DIMACS Technical Report 94-32
June 1994

Perfect Graphs are Kernel Solvable

by

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This research was supported in part by the Office of Naval Research (Grants N0001492F1375 and N0001492F4083) and by the Air Force Office of Scientific Research (Grant F49620-93-1-0041).

ABSTRACT

In this paper we prove that perfect graphs are kernel solvable, as it was conjectured by Berge and Duchet (1983). The converse statement, i.e. that kernel solvable graphs are perfect, was also conjectured in the same paper, and is still open. In this direction we prove that it is always possible to substitute some of the vertices of a non-perfect graph by cliques so that the resulting graph is not kernel solvable.

1 Introduction and Main Results

A directed graph $D = (V, A)$ is called a *super-orientation* of the simple graph $G = (V, E)$, if D is obtained from G by orienting all of its edges in an arbitrary way (for some or for all of the edges $(u, v) \in E$ both arcs $\overrightarrow{uv}$ and $\overrightarrow{vu}$ may be included in A).

A subset $S \subseteq V$ of the vertices of a directed graph $D = (V, A)$ is called a *kernel*, if it is a stable set and every vertex outside of S has a successor in S (i.e. if no arc $\overrightarrow{uv} \in A$ has both endpoints in S, and for every $u \notin S$ there exists a $v \in S$ such that $\overrightarrow{uv} \in A$). Let us note that if S is a kernel of a super-orientation of G, then S is a maximal stable set of G.

We shall say that a clique C of G has a kernel in a super-orientation D of G, if there is a vertex $v \in C$, which is the successor of all other vertices of C. Let us finally call a graph G *kernel solvable* if for every super-orientation D of G for which every clique has a kernel, D itself has a kernel.

Berge and Duchet studied kernels and kernel solvability (solvability), and conjectured that kernel solvability is equivalent with perfectness. More precisely, the following two conjectures were stated in [3]:

Conjecture 1. Perfect graphs are kernel solvable.

Conjecture 2. Kernel solvable graphs are perfect.

The first conjecture was proved for some special cases, including Gallai graphs in [16], line graphs in [17], and complements of strongly perfect graphs in [5]. Conjecture 2 was also shown to hold for line graphs in [17].

The main result of this paper is to prove the first conjecture, and give a partial answer for the second.

Theorem 1 *Perfect graphs are kernel solvable.*

We shall call a graph $G' = (V', E')$ a *blow-up* of $G = (V, E)$, if it is obtained from G by substituting some of the vertices by cliques, i.e. if there are subsets $C_v \subseteq V'$ for $v \in V$ such that $C_u \cap C_v = \emptyset$ if $u \neq v$, $V' = \bigcup_{v \in V} C_v$, and $(x, y) \in E'$ for $x \in C_u$ and $y \in C_v$ if and only if $u = v$ or $(u, v) \in E$. It was proved in [15] that if G is perfect then every blow-up G' of it is perfect, too. Thus, according to Theorem 1, any blow-up G' of a perfect graph G is kernel solvable. In the converse direction, as a partial answer for Conjecture 2, we can prove the following:

Theorem 2 *If the graph G is not perfect, then there exists a blow-up G' of G, which is not kernel solvable.*

In the next section we introduce a property of graphs, so called core solvability, and prove that it is equivalent with kernel solvability.

Since the proofs of Theorems 1 and 2 rely heavily on results of game theory, we introduce the necessary notions and results in Section 3. In Section 4 we show that core solvability of a graph G is equivalent with the stability of an associated effectivity function $\mathcal{E}_G$.

In Section 5 we show that perfectness of a graph is equivalent with a special property of its family of cliques, or in other words with the integrality of an associated polytope (see e.g. [6, 10]). Based on this equivalence, in Section 6, we show that perfectness of a graph G is equivalent with the g-stability of the associated family $\mathcal{K}_G$ of coalitions.

The connection between the two types of stability enables us to prove Theorems 1 and 2. A key ingredient in our proof is a result of [12] which is relatively hard to access. Therefore, we included an independent proof for Theorem 1 in Section 7, based on a result of [7] (see also [8, 9]).

In Section 8 we give an overview of the used notions, their relations, and summarize the main results of the paper. For the definitions see Section 3, and the beginning of Sections 2 and 4.

In the last two sections we discuss related results. In Section 9 we show further properties of kernel solvable graphs, and formulate new conjectures, equivalent with Conjecture 2. In Section 10 we generalize slightly the results of Section 5, and provide a characterization of integrality of a set-packing polytope, in general.

2 Sub-orientations and Core Solvability

A directed graph $D = (V, A)$ is called a *sub-orientation* of the simple graph $G = (V, E)$, if D is obtained from G by orienting some of the edges of G (for the sake of simplicity we shall assume that no edge is bidirected in a sub-orientation).

Let us note that there is an obvious one-to-one correspondence between sub- and super-orientations of a simple graph $G = (V, E)$. In a super-orientation of G edges can be uniquely or both ways oriented, while in a sub-orientation they can be uniquely or not at all oriented. Clearly, interchanging the operations "both way orient" and "not orient at all" will establish a one-to-one correspondence between sub- and super-orientations of G. For the clarity, let us say that a super-orientation $D^+ = (V, A^+)$ and a sub-orientation $D^- = (V, A^-)$ of G are *corresponding*, if $A^- \subseteq A^+$, and denoting the set of edges of G which are uniquely (and in the same way) oriented in both directed graphs by $E^1 = \{(u, v) | \overrightarrow{uv} \in A^-\}$ we have $A^+ = A^- \cup \{\overrightarrow{uv}, \overrightarrow{vu} \,|(u, v) \in E \setminus E^1\}$, i.e. edges belonging to $E \setminus E^1$ are not oriented in D^-, and are both way oriented in D^+.

Let us now fix a sub-orientation $D = (V, A)$ of the graph $G = (V, E)$. We shall say that a vertex v *rejects* (in D) a stable set $S \subseteq V$ if $\overrightarrow{uv} \in A$ for all edges $(u, v) \in E$ for which $u \in S$. We shall call a sub-orientation D of G *rejecting* if every stable set of G is rejected (in D) by some vertex. (Let us note that a non-maximal stable set S is always rejected by any of the vertices $v \notin S$ for which $S \cup \{v\}$ is also stable.) Finally, let us call a sub-orientation of G *clique acyclic* if it does not contain a directed cycle inducing a clique of G.

Let us observe the following easy relations.

Lemma 1 *Let $D^+ = (V, A^+)$ and $D^- = (V, A^-)$ be a corresponding pair of super- and sub-orientations of a simple graph $G = (V, E)$. Then*

(i) *D^- is rejecting if and only if D^+ has no kernel;*

(ii) *D^- is clique acyclic if and only if every clique of G has a kernel in D^+.*

Proof. (i). If a vertex v rejects in D^- a maximal stable set S, then, by definition, every edge of G between S and v are oriented uniquely toward v both in D^- and D^+. Hence v has no successor from S in D^+, i.e. S cannot be a kernel of D^+. This implies that if D^- is rejecting, i.e. if every maximal stable set of G is rejected by at least one vertex in D^-, then none of the maximal stable sets are kernels of D^+.

For the converse direction let us observe that if a maximal stable set S is not a kernel in D^+, then there must exist a vertex $v \notin S$ which has no successor in S, i.e. for which all the edges between S and v are directed uniquely toward v (there are such edges, since S is a maximal stable set). Then, all these edges appear with the same orientation in D^-, too, and hence v rejects S in D^-.

(ii). Let $C = \{v_0, v_1, ..., v_s\}$ be the vertex set of a directed cycle in D^- inducing a clique of G. Let us say $\overrightarrow{v_i v}_{i+1} \in A^-$ for $i = 0, ..., s-1$ and $\overrightarrow{v_s v}_0 \in A^-$. Then, since $A^- \subseteq A^+$ and no edge appearing in A^- is oriented both ways in A^+, none of the vertices of C is the successor of all the other, i.e. C has no kernel.

On the other hand, if D^- induces an acyclic subgraph on a clique C of G, then there exists a vertex $v \in C$ which has no successor in C in D^-. Such a vertex obviously is the successor of all other vertices of C in D^+. □

A simple graph G will be called *core solvable* if it has no clique acyclic rejecting sub-orientation. It follows from the previous lemma that core solvability and kernel solvability are equivalent.

