disks are added to $\Delta(k)$ disks, any other fixed number of disks can be added without substantial "damage" to $\delta(n)$ (asymptotically). Thus, for example, it would seem that if $\lim_{k\to\infty}\delta(n(k))$ exists for each of the class $n(k)=\Delta(2k)+1$, $\Delta(2k+1)+1$, and $\Delta(3k+1)+2$ (and the limits probably do exist), then all three limits are equal. In this sense the classes $\Delta(2k)+1$, $\Delta(2k+1)+1$, and $\Delta(3k+1)+2$ are "loose".

Similarly, the classes $n=\Delta(k+2)-2$ and $\Delta(2k+3)-3$ are "loose" in the sense that we can add one disk to the best (conjectured) packing without noticeable change of $\delta(n)$ for sufficiently large n. This follows from the fact that $\lim_{k\to\infty}\delta(\Delta(k)-p)\to 0$ for p fixed. The latter limit is obvious by noticing that a lower-bounding packing for $\Delta(k)-p$ disks when k is sufficiently large is simply the densest packing of $\Delta(k)$ disks with p disks removed; for such a packing, δ is asymptotically 0.

Formally, we say that an infinite class of packings of n disks, $n=n(1),n(2),...n(k),...$, is *loose*, if $\lim_{k\to\infty}[\delta(n(k)+1)-\delta(n(k))]=0$. Because we believe this limit exists for any class we consider, each class has to be either tight or loose.

We further conjecture that the classes $4\Delta(k)$ and $2\Delta(k+1)+2\Delta(k)-1$ are tight, similar to the class $\Delta(k)$.

Are there other tight classes? Here is our argument in favor of the existence of a countable *infinity* of distinct tight classes. We believe that if each densest packing of n disks for $n=n(1),n(2),...n(k),..$, consists of a *fixed* number, say r, of densely packed triangles $\Delta(.)$, then $\delta(n(k)+1)-\delta(n(k))$ is bounded away from zero as k goes to infinity.

Thus, we have the following task: for each r from some infinite set find a sequence $n(1),n(2),...n(k),..$, so that the densest packing of $n(k)$ disks for all k has the "same pattern" and consists of exactly r densely packed triangles. Note that *a priori* we are not able to define what the "same pattern" is (and hence we have no formal definition of what a "class" is); but after producing a class the pattern is usually clear.

Let us consider a two-parameter family of numbers $n=n_p(k)$, $p=1,2...$, $k=1,2...$, of

6 How good are the packings?

Let us compare the values of the best packings with the only bound currently available, namely one based on the inequality of Oler [O]. This inequality has the following form (see [FG] for a simple proof): Let K be a compact, convex subset of $\mathbb{E}^2$ with area $A(K)$ and perimeter $P(K)$. If $p(K)$ denotes the maximum number of points that can be placed in K so that any pair has mutual distance at least 1, then the following inequality holds:

$$p(K) \leq \frac{2}{\sqrt{3}} A(K) + \frac{1}{2} P(K) + 1 \ . \tag{1}$$

Inverting (1) and applying it with $p(K) = n$ and K being an equilateral triangle of side length $L(n)$, we obtain

$$L(n) \geq \frac{1}{2}(-3 + \sqrt{8n+1}) := t(n) \ . \tag{2}$$

In Fig 6.1 we plot the difference

$$\delta(n) := L(n) - t(n)$$

versus n for selected values of $n \leq 121$. A dot with a circle around it indicates that the corresponding value has been proved to be optimal; a dot without a surrounding circle or an open square indicates that the value is only conjectured to be optimal. For up to 37 disks there were only four values which did not fall into one of our infinite classes, namely, $n = 18$, 23, 32, and 33; those are indicated by open squares. The values which have been associated with classes are connected by lines, with a distinct type of line for each class.

We should point out that for each n, the value of the largest disk diameter $d(n)$ and the value of $L(n)$ are reciprocally related, i.e., $d(n)L(n) = 1$. Thus, $L(\frac{k(k+1)}{2}) = k - 1$ for $k \geq 1$. If our conjecture for $n = \Delta(k) + 1 = \frac{k(k+1)}{2} + 1$ is correct then it would follow that

$$\lim_{k\to\infty} \delta\left(\frac{k(k+1)}{2} + 1\right) \leq \frac{2}{\sqrt{3}} - 1 = 0.1547 \ \ldots \ .$$

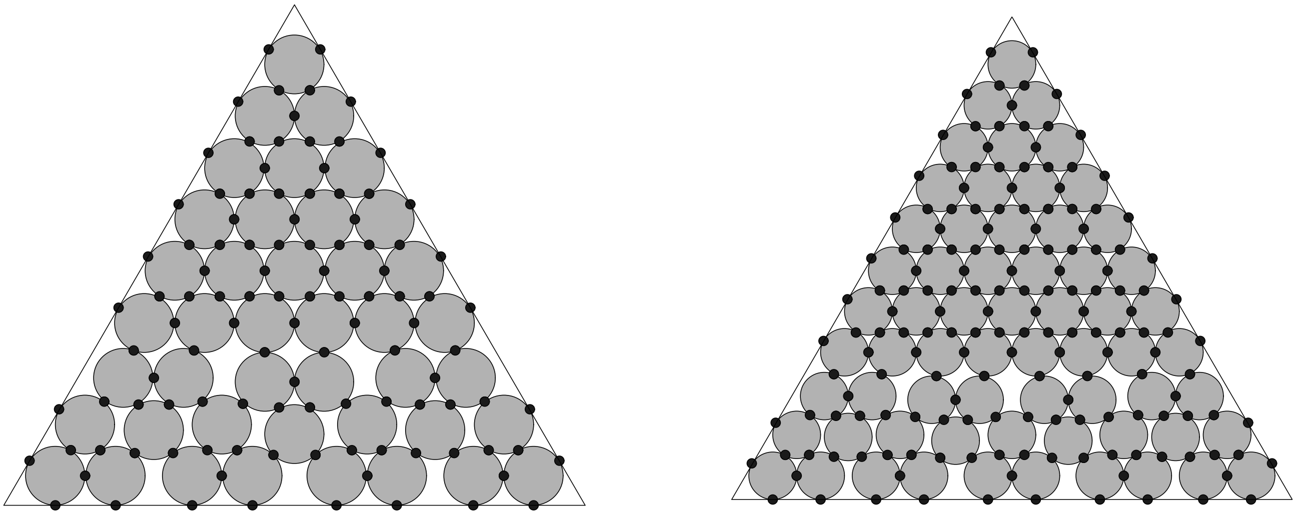
Using the explicit values for $d(4\Delta(k))$ and $d(2\Delta(k+1) + 2\Delta(k) - 1)$ given in Section 5, we have $L(4\Delta(k)) = 2k - 2 + \sqrt{3}$ and $L(2\Delta(k+1) + 2\Delta(k) - 1) = 2k - 1 + \sqrt{3}$, from which it follows that both $\lim_{k\to\infty} \delta(4\Delta(k))$ and $\lim_{k\to\infty} \delta(2\Delta(k+1) + 2\Delta(k) - 1)$ are at most $\sqrt{3} - 3/2 = 0.2321 \ \ldots$ with the distance between kth term and the limit being of the order of $1/k$.

We conjecture that for each of the classes $n = \Delta(2k) + 1$, $\Delta(2k+1) + 1$, and $\Delta(3k+1) + 2$ the value of $\delta(n)$ is bounded away from zero. In fact, we believe that for any fixed $c > 0$, triangle $\Delta(k)$ at the top, supported by two sparse rows of disks, each of which lacks a disk compared to what would be there in a perfect hexagonal packing. The top-$\Delta(k)$-plus-two-sparse-rows packing indeed exists for any $k \geq 3$ and is rigid. In particular, the pattern appears again in the best packings for $k = 4$ (t19a in Fig. 5.11).

However, the optimality of this pattern does not continue for $k > 4$, as can be seen in Fig. 3.5 where the best packing t26a has a different pattern. The top-$\Delta(k)$-plus-two-sparse-rows pattern is not even among the top four packings for $n = 26$. The pattern for t26a and t26b persists for the next term (see t34a and t34b in Fig. 3.11) but then roles become reversed for 43 disks (see t43a and t43b in Fig. 5.10). Will the pattern of packing t43a remain optimal for larger values of k? Unfortunately, our algorithm fails to obtain stable packings for 53 (or larger values of $\Delta(k+2)-2$) disks.

$\boldsymbol{\Delta}(\mathbf{3k+1})+\mathbf{2} = \boldsymbol{\Delta}(\mathbf{3k-1})+(\mathbf{2k+1})\boldsymbol{\Delta}(\mathbf{2})$. $30 = \Delta(5)+5\Delta(2)$ and packing t30a (Fig. 3.7) can be viewed as a $\Delta(5)$ triangle on top from which disk 8 fell off and became a rattler, supported by five triangles $\Delta(2)$ from below; t30a is the second term of the class. To produce this structure for the next value of k we add two triangles $\Delta(2)$ on the bottom and three more layers of disks to enlarge the top triangle to become a $\Delta(8)$ (and, in general, this procedure is repeated for larger values of k). Indeed our experiments with $n = 57$ ($k = 3$) and $n = 93$ ($k = 4$) did produce this structure in the best packings (see t57a and t93a in Fig. 5.12). For $k = 1$ we have $n = 12$, a degenerate case with no rattlers. The number of rattlers in this packing for general k is $k-1$.

$\boldsymbol{\Delta}(\mathbf{2k+3})-\mathbf{3} = \boldsymbol{\Delta}(\mathbf{2k})+(\mathbf{2k+1})\boldsymbol{\Delta}(\mathbf{2})$. The first term is packing t12a (Fig. 5.1) which also belongs to class $4\Delta(.)$. The second term is packing t25a (Fig. 3.4) which can be viewed as a $\Delta(2k)$ triangle on top supported by $2k+1$ alternating $\Delta(2)$ triangles below. This pattern of the second term, whose description also fits the first term, is more apparent in the third and fourth terms (see t42a and t63a shown in Fig. 5.9). According to our experiments the next terms t88a and t117a do not continue this pattern but give way to patterns which are somewhat similar to those of the class $\Delta(k+2)-2$ considered above. (We omit diagrams for both t88a and t117a.)



t42a

0.125451671189752 108 bonds

t63a

0.100080300212798 165 bonds

Figure 5.9: The best packings of 42 disks (t42a) and of 63 disks (t63a).

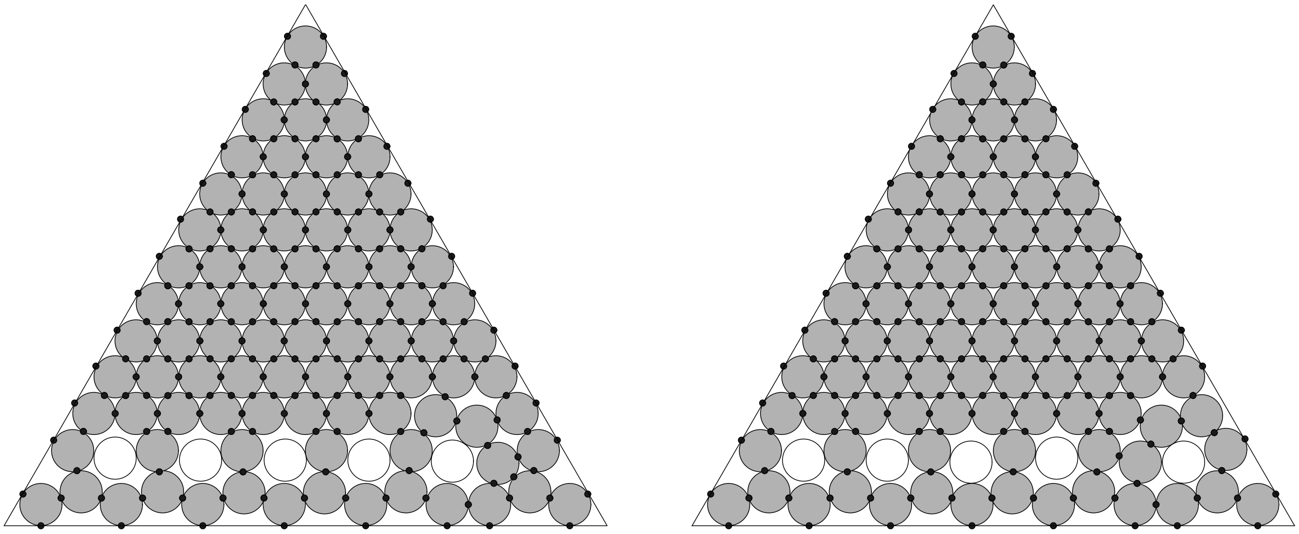
the diagram here.

Labels t29b63.2, t46b106.2, t67a161.2, t92a228.2, and t121a307.2 have the suffix 2 in them because there exist equivalent packings t29b63.1, t46b106.1, t67a161.1, t92a228.1, and t121a307.1, respectively. The latter differ from the former in the placement of only 4 disks. An easy way to explain this is to look at the second term of the class $n = \Delta(2k+1)+1$ for $n = 16$ disks (Fig. 5.5). In this case both modifications exist and both deliver the optimum, t16.a33.1 and t16.a33.2. They differ in the placement of disks 2, 3, 4, and 7.

The side rattler disk 15 in t16b can be considered a precursor for the side rattler 29 in t29d. The same side-rattler pattern was observed in lower ranked packings for $k = 4$ ($n = 46$), $k = 5$ ($n = 67$), $k = 6$ ($n = 92$), and $k = 7$ ($n = 121$).

Classes $\Delta(2k)+1$ and $\Delta(2k+1)+1$ are a parity complementary pair, similar to the pair of classes $4\Delta(k)$ and $2\Delta(k+1)+2\Delta(k)-1$ considered above.

