ON THE COMPLEXITY OF SOME BASIC PROBLEMS IN COMPUTATIONAL CONVEXITY: II. Volume and mixed volumes

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Introduction

This paper is the second part of a broader survey of computational convexity, an area of mathematics that has crystallized around a variety of results, problems and applications involving interactions among convex geometry, mathematical programming and computer science. The first part [GrK94a] discussed containment problems. This second part is concerned with computing volumes and mixed volumes of convex polytopes and more general convex bodies. In order to keep the paper self-contained we repeat some aspects that have already been mentioned in [GrK94a]. However, this overlap is limited to Section 1. For further background material and references, see [GrK94a], and for other parts of the survey see [GrK94b] and [GrK94c].

Our section headings are as follows.

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1. Preliminaries

1.1 What is Computational Convexity?

The subject of Computational Convexity draws its methods from discrete mathematics and convex geometry, and many of its problems from operations research, computer science, and other applied areas. In essence, it is the study of the computational and algorithmic aspects of high-dimensional convex bodies (especially polytopes), with a view to applying the knowledge gained to bodies that arise in other mathematical disciplines or in the mathematical modeling of problems from outside mathematics.

One of the requirements for turning a potential application into a real one is to have efficient algorithms for computing (or approximating or bounding) the functionals involved. The subject of computational convexity is centered on the search for such algorithms and on results concerning the intrinsic complexity of the functionals that may help to guide this search. Basic and typical problems deal with the efficient computation or approximation of geometric functionals such as the volume or the diameter of a polytope, or with the algorithmic reconstruction of a polytope from data concerning it, or with algorithmic versions of geometric theorems.

The name *Computational Convexity* is of recent origin, having first appeared in print in 1989 (see [GrK89]). However, results that retrospectively belong to this area go back a long way. In particular, many of the basic ideas of *Linear Programming* have an essentially geometric character and fit very well into our conception of Computational Convexity. The same is true of the subject of *Polyhedral Combinatorics* and of the *Algorithmic Theory of Polytopes and Convex Bodies*; see [GrK94a] for a brief survey of these areas and a list of references to related research papers, survey articles and books.

As opposed to the area that has come to be called *Computational Geometry* ([PrS85], [Ed87], [Ya90], [Ed93a], [Mu94], [Or94a]), the emphasis in Computational Convexity is on problems whose underlying structure is convexity in normed vector spaces of finite but generally *not* restricted dimension, rather than of fixed (low) dimension. This leads to much closer connections with the optimization problems that arise in a wide variety of disciplines.

In the study of Computational Convexity, the underlying model of computation is mainly the binary (Turing machine) model that is common in studies of computational complexity. This requirement is imposed by prospective applications, particularly in mathematical programming. For the study of algorithmic aspects of convex bodies the binary model is often augmented by additional devices called "oracles"; see Subsection 1.2 and [GrK94a]. Some cases of interest do involve other underlying models of computation, but the present paper focuses on aspects of computational convexity for which binary models seem most natural.

1.2 Presentations of polytopes and general convex bodies.

The setting for everything in this paper is a finite-dimensional real vector space $\mathbb{R}^n$. In the present context, $\mathbb{R}^n$ may be assumed to carry the usual Euclidean norm, thus forming the Euclidean n-space $\mathbb{E}^n = (\mathbb{R}^n, \|\ \ \|_2)$.

As the terms are used here, a *body* in $\mathbb{R}^n$ is a compact convex set and a *polytope* is a body that has only finitely many extreme points. (These objects are usually called *convex bodies* and *convex polytopes*, but we often omit the adjectives in the interest of brevity.) We use the symbols $\mathcal{K}^n$ and $\mathcal{P}^n$ to denote respectively the family of all bodies in $\mathbb{R}^n$ and the family of all polytopes in $\mathbb{R}^n$.

A body or a polytope in $\mathbb{R}^n$ is *proper* if it is n-dimensional and hence has nonempty interior. From an algorithmic point of view, polytopes are dealt with much more easily than general bodies, because polytopes can be presented in a finite manner. However, even for a polytope P the precise manner of presentation must be specified, and the difficulty of answering basic questions about P can be greatly influenced by the manner of presentation.

A $\mathcal{V}$*-presentation* of a polytope P consists of positive integers n and m, and m points $v_1, \dots, v_m$ in $\mathbb{R}^n$ such that $P = \mathrm{conv}\{v_1, \dots, v_m\}$. An $\mathcal{H}$*-presentation* of a polytope P consists of integers n and m with $m > n \geq 1$, a real $m \times n$ matrix A, and a vector $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

A *face* of a polytope P is P itself, the empty set, or the intersection of P with some supporting hyperplane. Faces of dimension i are called i*-faces* (with the convention that

$\dim(\emptyset) = -1$). The 0-faces, 1-faces, and $(n-1)$-faces of an n-polytope P are respectively its *vertices*, *edges*, and *facets*. For $i = -1, 0, \ldots, n$, $\mathcal{F}_i(P)$ denotes the set of all i-faces of P and $f_i(P) = \mathrm{card}(\mathcal{F}_i(P))$, the number of i-faces.

A $\mathcal{V}$- $\langle\mathcal{H}$-$\rangle$ presentation of P is *irredundant* if the omission of any of the points $v_1, \ldots, v_m$ $\langle$any of the inequalities in $Ax \le b\rangle$ changes the polytope, reducing it in the first case and enlarging it in the second. In geometric terms, a $\mathcal{V}$-presentation is irredundant if each point v_i is a vertex of P, and when P is n-dimensional, an $\mathcal{H}$-presentation is irredundant if each inequality induces a facet of P.

Each polytope $P \subset \mathbb{R}^n$ admits a $\mathcal{V}$-presentation and also admits an $\mathcal{H}$-presentation, and we refer to [Dy83], [Sw85], [Se87], [AvF91], and [ChHJ92] for algorithms that convert one sort of presentation into the other. However, since P may have many more vertices than facets, and vice-versa [Mc70] (see also [Gr67], [McS71], [Br83]), it can happen that the minimum size of one sort of presentation is much larger than the minimum size of the other sort.

Our discussion here is based mainly on the *binary* or *Turing machine* model of computation [GaJ79], in which the *size of the input* is defined as the length of the binary encoding needed to present the input data to a Turing machine and the *time-complexity* of an algorithm is also defined in terms of the operations of a Turing machine. For this model, each computation involving polytopes begins with a *rational* $\mathcal{V}$- or $\mathcal{H}$-presentation of a *rational* polytope P. The presentation's being rational means that $v_1, \ldots, v_m \in \mathbb{Q}^n$ if P is $\mathcal{V}$-presented, or that the matrix A has rational entries and $b \in \mathbb{Q}^m$ if P is $\mathcal{H}$-presented; integer presentations are defined in a similar way. The *(binary) size* of a rational $\mathcal{V}$- or $\mathcal{H}$-presentation – usually denoted by L (for "length") – is the number of binary digits needed to encode the data of the presentation; see e.g. [GrLS88].

As was mentioned earlier, our main emphasis is on questions involving the dimension n as part of the input. However, we will also mention corresponding results for fixed dimensions. As a notational convention, we restrict the use of L to the case of varying dimension; further, whenever the dimension is regarded as fixed, we explicitly say so. Thus when no assumption on the dimension is stated, the dimension n is always regarded as part of the input.

For algorithmic purposes it is usually not the rational polytope P as a *geometric* object that is relevant, but rather its *algebraic presentation.* We will speak of a *$\mathcal{V}$-polytope* P or of an *$\mathcal{H}$-polytope* P when a specific rational $\mathcal{V}$-presentation $(n, m; v_1, \ldots, v_m)$ or a specific rational $\mathcal{H}$-presentation $(n, m; A, b)$ is given. For most of the problems discussed here the focus is on polynomial-time computability or on various hardness results, and hence we may assume without loss of generality that presentations are irredundant. That is because, for a given $\mathcal{V}$- (or $\mathcal{H}$-) polytope P, linear programming can be used to produce, in polynomial time, an irredundant $\mathcal{V}$- (or $\mathcal{H}$-) presentation.