Corollary 1 *A simple graph is kernel solvable if and only if it is core solvable.*

□

Example 1. Let us consider odd holes and odd anti-holes. Let $V = \{v_0, v_1, ..., v_{n-1}\}$, where n is an odd integer, let

$$E = \{(v_i, v_{i+2 \bmod n}) | i = 0, ..., n-1\},$$

and let

$$E' = \{(v_i, v_{i+s \bmod n}) | i = 0, 1, ..., n-1,\ 2 \leq s \leq n-2\}.$$

Then, $G = (V, E)$ is an odd hole, and $G' = (V, E')$ is an odd anti-hole. Let us define now

$$A = \{\overrightarrow{v_i v}_{i+2 \bmod n} \ | i = 0, ..., n-1\}.$$

Then it is easy to verify that $D = (V, A)$ is a rejecting, clique acyclic sub-orientation of both G and G' (see Figure 1).

Let us remark that while odd holes have a unique clique acyclic rejecting sub-orientation, odd anti-holes may have several, non-isomorphic ones. As an example let us consider the sub-orientation of $\overline{C}_7$ given in Figure 2. This sub-orientation is clique acyclic, rejecting, and non-isomorphic to the one in Figure 1.

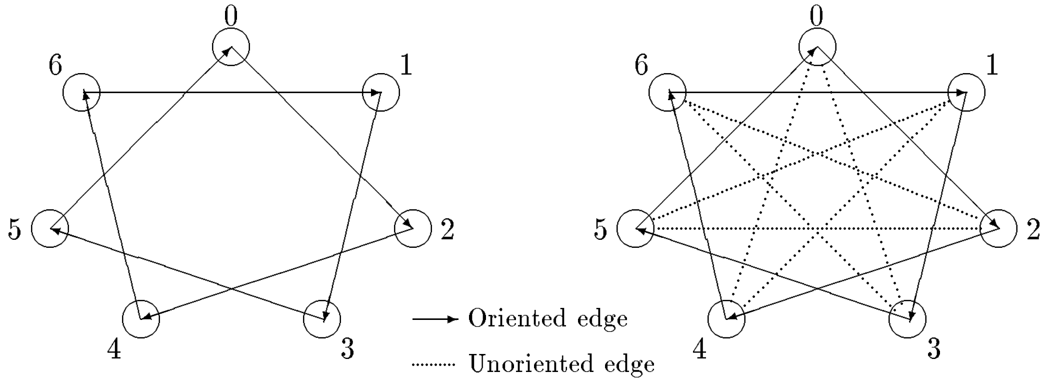

Figure 1: Clique acyclic rejecting sub-orientations of odd holes and anti-holes.

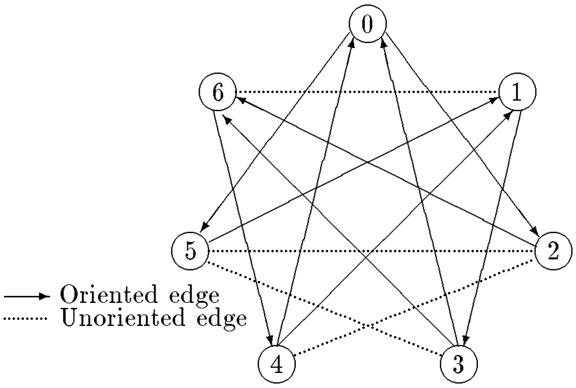

Figure 2: Another clique acyclic rejecting sub-orientation of $\overline{C}_7$.

3 Effectivity Functions and Stability

In this section we introduce definitions and recall basic results from game theory.

Let $\mathbf{I}$ denote the set of *players* and $\mathbf{A}$ denote the set of *outcomes*. Each player $i \in \mathbf{I}$ has a set of *strategies* denoted by X_i, and let $\mathbf{X} = \times_{i \in \mathbf{I}} X_i$. A game is described by two mappings, $g : \mathbf{X} \to \mathbf{A}$ and $u : \mathbf{I} \times \mathbf{A} \to \mathbb{R}$. The mapping g (sometimes called a *game form*) specifies the outcome of the game for every possible combinations of the startegies of the players. (Note that the mapping g is supposed to be surjective but not usually injective.) The real function u, called the *utility* (or *payoff*) describes the "value" of an outcome for an individual player. Player $i \in \mathbf{I}$ is said to *prefer* outcome $a_1 \in \mathbf{A}$ over $a_2 \in \mathbf{A}$ if $u(i, a_1) > u(i, a_2)$. We shall call in the sequel the quadruple $< \mathbf{I}, \mathbf{A}, \mathbf{X}, g >$ a *game form*, and the quintuple $< \mathbf{I}, \mathbf{A}, \mathbf{X}, g, u >$ a *game in normal form*.

Subsets of the players are called *coalitions* and subsets of the outcomes are called *blocks*. A coalition $K \subseteq \mathbf{I}$ is said to be *effective* for a block $B \subseteq \mathbf{A}$ if the players of K can guarantee the outcome of the game to belong to B. This relation can be represented as a Boolean mapping $\mathcal{E} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ called an *effectivity function*, i.e. $\mathcal{E}(K,B) = 1$ if and only if the coalition K is effective for the block B. We shall call the quadruple $< \mathbf{I}, \mathbf{A}, \mathcal{E}, u >$ a *game in effectivity function form*. Obviously, every game form $\Gamma =< \mathbf{I}, \mathbf{A}, \mathbf{X}, g >$ defines uniquely an effectivity function, denoted by $\mathcal{E}_\Gamma$ by seting $\mathcal{E}_\Gamma(K,B) = 1$ for a coalition $K \subseteq \mathbf{I}$ and a block $B \subseteq \mathbf{A}$ if and only if there are strategies $y_k \in X_k$ for every $k \in K$ such that $g(\mathbf{x}) \in B$, for every $\mathbf{x} \in \mathbf{X}$ for which $x_k = y_k$, $k \in K$.

Clearly, given the set of players $\mathbf{I}$ and the set of outcomes $\mathbf{A}$ any Boolean mapping $\mathcal{E} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ can be considered as an effectivity function, but not all effectivity functions are generated by game forms. The following theorem of [18] characterizes which effectivity functions are corresponding to game forms.

Theorem 3 *(Moulin and Peleg (1982)). A Boolean mapping $\mathcal{E} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ is the effectivity function of a game form Γ if and only if $\mathcal{E}$ satisfies the following conditions:*

Monotonicity: *$K \subseteq K' \subseteq \mathbf{I}$, $B \subseteq B' \subseteq \mathbf{A}$ and $\mathcal{E}(K,B) = 1$ imply that $\mathcal{E}(K',B') = 1$.*

Superadditivity: *$\mathcal{E}(K_1,B_1) = 1$, $\mathcal{E}(K_2,B_2) = 1$ and $K_1 \cap K_2 = \emptyset$ imply that $\mathcal{E}(K_1 \cup K_2, B_1 \cap B_2) = 1$.*

Boundary conditions: *$\mathcal{E}(\mathbf{I},B) = 1$ for all nonempty blocks $B \subseteq \mathbf{A}$, $\mathcal{E}(\mathbf{I},\emptyset) = 0$, furthermore $\mathcal{E}(K,\mathbf{A}) = 1$ for all nonempty coalitions $K \subseteq \mathbf{I}$, and $\mathcal{E}(\emptyset,\mathbf{A}) = 0$.*

An effectivity function $\mathcal{E}$ can also be specified by explicitly listing all pairs of coalitions K and blocks B for which K is effective for B. Conversely, given a list $\mathbf{T} =< (K_j, B_j) | j \in J >$, where J is a finite set of indices, $K_j \subseteq \mathbf{I}$ is a coalition and $B_j \subseteq \mathbf{A}$ is a block for $j \in J$, let $\mathcal{E}_{\mathbf{T}}$ denote the effectivity function corresponding to $\mathbf{T}$, i.e. $\mathcal{E}_{\mathbf{T}}(K,B) = 1$ if and only if $(K,B) = (K_j,B_j)$ for some $j \in J$.