$\boldsymbol{\Delta}(\mathbf{k}+\mathbf{2})-\mathbf{2}$. While the optimal packings of $\Delta(k+2)-1$ disks are always (apparently) perfectly hexagonal with a single disk removed, the removal of two disks from a hexagonal arrangement is never optimal. The first two terms t4a and t8a (Fig. 4.1) suggest no common pattern. Looking at the next case t13a (Fig. 5.1) suggests the pattern for $k \geq 3$ of a packed



t92a228.2

0.0801405249366249 228 bonds

t92b

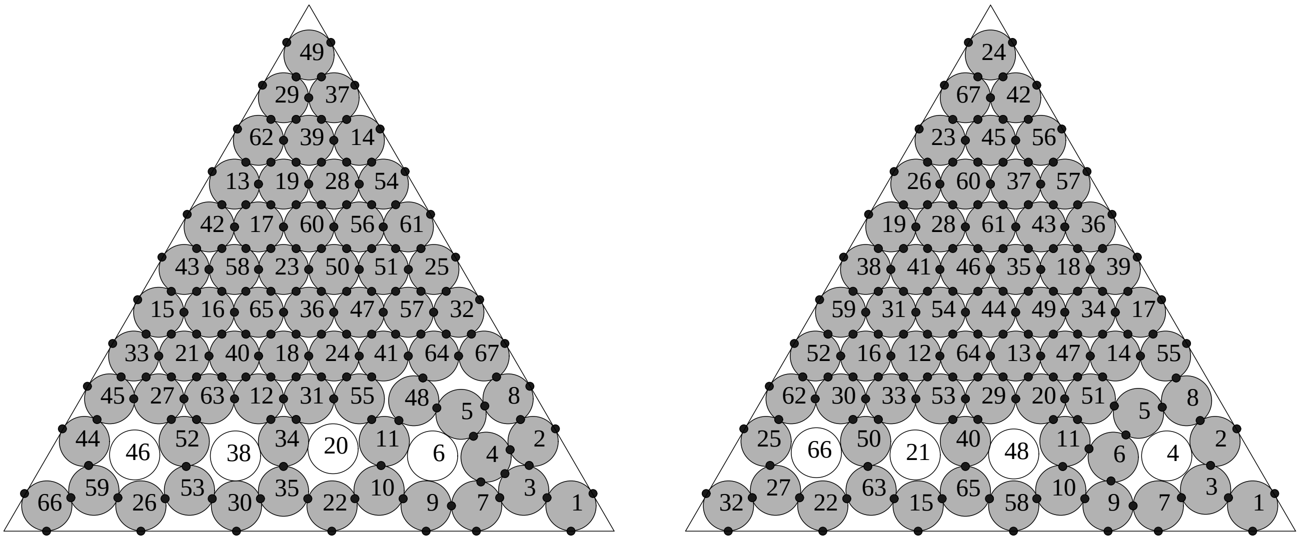
0.0801398309712360 228 bonds

Figure 5.8: A best (t92a228.2) and the next-best (t92b) packings of 92 disks.

t79a (194 bonds, 5 rattlers, $d(79) = 0.0871159038791759$), and t106a (267 bonds, 6 rattlers, $d(106) = 0.0742982999063026$). We do not reproduce the diagrams here for the latter two packings; their patterns are identical to the class description given above.

$\Delta(2k+1)+1$. When $m = 2k+1$, $k = 1, 2...$, the odd parity of m causes a more complex adjustment to the extra disk. The bottom row ripples in a non-symmetric way; the ripple creates k cages for rattlers and a cavity; see t29a (Fig. 3.6) for $k = 3$.

Notice that packing t29b63.2 (Fig. 3.6) has almost the same structure as t29a, except for the cage that consists of disks 2, 3, 7, 9, 6, 5, and 8, is depressed and disk 4 in t29b63.2 is not a rattler, and a nonrattler 6 in t29a becomes a rattler in t29b63.2. The same two modifications exist for $k = 4$ (i.e., $n = 46$), and the modification with the depressed cage, t46b106.2, is again inferior (Fig. 5.6). Beginning with $k = 5$ (i.e., $n = 67$), while both modifications exist, they exchange their roles: the depressed one becomes the best, t67a161.2, while the other one becomes the inferior one, t67b (Fig. 5.7). For example, t92a228.2 is the modification with the depressed cage, while t92b is the other one (Fig. 5.8). The same pattern is displayed by packing t121a307.2 (307 bonds, 6 rattlers, $d(121) = 0.0691630188894699$), for which we omit



t67a161.2

0.0952213722318656 161 bonds

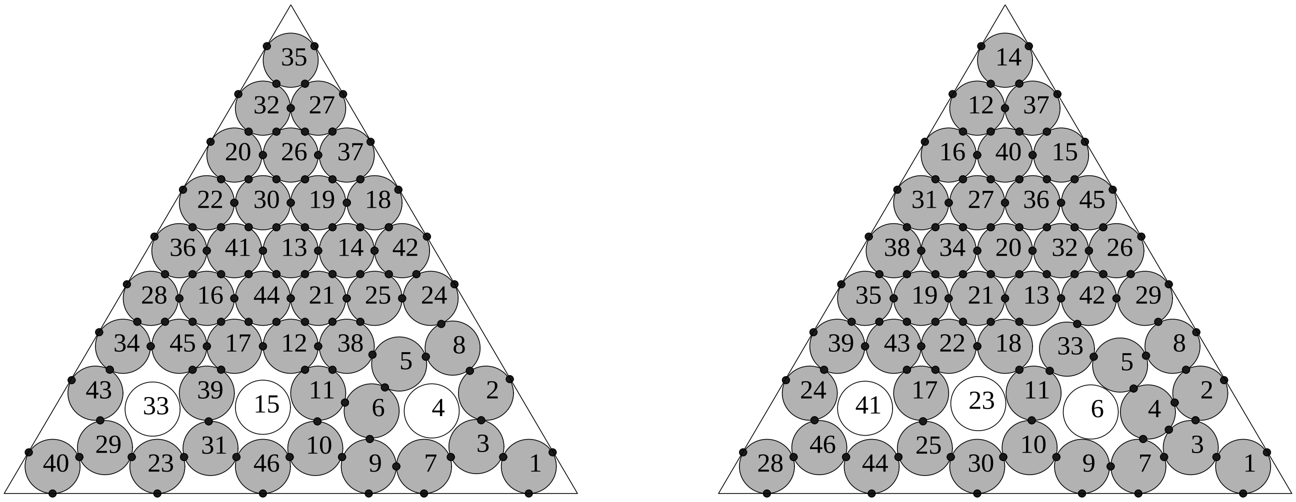
t67b

0.0952208239815187 161 bonds

Figure 5.7: A best (t67a161.2) and the next-best (t67b) packings of 67 disks.

The pattern of one of the parity-complementary class pair can be obtained from the pattern of the other class by a simple transformation. For example if we eliminate the eight bottom disks 2, 42, 6, 25, 15, 29, 32, 47, and the rattler 4 in t49a130 (Fig. 5.3), we obtain the pattern of t40a (Fig. 5.2).

$\boldsymbol{\Delta}(\mathbf{2k})+\mathbf{1}$. As we noted earlier, $\Delta(m)$ densely packed disks in an equilateral triangle form a perfect hexagonal lattice with m disks on a side. When $m = 2k$ is even, the structure with $\Delta(2k) + 1$ disks adjusts itself to one extra disk as follows. The top $2k - 1$ rows remain packed hexagonally, and the bottom row ripples to accommodate $2k + 1$ disks instead of $2k$. In this ripple of the bottom row, the 1st, 3rd, ... $(2k+1)$th disk beginning from the left corner remain attached to the bottom, while the 2nd, 4th, ... $2k$th disk rise and attach themselves to 2nd, 4th, ... $2k$th disks respectively, of the row above. Thus, $k - 1$ rigid cages are formed. The $k - 1$ disks of the row above to which no disk is attached from below fall off into these cages and become rattlers. The first seven terms of the class $\Delta(2k) + 1$ are: t4a ($k = 1$, no rattlers since $k - 1 = 0$, see Fig. 4.1), t11a (constructed in [M1] with one rattler; see Fig. 5.1), t22a (see Fig. 3.1) with two rattlers, t37 and t56 (Fig 5.4) with 3 and 4 rattlers, respectively,



t46a
0.117209943988839 106 bonds

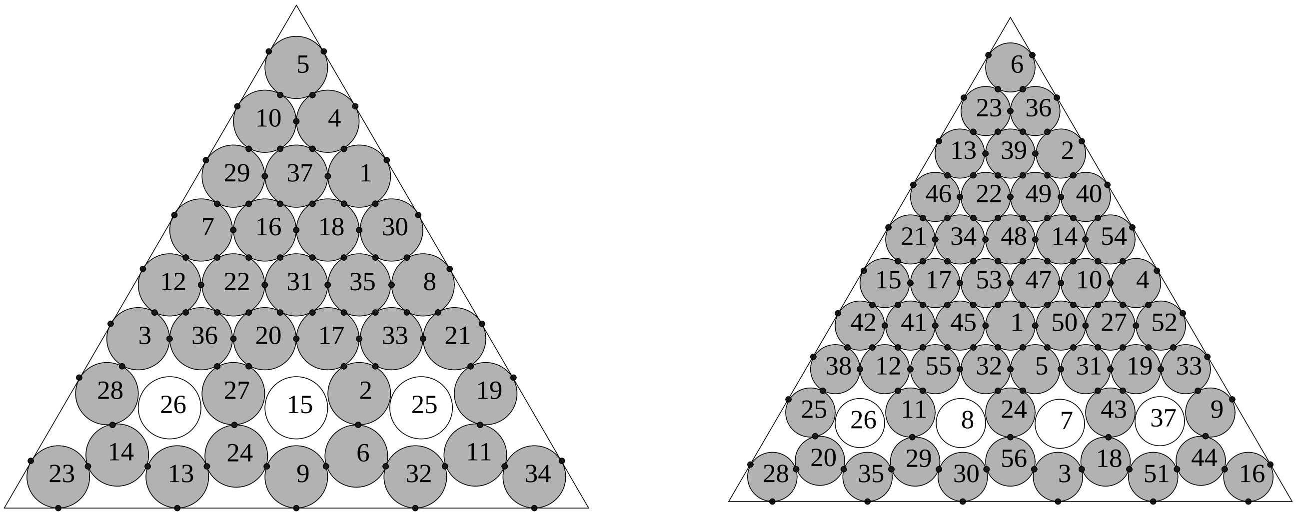
t46b106.2
0.117208402974392 106 bonds

Figure 5.6: The best (t46a) and a next-best (t46b106.2) packings of 46 disks.

The two larger triangles are defective: both coalesce a corner disk (disk 6 in t17a40, disk 4 in t31a79, disk 4 in t49a130). Packing t31a81.2 is obtained from t31a79 by the left larger triangle acquiring its corner disk 4 and pushing disks 31 and 29 from the left side of the other large triangle down into the cavity formed . If we push only disk 31, we obtain a packing t31a81.1 (not shown). If we push all three disks 31, 29, 10 into the cavity (and rotate the resulting structure to recover the symmetry with respect to the vertical axis), we obtain the fourth best packing t31a82.

A simple analysis of the patterns obtained implies that $d(2\Delta(k+1)+2\Delta(k)-1) = \frac{1}{2k-1+\sqrt{3}}$. Our experiments with $71 = 2\Delta(6) + 2\Delta(5) - 1$ disks produced the same patterns for the best packing.

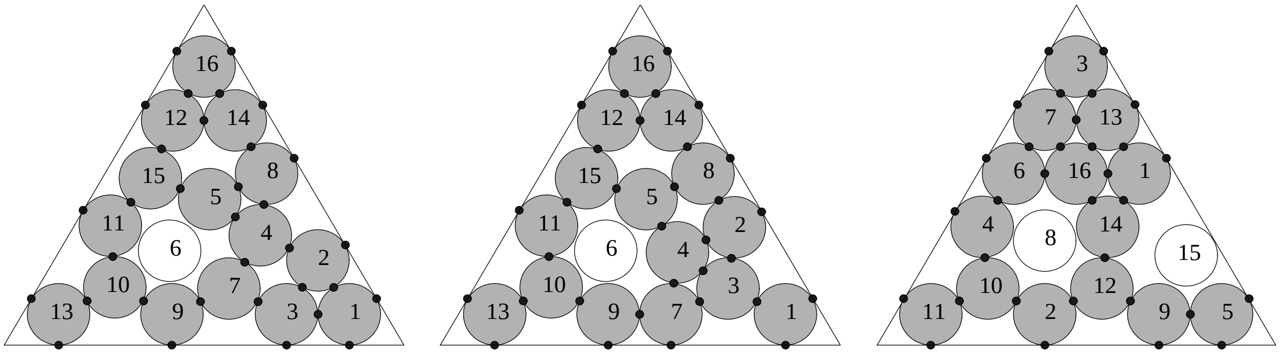
If we fit the members of class $2\Delta(k+1) + 2\Delta(k) - 1$ into the boundaries of the triangular periods, as we did for the class $4\Delta(k)$, we find that every other period has exactly one n of the form $2\Delta(k+1) + 2\Delta(k) - 1$ which lies almost exactly in the center of the period. Thus, classes $2\Delta(k+1) + 2\Delta(k) - 1$ and $4\Delta(k)$ are "parity-complementary" to each other. Beginning with the second triangular period 3 to 6, each period has exactly one term of one or the other class; even periods contain terms of the class $4\Delta(k)$ and odd periods contain terms of the class $2\Delta(k+1) + 2\Delta(k) - 1$.



t37a
0.132739196122383 84 bonds

t56a
0.105229325597427 133 bonds

Figure 5.4: The best packings of 37 disks (t37a) and 56 disks (t56a).



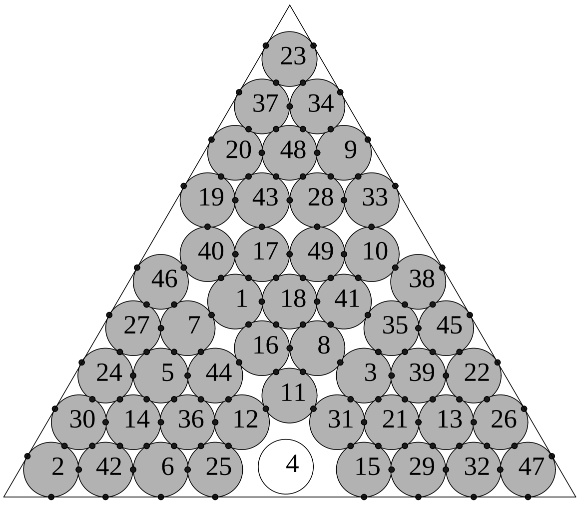
t16a33.1
0.216227269309782 33 bonds

t16a33.2
0.216227269309782 33 bonds

t16b
0.215843184970192 32 bonds

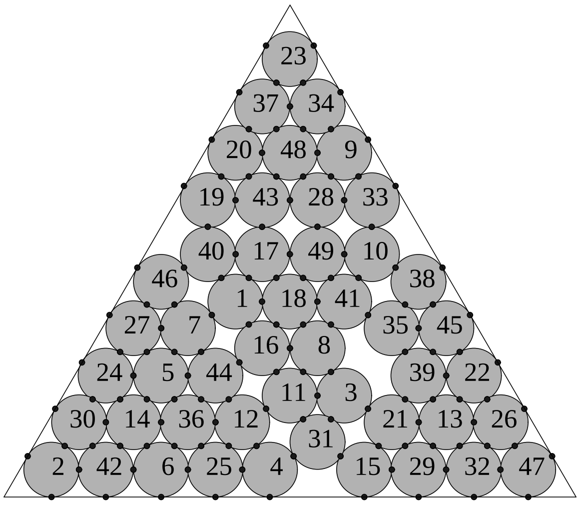
Figure 5.5: The best (t16a33.1, t16a33.2) and the next-best (t16b) packings of 16 disks.