In order to further illuminate the algorithmic differences between $\mathcal{V}$- and $\mathcal{H}$-polytopes and the difficulties that may be expected in attempting to transfer an algorithmic approach from one sort of presentation to the other, we mention that Linial [Li86] has established the $\#\mathbb{P}$-completeness (a strong measure of difficulty) of each of the following two problems:

> Given a positive integer n and an n-dimensional $\mathcal{V}$-polytope P, determine the number of facets of P.
>
> Given a positive integer n and an n-dimensional $\mathcal{H}$-polytope P, determine the number of vertices of P.

Among important special classes of polytopes, the zonotopes are particularly interesting because they can be so compactly presented. A *zonotope* is the vector sum (Minkowski sum) of a finite number of line segments. When the segments are $S_1, \dots, S_m$ and their centers are $c_1, \dots, c_m$, we have

$$\sum_{i=1}^{m} S_i \;=\; \left(\sum_{i=1}^{m} c_i\right) \;+\; \sum_{i=1}^{m}(S_i - c_i)$$

where the point $\sum_{i=1}^{m} c_i$ is the center of symmetry of the set $\sum_{i=1}^{m} S_i$ and each segment $S_i - c_i$ is centered at the origin. Hence it is convenient to define an *$\mathcal{S}$-presentation* of a zonotope in $\mathbb{R}^n$ as a sequence $(c; z_1, \dots, z_m)$ of points in $\mathbb{R}^n$, where c is the center and the z_i are the ends of segments centered at the origin 0, with one end listed for each segment. This sequence represents the zonotope

$$c + \sum_{i=1}^{m}[-1,1]z_i \;=\; c + \left\{\sum_{i=1}^{m} \lambda_i z_i : |\lambda_i| \leq 1 \text{ for all } i\right\}.$$

This zonotope is a polytope of dimension at most $\min\{m, n\}$, and it is a proper polytope if and only if there are n linearly independent points among the z_i. We speak of *integer* and *rational* $\mathcal{S}$-presentations, and define their sizes in the natural way. Sometimes we will also work with zonotopes whose relationship to the origin (and whose scaling) is different. Specifically, zontopes of the form $\sum_{i=1}^{m}[0,1]z_i$ are used quite frequently. To keep the notation simple, we refrain, however, from introducing an additional name for such a presentation.

Again, in our algorithmic model it is usually not the rational zonotope Z as a geometric object that is relevant, but rather its presentation; hence we speak of an *$\mathcal{S}$-zonotope Z* when a specific rational $\mathcal{S}$-presentation $(n, m; c; z_1, \dots, z_m)$ is given. Although each zonotope is a polytope, in general neither the vertices nor the facets of a zonotope are readily accessible from an $\mathcal{S}$-presentation. In fact, for zonotopes generated by m segments in general position, both the number of facets and the number of vertices grow exponentially as m increases.

A zonotope $Z = c + \sum_{i=1}^{m}[-1,1]z_i$ is called a *parallelotope* when the points $z_1, \dots, z_m$ are linearly independent. In contrast to the case of general zonotopes, the facial structure of a parallelotope is immediately accessible from an $\mathcal{S}$-presentation: The passage between a rational $\mathcal{S}$-presentation of a parallelotope P and a rational $\mathcal{H}$-presentation of P can be accomplished in polynomial time.

A different approach is required to deal with bodies K that are not polytopes, since an enumeration of all the extreme points of K or of K's polar is not possible. Sometimes K can be described explicitly in terms of an easily computable function. An example of this kind is the Euclidean unit ball $\mathbb{B}^n$ given in the form $\mathbb{B}^n = \{(\xi_1, \dots, \xi_n)^T : \xi_1^2 + \cdots + \xi_n^2 \leq 1\}$. However, such a description is often not available. A convenient way to deal with the general situation is to assume that the body in question is given by an algorithm (called an *oracle*) that answers certain sorts of questions about the body. All information about the specific body must be obtained from the oracle, which functions as a "black box." In other words, while it is assumed that the oracle's answers are always correct, nothing is assumed about the manner in which it produces those answers. This oracular approach has been extensively studied and utilized for combinatorial optimization problems by GRÖTSCHEL, LOVÁSZ & SCHRIJVER [GrLS81], [GrLS88]. In order to describe some oracles that have figured prominently in their work, let us recall that for $\epsilon \geq 0$ the *outer parallel body* and the *inner parallel body* of a convex body K are given respectively by

$$K(\epsilon) = K + \epsilon\mathbb{B}^n = \bigcup_{b\in\epsilon\mathbb{B}^n} (K + b) \quad \text{and} \quad K(-\epsilon) = K \setminus \bigcup_{b\in\epsilon\mathbb{B}^n} ((\mathbb{R}^n \setminus K) + b).$$

The three most important oracles of [GrLS88] are the ones that solve the following problems for proper bodies K.

WEAK MEMBERSHIP PROBLEM. *Given $K \in \mathcal{K}^n$, $y \in \mathbb{Q}^n$, and a rational number $\epsilon > 0$, conclude with one of the following:*

assert that $y \in K(\epsilon)$;
assert that $y \notin K(-\epsilon)$.

WEAK SEPARATION PROBLEM. *Given $K \in \mathcal{K}^n$, $y \in \mathbb{Q}^n$, and a rational number $\epsilon > 0$, conclude with one of the following:*

assert that $y \in K(\epsilon)$;
find a vector $z \in \mathbb{Q}^n$ such that $\| z \|_\infty = 1$ and $z^T x < z^T y + \epsilon$ for every $x \in K(-\epsilon)$.

WEAK (LINEAR) OPTIMIZATION PROBLEM. *Given $K \in \mathcal{K}^n$, a vector $c \in \mathbb{Q}^n$, and a rational number $\epsilon > 0$, conclude with one of the following:*

find a vector $y \in \mathbb{Q}^n \cap K(\epsilon)$ such that $c^T x \le c^T y + \epsilon$ for every $x \in K(-\epsilon)$;
assert that $K(-\epsilon) = \emptyset$.

If a proper body K is given by an algorithm that solves the weak membership problem, the weak separation problem, or the weak linear optimization problem, we say that K is described by a *weak membership oracle*, a *weak separation oracle* or a *weak (linear) optimization oracle*. The oracle is called *strong* if it solves the corresponding strong problem that is obtained by setting $\epsilon = 0$. A body K is called *circumscribed* or *well-bounded* if a positive rational number R or positive rational numbers r, R are given explicitly such that $K \subset R\mathbb{B}^n$ or, in addition, K contains a ball of radius r. If, further, we are given a vector $b \in \mathbb{Q}^n$ such that $b + r\mathbb{B}^n \subset K$, then K is called *centered.*

To place the weak linear optimization oracle in the perspective of classical convexity theory, recall Minkowski's useful functional approach to convex bodies by means of the *support function* $h : \mathcal{K}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$, which is defined for $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1} = \operatorname{bd}\mathbb{B}^n$ by

$$h(K, u) = \max_{x\in K}\langle x, u\rangle.$$

Note that presenting a $c \in \mathbb{Q}^n$ as a call to a weak optimization oracle for K provides us with an approximation of $h(K, c/\|c\|_2)$, and, in addition, with a "weak support point" in this direction.

The above three problems are very closely related in the sense that when the classes of proper bodies are appropriately restricted to those that are circumscribed, well-bounded, or centered, and when input sizes are properly defined, an algorithm that solves any one of the problems in polynomial time can be used as a subroutine to solve the others in polynomial time also; see Theorem 1.2.1. The definition of input size involves the size of ϵ, the dimension of K, the given a priori information, and the input required by the oracle. Suppose that K

is well-bounded with parameters r and R. Then the *input size* is defined as the sum of the following numbers:

size(ϵ);
size(K) $= n +$ size(r) $+$ size(R);
an additional term size(b) when the oracle is centered with center b;
an additional term size(y) for the membership problem and the separation problem, and size(c) for the optimization problem.

The following theorem contains a list of the precise relationships among the three basic oracles for bodies (see [GrLS88]). The notation "$(\mathcal{A}; prop) \rightarrow_\pi \mathcal{B}$" indicates the existence of an (oracle-) polynomial-time algorithm that solves problem $\mathcal{B}$ for every body that is given by the oracle $\mathcal{A}$ and has all the properties specified in *prop*. (*prop*$= \emptyset$ means that the statement holds for general bodies.)