Let us call an effectivity function $\mathcal{E}$ *playable* if there exists a game form Γ for which $\mathcal{E} \le \mathcal{E}_\Gamma$, i.e. for which $\mathcal{E}(K,B) = 1$ implies $\mathcal{E}_\Gamma(K,B) = 1$.

The following theorem of [12] provides a necessary and sufficient condition for the playability of an effectivity function.

Theorem 4 *(Gurvich and Vasin, 1978). An effectivity function $\mathcal{E}$ defined by the list $\mathbf{T} =< (K_j, B_j) | j \in J >$ is playable if and only if for every subset $J' \subseteq J$ for which the corresponding coalitions are pairwise disjoint (i.e., $K_j \cap K_{j'} = \emptyset$ for every $j, j' \in J'$, $j \ne j'$), the corresponding blocks have a nonempty intersection, i.e. $\cap_{j \in J'} B_j \ne \emptyset$.*

Let us remark that Theorem 4 can be derived directly from Theorem 3.

Let us now fix an effectivity function $\mathcal{E} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ and a utility $u : \mathbf{I} \times \mathbf{A} \to \mathbb{R}$. We shall say that a coalition $K \subseteq \mathbf{I}$ *can reject* an outcome $a \in \mathbf{A}$ if there exists a block $B \subseteq \mathbf{A}$ such that $\mathcal{E}(K,B) = 1$ and $u(k,b) > u(k,a)$ for every player $k \in K$ and outcome

$b \in B$. In other words, K can reject a if K is effective for a block B in which every outcome is strictly and unanimously prefered to a by all players of K. Let

$$R(K,B) = \{a \in \mathbf{A} | \forall k \in K\, \forall b \in B\, u(k,b) > u(k,a)\}. \tag{1}$$

Given a family $\mathcal{K} \subseteq 2^{\mathbf{I}}$ of coalitions, the $\mathcal{K}$-*core* of the game $< \mathbf{I}, \mathbf{A}, \mathcal{E}, u >$ in effectivity function form is defined as

$$C(\mathcal{E}, u, \mathcal{K}) = \mathbf{A} \setminus \bigcup_{\substack{K \in \mathcal{K} \\ B \subseteq \mathbf{A} \\ \mathcal{E}(K,B)=1}} R(K,B). \tag{2}$$

Given the set of players $\mathbf{I}$ and the set of outcomes $\mathbf{A}$ an effectivity function $\mathcal{E}$ is called *stable* if $C(\mathcal{E}, u, 2^{\mathbf{I}}) \neq \emptyset$ for every utility function $u : \mathbf{I} \times \mathbf{A} \to \mathbb{R}$.

Given an effectivity function $\mathcal{E}$, let us consider a table of triples $\mathbf{R} =< (K_j, B_j, R_j) | j \in J >$, where $K_j \subseteq \mathbf{I}$, $B_j \subseteq \mathbf{A}$, and $R_j \subseteq \mathbf{A}$ for all $j \in J$. We shall call $\mathbf{R}$ a *rejecting table* of $\mathcal{E}$, if $\mathcal{E}(K_j, B_j) = 1$ and $R_j \cap B_j = \emptyset$ for all $j \in J$, and $\mathbf{A} = \cup_{j \in J} R_j$. The ordered subset $\{j_1, j_2, ..., j_s\} \subseteq J$ of the indices is called a *common player cycle* of the rejecting table $\mathbf{R}$ if $R_{j_l} \cap B_{j_{l+1}} \neq \emptyset$ for $l = 1, ..., s-1$, $R_{j_s} \cap B_{j_1} \neq \emptyset$ and $\emptyset \neq \cap_{l=1}^{s} K_{j_l}$.

The following theorem of [14] characterizes stability of effectivity functions in terms of their rejecting tables.

Theorem 5 *(Keiding, 1985) An effectivity function is stable if and only if every rejecting table of it contains a common player cycle.*

Given the set of players, $\mathbf{I}$, a family $\mathcal{K} \subseteq 2^{\mathbf{I}}$ of coalitions is called *g-stable* if $\emptyset \neq C(\mathcal{E}_\Gamma, u, \mathcal{K})$ for every game $\Gamma =< \mathbf{I}, \mathbf{A}, \mathbf{X}, g, u >$ in normal form.

Let us consider a family $\mathcal{K}$ of coalitions, and a nonnegative integer valued weight function on it, $w : \mathcal{K} \to \mathbb{Z}_+$. The function w is called a *balanced weighting* of $\mathcal{K}$ with multiplicity σ if for every player $i \in \mathbf{I}$ the equation

$$\sum_{\substack{K \in \mathcal{K} \\ K \ni i}} w(K) = \sigma \tag{3}$$

holds. The subfamily $\mathcal{K}_w = \{K \in \mathcal{K} | w(K) > 0\}$ is called the *support* of w. A balanced weighting w with multiplicity $\sigma = 1$ is called a *partition weighting* (or simply a partition), for the support $\mathcal{K}_w$ of such a weighting w is a partition of $\mathbf{I}$. We shall say that a family $\mathcal{K}$ has property *BSP* (i.e. has the property that every **b**alanced **s**ubfamily of $\mathcal{K}$ is **p**artitionable), if every balanced weighting of $\mathcal{K}$ is the nonnengative integral combination of partitions. In other words, $\mathcal{K}$ has property BSP if and only if for every balanced weighting $w : \mathcal{K} \to \mathbb{Z}_+$ there are partitions $w^i : \mathcal{K} \to \{0,1\}$ and nonnegative integers $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., l$, such that $w = \sum_{i=1}^{l} \alpha_i w^i$. Given a finite set S, let us denote finally by $[S]$ the family of all one element subsets (singletons) of S, i.e. $[S] = \{\{i\} | i \in S\}$.

The following theorem of [12] (see also [21]) characterizes the g-stability of a family of coalitions.

Theorem 6 *(Gurvich and Vasin, 1978). The family of coalitions $\mathcal{K}$ is g-stable if and only if $\mathcal{K} \cup [\mathbf{I}]$ has property BSP.*

An effectivity function $\mathcal{E} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ is called *B-monotone* if $\mathcal{E}(K,B) = 1$ and $B' \supseteq B$ imply $\mathcal{E}(K,B') = 1$. Let us call $\mathcal{E}$ *balanced* if for every partition

$$\mathbf{A} = \bigcup_{i \in \mathbf{I}} D_i \tag{4}$$

(i.e. where $D_i \cap D_j = \emptyset$ whenever $i \neq j$, $i,j \in \mathbf{I}$), and for every balanced weighting $w : \mathcal{K} \to \mathbb{Z}_+$ there exists a coalition $K \in \mathcal{K}_w$ such that

$$\mathcal{E}(K, \bigcup_{i \notin K} D_i) = 0.$$

Balancedness of an effectivity function is a strong property, and according to the following theorem, it is essentially a sufficient condition for its stability (see [7, 8, 9]).

Theorem 7 *(Danilov and Sotskov, 1987). B-monotone balanced effectivity functions are stable.*

We have to remark here that the above theorem was originally stated only for effectivity functions which are generated by game forms (see Theorem 3). The proof used in [7], however, is in fact valid for any B-monotone effectivity function, as stated above.

We also would like to point out that both theorems, 6 and 7, build heavily on a result by Scarf, [20].

4 Effectivity Functions of Graphs

Given a graph $G = (V,E)$ we shall consider games in which players are maximal cliques, and outcomes are maximal stable sets. Let $\mathbf{I}_G$ and $\mathbf{A}_G$ denote respectively the families of maximal cliques and maximal stable sets of G. For every vertex v let $K_v \subseteq \mathbf{I}_G$ denote the family of all maximal cliques of G containing v, and let $B_v \subseteq \mathbf{A}_G$ denote the family of all maximal stable sets of G which contain vertex v. Finally, let $\mathcal{K}_G = \{K_v | v \in V\}$, $\mathbf{T}_G = < (K_v, B_v) | v \in V >$, and let $\mathcal{E}_G$ denote the effectivity function defined by $\mathbf{T}_G$.