There are three different numbers of bonds in these packings; the smallest number of bonds is in the packing with a rattler, then $k-2$ packings each of which has a "cavity" and the same number of bonds, and finally one more packing without a rattler or a cavity with the largest number of bonds. The four triangles, two small and two large, that illustrate the expression $2\Delta(k+1)+2\Delta(k)-1$, can be seen in the packings with a rattler (t17a40, t31a79, t49a130).



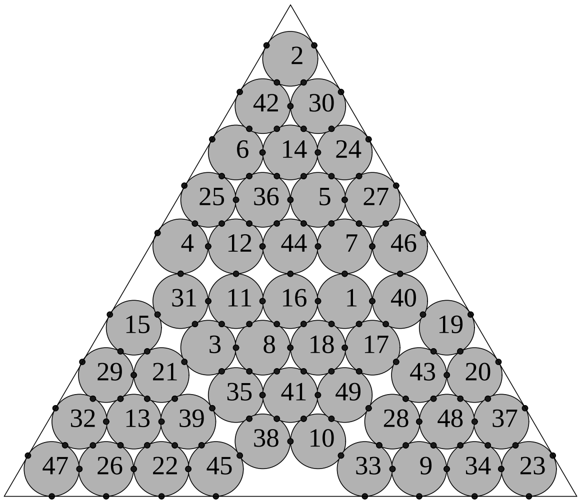
t49a130

0.114520634618068 130 bonds



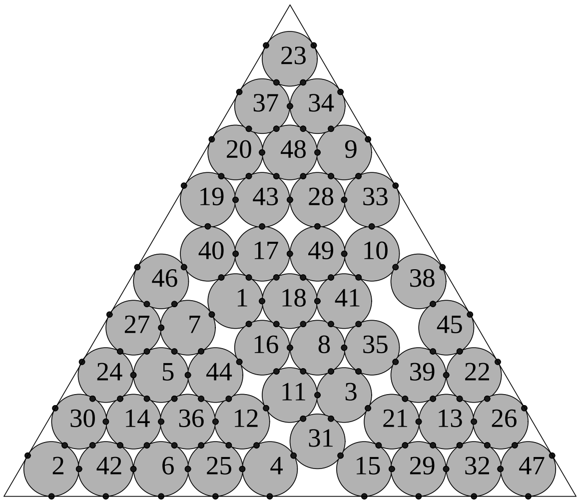
t49a132.2

0.114520634618068 132 bonds



t49a133

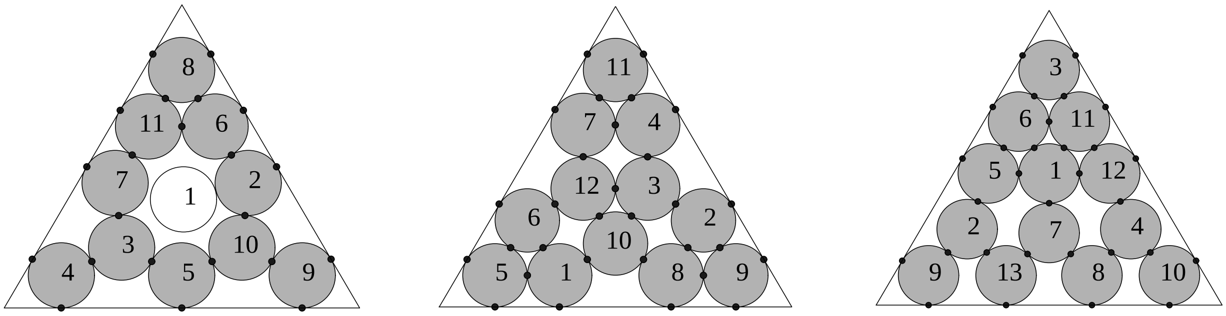
0.114520634618068 133 bonds



t49a132.3

0.114520634618068 132 bonds

Figure 5.3: Four (out of the five existing) best packings of 49 disks.

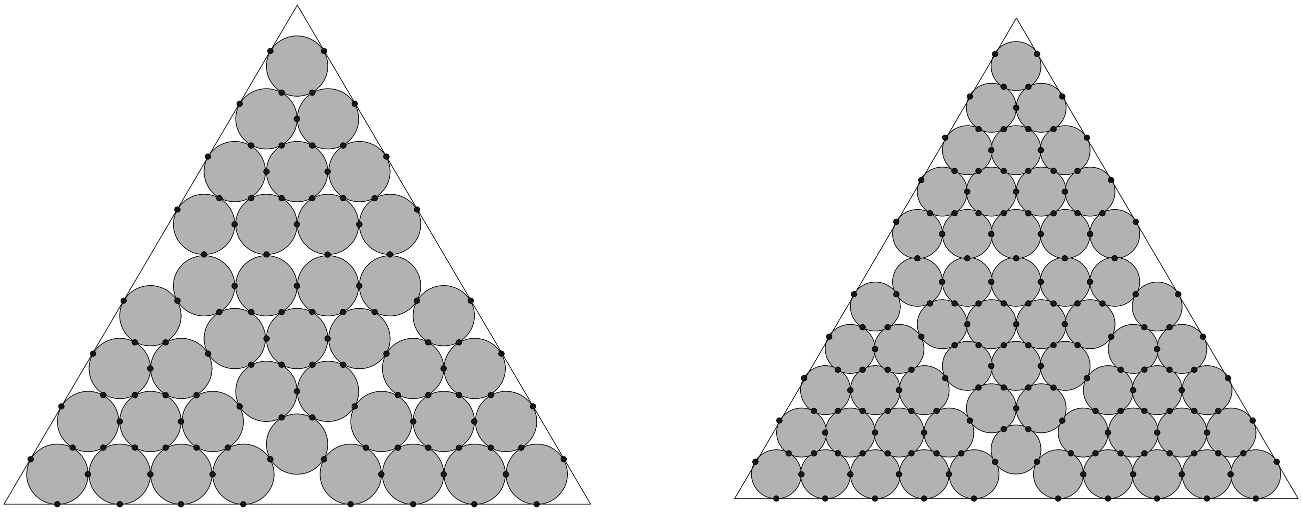


t11a
0.275255128608411 22 bonds

t12a
0.267949192431123 30 bonds

t13a
0.251813236653061 30 bonds

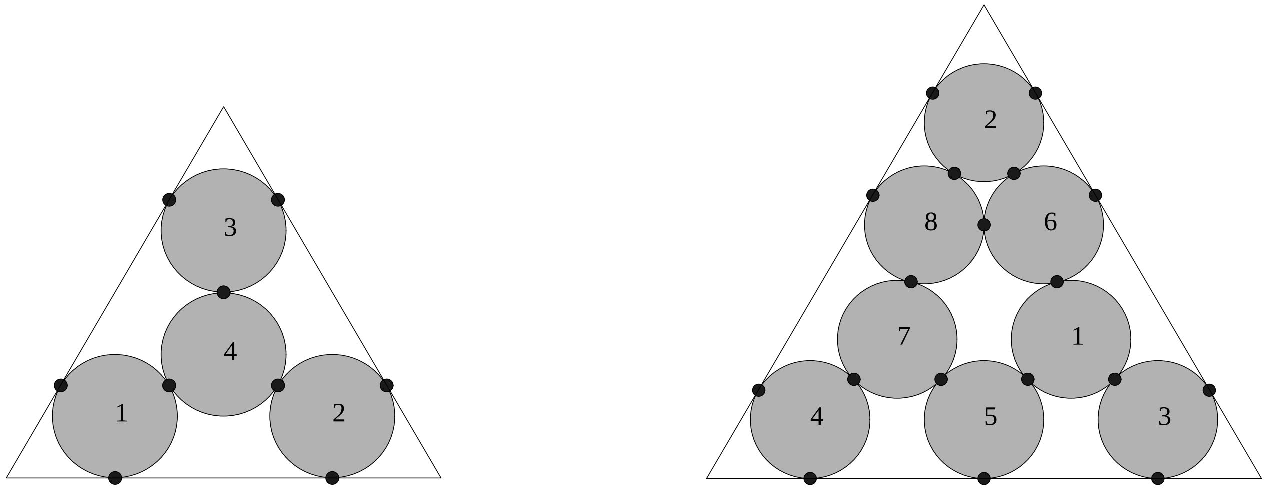
Figure 5.1: The best packings of 11 disks (t11a), of 12 disks (t12a), and of 13 disks (t13a).



t40a
0.129331793710034 108 bonds

t60a
0.102753265449690 154 bonds

Figure 5.2: The best packings of 40 disks (t40a) and 60 disks (t60a).



t4a

0.577350269189626 9 bonds

t8a

0.343070330817254 18 bonds

Figure 4.1: The best packing of 4 disks (t4a) and 8 disks (t8a).

and $60 = 4\Delta(5)$ disks (see Fig. 5.2), and also with $84 = 4\Delta(6)$ and $112 = 4\Delta(7)$ disks have the same structure as well.

A simple analysis of the patterns obtained implies that $d(4\Delta(k)) = \frac{1}{2k-2+\sqrt{3}}$.

If we fit members of class $4\Delta(.)$ within the boundaries of the triangular periods, i.e., among members of the class $\Delta(.)$, then every other period has exactly one n of the form $4\Delta(k)$ lying almost exactly at the middle of the period.

$\mathbf{2\Delta(k+1)+2\Delta(k)-1.}$ For each k there are $k+1$ distinct best packings: two for 7 disks (Fig. 1.1), three for 17 disks (Fig. 2.1), four for 31 disks (three of these four are shown in Fig. 3.8), and five for 49 disks (four of these five are shown in Fig. 5.3).

4 Conjectures for individual packings

Each packing diagram we give can imply several different statements:

(I) There exists a valid configuration of nonoverlapping disks with all pairwise distances marked by bonds equal to zero, and those not marked by bonds strictly positive, and with disk diameter equal to the indicated value with a relative error of less than 10^{-14}.

(II) The configuration is rigid: no disk or set of disks except for rattlers can be continuously displaced from the indicated positions without overlaps.

(III) The configurations are correctly ranked. That is, the a-packing really is optimal, the b-packing is second best, etc.

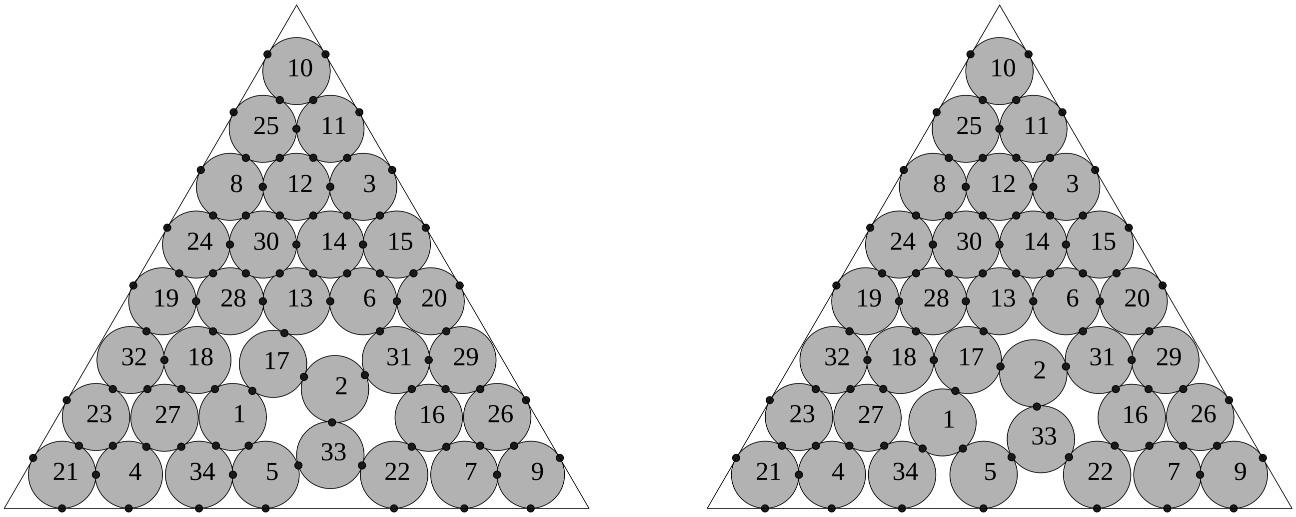
We believe (I) and (II) are correct. As to (III), we hope the statement is correct with respect to the a-packings. In other words, we believe these are the optimal packings. We are less confident for the lower ranked packings. For example, if someone finds a new packing in between our c- and d-packings, we will not be astounded. We provide these mainly for comparison purposes, and to serve as benchmarks for other packing algorithms.

It would also not be surprising to discover a nonisomorphic packing to one we have presented which has exactly the same disk diameter and the same number of bonds (e.g., as in t17b42ns and t17b42s in Fig. 2.1).

5 Conjectures for infinite classes

Dense packings in an equilateral triangle seem to “prefer” to form blocks of dense triangles and arrangements that are nearly so. In this section we describe seven infinite classes where we think we have found the optimal packings. Each class has its individual pattern of the optimal packings, which is different from the patterns for other classes. However, since they are the result of the particular packings we found, which themselves are only conjectured to be optimal, then the general conjectures have even less reliability. We still think they might serve as useful organizers for the maze of published dense packings.