1.2.1 (Weak Membership; centered, well-bounded) $\rightarrow_\pi$ Weak Separation;
(Weak Membership; centered, well-bounded) $\rightarrow_\pi$ Weak Optimization;
(Weak Separation; $\emptyset$) $\rightarrow_\pi$ Weak Membership;
(Weak Separation; circumscribed) $\rightarrow_\pi$ Weak Optimization;
(Weak Optimization; $\emptyset$) $\rightarrow_\pi$ Weak Membership;
(Weak Optimization; $\emptyset$) $\rightarrow_\pi$ Weak Separation.

We want to emphasize the following fact, for it implies that the "oracular" approach to convex bodies is in an important sense the most general sort of presentation introduced in this subsection. It also helps to clarify the way in which the formulation in terms of oracles leads to an efficient modular approach to the problems of computational convexity.

1.2.2 *There are polynomial-time algorithms which, accepting as input a proper $\mathcal{V}$-polytope or a proper $\mathcal{H}$-polytope P, or a proper $\mathcal{S}$-zonotope Z, produce membership, separation and optimization oracles for P and Z, and also compute lower bounds on P's and Z's inradius, upper bounds on P's and Z's circumradius, and "centers" b_P and b_Z for P and Z respectively.*

This implies that if an algorithm performs certain tasks for bodies given by some of the above (appropriately specified) oracles, then the same algorithm can also serve as a basis for procedures that perform these tasks for $\mathcal{V}$- or $\mathcal{H}$-polytopes and for $\mathcal{S}$-zonotopes. (On the other hand, it is possible to derive some lower bounds on the performance of approximate algorithms for the oracle model that do not carry over to the case of $\mathcal{V}$- or $\mathcal{H}$-polytopes or $\mathcal{S}$-zonotopes. Examples of this kind can be found in Subsection 6.3.)

Let us end this section by stating some basic algorithmic problems of volume computation. Other variants of these problems (including those asking for weak approximations) will be introduced later. Here are the problems.

Volume Computation.

Instance: *A positive integer n, an $\mathcal{H}$-polytope $\langle$or a $\mathcal{V}$-polytope, or an $\mathcal{S}$-zonotope$\rangle$ P.*
Task: *Compute the volume $V(P)$ of P.*

Weak Volume Computation.

Instance: *A positive integer n, a body K in $\mathbb{R}^n$ that is given by a weak optimization oracle $\langle$or a weak membership oracle or a weak separation oracle; a rational* *vector* $b \in \mathbb{R}^n$ *and positive rationals* r, R *such that* $b + r\mathbb{B}^n \subset K \subset R\mathbb{B}^n\rangle$; *a positive rational* λ.

Task: *Compute a rational* μ *such that* $|\mu - V(K)| \leq \lambda$.

It should be emphasized that this survey concentrates on providing some idea of the principal methods that are available for computing or approximating volumes and mixed volumes, and sketches (or, much less frequently, details) of proofs are given only for the purpose of enhancing the intuitive understanding of these underlying ideas and concepts. To further research in this fruitful area of computational convexity we have formulated unsolved problems that seem especially natural or important, and in some cases of particular interest we have even included some "speculative" material, speculative in the sense that we show how certain procedures (which may not be available at present) could in principle be used to solve certain other problems efficiently. Finally, it should be mentioned that much of this survey is "qualitative" in the sense that the primary distinction in computational complexity is that between polynomial-time solvability on the one hand and $\mathbb{NP}$-hardness or $\#\mathbb{P}$-hardness on the other hand. We recognize that this classification is only a first step toward finding optimal algorithms, but we believe it to be a useful guide for the latter effort.

2. Foundations

In the present section, Subsection 2.1 fixes some terminology and Subsections 2.3–2.5 discuss the aspects of volume and mixed volumes that are most relevant to what follows. Subsection 2.2 represents a deviation (but a fascinating one) from our main line of discussion. Much of the material in 2.2 has been treated in books by Boltyanskii [Bo78] and Wagon [Wa85]. The material in 2.3–2.5 has been the subject of various books and survey articles, including the book by Hadwiger [Ha57], the survey by McMullen & Schneider [McS83], and a recent handbook article by McMullen [Mc93]. For this reason, and also because the present article is concerned mainly with algorithmic aspects, we will be rather brief in this section. More details can be found in the cited references.

2.1 Background and terminology.

We could begin by simply noting that convex bodies are Lebesgue measurable, and that our term *volume* is synonymous to *Lebesgue measure*. However, when restricted to bodies and especially when restricted to polytopes, Lebesgue measure exhibits many properties that are of fundamental geometric significance. Further, these properties can in some cases be formulated in an "elementary" way – i.e., without recourse to limiting processes – and it turns out that some of the notation and terminology needed to describe the properties is also useful for algorithmic studies.

We speak of a *dissection* of an n-polytope P into n-polytopes $P_1, \ldots, P_k$ if

$$P = \bigcup_{i=1}^{k} P_i$$

$$\operatorname{int}(P_i \cap P_j) = \emptyset \qquad \text{for } i, j = 1, \ldots, k,\ i \neq j.$$

With respect to a subgroup G of the group of all affine automorphisms of $\mathbb{R}^n$, two polytopes $P, Q \subset \mathbb{R}^n$ are said to be *G-equidissectable* (or *equidissectable under* G) if (for some k) there exist dissections $P_1, \ldots, P_k$ of P and $Q_1, \ldots, Q_k$ of Q and elements $g_1, \ldots, g_k$ of G such that $P_i = g_i(Q_i)$ for all i.

A related notion is that of equicomplementability. Two polytopes $P, Q \subset \mathbb{R}^n$ are called *G-equicomplementable* if there are polytopes P_1, P_2 and Q_1, Q_2 such that P_2 is dissected into P and P_1, Q_2 is dissected into Q and Q_1, P_1 and Q_1 are G-equidissectable, and P_2 and Q_2 are G-equidissectable. HADWIGER [Ha57, p.47] showed that two polytopes are G-equidissectable if and only if they are G-equicomplementable.

Let $\mathcal{S}^n$ be a family of subsets of $\mathbb{R}^n$. A functional $\varphi : \mathcal{S}^n \to \mathbb{R}$ is called a *valuation* on $\mathcal{S}^n$ if

$$\varphi(S_1) + \varphi(S_2) = \varphi(S_1 \cup S_2) + \varphi(S_1 \cap S_2) \qquad \text{whenever } S_1, S_2, S_1 \cup S_2, S_1 \cap S_2 \in \mathcal{S}^n.$$

The families of principal interest to us here are $\mathcal{P}^n$ and $\mathcal{K}^n$. A valuation φ is called *G-invariant* if

$$\varphi(S) = \varphi(g(S)) \text{ for all } S \in \mathcal{S}^n \text{ and } g \in G,$$

simple if

$$\varphi(S) = 0 \text{ whenever } S \in \mathcal{S}^n \text{ and } S \text{ is contained in a hyperplane,}$$

and *monotone* if

$$\varphi(S_1) \leq \varphi(S_2) \text{ whenever } S_1, S_2 \in \mathcal{S}^n \text{ with } S_1 \subset S_2.$$

Obviously, if φ is a G-invariant simple valuation on $\mathcal{P}^n$ and P and Q are G-equidissectable (or G-equicomplementable) then $\varphi(P) = \varphi(Q)$. HADWIGER [Ha57] showed that this leads already to a characterization of G-equidissectability of polytopes.

2.1.1 *Two polytopes P and Q are G-equidissectable if and only if $\varphi(P) = \varphi(Q)$ for all G-invariant simple valuations on $\mathcal{P}^n$.*

2.2 "Elementary" approaches to volume.

The present subsection states some results on isometry-based elementary approaches to volume and contrasts them later with a result on an affinity-based approach.

The most famous result concerning equidissectability involves the group D of isometries. It is the following *Bolyai-Gerwien theorem* (see [Ge1833], [Bo78], [Wa85]).