Lemma 2 *For every graph G the associated effectivity function $\mathcal{E}_G$ is playable.*

Proof. Indeed, the conditions of Theorem 4 are fulfilled. To see this let us consider an arbitrary subset $W \subseteq V$ of the vertices, for which the coalitions K_w, $w \in W$ are pairwise disjoint. Clearly, W is a stable set of G. Let S be a maximal stable set of G which contains W. Then, $S \in B_w$ for all $w \in W$, by the definition of the blocks B_v, $v \in V$. This implies that $\emptyset \neq \cap_{w \in W} B_w$. □

Next we show that the core solvability of G is equivalent with the stability of $\mathcal{E}_G$.

Lemma 3 *A simple graph G is core solvable if and only if the associated effectivity function $\mathcal{E}_G$ is stable.*

Proof. Let us assume first that $\mathcal{E}_G$ is not stable. Then by Theorem 5 there exists a rejecting table $\mathbf{R} =< (K_j, B_j, R_j)|j \in J >$ having no common player cycles. Let us observe first that $\mathcal{E}_G(K, B) = 0$ unless $K = K_v$ and $B = B_v$ for some $v \in V$, by the definition of $\mathcal{E}_G$. This implies, by the definition of rejecting tables that $J \subseteq V$. Since the addition of triples $(K_v, B_v, \emptyset)$, $v \in V \setminus J$ to $\mathbf{R}$ keeps $\mathbf{R}$ a rejecting table of $\mathcal{E}_G$, we may assume $J = V$, as well.

Let us define now an orientation A of some of the edges of G as follows: For every maximal stable set $S \in R_v$, $v \in V$ and for every vertex $u \in S$ for which $(u, v) \in E$ let us include the arc $\overrightarrow{uv} \in A$.

First, we claim that $D = (V, A)$ is clique acyclic, thus in particular no edge of G is oriented both ways in D, and hence D is a clique acyclic suborientation of G. Let us assume on the contrary that $\{v_1, v_2, ..., v_s\}$ is a clique cycle, i.e. $\overrightarrow{v_iv_{i+1}} \in A$ for $i = 1, .., s-1$ and $\overrightarrow{v_sv_1} \in A$, and $\{v_1, ..., v_s\} \subseteq V$ is a clique of G. Let C be a maximal clique of G containing vertices v_i, $i = 1, ..., s$. The existence of an arc $\overrightarrow{v_iv_j}$ implies the existence of a maximal stable set $S \in R_{v_i}$ containing v_j, and hence $\emptyset \neq B_{v_j} \cap R_{v_i}$. Since $C \in K_{v_j}$, $j = 1, ..., s$, the above together would impy that $\{v_1, v_2, ..., v_s\}$ is a common player cycle in $\mathbf{R}$, a contradiction which proves the claim.

We claim next that $D = (V, A)$ is a rejecting suborientation of G. This is quite immediate, since $\mathbf{R}$ is a rejecting table, thus, for every maximal stable set $S \in \mathbf{A}_G$ there exists a vertex v for which $S \in R_v$, and therefore, by the definition of the arcs in D, the set S is rejected at vertex v. Therefore, D is a clique acyclic rejecting sub-orienation of G, and thus G is not core solvable.

For the converse direction, let us assume now that G is not core solvable. This implies that there exists a clique acyclic rejecting suborientation $D = (V, A)$ of G. Let us define the collection R_v of maximal stable sets by including all those which are rejected at vertex v. Clearly, $v \notin S$ for all $S \in R_v$, and hence $B_v \cap R_v = \emptyset$, $v \in V$. Since every maximal stable set of G is rejected by D, $\mathbf{A}_G = \cup_{v \in V} R_v$ holds, and thus $\mathbf{R} =< (K_v, B_v, R_v)|v \in V >$ is a rejecting table for $\mathcal{E}_G$.

We claim that this rejecting table has no common player cycles, and thus it proves that $\mathcal{E}_G$ is not stable. To see this claim, let us assume on the contrary that $\{v_1, ..., v_s\}$ forms a common player cycle, i.e. $R_{v_i} \cap B_{v_{i+1}} \neq \emptyset$ for $i = 1, ..., s-1$, $R_{v_s} \cap B_{v_1} \neq \emptyset$ and $\cap_{i=1}^{s} K_{v_i} \neq \emptyset$. The existence of maximal stable sets $S \in R_{v_i} \cap B_{v_j}$ shows that $\overrightarrow{v_jv_i} \in A$, and hence $\{v_1, ..., v_s\}$ is a directed cycle in $D = (V, A)$. The existence of a maximal clique $C \in \cap_{i=1}^{s} K_{v_i}$ shows that this cycle is a clique cycle, contradicting to the clique acyclicity of D. This contradiction proves the claim, and finishes the proof of the lemma. □

Corollary 2 *A graph G is kernel solvable if and only if the associated effectivity function $\mathcal{E}_G$ is stable.*

Proof. It follows immediately by Corollary 1 and by Lemma 3. □

5 Perfectness and Partitionable Balanced Families

In this section we prove that perfectness of a graph G is equivalent with the g-stability of the associated family $\mathcal{K}_G$ of coalitions. The main technical result of this section, Lemma 5 is essentially equivalent with a result of [6, 10] about the integrality of the fractional vertex packing polytope. For completeness and clarity we give a simple proof below.

Let us observe first the following easy but helpful statement.

Lemma 4 *A family $\mathcal{H} \subseteq 2^S$ of subsets of S has property BSP if and only if for every balanced weighting $w : \mathcal{H} \to \mathbb{Z}_+$ of $\mathcal{H}$ the support $\mathcal{H}_w = \{H \in \mathcal{H} | w(H) > 0\}$ of w contains a partition of S.*

Proof. Let us assume first that $\mathcal{H}$ has property BSP. Then, by definition of the BSP property, for every balanced weighting $w : \mathcal{H} \to \mathbb{Z}_+$ there are positive integers α_i and partitions $w^i : \mathcal{H} \to \{0,1\}$, such that $w = \sum_{i=1}^{s} \alpha_i w^i$. Then $w(H) \geq w^i(H)$ for every subset $H \in \mathcal{H}$, and hence the partition $\mathcal{P}^1 = \{H \in \mathcal{H} | w^1(H) = 1\}$ is clearly included in the support $\mathcal{H}_w$ of w.

Let us assume next that for every balanced weighting $w : \mathcal{H} \to \mathbb{Z}_+$ of $\mathcal{H}$ the support $\mathcal{H}_w = \{H \in \mathcal{H} | w(H) > 0\}$ contains a partition $\mathcal{P}$ of S. Let us prove now by induction on the multiplicity of balanced weightings that every balanced weighting is the sum of partitions. Clearly, if w is a balanced weighting with multiplicity 1, then it is a partition itself. So, let us assume that we already have shown this for balanced weightings with multiplicity not more than $\sigma - 1$, and let us consider a balanced weighting $w : \mathcal{H} \to \mathbb{Z}_+$ of multiplicity σ. Let us consider the weight function w' defined by $w'(H) = w(H) - 1$ if $H \in \mathcal{P}$ and $w'(H) = w(H)$ otherwise, and define another weight function w'' by seting $w''(H) = 1$ if $H \in \mathcal{P}$ and $w''(H) = 0$ otherwise. Clearly, $w = w' + w''$ and w'' is a partition. It is also clear that w' is a balanced weighting with multiplicity $\sigma - 1$. Hence, by our inductive assumption, w' is the sum of partitions. Thus, $w = w' + w''$ is also the sum of partitions. □

Lemma 5 *A graph G is perfect if and only if the family $\mathcal{K}_G \cup [\mathbf{I}_G]$ has property BSP.*

Proof. Let us assume first that G is perfect. To prove that $\mathcal{K}_G \cup [\mathbf{I}_G]$ has property BSP it is enough, by Lemma 4, to show that the support of every balanced weighting w contains a partition of $\mathbf{I}_G$. For this end, let us define $W = \{v \in V | w(K_v) > 0\}$ and let us consider the induced subgraph $G' = (W, E')$ of G. Let us associate the weight $z_v = w(K_v)$ to $v \in W$. Since G' is perfect, it contains a stable set $S \subseteq W$ intersecting all maximum weight cliques of G' (with respect to the weights z_v, $v \in W$, see [15]). Let us observe now that for any maximal clique $C \in \mathbf{I}_G$ of G, the clique $W \cap C$ is either a maximum weight clique of G' (with respect to the weights z_v, $v \in W$), in which case $C \cap S \neq \emptyset$, or $w(\{C\}) > 0$ follows by the balancedness of w. Let us define then $\mathcal{P} = \{K_v | v \in W\} \cup \{\{C\} \in [\mathbf{I}_G] | C \cap S = \emptyset\}$. Clearly, $\mathcal{P}$ is a partition of $\mathbf{I}_G$ and it is included in the support of w.