$4\Delta(k)$. The best packing (we found) of $24 = 4\Delta(3)$ disks in Fig. 3.3 consists of four triangles, each with $\Delta(3) = 6$ disks. The best packings of 12 disks (in [M1]) in Fig. 5.1, and even 4 disks in Fig. 4.1 have the same form. The packings we obtained while experimenting with $40 = 4\Delta(4)$

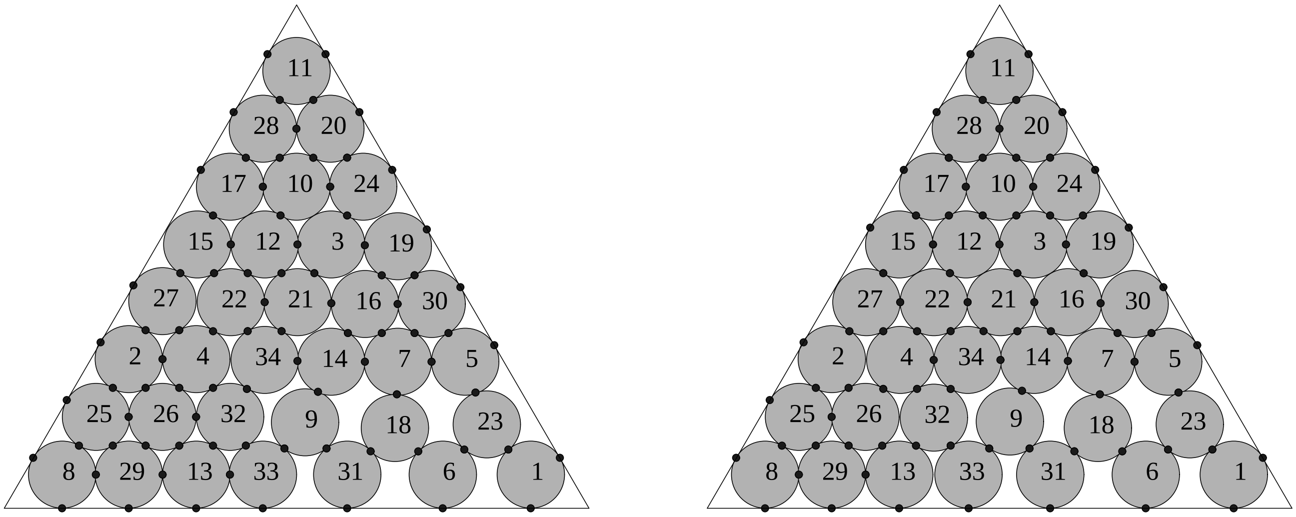


t34a

0.142869646754496 84 bonds

t34b

0.142867647681844 83 bonds



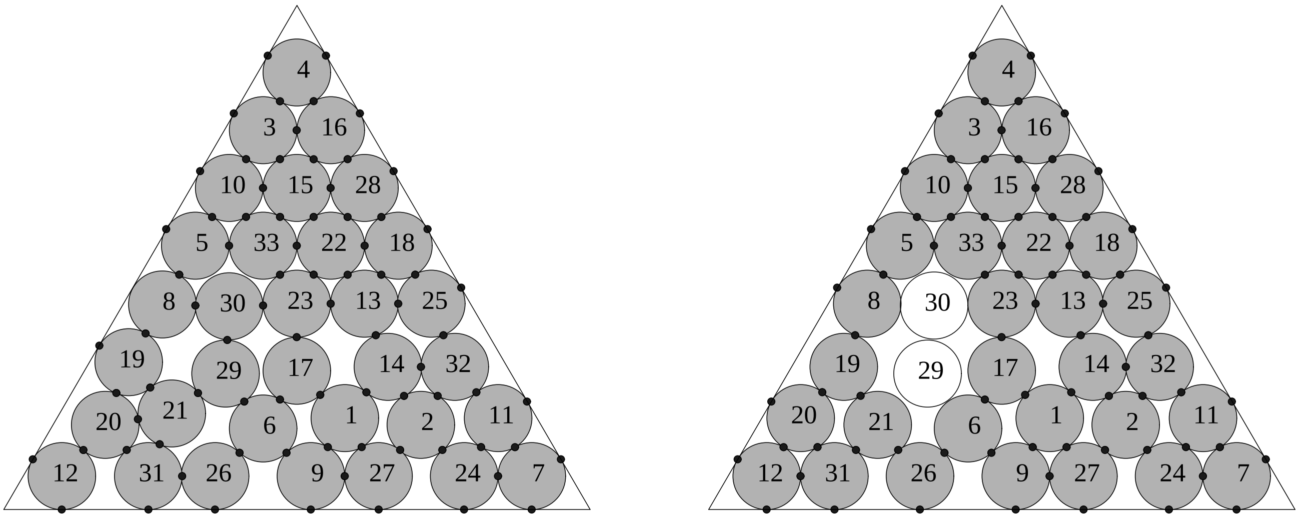
t34c

0.142866887845831 84 bonds

t34d

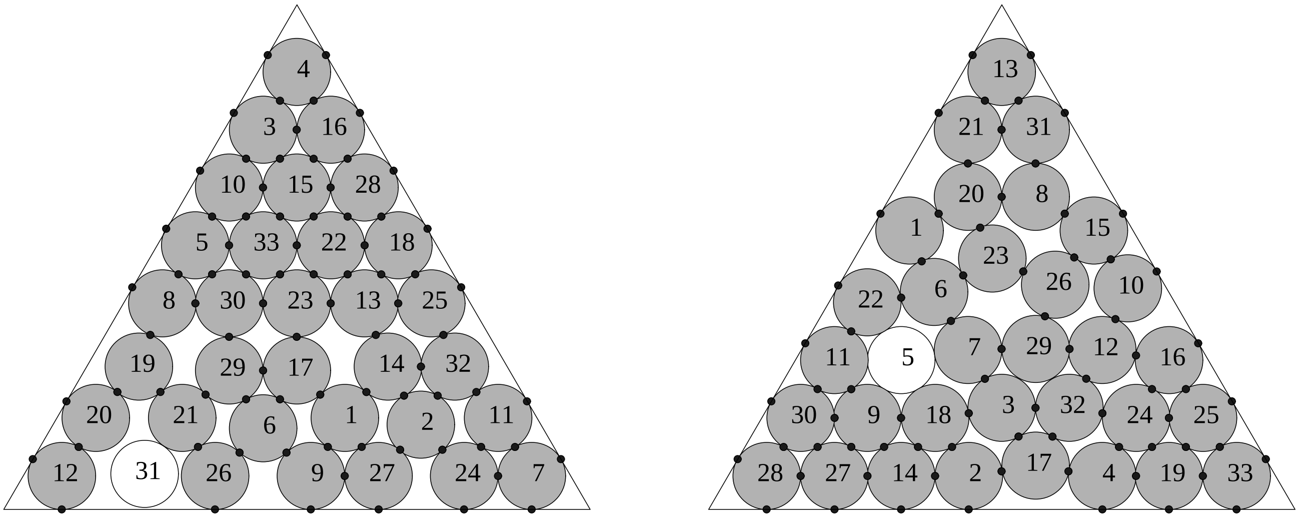
0.142866698669904 83 bonds

Figure 3.11: The best (t34a), the next-best (t34b), the third-best (t34c), and the fourth-best (t34d82) packings of 34 disks.



t33a
0.143447408371201 79 bonds

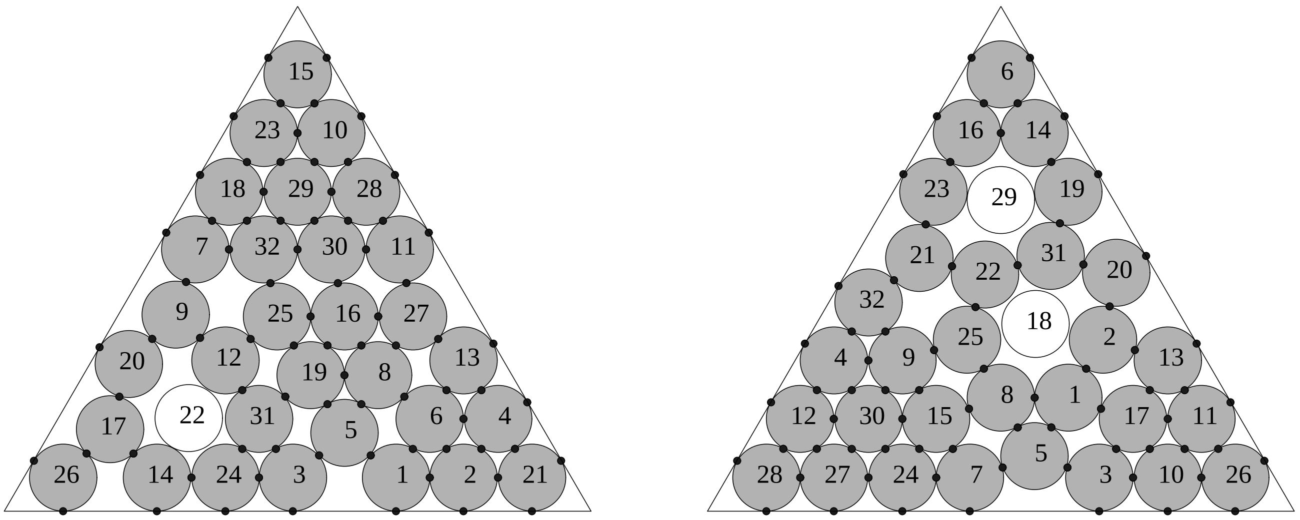
t33b75
0.143447385418276 75 bonds



t33b79
0.143447385418276 79 bonds

t33c
0.143309997215537 74 bonds

Figure 3.10: The best (t33a), the next-best (t33b75, t33b79), and the third-best (t33c) packings of 33 disks.

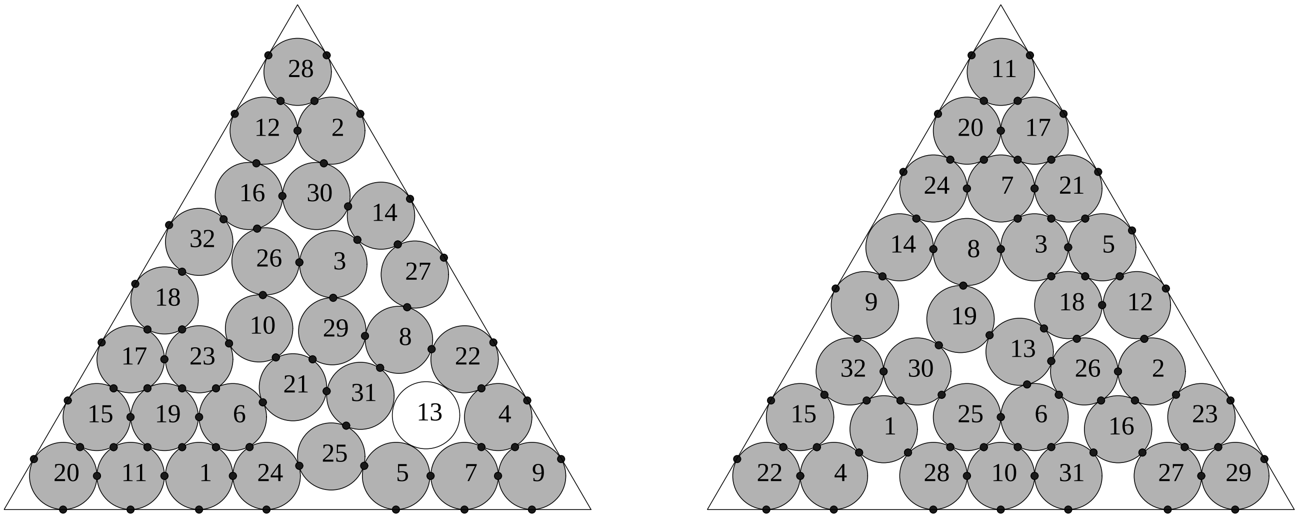


t32a

0.145102169183849 76 bonds

t32b

0.144984727468812 72 bonds



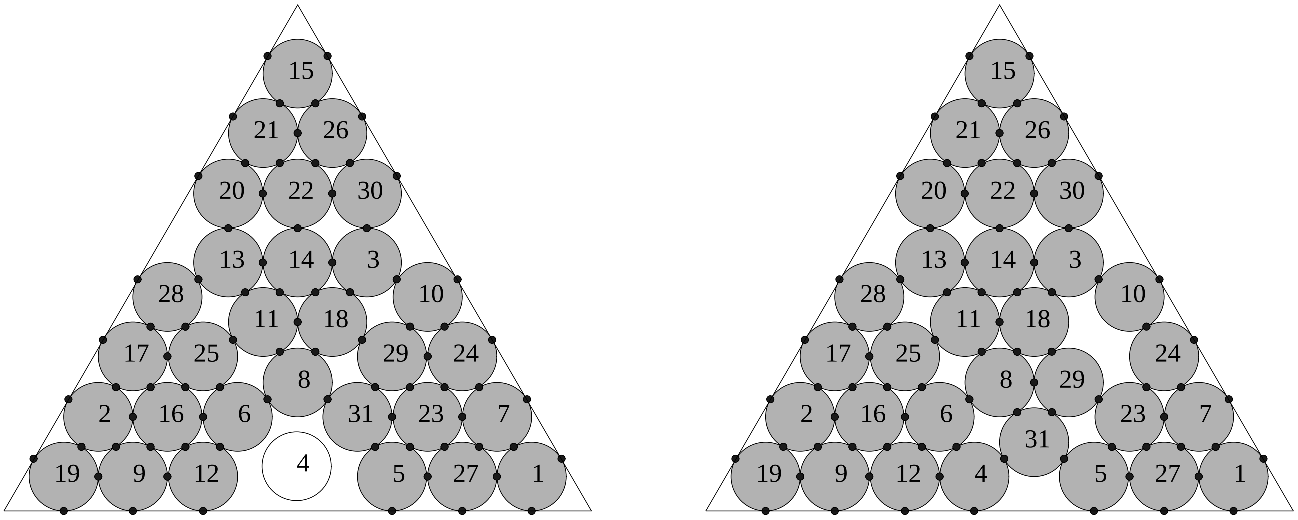
t32c

0.144651911950450 71 bonds

t32d

0.144616419018845 77 bonds

Figure 3.9: The best (t32a), the next-best (t32b), the third-best (t32c), and the fourth-best (t32d) packings of 32 disks.