2.2.1 *Two plane polygons are of equal area if and only if they are D-equidissectable.*

If one agrees that an a-by-b rectangle should have area ab, and also agrees that the area function should be a D-invariant simple valuation, it then follows from 2.2.1 that the area of any plane polygon P can be determined (at least in theory) by finding a rectangle R to which P is equidissectable. This provides a satisfyingly geometric theory of area that does not require any limiting considerations. Several refinements of the Bolyai-Gerwien theorem have been established. For example, rather than using the group of all isometries, it suffices to use translations and half-turns. Also, the pairs (P,Q) of polygons that are equidissectable under translations alone have been characterized by HADWIGER & GLUR [HaG51]. (See [Bo78] for this and other refinements.) Although the original proof of 2.2.1 was algorithmic in nature, there remain open questions concerning how rapidly, under various hypotheses, one can find an equidissection of two given polygons of equal area, and there are also open questions concerning the minimum number of pieces needed in certain equidissections. See [KoK94] for some of the algorithmic aspects, and see [Mo91, p.215] for a problem concerning minimum dissections.

The third problem of HILBERT [Hi00] asked, in effect, whether the Bolyai-Gerwien result extends to 3-polytopes. A negative answer was supplied by DEHN [De00], who showed that a regular tetrahedron and a cube are not equidissectable. His work led to the notion of a *Dehn invariant* of a 3-polytope P. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary additive function such that $f(\pi) = 0$ but f is not identically zero. (This implies that f is discontinuous.) For each such f, and for each 3-polytope P, let

$$f^*(P) = \sum_{i=1}^{k} \sigma_i f(\alpha_i),$$

where $\sigma_1, \ldots, \sigma_k$ are the lengths of the various edges of P and $\alpha_1, \ldots, \alpha_k$ are the radian measures of the corresponding dihedral angles. Then the number $f^*(P)$ is known as the *Dehn invariant* of P associated with the functional f. In the following result, the necessity was proved by DEHN [De00] and the sufficiency by SYDLER [Sy65] 65 years later.

2.2.2 *For two 3-polytopes P and Q to be equidissectable under the group of all isometries of $\mathbb{R}^3$, it is necessary and sufficient that $f^*(P) = f^*(Q)$ for each additive function f such that $f(\pi) = 0$.*

Dehn's necessary conditions (for equidissectability of proper polytopes in 3-space) were extended to n-space by HADWIGER (see [Bo78] and [Sa79] for references), and the sufficiency of the extended conditions was proved by JESSEN [Je68], [Je72] when $n = 4$. However, the case of $n \geq 5$ is still unsettled. See [Bo78] and [Sa79] for expositions of Jessen's proof, and see SAH [Sa79] for an account of algebraic studies that have been inspired by Hilbert's third problem.

A notion related to G-equidissectability is that of *G-equidecomposability*, where a *decomposition* of a set X is a way of expressing X as a union $X_1 \cup \cdots \cup X_k$ of a finite number of pairwise disjoint sets X_i. In contrast to the notion of a dissection, these sets X_i are not even permitted to have boundary points in common, and there is no restriction on the nature of the individual sets (they may even be nonmeasurable). Hence the study of equidecomposability is far from our algorithmic approach. Nevertheless, we feel that its principal results should at least be mentioned here because they are the most striking of all results related to volume.

Even though equidecomposability does not require measurability of the sets in the decomposition, in the following result these sets may be taken as open triangles together with nice portions of their boundaries.

2.2.3 *Two plane polygons are of equal area if and only if they are D-equidecomposable.*

Theorem 2.2.3 is due to TARSKI (see [BaT24]), and it led to the question as to which pairs of nonpolygonal plane bodies are D-equidecomposable. Although the sets in a decomposition need not be measurable, equidecomposability of two plane bodies does imply that the bodies are of equal area. That is a consequence of the following result.

2.2.4 *Lebesgue measure on the line or in the plane can be extended to a D-invariant simple monotone valuation μ that is defined and finite for all bounded sets.*

In fact, μ can also be required to multiply properly with respect to all similarity transformations of $\mathbb{R}^2$. See [Wa85] for references to proofs of 2.2.4, and for a discussion of further ramifications of the theorem.

LACZKOVICH [La90] sharpened 2.2.3 as follows.

2.2.5 *Any two plane polygons of equal area are equidecomposable under the group of translations.*

He also settled Tarski's old problem of "squaring the circle" by showing that a square and a circular disk of equal area are equidecomposable. There too, he needed only translations. (See [GaW89] for an expositon of the methods and theorems of [La90].) On the other hand, a theorem of DUBINS, HIRSCH & KARUSH [DuHK63] implies that a disk and a square cannot be "scissors congruent"; i.e., there is no equidissection (with respect to rigid motions) into pieces which, roughly speaking, could be cut out with a pair of scissors.

The relationship of equidecomposability to volume in $\mathbb{R}^n$ changes dramatically with the passage from $n = 2$ to $n = 3$. That is clear from the following result, which is known as the *Banach-Tarski paradox* [BaT24].

2.2.6 *If X and Y are bounded subsets of $\mathbb{R}^n$ (with $n \geq 3$), and each set has nonempty interior, then X and Y are D-equidecomposable.*

The essential difference between the cases $n \leq 2$ and $n \geq 3$ lies in the fact that for $n \leq 2$ the group of all isometries of $\mathbb{R}^n$ is *solvable* (a condition of near-commutativity), while for $n \geq 3$ it contains a free nonabelian subgroup and hence is not solvable. Thus, for example, while it is clear from 2.2.6 that 2.2.4 does not (as stated) extend to $\mathbb{R}^n$, it does extend when the group D of all isometries is replaced by the group of translations. See WAGON [Wa85] for a survey of the Banach-Tarski paradox and of several other results and problems related to the results stated in this subsection; see also [St79].

We see from 2.2.1 that an "elementary" theory of the area of plane polygons (i.e., a theory free of limiting processes) can be based on isometries and equidissectability, and we see from 2.2.3 that such a theory can also be based on isometries and equidecomposability. When the underlying group is the group D of isometries, both of these statements fail in $\mathbb{R}^n$ for each $n \geq 3$. Equidecomposability fails because (by 2.2.6) the associated equivalence class is far too large, having no connection with equality of volume. Equidissectability fails because, although this condition implies equality of volume, proper polytopes of equal volume can fail to be equidissectable (see 2.2.2). A pleasant contrast to these difficulties is provided by the following result (see [Mc93, p.966]), which is valid for all n. It is based on volume-preserving affinities rather than on isometries.

2.2.7 *Under the group of all volume-preserving affinites of $\mathbb{R}^n$, two n-polytopes are equidissectable if and only if they are of equal volume.*

For this result, as for 2.2.1, it seems that little is known about minimizing the number of pieces in an equidissection or about the computational complexity of finding an equidissection.

2.3 Characterizations of the volume.

The volume function can be characterized as follows in terms of valuations.

2.3.1 *If φ is a translation-invariant, nonnegative simple valuation on $\mathcal{P}^n$ $\langle \mathcal{K}^n \rangle$, then there exists a nonnegative real α such that $\varphi = \alpha V$.*

2.3.2 *A translation-invariant valuation on $\mathcal{P}^n$ which is homogeneous of degree n is a constant multiple of the volume.*

2.3.3 *A continuous rigid-motion-invariant simple valuation on $\mathcal{K}^n$ is a constant multiple of the volume.*

2.3.4 *A nonnegative simple valuation on $\mathcal{P}^n$ $\langle \mathcal{K}^n \rangle$ which is invariant under all volume-preserving linear maps of $\mathbb{R}^n$ is a constant multiple of the volume.*

Proofs of 2.3.1, 2.3.2 and 2.3.3 can be found in the book of HADWIGER [Ha57], in Section 2.1.3 and on pages 79 and 221. Theorem 2.3.4 is also due to HADWIGER [Ha70]. See, in addition, the surveys [McS83] and [Mc93].

It is unknown whether, in 2.3.3, $\mathcal{K}^n$ can be replaced by $\mathcal{P}^n$.

2.4 Mixed volumes.

The study of mixed volumes, the *Brunn-Minkowski theory*, forms the backbone of classical convexity theory and is also useful for applications in various other areas including combinatorics and algebraic geometry (see Section 9). SCHNEIDER [Sc93] has an excellent treatment of the theory that includes proofs for all the results presented in this section.

The starting point for the Brunn-Minkowski theory is the following famous theorem of MINKOWSKI [Mi11] (see [BoF34], [Sa93], [Sc93]):

2.4.1 *Let $K_1, \ldots, K_r$ be convex bodies in $\mathbb{R}^n$, and let $\xi_1, \ldots, \xi_r$ be nonnegative reals. Then the function $V\big(\sum_{i=1}^r \xi_i K_i\big)$ is a homogeneous polynomial of degree n in the variables $\xi_1, \ldots, \xi_r$, and can be written in the form*

$$V\left(\sum_{i=1}^{r} \xi_i K_i\right) = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \cdots \sum_{i_n=1}^{r} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n} V(K_{i_1}, K_{i_2}, \ldots, K_{i_n}),$$

where the coefficients $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ are order-independent, i.e. invariant under permutations of their argument.