Let us assume next that the family $\mathcal{K}_G \cup [\mathbf{I}_G]$ has property BSP. To show that G is perfect it is enough to show that in any induced subgraph $G' = (W, E')$ of G induced by

$W \subseteq V$ there is a stable set $S \subseteq W$ which intersects all maximum cliques of G' (see [15]). To see this let us define first a weighting w of the family $\mathcal{K}_G \cup [\mathbf{I}_G]$ by seting $w(K_v) = 1$ for $v \in W$, $w(K_v) = 0$ for $v \in V \setminus W$ and $w(\{C\}) = \omega(G') - |W \cap C|$ for all maximal cliques $C \in \mathbf{I}_G$, where $\omega(G')$ is the size of the maximum clique in G'. It is easy to see then that w is a balanced weighting of $\mathcal{K}_G \cup [\mathbf{I}_G]$ of multiplicity $\omega(G')$. By Lemma 4 the support of w contains a partition $\mathcal{P}$ of $\mathbf{I}_G$. Let $S = \{v \in V | K_v \in \mathcal{P}\}$. Clearly S is a stable set and $S \subseteq W$ since $w(K) > 0$ must hold for every family $K \in \mathcal{P}$. If $C \in \mathbf{I}_G$ is a maximal clique for which $|W \cap C| = \omega(G')$, then $w(\{C\}) = 0$ by the definition of w, thus C must be covered by some $K_v \in \mathcal{P}$, and hence $v \in C \cap S$ follows, showing that S intersects C. This proves that S intersects all maximum cliques of G'. Since this holds for arbitrary induced subgraphs of G, it follows that G is perfect. □

Corollary 3 *A graph G is perfect if and only if $\mathcal{K}_G$ is g-stable.*

Proof. Follows immediately from Theorem 6 and Lemma 5. □

6 Stability and g-Stability in Graphs

We are ready now to connect the stability of $\mathcal{E}_G$ with the g-stability of $\mathcal{K}_G$ for a graph G.

Lemma 6 *Let G be a simple graph, $\mathcal{E}_G$ the associated effectivity function and $\mathcal{K}_G$ the associated family of coalitions, as before. Then, if $\mathcal{K}_G$ is g-stable, then $\mathcal{E}_G$ is stable. Furthermore, if $\mathcal{E}_{G'}$ is stable for every blow up G' of G, then $\mathcal{K}_G$ is g-stable.*

Proof. Let us assume first that $\mathcal{K}_G$ is g-stable. Since $\mathcal{K}_G$ is always playable by Lemma 2, there exists a game form $\Gamma =< \mathbf{I}_G, \mathbf{A}_G, \mathbf{X}, g >$ for which the corresponding effectivity function $\mathcal{E}_\Gamma$ is a majorant of $\mathcal{E}_G$, i.e. $\mathcal{E}_G \leq \mathcal{E}_\Gamma$. This implies by (2) that

$$C(\mathcal{E}_G, u, \mathcal{K}_G) \supseteq C(\mathcal{E}_\Gamma, u, \mathcal{K}_G) \tag{5}$$

holds for every utility function $u : \mathbf{I}_G \times \mathbf{A}_G \to \mathbb{R}$. Since $\mathcal{E}_G(K, B) = 0$ whenever $K \notin \mathcal{K}_G$, by (2) it follows also that

$$C(\mathcal{E}_G, u, 2^{\mathbf{I}_G}) = C(\mathcal{E}_G, u, \mathcal{K}_G). \tag{6}$$

Since $\mathcal{K}_G$ is g-stable, the $\mathcal{K}_G$-core $C(\mathcal{E}_\Gamma, u, \mathcal{K}_G)$ of Γ is never empty. This implies by (5) and (6) that $C(\mathcal{E}_G, u, 2^{\mathbf{I}_G}) \neq \emptyset$ for all utility functions u, and thus proves that $\mathcal{E}_G$ is stable.

Let us assume next that $\mathcal{K}_G$ is not g-stable. We shall show that there exists a blow up $\widetilde{G}$ of G for which $\mathcal{E}_{\widetilde{G}}$ is not stable.

Since now $\mathcal{K}_G$ is not g-stable, there exists a finite game form $\Gamma =< \mathbf{I}_G, \mathbf{A}, \mathbf{X}, g >$ ($\mathbf{A} \neq \mathbf{A}_G$ in general), for which the $\mathcal{K}_G$-core $C(\mathcal{E}_\Gamma, u, c\mathcal{K}_G)$ is empty. Let us define now a new effectivity function $\mathcal{E}$ by seting $\mathcal{E}(K, B) = \mathcal{E}_\Gamma(K, B)$ whenever $K \in \mathcal{K}_G$, $B \subseteq \mathbf{A}$ and let $\mathcal{E}(K, B) = 0$ otherwise. Clearly, $C(\mathcal{E}, u, 2^{\mathbf{I}_G}) = C(\mathcal{E}, u, \mathcal{K}_G) = C(\mathcal{E}_\Gamma, u, \mathcal{K}_G) = \emptyset$, and therefore $\mathcal{E}$ is not stable. Thus, by Theorem 5, there exists a rejecting table $\mathbf{R} =< (K_j, B_j, R_j) | j \in J >$ of $\mathcal{E}$ having no common player cycles, i.e. $\mathcal{E}(K_j, B_j) = 1$, $B_j \cap R_j = \emptyset$ for $j \in J$, and $\cup_{j \in J} R_j = \mathbf{A}$. Since $\mathcal{E}(K, B) = 0$ for $K \notin \mathcal{K}_G$, all the coalitions K_j, $j \in J$ must belong to $\mathcal{K}_G$. Since the inclusion of triples $(K_v, \mathbf{A}, \emptyset)$ in $\mathbf{R}$ is not changing the fact that $\mathbf{R}$ is a rejecting table of $\mathcal{E}$, we may assume as well that $K_v = K_j$ for at least one index $j \in J$ for every vertex $v \in V$.

Let us consider now the intersection graph $\widetilde{G} = (J, \widetilde{E})$ of the family $\{K_j | j \in J\}$, i.e. $(j_1, j_2) \in \widetilde{E}$ exactly when $K_{j_1} \cap K_{j_2} \neq \emptyset$. Clearly, $\widetilde{G}$ is a blow-up of G. Let us associate now to any maximal stable set S of $\widetilde{G}$ an outcome $a_S \in \mathbf{A}$ as follows: The coalitions K_j, $j \in S$ are pairwise disjoint, because S is a stable set of $\widetilde{G}$. Therefore, $\cap_{j \in S} B_j \neq \emptyset$ follows by Theorem 4, since $\mathcal{E}$ is a playable effectivity function by its definition. Let us choose then $a_S \in \cap_{j \in S} B_j$ arbitrarily. It follows from this definition that every B_j ($j \in J$) contains all outcomes a_S associated to maximal stable sets of $\widetilde{G}$ which contain j. It follows also that if $a_S \in R_j$ for some maximal stable set $S \subseteq J$ and vertex $j \in J$, then $j \notin S$.

Let us define then an orientation $\widetilde{D} = (J, \widetilde{A})$ of the edges of $\widetilde{G}$, by including all arcs $\overrightarrow{lj} \in \widetilde{A}$ for all $l \in S$ and $j \in J$ for which $a_S \in R_j$ and $(j, l) \in \widetilde{E}$. We shall show that $\widetilde{D}$ is a clique acyclic rejecting suborientation of $\widetilde{G}$, which will conclude the proof of the theorem.