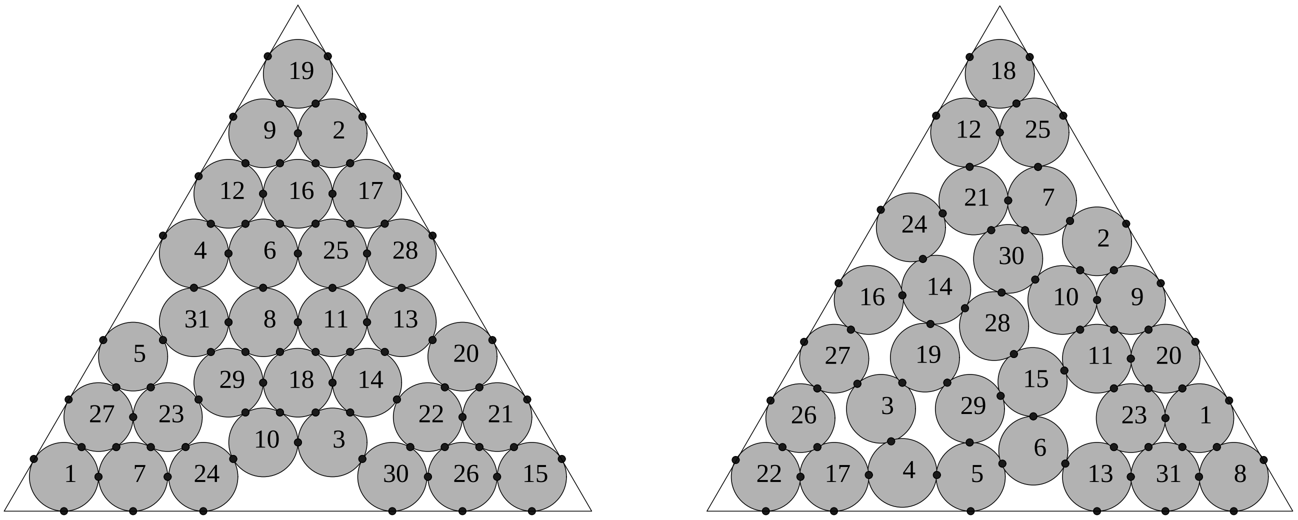


t31a79

0.148543145110506 79 bonds

t31a81.2

0.148543145110506 81 bonds



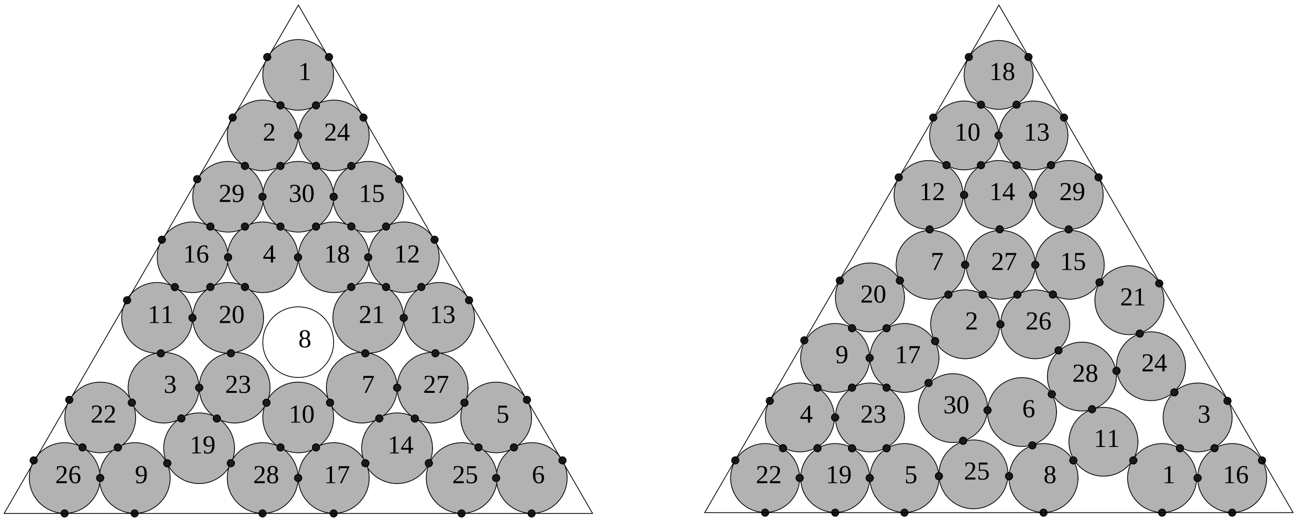
t31a82

0.148543145110506 82 bonds

t31b

0.146207330881888 71 bonds

Figure 3.8: The best (t31a79, t31a81.2, t31a82) and the next-best (t31b) packings of 31 disks.

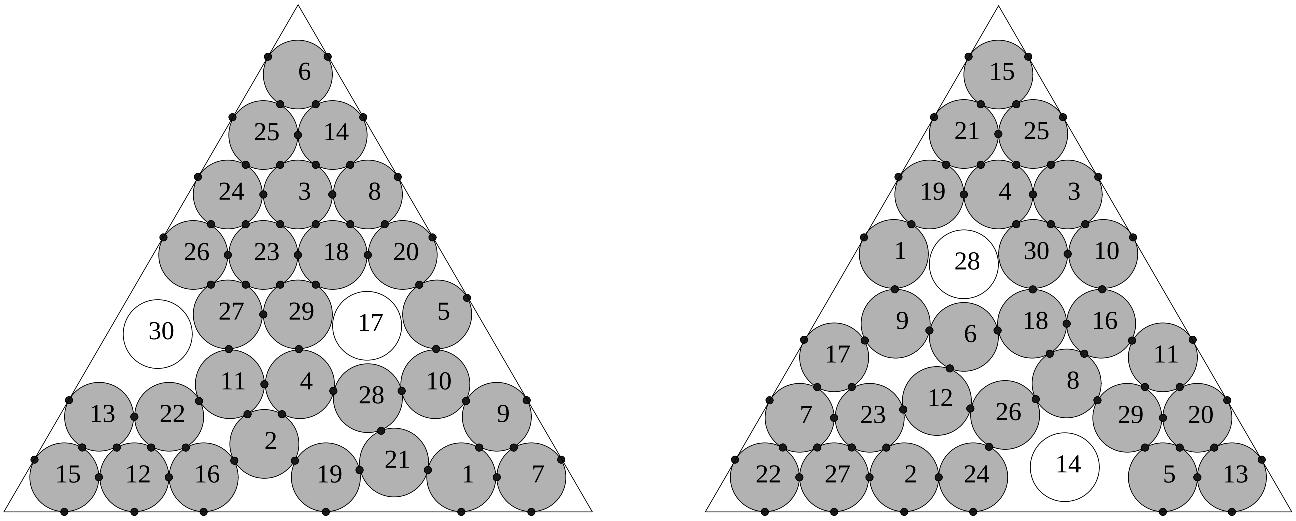


t30a

0.150761500215428 73 bonds

t30b

0.149057883638389 71 bonds



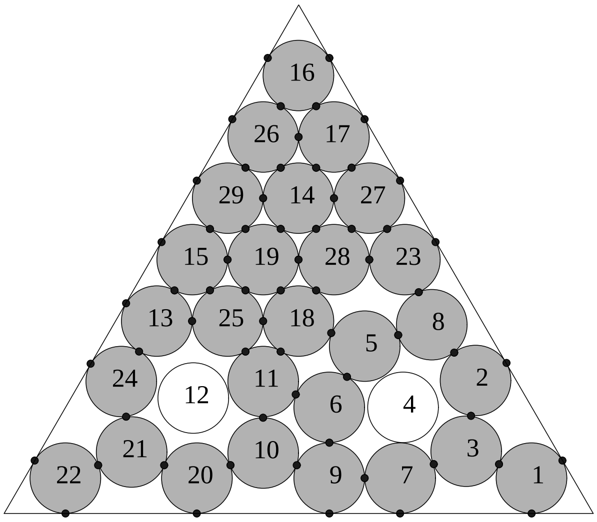
t30c

0.149047199036657 68 bonds

t30d

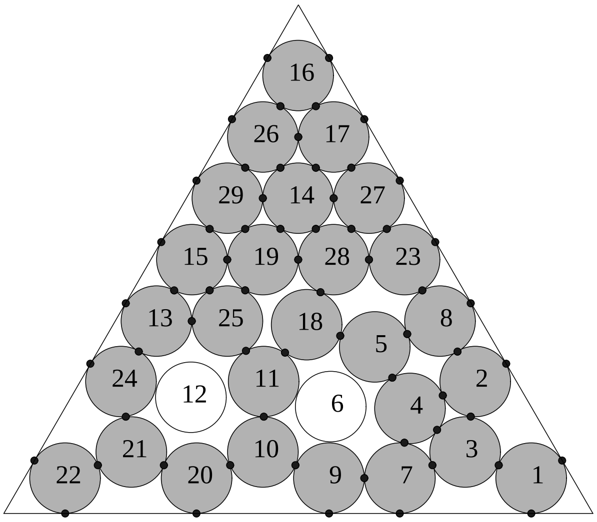
0.148951506290246 67 bonds

Figure 3.7: The best (t30a), the next-best (t30b), the third-best (t30c), and the fourth-best (t30d) packings of 30 disks.



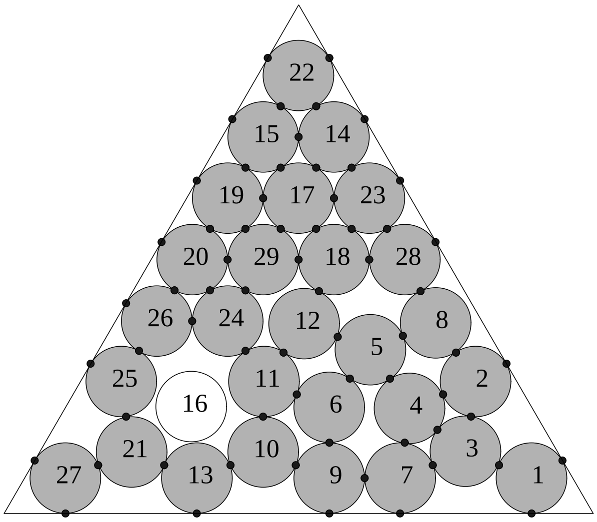
t29a

0.152189614060732 63 bonds



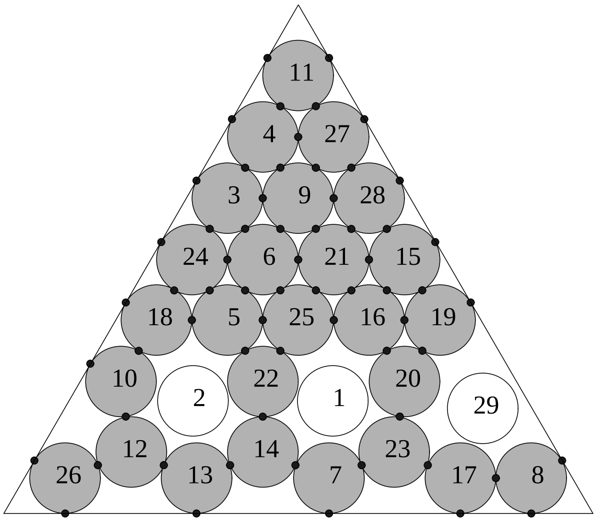
t29b63.2

0.152172645377571 63 bonds



t29c

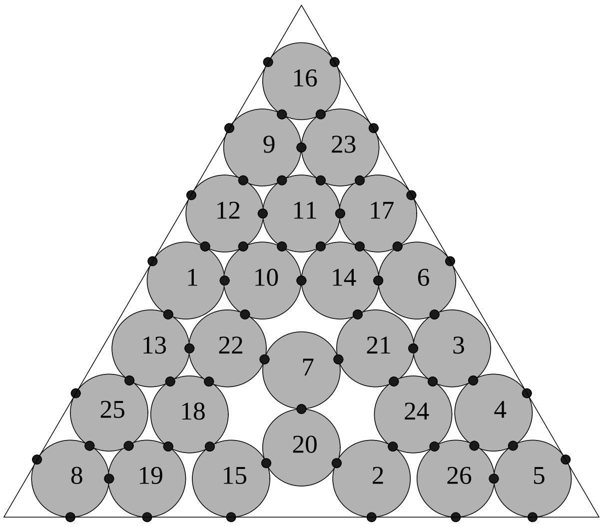
0.152161553910934 65 bonds



t29d

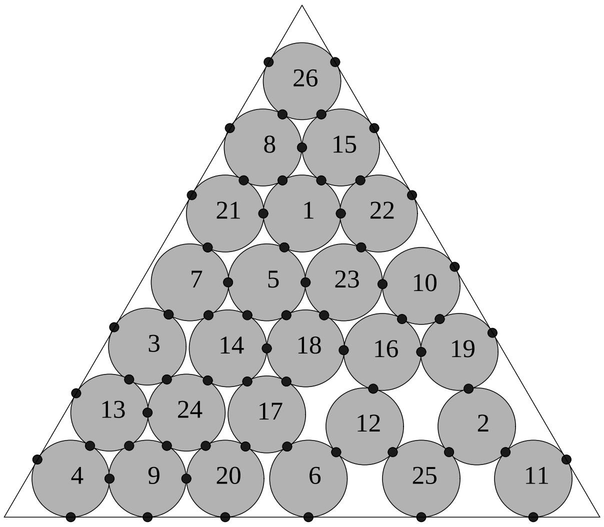
0.152109020552728 63 bonds

Figure 3.6: The best (t29a), a next-best (t29b63.2), the third-best (t29c), and the fourth-best (t29d) packings of 29 disks.