The coefficient $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$ is called the *mixed volume* of $K_{i_1}, K_{i_2}, \ldots, K_{i_n}$. We will also use the term *mixed volume* for the functional

$$\begin{aligned} V : \overbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}^{n} &\to \mathbb{R} \\ K_1, \ldots, K_n &\mapsto V(K_1, \ldots, K_n) \end{aligned}$$

as well as for restrictions of this functional to certain subsets of $\mathcal{K}^n \times \cdots \times \mathcal{K}^n$.

Here are some of the most important properties of mixed volumes.

2.4.2 *Let $K_1, \dots, K_n \in \mathcal{K}^n$. Then the mixed volumes have the following properties.*

(i) *Mixed volumes are nonnegative, monotone, multilinear, and continuous valuations.*

(ii) $V(K_1) = V(\overbrace{K_1, \dots, K_1}^{n})$.

(iii) *If A is an affine transformation, then*

$$V(A(K_1), \dots, A(K_n)) = |\det(A)| V(K_1, \dots, K_n).$$

Property 2.4.2 (ii) shows that mixed volumes directly generalize the ordinary volume. This implies that in general, computing mixed volumes of polytopes is no easier than computing volumes of polytopes.

The multilinearity of the mixed volume is important for certain algorithmic approaches outlined in Subsection 7.2. It says that mixed volumes can again be expanded into mixed volumes, or, more explicitly, that for $C_1, \dots, C_r \in \mathcal{K}^n$, $\xi_1, \dots, \xi_r \in [0, \infty[$, $K = \sum_{i=1}^r \xi_i C_i$ and $k \in \mathbb{N}$, $k \leq n-1$,

$$V(\overbrace{K, \dots, K}^{k}, K_{k+1}, \dots, K_n) = \\ = \sum_{i_1=1}^r \sum_{i_2=1}^r \cdots \sum_{i_k=1}^r \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k} V(C_{i_1}, \dots, C_{i_k}, K_{k+1}, \dots, K_n).$$

One of the most famous inequalities in convexity theory is the following, known as the *Aleksandrov-Fenchel inequality*, [Al37], [Al38], [Fe36].

2.4.3 *For $K_1, \dots, K_n \in \mathcal{K}^n$, it is true that*

$$V(K_1, K_2, K_3, \dots, K_n)^2 \geq V(K_1, K_1, K_3, \dots, K_n)\ V(K_2, K_2, K_3, \dots, K_n).$$

The cases for which equality holds in 2.4.3 have not been fully characterized; see [Sa93], [Sc93].

A famous consequence of Theorem 2.4.3 is the following *Brunn-Minkowski theorem* (see [Sa93], [Sc93]):

2.4.4 *If $K_0, K_1 \in \mathcal{K}^n$ and $\lambda \in [0, 1]$, then*

$$V^{\frac{1}{n}}((1-\lambda)K_0 + \lambda K_1) \geq (1-\lambda) V^{\frac{1}{n}}(K_0) + \lambda V^{\frac{1}{n}}(K_1).$$

Let us close this subsection by introducing the *quermassintegrals* and the *intrinsic volumes* of a body K.

The quermassintegrals $W_0, \dots, W_n$ are defined on $\mathcal{K}^n$ by

$$W_i(K) = V(\overbrace{K, \dots, K}^{n-i}, \overbrace{\mathbb{B}^n, \dots, \mathbb{B}^n}^{i}),$$

and the intrinsic volumes $V_0, \dots, V_n$, are given by

$$\kappa_{n-i} V_i = \binom{n}{i} W_{n-i},$$

where κ_k denotes the k-volume of $\mathbb{B}^k$. (See [Ha57], [Mc75], [Mc77], [Mc93], [Sa93], [Sc93]). In particular, V_n is the volume V, V_{n-1} is half the surface area, and $V_0 = 1$. Note that the intrinsic volumes are *dimension-invariant* in the sense that $V_i(K)$ depends only on the body K and not on the dimension of the space in which K is embedded. This implies, in particular, that for an i-dimensional body K the ith intrinsic volume $V_i(K)$ is just the i-volume of K.

Note that for $\xi \geq 0$, the body $K + \xi\mathbb{B}^n$ is the outer parallel body (already introduced in Subsection 1.2) and the mixed volume expansion becomes the *Steiner-formula* [St1840]:

$$V(K + \xi\mathbb{B}^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\xi^i = \sum_{i=0}^{n} \kappa_{n-i}V_i(K)\xi^{n-i}.$$

2.5 Characterizations of mixed volumes.

The following characterization is due to FIREY [Fi76] for $k = 1$ and to MCMULLEN [Mc90] for $k = n - 1$.

2.5.1 *Let $k = 1$ or $k = n - 1$. If φ is a monotone translation-invariant valuation on $\mathcal{K}^n$ and is homogeneous of degree k, then there exist bodies $K_{k+1}, \ldots, K_n$ such that*

$$\varphi(K) = V(\overbrace{K, \ldots, K}^{k}, K_{k+1}, \ldots, K_n).$$

The problem of extending 2.5.1 to general k is wide open, and examples of GOODEY (private communication) show that the general situation is more complicated than that for $k = 1$ and $k = n - 1$. For instance, let X and Y be orthogonal 2-spaces in $\mathbb{E}^4$ and for each $K \in \mathcal{K}^n$ set $\varphi(K) = V_2(\Pi_X(K)) + V_2(\Pi_Y(K))$, where Π_X and Π_Y denote the orthogonal projections onto X and Y respectively. The valuation φ is monotone, translation-invariant and homogeneous of degree 2, but it cannot be expressed as a mixed volume. Goodey gives similar examples in arbitrary dimensions.

Let us close this section with the famous characterization theorems of HADWIGER [Ha57, Section 6.1.10], showing that the *quermassintegrals* or *intrinsic volumes* form a basis for a certain space of valuations. (Recall that the intrinsic volume V_i is a continuous valuation, invariant under rigid motions and homogeneous of degree i.)

2.5.2 *If φ is a continuous valuation that is invariant under rigid motions, then there are constants $\alpha_0, \ldots, \alpha_n$ such that*

$$\varphi(K) = \sum_{i=0}^{n} \alpha_i V_i(K) \qquad \textit{for all } K \in \mathcal{K}^n.$$

2.5.3 *If φ is a monotone valuation that is invariant under rigid motions, then there are nonnegative constants $\alpha_0, \ldots, \alpha_n$ such that*

$$\varphi(K) = \sum_{i=0}^{n} \alpha_i V_i(K) \qquad \textit{for all } K \in \mathcal{K}^n.$$

The following example indicates the manner in which the characterization results 2.5.2 and 2.5.3 can be used to identify certain functionals as intrinsic volumes. For a body K in $\mathbb{R}^n$, and for $u \in \mathbb{S}^{n-1}$, the *breadth* of K in the direction u is defined as the number

$$b_u(K) = \max_{y \in K} \langle u, y \rangle - \min_{y \in K} \langle u, y \rangle;$$

this is just the distance between the two supporting hyperplanes of K that are orthogonal to the line $\mathbb{R}u$. The *width* of K is the minimum of the numbers $b_u(K)$. The *mean width* of K is obtained by integrating the function $b_u(K)$ over $u \in \mathbb{S}^{n-1}$ and then dividing the result by the $(n-1)$-measure of $\mathbb{S}^{n-1}$. As a consequence of 2.5.3 we see that up to a positive factor the mean width of K is just $V_1(K)$.

3. Deterministic methods for volume computation

The problem of computing polytope volume has been studied by many authors. The present section will summarize several of the basic ideas for deterministic volume computation. In addition to the papers that are mentioned below in connection with the various methods, the reader may be interested in the following papers that are not mentioned below: [AlS86], [BaS79], [CoH79], [Ka94], [Ko82], [La83], [LeR82a], [LeR82b], [ShH54], [Sp86].