Exactly as in the proof of Lemma 3 we show first that any clique cycle $\{j_1, ..., j_s\}$ of $\widetilde{D}$ corresponds to a common player cycle of $\mathbf{R}$. Since $\mathbf{R}$ has no common player cycles, it will follow that $\widetilde{D}$ is clique acyclic, and in particular that its edges are not oriented both ways, thus it is a clique acyclic suborientation of $\widetilde{G}$.

To see this let $\widetilde{C}$ be a maximal clique of $\widetilde{G}$ containing vertices $\{j_1, ..., j_s\}$. Then, since $\widetilde{G}$ is a blow-up of G, the corresponding maximal clique C of G must belong to each coalition K_{j_l}, $l = 1, ..., s$, thus they have a common player. From the definition of the arcs of $\widetilde{D}$ it follows that, if there is an arc $\overrightarrow{lj} \in \widetilde{A}$, then there exists a maximal stable set S of $\widetilde{G}$ for which $a_S \in R_j \cap B_l$. These imply that $\{j_1, ..., j_s\}$ is a common player cycle of $\mathbf{R}$.

Let us show finally that $\widetilde{D}$ is a rejecting suborientation of $\widetilde{G}$. Again, just as in the proof of Lemma 3, this follows immediately from the definition: For any maximal stable set S of $\widetilde{G}$ the corresponding outcome a_S is rejected by some coalitions K_j, (i.e. $a_S \in R_j$), since $\mathbf{R}$ is a rejecting table, and thus S is rejected at vertex $j \in J$ of $\widetilde{G}$ by the definition of arcs of $\widetilde{D}$.

Thus, we can conlude that the blow up $\widetilde{G}$ of G is not core solvable, since we could construct a clique acyclic rejecting sub-orientation $\widetilde{D}$ of it. Therefore, $\mathcal{E}_{\widetilde{G}}$ is not stable by Lemma 3. □

Using the above lemma, we can now prove our main theorems.

Proof of Theorem 1. If G is a perfect graph, then $\mathcal{K}_G$ is g-stable by Corollary 3, therefore, $\mathcal{E}_G$ is stable by Lemma 6. Thus G is kernel solvable by Corollary 2. □

Proof of Theorem 2. If G is not perfect, then $\mathcal{K}_G$ is not g-stable by Corollary 3, therefore there exists a blow up $\widetilde{G}$ of G, for which $\mathcal{E}_{\widetilde{G}}$ is not stable by Lemma 6. Then, by Corollary 2, the blow up $\widetilde{G}$ is not kernel solvable. □

7 Another proof for Theorem 1

In this section we present an alternative proof for Theorem 1, based on Theorem 7.

Let us consider a graph $G = (V, E)$ as before, and let us denote by $\hat{\mathcal{E}}_G$ the B-monotone extension of $\mathcal{E}_G$, i.e. $\hat{\mathcal{E}}_G(K, B) = 1$ iff $K = K_v$ and $B \supseteq B_v$ for some vertex $v \in V$, or if $B = \mathbf{A}_G$ and $K \neq \emptyset$.

We prove first that the balancedness of $\hat{\mathcal{E}}_G$ is equivalent with the perfectness of G.

Lemma 7 *The effectivity function $\hat{\mathcal{E}}_G$ is balanced if and only if the graph G is perfect.*

Proof. Let us assume first that G is not perfect, and let $W \subseteq V$ be a subset of vertices for which the induced subgraph $G' = G|_W$ is minimally non-perfect.

Let us define now a balanced weighting $w : 2^{\mathbf{I}_G} \to \mathbb{Z}_+$ of the coalitions as follows: let $w(K_v) = 1$ for all $v \in W$, let $w(\{i\}) = \omega(G') - |i \cap W|$ for all maximal cliques $i \in \mathbf{I}_G$, and let $w(K) = 0$ for all other coalitions $K \subseteq \mathbf{I}_G$. Clearly, w is a balanced weighting with multiplicity $\sigma = \omega(G')$.

Let us define next a partition $\mathbf{A}_G = \bigcup_{i \in \mathbf{I}_G} D_i$, $D_i \cap D_j = \emptyset$ for all $i \neq j$, $i, j \in \mathbf{I}_G$. Since G' is minimally imperfect, for every stable set $S' \subseteq W$ there exists a maximal clique $C' \subseteq W$ of G' for which $S' \cap C' = \emptyset$ (see e.g. [19]). This implies that for every maximal stable set $S \in \mathbf{A}_G$ there exists a maximal clique $i \in \mathbf{I}_G$, which is a maximum clique of the induced subgraph G', i.e. for which $|i \cap W| = \omega(G')$, and which has no common point with S in W, i.e. for which $i \cap W \cap S = \emptyset$. Let $S \in D_i$ for such a clique $i \in \mathbf{I}_G$.

We claim that with these definitions

$$\hat{\mathcal{E}}_G(K, \bigcup_{i \notin K} D_i) = 1 \tag{7}$$

for every coalition for which $w(K) > 0$, implying thus that $\hat{\mathcal{E}}_G$ is not balanced. To see this end, let us observe that if $w(K) > 0$ for a coalition then either $K = K_v$ for some vertex $v \in W$, or $K = \{i\}$ for some maximal clique $i \in \mathbf{I}_G$.

In the first case, $i \cap S \neq \emptyset$ for every stable set $S \in B_v$ and for every clique $i \in K_v$, and thus by the definition of D_is we have $S \notin D_i$. Hence, $\bigcup_{i \notin K_v} D_i \supseteq B_v$ follows, and thus (7) holds.

In the second case, $w(\{i\}) > 0$ implies $|i \cap W| < \omega(G')$ by the definition of w, and thus $D_i = \emptyset$ by the definition of the partition. Hence, $\bigcup_{j \notin \{i\}} D_i = \mathbf{I}_G$ and (7) follows by the definition of B-monotonicity.

Let us assume secondly that G is perfect. To show that $\hat{\mathcal{E}}_G$ is balanced it is enough to prove that for any balanced weighting w and for any partition $\mathbf{A}_G = \bigcup_{i \in \mathbf{I}_G} D_i$ there exists a coalition $K \in \mathcal{K}_w = \{K \subseteq \mathbf{I}_G | w(K) > 0\}$ for which

$$\hat{\mathcal{E}}_G(K, \bigcup_{i \notin K} D_i) = 0. \tag{8}$$

To this end let us consider a balanced weighting w and a partition $\mathbf{A}_G = \bigcup_{i \in \mathbf{I}_G} D_i$ of the outcomes, and let us define $W = \{v \in V | w(K_v) > 0\}$. Since G is perfect by our assumption now, the induced subgraph $G' = G|_W$ is perfect, too.

Let us observe that if there exists an $i \in K$, $D_i \neq \emptyset$ for a coalition K for which $w(K) > 0$ and $K \neq K_v$ for any $v \in W$, then $\bigcup_{i \notin K} D_i \neq \mathbf{I}_G$, and thus (8) follows for this K.

If there is no such pair $i \in K$, then $D_i \neq \emptyset$ implies

$$\sum_{v \in i \cap W} w(K_v) = \sigma,$$

where σ denotes the common multiplicity of cliques in the balanced weighting w. In other words, $D_i \neq \emptyset$ is possible only for cliques $i \in \mathbf{I}_G$ for which $i \cap W$ is a maximum weight clique of G' with respect to the weights $w(K_v)$ associated to the vertices $v \in W$. Since G' is perfect, there exists in this case a stable set $S' \subseteq W$ which intersects all maximum weight cliques of G' (see e.g. [15]), and thus for any maximal stable set S of G for which $S \cap W = S'$, $S \in D_i$ implies $S \cap i \cap W \neq \emptyset$. Let $v \in S \cap i \cap W$ be an arbitrary vertex now. For this vertex v $\bigcup_{j \notin K_v} D_j \not\supseteq B_v$, and thus (8) holds again.