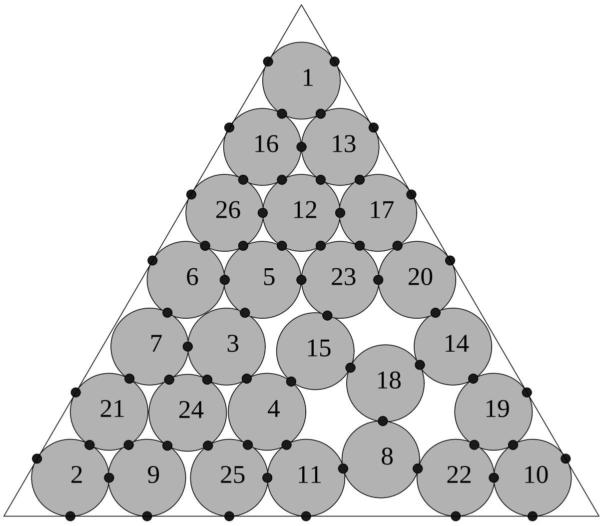
t26a

0.166738399395271   63 bonds



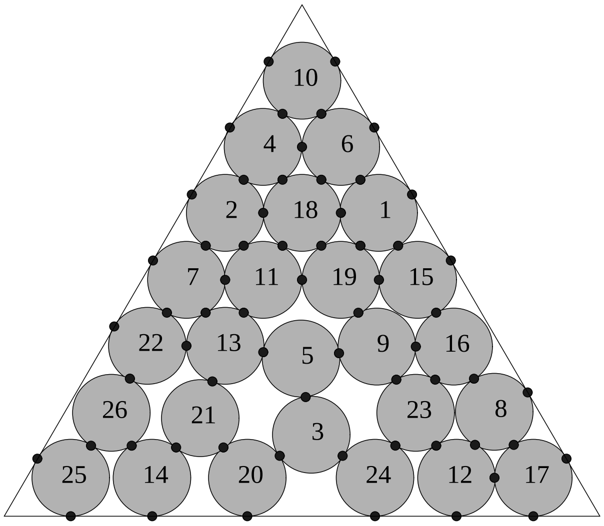
t26b

0.166732017260692   63 bonds



t26c

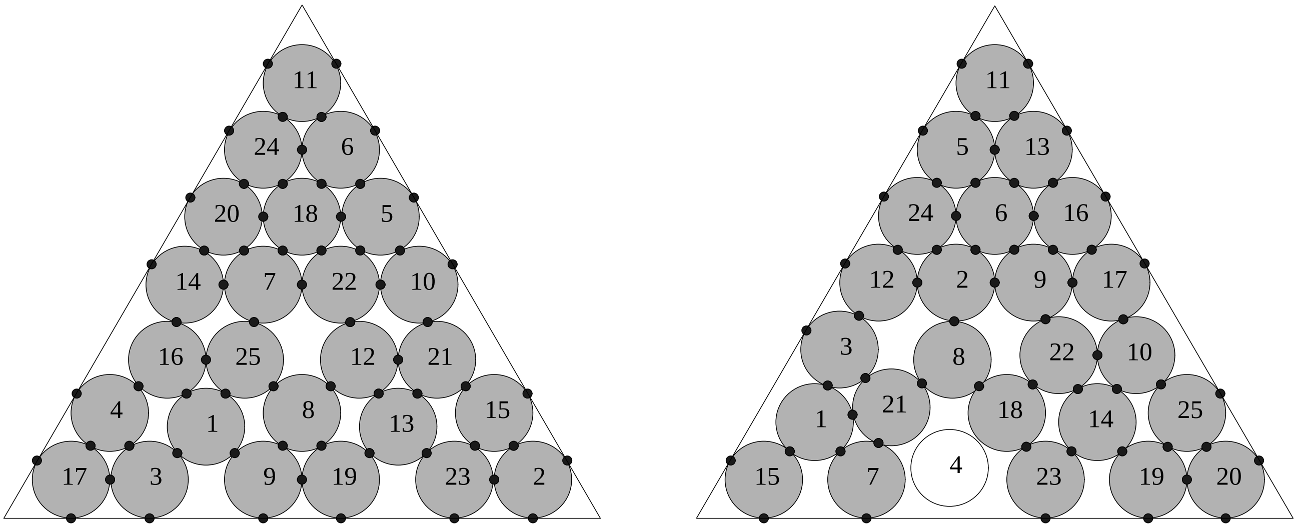
0.166698218018074   63 bonds



t26d

0.166695724000574   62 bonds

Figure 3.5: The best (t26a), the next-best (t26b), the third-best (t26c), and the fourth-best (t26d) packings of 26 disks.

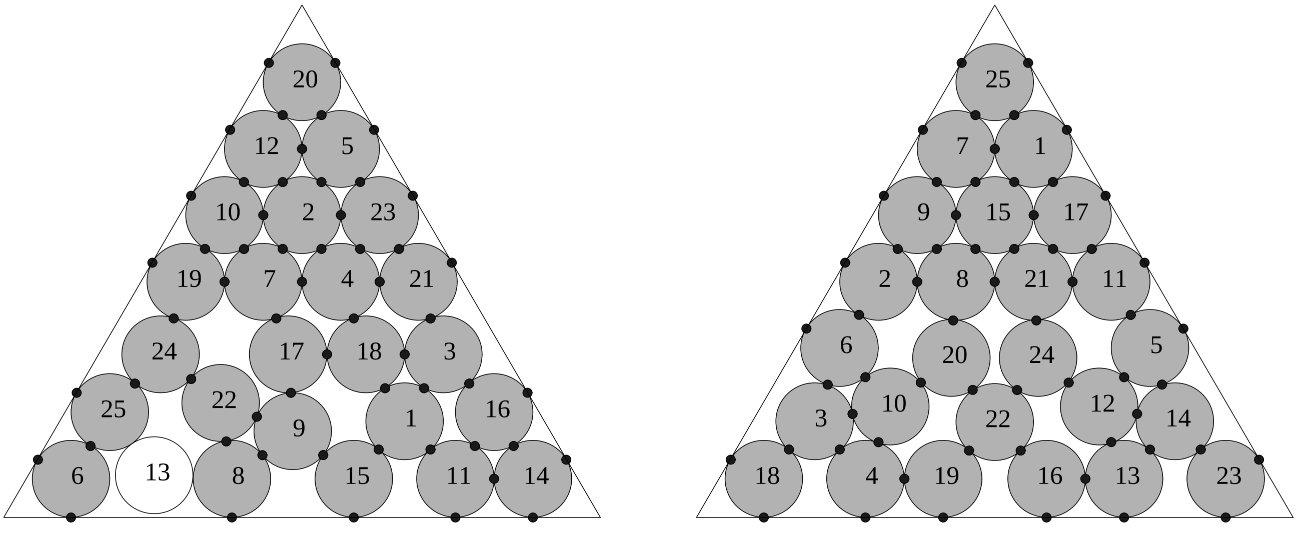


t25a

0.169065874417891 63 bonds

t25b

0.167753380810744 58 bonds



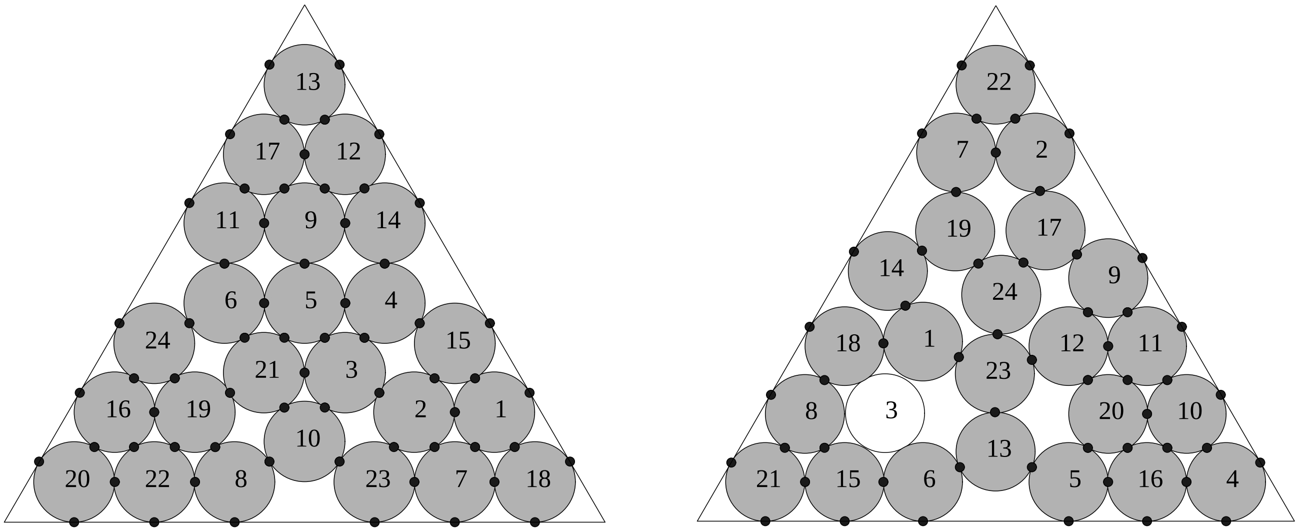
t25c

0.167714758299772 57 bonds

t25d60

0.167685455574367 60 bonds

Figure 3.4: The best (t25a), the next-best (t25b), the third-best (t25c), and a fourth-best (t25d60) packings of 25 disks.

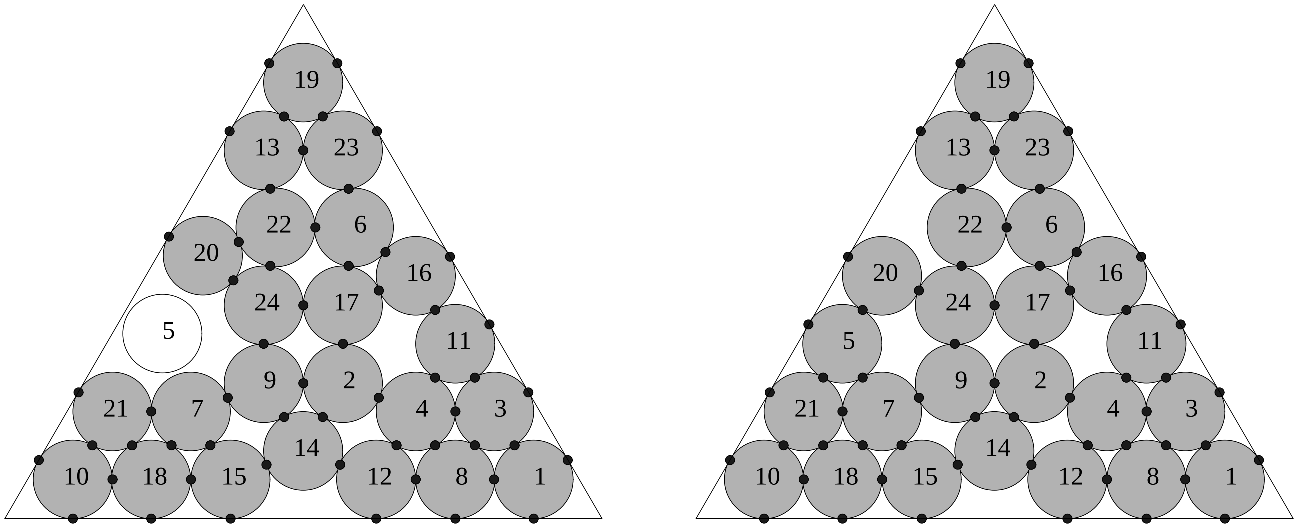


t24a

0.174457630187010 63 bonds

t24b

0.171024411616889 53 bonds



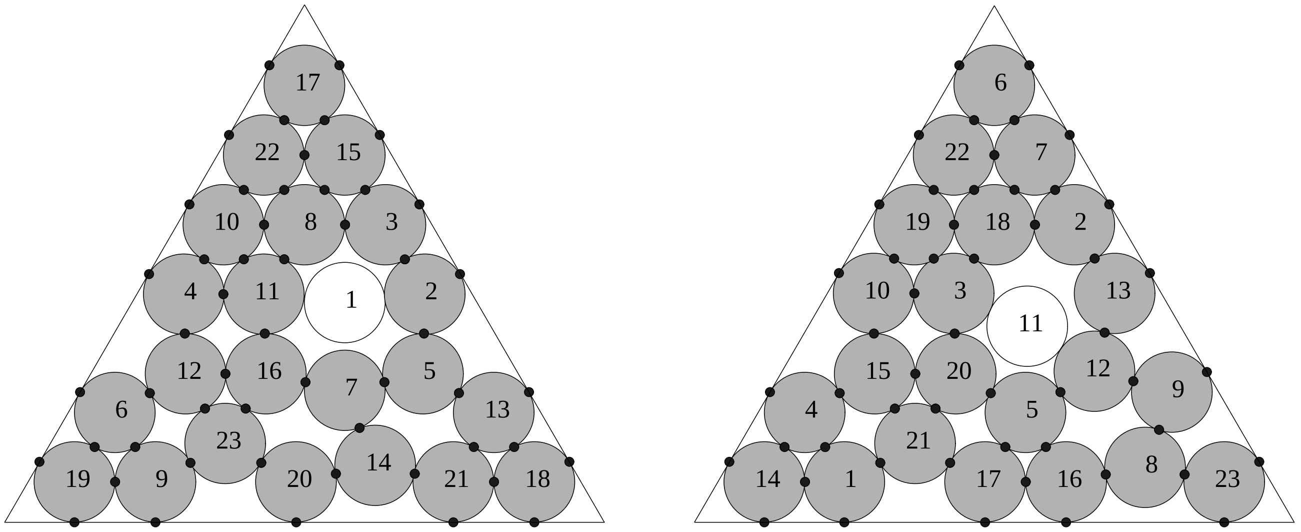
t24c56

0.170613243353863 56 bonds

t24c59

0.170613243353863 59 bonds

Figure 3.3: The best (t24a), the next-best (t24b), and the third-best (t24c56, t24c59) packings of 24 disks.