3.1 Triangulation.

If $v_0, \ldots, v_n$ are affinely independent points of $\mathbb{R}^n$, and $T = \mathrm{conv}\{v_0, \ldots, v_n\}$, then

$$V(T) = \frac{1}{n!} |\det(v_1 - v_0, \ldots, v_n - v_0)|.$$

Thus computing the volume of an n-simplex is equivalent to computing the determinant of an $n \times n$ matrix and can be handled very efficiently by means of *Gaussian elimination*. Other formulas for computing the volume of a simplex are stated in Subsection 3.6.

Since simplex volumes can be computed so easily, the most natural approach to the problem of computing the volume of a polytope P is to produce a dissection of P into simplices. Then compute the volumes of the individual simplices and add them up to find the volume of P. (This uses the fact that the volume is a simple valuation.)

In fact, we shall tell, for both $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes, how to produce a triangulation. As the term is used here, a *triangulation* is not merely a dissection into simplices, but it has the additional property that the intersection of any two simplices in the dissection is a face of each.

We will first outline an "incremental" algorithm that constructs a triangulation of a $\mathcal{V}$-polytope. The case of $\mathcal{H}$-polytopes is treated later in this subsection.

The problem of constructing triangulations of $\mathcal{V}$-polytopes is intimately related to the task of computing the *face-lattice* of the convex hull of a given finite point set, and this is a fundamental task in *computational geometry*; see Edelsbrunner [Ed87], [Ed93] and Chazelle [Ch93]. The *incremental* method, a paradigmatic procedure in computational geometry, uses the *beneath-beyond* approach that goes back to Grünbaum [Gr67, p.78].

The basic strategy is to add one of the given points at a time and hence compute the convex hull step by step. This requires the use of an ordering of the input vectors.

The following algorithm is based on a convex hull algorithm due to SEIDEL [Se91], and can also be found in [Ed87] and [Ed93].

Let $v_1, \ldots, v_m \in \mathbb{Q}^n$ be given, and suppose that

(i) $\operatorname{aff}\{v_1, \ldots, v_{n+1}\} = \mathbb{R}^n$, and

(ii) a rational vector $z_0 \in \mathbb{R}^n$ is given such that $\langle v_1, z_0\rangle < \langle v_2, z_0\rangle < \cdots < \langle v_{m-1}, z_0\rangle < \langle v_m, z_0\rangle$.

Then the incremental algorithm proceeds as follows:

- Let $P_{n+1} = \operatorname{conv}\{v_1, \ldots, v_{n+1}\}$, set $\mathcal{T}_{n+1} = \{P_{n+1}\}$, and assume that for some $i \geq n+1$ a triangulation $\mathcal{T}_i$ of the convex hull P_i of $\{v_1, \ldots, v_i\}$ has already been constructed.
- Let $\mathcal{B}_i$ denote the induced triangulation of the boundary $\operatorname{bd}(P_i)$. (Note that P_{n+1} is a simplex, whence $\mathcal{B}_{n+1} = \mathcal{F}_{n-1}(P_{n+1})$.)
- Let $\hat{\mathcal{B}}_i$ denote the set of all $(n-1)$-simplices in $\mathcal{B}_i$ that are visible from v_{i+1} with respect to P_i, i.e. the affine hull of a simplex in $\mathcal{B}_i$ separates v_{i+1} from P_i.
- Finally, set $P_{i+1} = \operatorname{conv}(\{v_{i+1}\} \cup P_i)$ and $\mathcal{T}_{i+1} = \mathcal{T}_i \cup \{\operatorname{conv}(\{v_{i+1}\} \cup F) : F \in \hat{\mathcal{B}}_i\}$.

Before mentioning some complexity issues of the main algorithm, let us comment on the assumptions (i) and (ii). From a theoretical point of view, none of these assumptions constitutes any restriction of generality. However, since we are here interested in *algorithmic* questions, we need efficient computational procedures for satisfying the assumptions in order to conclude that they are not too restrictive for our purposes.

Using Gaussian elimination, we can determine a maximal affinely independent subset of $\{v_1, \ldots, v_m\}$ in polynomial time, and (possibly after reordering) we may assume that it consists of the first k vectors. If $k < n+1$ we may terminate the algorithm (in view of the fact that we are here interested in triangulations only as a tool for volume computation), or we may decide to continue in $\operatorname{aff} V$. In any case, Assumption (i) is "algorithmically acceptable."

As to Assumption (ii), it is possible as follows to obtain such a hyperplane $H_0 = \{x : \langle x, z_0\rangle = 0\}$ through the origin with the property that no line determined by two of the vectors of V is parallel to H_0. For any pair (v_i, v_j) of different vectors of V, let $\delta_{ij} \in \{-1, 1\}$ and $y_{ij} = \delta_{ij}(v_i - v_j)$, where the choice of the sign δ_{ij} is such that the first nonzero coordinate of y_{ij} is positive. Then the positive hull of all such vectors y_{ij} is a pointed convex cone, and we can use linear programming to find (in polynomial time) a vector z_0 such that $\langle y_{ij}, z_0\rangle \geq 1$ for all such vectors y_{ij}. Clearly, Assumption (ii) is then satisfied by z_0 after a suitable sorting of the inner-product values $\langle v_i, z_0\rangle$, and hence is "algorithmically acceptable" as well.

(It is also possible to deal with Assumption (ii) by choosing any hyperplane and then "simulating" a perturbation of the input points [EdM90], an approach similar to the lexicographic rule of the simplex algorithm, see [Da63].)

Note that the ordering of $\{v_1, \ldots, v_m\}$ implies that the segment $\operatorname{conv}\{v_i, v_{i+1}\}$ meets P_i only in the point v_i, and hence v_i belongs to an $(n-1)$-simplex of $\mathcal{B}_i$ that is visible from v_{i+1}. This allows us in the main algorithm to find an element of $\hat{\mathcal{B}}_i$ by looking at all simplices of $\mathcal{B}_i$ that contain v_i; and then we proceed by looking at neighboring boundary simplices.

It is not hard to see that the running time of the above incremental algorithm is of the order $O(\pi(L) m^{\lfloor (n+1)/2 \rfloor})$, where π denotes a suitable polynomial in L. Observe that this bound is polynomial only in the case of *fixed* dimension; for general $\mathcal{V}$-polytopes, the number of simplices in a triangulation is indeed exponential in n since the number of facets may already be exponential in the dimension.

Let us mention in passing that triangulations with special properties are studied prominently in computational geometry, see [Ed87], [Ed93]. A particular class of triangulations that has received a lot of attention because of its wide range of applications is the class of *Delauney-triangulations* that are "dual" to the *Voronoi-diagrams.* Properties of triangulations that are important for practical application (for instance in "surface-design" in the automobile industry) include "good conditioning" in the sense that the ratio of a longest to a shortest edge of the triangulation is bounded above by a reasonably small constant.

For some structural properties of triangulations and dissections, and a related bibliography, see BAYER & LEE [BaL93].

For a given $\mathcal{H}$-polytope we could, of course, compute all vertices and then proceed just as before. However, we will outline an algorithmically different approach that is based on the fact that linear programming problems can be solved in polynomial time. It will allow us to derive an additional result in the case when the dimension is part of the input (and then, of course, rational $\mathcal{V}$-presentations and rational $\mathcal{H}$-presentations are no longer "polynomially equivalent").

Suppose that P is an $\mathcal{H}$-polytope given by the irredundant presentation $(n, m; A, b)$. A triangulation $\mathcal{T}(P)$ of P can be computed recursively as follows:

- Determine a vertex v of P. This can be done in polynomial time by an application of the ellipsoid algorithm or by interior-point methods.
- Determine the set $\mathcal{F}$ of (irredundant $\mathcal{H}$-presentations of) facets of P that do not contain v. This can again be done by linear progamming. (Note that from $(n, m; A, b)$ we can easily obtain $\mathcal{H}$-presentations for the facets in $\mathcal{F}$, and the subsequent redundancy tests require the solution of at most $O(m^2)$ suitable linear programs; hence this step can be done in time that is polynomial in the size of the original input.)

The same step is now repeated for the facets in $\mathcal{F}$ and so on, and the results are stored in a layered graph. The 0th node is the pair $(\emptyset, P)$ and the nodes of layer j are pairs S and F, where S is a set of j vertices and F is a face of dimension $n-j$. The recursive process stops with the $(n+1)$st layer. Then the respective faces are all empty, and the sets S contain the vertices of the simplices of the so constructed triangulation. (A close relative of this method appears in a paper by VON HOHENBALKEN [Vo81].)