Since the above could be repeated for any balanced weighting and for any partition of the outcomes, the balancedness of $\hat{\mathcal{E}}_G$ follows by the definition. $\square$

Second proof of Theorem 1. If G is a perfect graph, then $\hat{\mathcal{E}}_G$ is balanced by Lemma 7 and thus $\hat{\mathcal{E}}_G$ is stable by Theorem 7. Since $C(\hat{\mathcal{E}}_G, u, 2^{\mathbf{I}_G}) \supseteq C(\mathcal{E}_G, u, 2^{\mathbf{I}_G})$ holds for every utility functions u by (2), the stability of $\mathcal{E}_G$ is implied, too, and thus the kernel solvability of G follows by Corollary 2. $\square$

8 Summary of results

In this section we summarize the obtained results to give an overview of the various notions and their relations.

Corollary 4 *Let $G = (V, E)$ be a graph, and let us consider the following properties:*

(a) *G is perfect;*

(b) *$\hat{\mathcal{E}}_G$ is balanced;*

(c) *$\mathcal{K}_G \cup [\mathbf{I}_G]$ has property BSP;*

(d) *$\mathcal{K}_G$ is g-stable;*

(e) *$\mathcal{E}_G$ is stable;*

(f) *G is core solvable;*

(g) *G is kernel solvable.*

Then we have

$$\textbf{(a)} \Longleftrightarrow \textbf{(b)} \Longleftrightarrow \textbf{(c)} \Longleftrightarrow \textbf{(d)} \Longrightarrow \textbf{(e)} \Longleftrightarrow \textbf{(f)} \Longleftrightarrow \textbf{(g)}.$$

Proof. The first group of equivalences follow from Lemma 7, Lemma 5 and Theorem 6. The second group of equivalences follow, in this order, from Lemma 3 and Lemma 1. Finally the implication follows from Lemma 6. $\Box$

9 Further properties of kernel solvable graphs

Let us remark first that the notion of rejecting suborientation can slightly be generalized, yielding a stronger characterization of kernel solvability than Corollary 2.

Let us consider a directed graph $D = (V, A)$, a simple graph $G = (V, E)$, and a subset $W \subseteq V$. Let $D|_W$ and $G|_W$ denote the induced subgraphs of D and G, respectively, furthermore, let $A|_W \subseteq A$ and $E|_W \subseteq E$ denote the subset of arcs and edges, respectively, which have both endpoints in W. It is clear that if D is a sub-orientation of G, then for every subset $W \subseteq V$, $D|_W$ is a sub-orientation of $G|_W$. Let us call a sub-orientation $D = (V, A)$ of $G = (V, E)$ *minimal* if it is rejecting, and if there is no proper subset $W \subset V$ for which $D|_W$ is a rejecting sub-orientation of $G|_W$. Let us call the sub-orientation D *strongly rejecting* if every stable set S of G is rejected in D by some vertex v for which $S \cup \{v\}$ is not stable.

Lemma 8 *Every minimal sub-orientation is strongly rejecting.*

Proof. Let $D = (V, A)$ be a minimal sub-orientation of $G = (V, E)$, and let us assume indirectly that $S \subseteq V$ is a stable set which is not rejected.

Let N_S denote the set of neighbouring vertices of S, i.e. $N_S = \{v \in V | \exists u \in S$ for which $(u, v) \in E\}$, and let $W = V \setminus (S \cup N_S)$.

We shall show that $D|_W$ is a rejecting sub-orientation of $G|_W$, contradicting thus the minimality of D, and hence proving the lemma.

Since D is rejecting, S is not a maximal stable set of G, and thus $W \neq \emptyset$. Let $P \subseteq W$ be any maximal stable set of $G|_W$. Then $S \cup P$ is a maximal stable set of G. Since D is a rejecting sub-orientation of G, $P \cup S$ is rejected by some vertex $u \in V \setminus (S \cup P)$.

We claim that $u \in W$. Since S is not rejected in D, for every vertex $v \in N_S$ there must be a neighbour $w \in S$ for which $\overrightarrow{wv} \notin A$. Therefore, none of the vertices of N_S can reject $S \cup P$, and hence u belongs to W. Thus, by the definition of rejection, u rejects P in $D|_W$, too. $\Box$

Lemma 9 *If a connected graph $G = (V, E)$ is not core solvable, then it has a strongly rejecting, clique acyclic sub-orientation.*

Proof. Let $D = (V, A)$ be a rejecting and clique acyclic sub-orientation of G.

Let us choose now a subset $W \subseteq V$ which is minimal for the property that $D|_W$ is a rejecting sub-orientation of $G|_W$. Then, according to Lemma 8, $D|_W$ is a strongly rejecting sub-orientation of $G|_W$. Since, for any subset $S \subseteq V$ the sub-orientation $D|_S$ of $G|_S$ will obviously be again clique acyclic, the sub-orientation $D|_W$ is also a clique acyclic sub-orientation of $G|_W$. Thus, $D|_W$ will be both a strongly rejecting and clique acyclic sub-orientation of $G|_W$.

Let us define now a new suborientation of G. Let $W = \{w_1, ..., w_s\}$ and let $V \setminus W = \{v_1, v_2, ..., v_{n-s}\}$ such that every vertex v_j has a neighbour in $W \cup \{v_i | i > j\}$. Such a labeling of the vertices of $V \setminus W$ can be obtained e.g. by puting them in decreasing order of their distances to the set W. Then, let

$$\widetilde{A} = A|_W \cup \{\overrightarrow{v_i v_j} \; |(v_i, v_j) \in E, \, i < j\} \cup \{\overrightarrow{v_i w_j} \; |(v_i, w_j) \in E\}. \tag{9}$$

It is easy to check that $\widetilde{D} = (V, \widetilde{A})$ is a strongly rejecting, clique acyclic sub-orientation of G. □

Lemmas 8 and 9 yield the following characterization of kernel solvability.

Corollary 5 *A connected graph $G = (V, E)$ is not kernel solvable if and only if it has a strongly rejecting clique acyclic suborientation.*

.

Even more, Lemma 9 implies the following property of not kernel solvable graphs.

Lemma 10 *If the graph $G = (V, E)$ contains a non kernel solvable induced subgraph, then it has a rejecting and clique acyclic suborientation, thus, in particular, G itself is not kernel solvable.*

Proof. Let $G' = (W, E')$ be a non kernel solvable induced connected subgraph of G, which obviously exists by the assumption. Then, by Corollary 5 there is a strongly rejecting clique acyclic suborientation $D = (W, A)$ of G'.

Let us repeat the construction in the proof of Lemma 9, and build a suborientation $\widetilde{D}$ of G itself, as follows. Let us fix a linear order of the vertices not in W, i.e. let $V \setminus W = \{v_1, ..., v_s\}$. Let us define then

$$\widetilde{A} = A \cup \{\overrightarrow{v_i v_j} \; |1 \leq i < j \leq s\} \cup \{\overrightarrow{v_i w} \; |w \in W, \text{ and } 1 \leq i \leq s\}.$$

It is quite immediate now to see that $\widetilde{D} = (V, \widetilde{A})$ is a rejecting and clique acyclic suborientation of G. □

Let us remark that if G itself is connected in the above lemma, then by choosing the appropriate labeling, as in Lemma 9, one can obtain a strongly rejecting clique acyclic sub-orientation of G, too. Let us remark also that the suborientations constructed in Example 1 are actually strongly rejecting clique acyclic sub-orientations of the odd holes and anti-holes. This implies immediately the following statement.

Corollary 6 *If the (connected) graph G contains an odd hole or an odd anti-hole, as an induced subgraph, then it has a (strongly) rejecting, clique acyclic sub-orientation, thus, in particular, G is not kernel solvable.*

□

The above may suggest that clique acyclic minimal sub-orientations always induce a strongly rejecting sub-orientation of an induced odd hole or odd anti-hole. This however is not true as the example in Figure 3 shows.

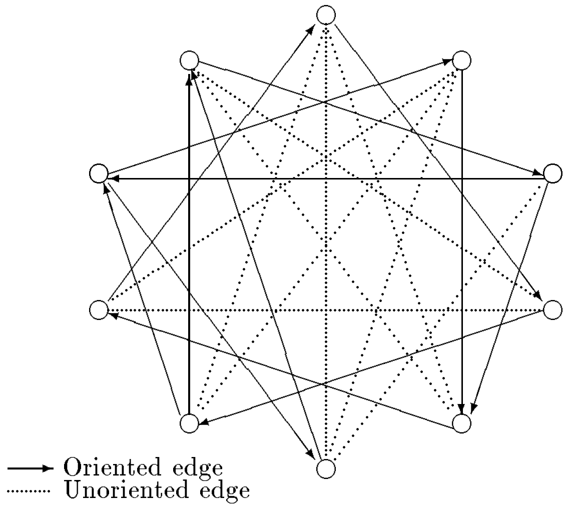

Figure 3: Another example for a minimal, clique acyclic and rejecting (and thus strongly rejecting) sub-orientation of a non-perfect graph.