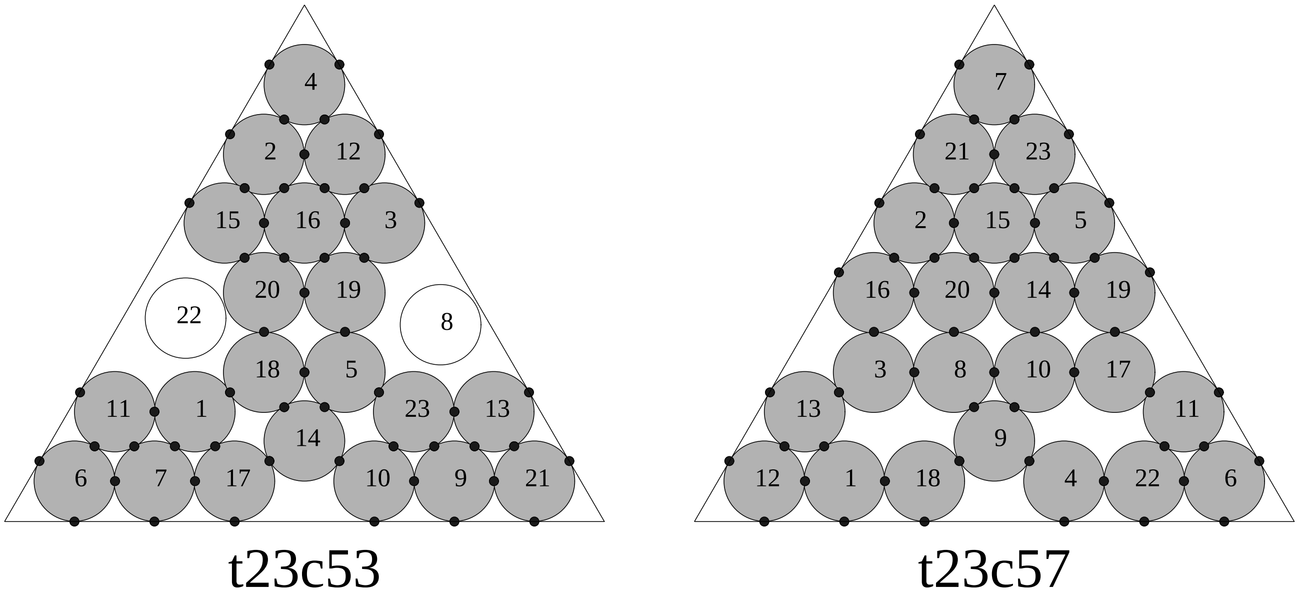


t23a

0.175153309170525 52 bonds

t23b

0.174962364462008 52 bonds



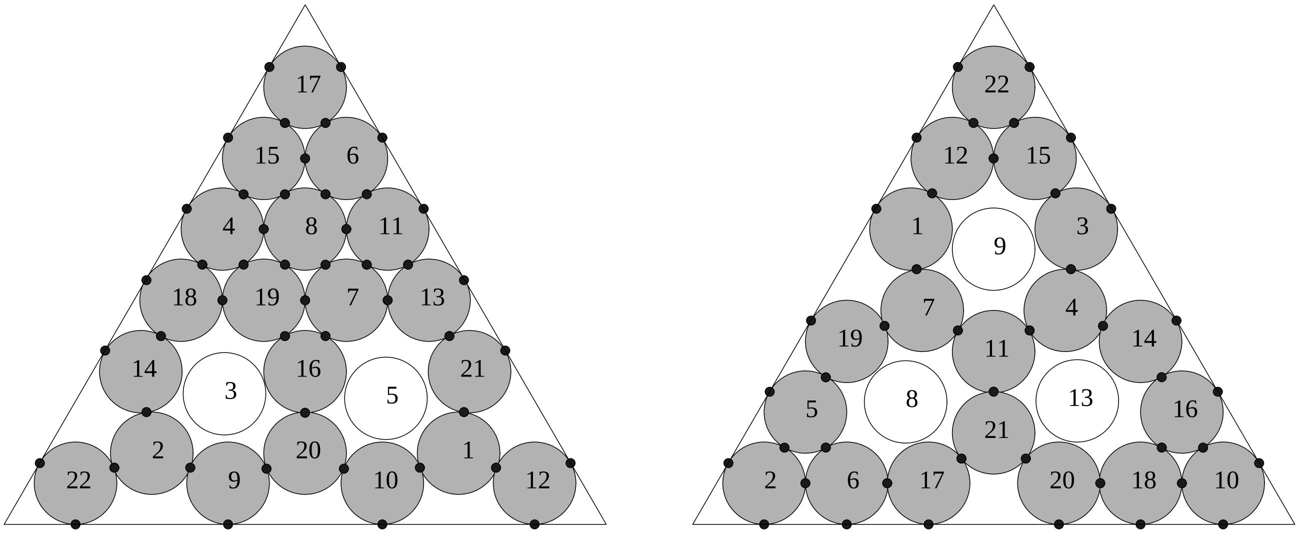
t23c53

0.174457630187009 53 bonds

t23c57

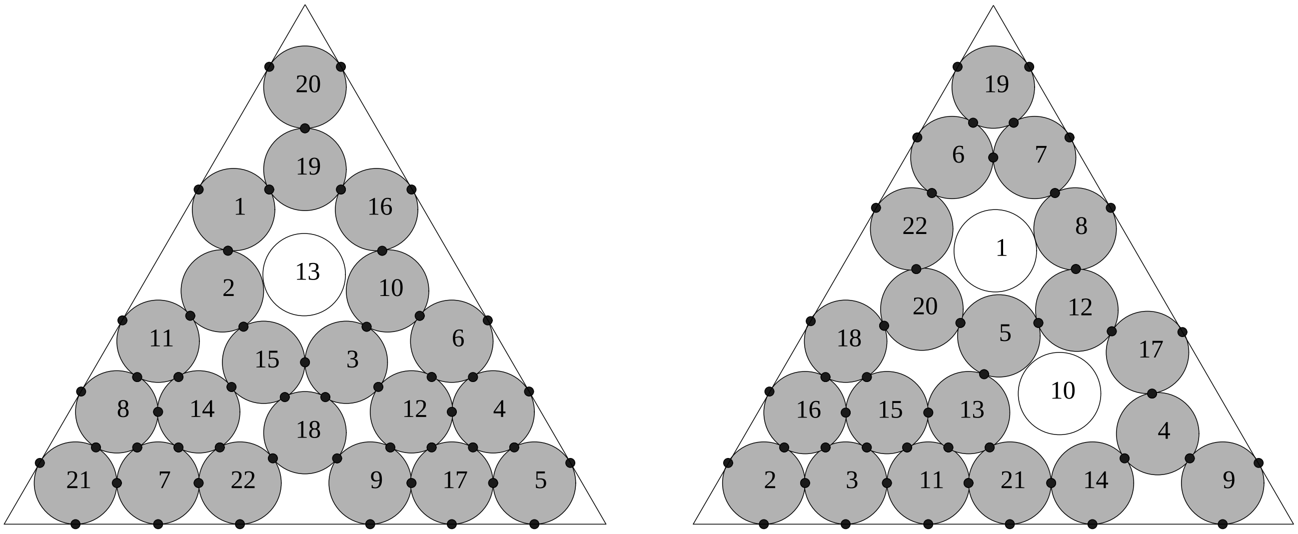
0.174457630187009 57 bonds

Figure 3.2: The best (t23a), the next-best (t23b), and the third-best (t23c53, t23c57) packings of 23 disks.



t22a
0.179396908611866 47 bonds

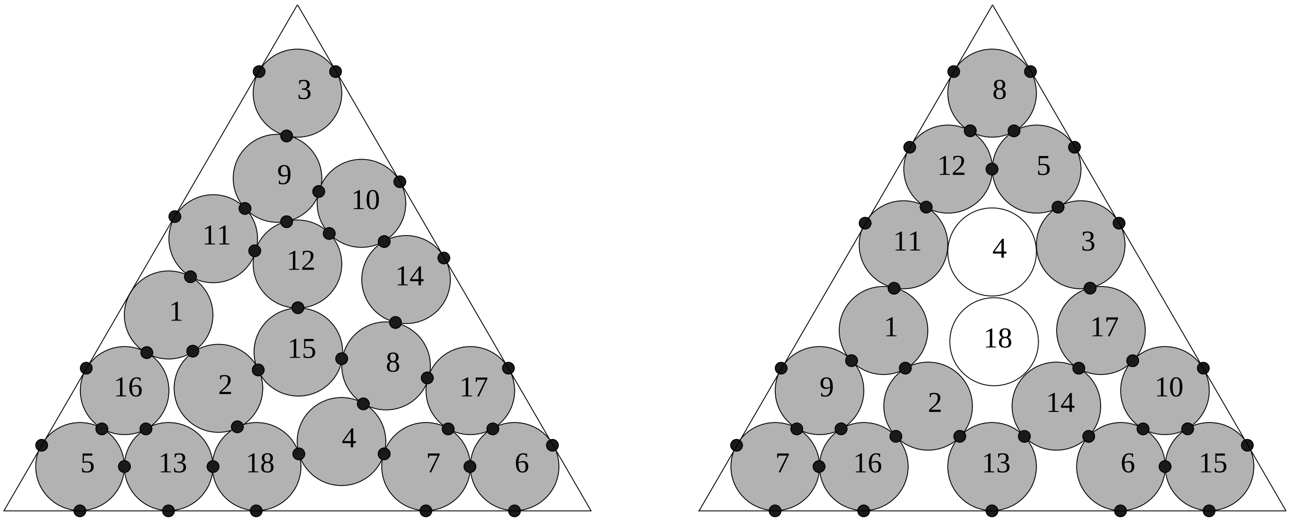
t22b42
0.179132453213560 42 bonds



t22b50
0.179132453213560 50 bonds

t22c
0.178763669382058 46 bonds

Figure 3.1: The best (t22a), the next-best (t22b42, t22b50), and the third-best (t22c) packings of 22 disks.

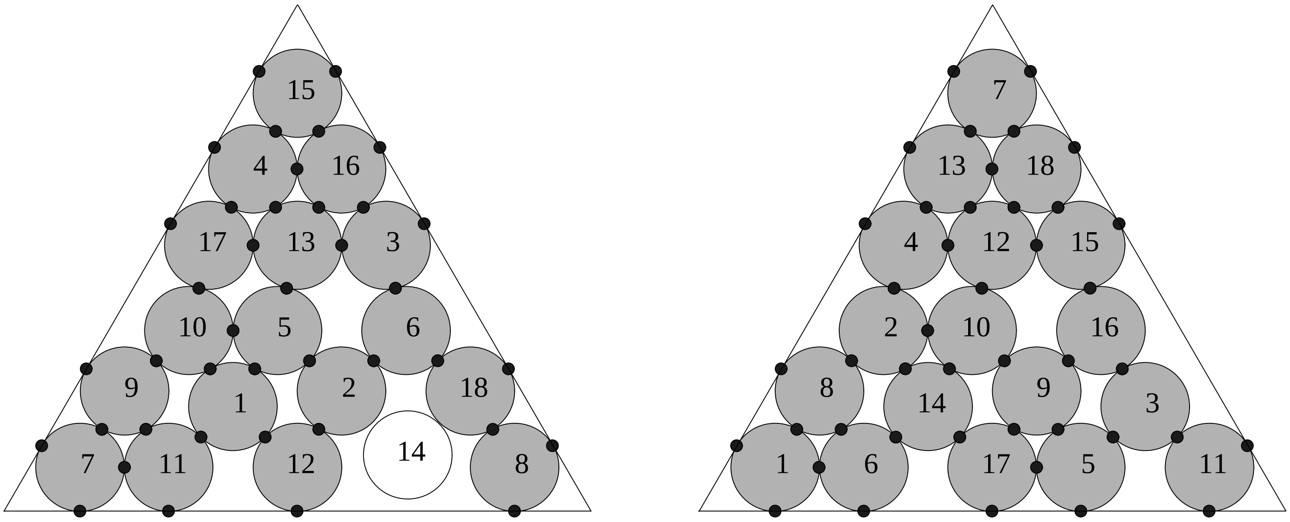


t18a

0.203465240539124   40 bonds

t18b36

0.203464834591373   36 bonds



t18b40

0.203464834591373   40 bonds

t18b43

0.203464834591373   43 bonds

Figure 2.2: The best (t18a) and the next best (t18b36, t18b40, t18b43) packings of 18 disks.

Fig. 3.3: Disk 20 in t24c56 and in t24c59 is locked in place because its center is strictly inside the triangle formed by the three bonds of disk 20. In both packings, the distance of the disk center to the boundary of this enclosing triangle is the distance to the line between bonds with the left side of the triangle and disk 24, and is 0.0317185.. of the disk diameter.

Fig. 3.4: The given d-packing of 25 disks t25d60 is symmetric with respect to the vertical axis. An equivalent non-symmetric d-packing t25d53 was also obtained in which all disks are located in the same places as in t25d60, except for disks 5, 12, 13, 14, 23, and 24. These six disks form a pattern which is roughly equivalent to that formed by disks 10, 14, 19, 25, 20, and 22, respectively, in t25b. Disk 24 in t25d53 is a rattler.

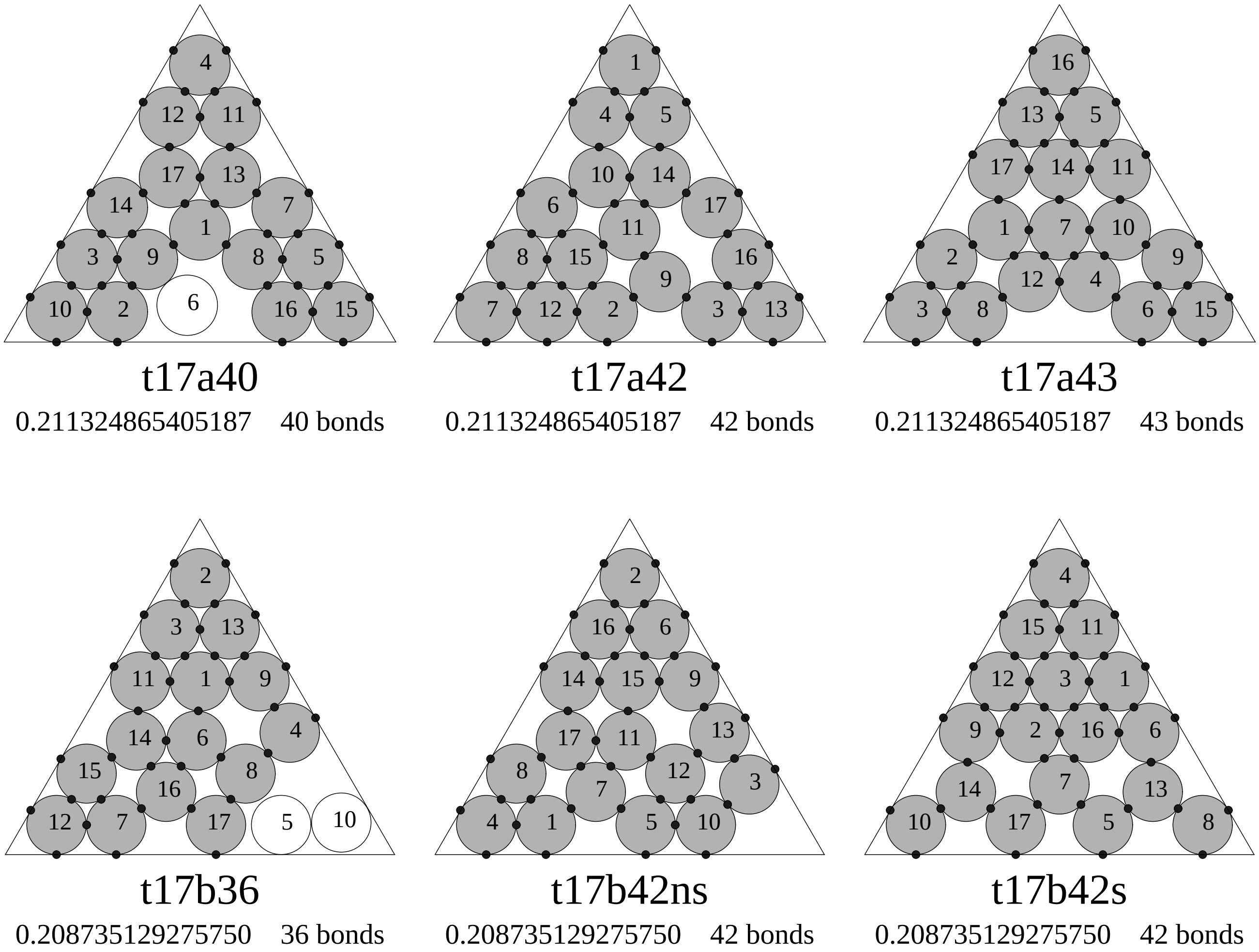
Fig. 3.6: Only one of the two b-packings of 29 disks we found is shown, namely, t29b63.2. The other b-packing, t29b63.1, differs in the placements of only disks 2, 3, 4, 7 as explained in Section 4.