A first observation confirms the result that for *fixed* n, the volume of an $\mathcal{H}$-polytope can be computed in polynomial time. (Note that this follows already from the triangulation routine for $\mathcal{V}$-polytopes that was outlined earlier, in conjunction with the fact that, when the dimension is fixed, a passage from a rational $\mathcal{V}$-presentation of a polytope to a rational $\mathcal{H}$-presentation of the same polytope can be carried out in polynomial time.)

3.1.1 *When the dimension n is fixed, the volume of $\mathcal{V}$-polytopes and $\mathcal{H}$-polytopes can be computed in polynomial time.*

Clearly, the above algorithm may require time that is exponential in n. However, for the case of $\mathcal{H}$-polytopes that are simplicial, the algorithm runs in polynomial time even when the dimension n is part of the input. To see this, observe first that the problem of deciding whether P has volume 0 can be solved by way of linear programming. So, suppose that P is n-dimensional. Further, note that the number of simplices of $\mathcal{T}(P)$ is bounded by m, the number of constraints in the given $\mathcal{H}$-presentation. We can then use linear programming to identify for each facet F of P the constraint hyperplanes of the given presentation that contain F's facets (which are $(n-2)$-dimensional faces of P), and then it is easy to derive irredundant $\mathcal{H}$-presentations for all simplices of $\mathcal{T}(P)$. For simplices, one presentation can be converted easily into the other, so we obtain, in time that is polynomial in L, rational $\mathcal{V}$-presentations of all simplices in $\mathcal{T}(P)$. As the final step we compute the volumes of these simplices and add them all up to obtain the volume of P. This result can be easily extended to "near-simplicial" $\mathcal{H}$-polytopes. To be more precise, let σ be a nonnegative integer and define the class $\mathcal{P}_{\mathcal{H}}(\sigma)$ by

$$\mathcal{P}_{\mathcal{H}}(\sigma) = \bigcup_{n\in\mathbb{N}} \{P \in \mathcal{P}^n : P \text{ is an } \mathcal{H}\text{-polytope, and } f_0(F) \leq n+1+\sigma \text{ for each facet } F \text{ of } P\}.$$

Then we obtain the following result:

3.1.2 *Let σ be a nonnegative integer constant. When restricted to $\mathcal{P}_{\mathcal{H}}(\sigma)$, the problem of volume computation can be solved in time that is bounded by a polynomial in L.*

Let us conclude with a result about the binary size of the volume of $\mathcal{V}$-polytopes. Clearly, when the dimension is fixed, the volume of a ($\mathcal{V}$- or $\mathcal{H}$-) polytope P can be computed in polynomial time and its size is therefore polynomial in the size of the input. It is not clear *a priori* whether this property of the volume persists when the dimension is part of the input. It is true that each vertex of P is rational of size that is bounded above by a polynomial in L, and that each simplex in a triangulation has volume of size that is again bounded by a polynomial in L. However, it is also true that there may be exponentially many simplices in any possible triangulation, and thus it is conceivable that $V(P)$ (the sum of all the simplex-volumes) may be of exponential size. (Remember that we are speaking here of the size or length of the volume as a binary number, and not of its magnitude as a real number!) As we will see in the next subsection, this may actually be the case for $\mathcal{H}$-polytopes. On the other hand, it is easy to see (by multiplying with the common denominator or simply with the product of all denominators of the rational entries $(v_1, \ldots, v_m)$), that the size of the volume of a $\mathcal{V}$-polytope is indeed bounded above by a polynomial in L.

3.1.3 *If P is a $\mathcal{V}$-polytope, then the binary size of $V(P)$ is bounded above by a polynomial in the size L of P's $\mathcal{V}$-presentation.*

3.2 Sweeping-plane formulas.

Another approach that has become a standard tool for many algorithmic questions in geometry is the *sweeping-plane* technique. It goes back (at least) to Hadwiger [Ha55], who used it in the context of the Euler characteristic on the convex ring. It has been applied to volume computation by Bieri & Nef [BiN83], Lawrence [La91] and Khachiyan [Kh88], [Kh89], [Kh93].

The general idea is to "sweep" a hyperplane through a polytope P, keeping track of the changes that occur when the hyperplane sweeps through a vertex. Let us illustrate this idea for the problem of computing the volume of a triangle $T = \{v_0, v_1, v_2\}$ in the plane. Let $c \in \mathbb{R}^2$ be a rational vector such that

$$\langle c, v_0\rangle < \langle c, v_1\rangle < \langle c, v_2\rangle.$$

For $\tau \in \mathbb{R}$, let

$$H(\tau) = \{x : \langle c, x\rangle \leq \tau\} \quad \text{and} \quad \varphi(\tau) = V(T \cap H(\tau)).$$

Clearly, $\varphi(\tau) = 0$ for $\tau \leq \langle c, v_0\rangle$ and $\varphi(\tau) = V(T)$ for $\tau \geq \langle c, v_2\rangle$. Now define the following three cones:

$$C_0 = v_0 + \operatorname{pos}\{v_1 - v_0, v_2 - v_0\}, \quad C_1 = v_1 + \operatorname{pos}\{v_1 - v_0, v_2 - v_1\},$$
$$C_2 = v_2 + \operatorname{pos}\{v_2 - v_0, v_2 - v_1\}.$$

Note that $C_i \cap H(\tau)$ is bounded for each i, whence (as an easy case of the *inclusion-exclusion principle*)

$$\varphi(\tau) = V(C_0 \cap H(\tau)) - V(C_1 \cap H(\tau)) + V(C_2 \cap H(\tau)).$$

Further, for $i = 0, 1, 2$,

$$V(C_i \cap H(\tau)) = \begin{cases} 0 & \text{for } \tau \leq \langle c, v_i\rangle; \\ \gamma_i(\tau - \langle c, v_i\rangle)^2 & \text{for } \tau \geq \langle c, v_i\rangle, \end{cases}$$

where the γ_i are suitable (easily computable) constants. Hence for $\tau \geq \langle c, v_2\rangle$ we obtain

$$V(T) = \sum_{i=0}^{2} (-1)^i \gamma_i(\tau - \langle c, v_i\rangle)^2.$$

Clearly, this approach can be generalized to arbitrary polytopes, and it yields a volume formula that does not explicitly involve triangulations. This formula was first derived by Bieri & Nef [BiN83] (even for more general bounded polyhedral sets); other proofs are due to Lawrence [La91] and Filliman [Fi92]. We will give Lawrence's [La91] statement of the volume formula (under additional restrictions) since it is formulated in terms of the standard ingredients of the simplex tableau of linear programming. Later we will comment on some generalizations.

Suppose that $(n, m; A, b)$ is an irredundant $\mathcal{H}$-presentation of a *simple* polytope P. Recall that P's being simple means geometrically that each vertex of P is contained in precisely n facets. Let $b = (\beta_1, \ldots, \beta_m)^T$ and denote the row-vectors of A by $a_1^T, \ldots, a_m^T$. Let $M = \{1, \ldots, m\}$ and for each nonempty subset I of M, let A_I denote the submatrix of A of rows with indices in I and let b_I denote the corresponding right-hand side. For each vertex v of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ there is a subset $I = I_v \subset M$ of cardinality n such that $A_I v = b_I$ and $A_{M\setminus I} v \leq b_{M\setminus I}$. Since P is assumed to be simple and its $\mathcal{H}$-presentation to be irredundant, the set I_v is unique. Stated in the terminology of the (dual) simplex algorithm this means that the basic feasible solutions of $Ax \leq b$ are in one-to-one correspondence with the vertices of P, and hence the corresponding linear program is nondegenerate.

Let $c \in \mathbb{R}^n$ such that $\langle c, v_1\rangle \neq \langle c, v_2\rangle$ for any pair of vertices v_1, v_2 that form an edge of P, and set $H(\tau) = \{x \in \mathbb{R}^n : \langle c, x\rangle \leq \tau\}$ for $\tau \in \mathbb{R}$.