Let us note that, by Lemma 6, Conjecture 2 is equivalent with the following conjecture.

Conjecture 3. Every blow up of a kernel solvable graph is kernel solvable.

Proof of equivalence between Conjecture 2 and 3. Let us assume Conjecture 2 to hold. Then, by Lemma 6 perfectness and kernel solvability are equivalent. Thus Conjecture 2 follows by the fact that every blow up of a perfect graph is perfect.

Conversely, if we assume Conjecture 3 to hold, then every graph which has a non kernel solvable blow up must itself be non kernel solvable. Thus Lemma 6 implies now Conjecture 2. □

It is interesting to note that the following, seemingly unrelated statement is also equivalent with Conjecture 2.

Conjecture 4. A graph G is kernel solvable if and only if its complement $\overline{G}$ is kernel solvable.

Before proving the above equivalence we need the following technical lemma.

Lemma 11 *Let $G = (V, E)$ be a simple graph, $V = \{v_1, ..., v_n\}$, and let S_i, $i = 1, ..., n$ be pairwise disjoint subsets. Let furthermore, $G' = (S_1 \cup \cdots \cup S_n, E')$ be the graph obtained from G by substituting vertex v_i by the set S_i as a stable set, for $i = 1, ..., n$, i.e. $(x, y) \in E'$ for some $x \in S_i$ and $y \in S_j$ if and only if $i \neq j$ and $(v_i, v_j) \in E$. Then, G is not kernel solvable whenever G' is not kernel solvable.*

Proof. Let $D' = (S_1 \cup \cdots \cup S_n, A')$ be a clique acyclic, rejecting sub-orientation of G'. Let us define now a suborientation of G as follows:

$$A = \{\overrightarrow{v_i v_j} \mid \exists y \in S_j \, \forall x \in S_i \; \overrightarrow{xy} \in A'\}.$$

Let us remark first that $D = (V, A)$ is obviously a clique acyclic sub-orientation of G, since every clique cycle in D would naturally correspond to a clique cycle of D'.

To see that it is rejecting, let us consider a maximal stable set S of G, and let

$$S' = \bigcup_{i: v_i \in S} S_i.$$

Clearly, S' is a maximal stable set of G', and thus it is rejected by a vertex x in D'. It is easy now to verify that if $x \in S_i$, then v_i must reject S in D. $\square$

Proof of equivalence of Conjectures 2 and 4. Let us assume first that Conjecture 2 holds. Then, by Lemma 6 perfectness and kernel solvability are equivalent, thus Conjecture 4 follows by [15].

Let us assume now Conjecture 4 to be true, and let us consider a non perfect graph G. Then its complement, $\overline{G}$ is also non perfect (see [15]), and thus $\overline{G}$ has a blow up, $\overline{G}'$ which is not kernel solvable, by Lemma 6. Then, by Conjecture 4 the complement $\overline{\overline{G}'}$ is also non kernel solvable, and hence G itself is not kernel solvable, by Lemma 11. $\square$

10 Integrality of the set packing polytope

Let us remark finally that the connection between property BSP and perfectness, i.e. Lemma 5, can also be generalized in the following way.

Given a finite set $\mathbf{I}$ let $\mathcal{K} = \{K_j | j \in J\}$ be a family of subsets of $\mathbf{I}$. We shall say that the family has the *Helly property* if for any subfamily $\mathcal{K}' \subseteq \mathcal{K}$, the members of which are pairwise intersecting, all the members of $\mathcal{K}'$ have an element in common. In other words, if for every $J' \subseteq J$ for which $K_j \cap K_l \neq \emptyset$ whenever $j, l \in J'$, we have $\cap_{j \in J'} K_j \neq \emptyset$. Let us denote furthermore by $G_{\mathcal{K}}$ the intersection graph of $\mathcal{K}$, i.e. $V(G_{\mathcal{K}}) = J$ and for $j, l \in J$ we have $(j, l) \in E(G_{\mathcal{K}})$ iff $K_j \cap K_l \neq \emptyset$.

Theorem 8 *Let $\mathcal{K}$ be a family of subsets of a finite set $\mathbf{I}$. The family $\mathcal{K} \cup [\mathbf{I}]$ has property BSP if and only if $\mathcal{K}$ has the Helly property and the intersection graph $G_{\mathcal{K}}$ is perfect.*

Proof. Let us show first that the Helly property of $\mathcal{K}$ is necessary for $\mathcal{K} \cup [\mathbf{I}]$ to have property BSP. For if not, let $\mathcal{K}' = \{K_j | j \in J'\}$ be a minimal non Helly subfamily, i.e. for which $\mathcal{K}'' = \{K_j | j \in J''\}$ has the Helly property for any proper subset $J'' \subset J'$. Then $\cap_{j \in J'} K_j = \emptyset$, and there are elements $x_j \in \mathbf{I}$, $j \in J'$ such that $x_j \in K_l$ for every $j, l \in J'$, $l \neq j$. Let us define now a weighting w of $\mathcal{K} \cup [\mathbf{I}]$ as follows. Let $w(K_j) = 1$ for $j \in J'$, and let $w(K_j) = 0$ otherwise. Furthermore, let $w(\{x\}) = |J'| - 1 - |\{j \in J' | K_j \ni x\}|$, for every $x \in \mathbf{I}$. Then w is a balanced weighting of $\mathcal{K} \cup [\mathbf{I}]$ of multiplicity $|J'| - 1$.

It is easy to see that the support of w does not contain a partition. For this, let us observe that $w(\{x_j\}) = 0$ for $j \in J'$, thus any partition must contain at least one of the sets of $\mathcal{K}'$. Since the sets of $\mathcal{K}'$ are pairwise intersecting, no partition can contain two or more of these sets, therefore a partition must contain exactly one, say K_j for some $j \in J'$. But then $x_j \notin K_j$ could not be covered by sets of the support of w. This proves that the support of the balanced weighting w does not contain a partition, and hence it follows by Lemma 4 that $\mathcal{K} \cup [\mathbf{I}]$ cannot have property BSP.

Let us associate now to every element $i \in \mathbf{I}$ a clique $C(i)$ of $G_{\mathcal{K}}$ by seting $C(i) = \{j \in J | K_j \ni i\}$. Let us observe next that if $\mathcal{K}$ has the Helly property, then the collection of cliques $\{C(i) | i \in \mathbf{I}\}$ contains all maximal cliques of $G_{\mathcal{K}}$.

Let us call an element $i \in \mathbf{I}$ *passive* if there is another element $i' \in \mathbf{I}$ such that the family of sets $K \in \mathcal{K}$ containing i is a proper subfamily of those sets $K' \in \mathcal{K}$ which contain i'. Let us observe that the restriction of the sets of $\mathcal{K}$ on the set of non-passive elements yields a family $\mathcal{K}^*$ for which $\mathcal{K}^* \cup [\mathbf{I}]$ has property BSP if and only if $\mathcal{K} \cup [\mathbf{I}]$ has property BSP.

The above show that the statement is reduced to the equivalence of the perfectness of $G_{\mathcal{K}}$ with the BSP property of $\mathcal{K}^* \cup [\mathbf{I}]$, where $\mathcal{K}^* = \mathcal{K}_{G_{\mathcal{K}}}$. Thus by applying Lemma 5 we can conclude our proof. $\square$

It is an easy exercise to see that property BSP is in fact equivalent with the integrality of the associatyed set-packing polytope. In this way the above theorem generalizes the results by [6, 10].

Acknowledgement. The authors are thankful to Salvador Barbera, Pierre Duchet, Peter Hammer, Frederic Maffray, Herve Moulin and Bezalel Peleg for the helpful questions, comments and/or discussions.

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