Fig. 3.8: Four a-packings of 31 disks exist; only three are shown in the figure; the fourth one, t31a81.1, is described in Section 4.

Fig. 3.10: In t33a, the gap between disk 8 and left side is 0.0017032... of the disk diameter. In t33c, disk 7 is stably locked by its bonds with 3, 6, and 29. However, the distance from disk 7 center to the line on bonds with disks 3 and 6 is only 0.0002097575.... of the disk diameter. As a result, the cage of rattler disk 5 in t33c is very tight: the gap between disk 22 and disk 5 or disk 18 and disk 5 does not exceed 4×10^{-9} of the disk diameter.

Fig. 3.11: In t34a, the small gaps between "almost" touching pairs disk-disk or disk-wall take on only three values (relative to the disk size): in pairs 20–31, 16–26, 23–27, 18–19, 1–27 the gap is 0.021359..., in pairs left-32, right-29 it is 0.024750..., and in pairs 4–34, 7–22, it is 0.042561... Similarly, there are only three values of gaps in each of t34b, t34c, and t34d.

> t34b: in pairs 18–19, 23–27, 17–28, 20–31, 16–26 the gap is 0.019583...; in pairs left-32, right-29 it is 0.022686...; in pairs 4–34, 7–22, it is 0.039035...
>
> t34c: in pairs 12–17, 22–27, 3–10, 14–21, 4–34. 3–16 the gap is 0.018864..; in pair left-15 it is 0.021850...; in pair 19-24 it is 0.037606...
>
> t34d: in pairs 2–4, 26–32, 15–22, 12–21, 3–16, 7–16 the gap is 0.018681...; in pair left-27 it is 0.021637...; in pairs 13–33, 19–30 it is 0.037242...



t17a40
0.211324865405187 40 bonds

t17a42
0.211324865405187 42 bonds

t17a43
0.211324865405187 43 bonds

t17b36
0.208735129275750 36 bonds

t17b42ns
0.208735129275750 42 bonds

t17b42s
0.208735129275750 42 bonds

Figure 2.1: The best (t17a40, t17a42, t17a43) and the next-best (t17b36, t17b42ns, t17b42s) packings of 17 disks.

same disk diameter; for example, t17b42ns and t17b42s in Fig. 2.1, where the provisional "ns" stands for "non-symmetric" and "s" for "symmetric."

We point out that the a-packings of 17 and 18 disks that we show have previously been given by Melissen and Schuur [MS], who also conjecture their optimality.

3 Additional comments

Fig. 3.2: Two more c-packings for 23 disks that are not shown in the figure were generated: t23c55.1 and t23c55.2. Both have 55 bonds. t23c55.1 can be obtained by combining the left side of t23c53 with the right side of t23c57. t23c55.2 is a variant of t23c55.1.

($n = \Delta(k) - 1$). The bulk of our efforts concentrated on the other 11 values of n, for $21 \leq n \leq 36$. These are presented in Figures 3.1 to 3.11.

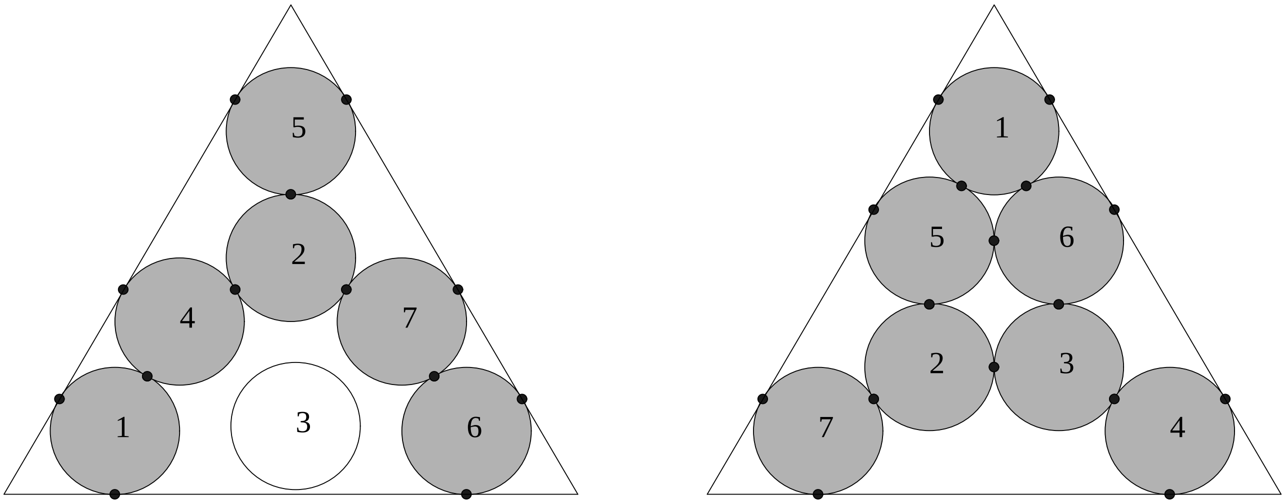
To navigate among the various packing presented we will use the labeling system illustrated by Fig. 3.1 t22a. Here, $n = 22$, "a" denotes that the packing is the best we found, "b" would be the second best (as in t23b in Fig. 3.2), "c" would be third best, and "d" would be fourth best.

Small black dots in the packing diagrams are "bonds" whose number is also entered by each packing. For example, there are 47 bonds in t22a. A bond between two disks or between a disk and a boundary indicates that the distance between them is zero. The *absence* of a bond in a spot where disk-disk or disk-wall are apparently touching each other means that the corresponding distance is strictly *positive*, though perhaps too small for the resolution of the drawing to be visible. For example, there is no bond between disk 1 and the left side of the triangle in t18a (Fig. 2.2); according to our computations, the distance between disk 1 and the side is 0.0048728... of the disk diameter. (Packing t18a was constructed in [M1].) Each disk in most of the packings is provided with a label which uniquely identifies the disk in the packing. This labeling is nonessential; it is assigned in order to facilitate referencing.

Each disk normally has at least three bonds attached. The polygon formed by these bonds as vertices contains the center of the disk strictly inside. This is a necessary condition for packing "rigidity". In [LS], where the packing algorithm was applied to a similar problem, the disks without bonds were called "rattlers." A rattler can move freely within the confines of the "cage" formed by its rigid neighbors and/or boundaries. (If we "shake" the packing, the rattler will "rattle" while hitting its cage.) t22a has two rattlers, disks 3 and 5. In the packing diagrams, all disks, except for the rattlers, are shaded.

A number with 15 significant digits is indicated for each packing in the figures, e.g., the number 0.17939 69086 11866 for packing t22a. This number is the disk diameter $d(n)$ which is measured in units equal to the side of the smallest equilateral triangle that contains the *centers* of all disks. For packing t22a such a triangle is the one with vertices at the centers of disks 22, 17, and 12. This unit of measure for $d(n)$ conforms with previously published conventions.

Sometimes several packings exist for the same disk diameter. An example is t7a13 and t7a16 in Fig.1.1. Thus, we distinguish such packings by suffixing their labels with the number of bonds. Other examples are t17a40, t17a42, and t17a43 in Fig. 2.1, t22b42 and t22b50 in Fig. 3.1. However, even the number of bonds may not distinguish different packings of the new packings we obtain are also optimal. The new packings cover two “triangular periods”: $21 = \Delta(6)$ to $\Delta(7)$ to $\Delta(8) = 36$.



t7a13
0.366025403784439 13 bonds

t7a16
0.366025403784439 16 bonds

Figure 1.1: Two equivalent but nonisomorphic densest packings of 7 disks.

In addition, we conjecture optimal packings for seven infinite classes of n, namely, $n = \Delta(2k)+1$, $\Delta(2k+1)+1$, $\Delta(k+2)-2$, $\Delta(2k+3)-3$, $\Delta(3k+1)+2$, $4\Delta(k)$, and $2\Delta(k+1)+2\Delta(k)-1$, where $k = 1, 2....$ Each class has its individual pattern of the optimal packings which is different from patterns for other classes. These were suggested by the preceding packings, and we give packings for some additional values of these forms, namely, $n = 37$, 40, 42, 43, 46, 49, 56, 57, 60, 63, 67, 71, 79, 84, 92, 93, 106, 112, 121, and 254, as well as for $n = 58$, 95, 108, 175, 255, 256, 258, and 260.

We say that an infinite class of packings of n disks, $n = n(1), n(2), ...n(k), ...$, is *tight* , if $[1/d(n(k)+1) - 1/d(n(k))]$ is bounded away from zero as k goes to infinity. We conjecture that some of our infinite classes are tight, others are not, and that there are infinitely many tight classes.

2 The packings

We performed a small number of runs with $n = 21$, 27, 28, 35 and 36 disks. In every case, the resulting packings were consistent with the existing results ($n = \Delta(k)$) and conjectured

Dense Packings of Equal Disks in an Equilateral Triangle: From 22 to 34 and Beyond

R. L. Graham
B. D. Lubachevsky
AT&T Bell Laboratories,
Murray Hill, New Jersey 07974

1 Introduction

Geometrical packing problems have a long and distinguished history in combinatorial mathematics. In particular, such problems are often surprisingly difficult. In this note, we describe a series of computer experiments designed to produce dense packings of equal nonoverlapping disks in an equilateral triangle. It was first shown by Oler in 1961 [O] that the densest packing of $n = \Delta(k) := \frac{k(k+1)}{2}$ equal disks is the appropriate triangular subset of the regular hexagonal packing of the disks (well known to pool players in the case of $n = 15$). It has also been conjectured by Newman [N] (among others) that the optimal packing of $\Delta(k) - 1$ disks is always obtained by removing a single disk from the best packing for $\Delta(k)$, although this statement has not yet been proved. The only other values of n (not equal to $\Delta(k)$) for which optimal packings are known are $n = 2, 4, 5, 7, 8, 9, 11$ and 12 (see Melissen [M1], [M2] for a survey).

As the number n of packed disks increases, it becomes not only more difficult to *prove* optimality of a packing but even to *conjecture* what the optimal packing might be. In this paper, we present a number of conjectured optimal packings. These packings are produced on a computer using a so-called "billiards" simulation algorithm. A detailed description of the philosophy, implementation and applications of this event-driven algorithm can be found in [L], [LS]. Essentially, the algorithm simulates a system of n perfectly elastic disks. In the absence of gravitation and friction, the disks move along straight lines, colliding with each other and the region walls according to the standard laws of mechanics, all the time maintaining a condition of no overlap. To form a packing, the disks are uniformly allowed to gradually increase in size, until no significant growth can occur. Not infrequently, it can happen at this point that there are disks which can still move (e.g., disk 3 in t7a13 (see Fig. 1.1).

Every packing of n disks occurring in the literature for n different from $\Delta(k)$ and $\Delta(k) - 1$ which has been conjectured or proved to be optimal was also found by our algorithm. These occur for $n = 13, 16, 17, 18,$ and 19 (see [M1], [MS]). This increases our confidence that the

Dense Packings of Equal Disks in an Equilateral Triangle: From 22 to 34 and Beyond

R. L. Graham
B. D. Lubachevsky
AT&T Bell Laboratories,
Murray Hill, New Jersey 07974

August 19, 1994

ABSTRACT

Previously published packings of equal disks in an equilateral triangle have dealt with up to 21 disks. We use a new discrete-event simulation algorithm to produce packings for up to 34 disks. For each n in the range $22 \leq n \leq 34$ we present what we believe to be the densest possible packing of n equal disks in an equilateral triangle. For these n we also list the second, often the third and sometimes the fourth best packings among those that we found. In each case, the structure of the packing implies that the minimum distance $d(n)$ between disk centers is the root of polynomial P_n with integer coefficients. In most cases we do not explicitly compute P_n but in all cases we do compute and report $d(n)$ to 15 significant decimal digits.

Disk packings in equilateral triangles differ from those in squares or circles in that for triangles there are an infinite number of values of n for which the exact value of $d(n)$ is known, namely, when n is of the form $\Delta(k) := \frac{k(k+1)}{2}$. It has also been conjectured that $d(n-1) = d(n)$ in this case. Based on our computations, we present conjectured optimal packings for seven other infinite classes of n, namely

$$\begin{aligned} n &= \Delta(2k)+1,\ \Delta(2k+1)+1, \Delta(k+2)-2,\ \Delta(2k+3)-3, \\ &\quad \Delta(3k+1)+2,\ 4\Delta(k),\ \text{ and }\ 2\Delta(k+1)+2\Delta(k)-1\ . \end{aligned}$$

We also report the best packings we found for other values of n in these forms which are larger than 34, namely, $n =$ 37, 40, 42, 43, 46, 49, 56, 57, 60, 63, 67, 71, 79, 84, 92, 93, 106, 112, 121, and 254, and also for $n =$ 58, 95, 108, 175, 255, 256, 258, and 260. We say that an infinite class of packings of n disks, $n = n(1), n(2), ...n(k), ...$, is *tight* , if $[1/d(n(k)+1) - 1/d(n(k))]$ is bounded away from zero as k goes to infinity. We conjecture that some of our infinite classes are tight, others are not tight, and that there are infinitely many tight classes.