3.2.1 *If the polytope P is simple, and $(n, m; A, b)$ is an irredundant $\mathcal{H}$-presentation, then (with the above notation)*

$$V(P \cap H(\tau)) = \frac{(-1)^n}{n!} \sum_{v \in \mathcal{F}_0(P)} \frac{(\max\{0, \tau - \langle c, v\rangle\})^n}{\prod_{i=1}^n e_i^T A_{I_v}^{-1} c|\det(A_{I_v})|},$$

where $e_1, \ldots, e_n$ denote the standard unit vectors of $\mathbb{R}^n$.

Consequently,

$$V(P) = \frac{(-1)^n}{n!} \sum_{v \in \mathcal{F}_0(P)} \frac{(\tau - \langle c, v\rangle)^n}{\prod_{i=1}^n e_i^T A_{I_v}^{-1} c |\det(A_{I_v})|}$$

whenever $\tau \geq \max_{x \in P} \langle c, v\rangle$. It follows that as a polynomial in τ, the right-hand side of this formula is constant. Evaluation at $\tau = 0$ yields the following volume formula.

3.2.2 *If the polytope P is simple, and $(m, n; A, b)$ is an irredundant $\mathcal{H}$-presentation, then (with the above notation)*

$$V(P) = \frac{1}{n!} \sum_{v \in \mathcal{F}_0(P)} \frac{\langle c, v\rangle^n}{\prod_{i=1}^n e_i^T A_{I_v}^{-1} c |\det(A_{I_v})|}.$$

As was mentioned earlier, the ingredients of this volume formula are those which are computed in the (dual) simplex algorithm. More precisely, $\langle c, v\rangle$ is just the value of the objective function at the current basic feasible solution v, $\det(A_{I_v})$ is the determinant of the current basis, and $A_{I_v}^{-1} c$ is the vector of reduced costs, i.e. the (generally infeasible) dual point that belongs to v. Note that the signs that were present in our introductory example (which come from the inclusion-exclusion principle, or from Gram's relation) are now hidden in $A_{I_v}^{-1} c$; each dually infeasible component contributes a negative sign.

For practical computations, 3.2.2 has to be combined with some vertex enumeration technique. Its closeness to the simplex algorithm suggests the use of a *reverse search* method of Avis & Fukuda [AvF91], which is based on the simplex method with Bland's pivoting rule.

As it stands, the volume formula 3.2.2 does not involve triangulation. However, if we interpret it in a polar setting, it becomes clear that we are really dealing with the faces of the simplicial polytope P°. Accordingly, generalization to nonsimple polytopes involves polar triangulation. In fact, for general polytopes P, [BiN83] and [La91] suggest a "lexicographic rule" to move from one basis to another, and this is just a particular triangulation of P°; see also Filliman [Fi92, Theorem 1].

As an application of formula 3.2.2, Lawrence [La91] derives the following negative result for the binary size of $V(P)$ for $\mathcal{H}$-polytopes P.

3.2.3 *The binary size of the volume of $\mathcal{H}$-polytopes is in general not bounded above by a polynomial in L.*

This result is in striking contrast to the case of $\mathcal{V}$-polytopes (3.1.3) and answers a question of Dyer & Frieze [DyF88].

The example given in [La91] is a projective image of the standard cube. More precisely, let

$$C_n = [0,1]^n = \{x = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n : 0 \leq \xi_1, \ldots, \xi_n \leq 1\},$$

let

$$a = \frac{1}{2^n}(2^{n-1}, 2^{n-2}, \ldots, 2^0)^T,$$

and consider the projective transformation π_a defined by

$$\pi_a(x) = \frac{x}{1 + \langle a, x\rangle}.$$

Then $P = \pi_a(C_n)$ is a polytope which is defined by the inequalities

$$\begin{aligned} \xi_i + \langle a, x\rangle \le 1 \qquad & i = 1, \ldots, n \\ \xi_i \ge 0 \qquad & i = 1, \ldots, n, \end{aligned}$$

and this is a rational $\mathcal{H}$-presentation of P of size that is polynomial in n. However,

$$V(P) = \frac{1}{n!} 2^{(n^2+3)/2} \sum_{i=2^n}^{2^{n+1}-1} (-1)^{\omega(i)-1} \frac{1}{i},$$

where $\omega(i)$ is the number of 1's in i's binary representation. Now, write this number as a rational β/γ in coprime representation (with $\beta, \gamma > 0$), and let p be a prime with $2^n < p < 2^{n+1}$. Note that p divides a denominator i of a summand if and only if $p = i$. Let $\tau = \prod q^k$, where the product extends over all primes q and q^k is the highest power of q that divides at least one of the integers between 2^n and $2^{n+1} - 1$; certainly γ divides τ. Then

$$\sum_{i=2^n}^{2^{n+1}-1} (-1)^{\omega(i)-1} \frac{1}{i} = \sum_{i=2^n}^{2^{n+1}-1} (-1)^{\omega(i)-1} \frac{\tau/i}{\tau},$$

and p divides each τ/i with $p \ne i$ but does not divide τ/p. Thus the numerator

$$\sum_{i=2^n}^{2^{n+1}-1} (-1)^{\omega(i)-1} \frac{\tau}{i}$$

is not divisible by p and p is not factored out when producing the coprime representation β/γ. This implies that

$$\gamma \ge \prod \{p : p \text{ is a prime with } 2^n < p < 2^{n+1}\}.$$

By the prime number theorem (see e.g. [HaW68]) there are asymptotically $2^n/n$ such primes and thus γ is an integer of order 2^{2^n}. Hence the binary size of the denominator of $V(P)$ is not bounded by a polynomial in n.

3.3 Exponential integrals.

Another possibility to compute the volume of a polytope P – at least if P belongs to some special classes of polytopes – is to study the *exponential integral*

$$\int_P e^{\langle c, x\rangle} dx,$$

where c is an arbitrary vector of $\mathbb{R}^n$. (Note that for $c = 0$, the above integral gives just the volume of P.) Exponential integrals satisfy certain relations, some of which are stated later, that make it possible to compute the integrals efficiently in some important cases.

Let us begin by stating formulas for the cube $C_n = [0,1]^n$ and for the regular $(n-1)$-simplex $T_n = C_n \cap H$, where $H = \{x \in \mathbb{R}^n : \xi_1 + \cdots + \xi_n = 1\}$, that is embedded in $\mathbb{R}^n$; see [Ba93a]. Let $c = (\gamma_1, \ldots, \gamma_n)^T$. Then in the first case we have

$$\int_{C_n} e^{\langle c,x\rangle} dx = \prod_{i=1}^{n} \alpha_i \qquad \text{where} \quad \alpha_i = \begin{cases} 1 & \text{if } \gamma_i = 0; \\ \frac{e^{\gamma_i}-1}{\gamma_i} & \text{else.} \end{cases}$$

In the second case, let μ denote the Lebesgue measure on H induced from $\mathbb{R}^n$. Then

$$\int_{T_n} e^{\langle c,x\rangle} d\mu = \sqrt{n} \sum_{i=1}^{n} e^{\gamma_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{\gamma_i - \gamma_j},$$

for all $c \in \mathbb{R}^n$ with pairwise distinct coordinates. This result is due to PODKORYTOV [Po80] and a different proof was given by BARVINOK [Ba93a]. The following proposition stems from [Ba93a].

3.3.1 *Let C be an n-dimensional line-free polyhedral cone in $\mathbb{R}^n$. Further, let $z_1, \ldots, z_m$ be a minimal set of vectors that generate all extreme rays of C, and let $H_i = \{x : \langle z_i, x\rangle = 0\}$. Then the integral*

$$\hat{\sigma}_C(c) = \int_C e^{\langle c,x\rangle} dx$$

exists for all vectors c that are contained in the interior of C's polar C°, and the function $\hat{\sigma}_C$ is rational in c. Further, $\hat{\sigma}_C$ can be naturally extended to a rational function σ_C on $\mathbb{C}^n$, with singularities precisely in $H_1 \cup \cdots \cup H_m$.

The following theorem is the central result in this context. It is due to BRION [Br88] for rational polytopes, and was later extended to the form stated below by KHOVANSKII & PUHLIKOV (1989, unpublished) and BARVINOK [Ba91], [Ba93a].

3.3.2 *Let P be an n-polytope, and for each vertex v of P let C_v denote the cone $v + \mathrm{pos}(P - v)$. Further, let σ_{C_v} be the functions defined in 3.3.1 (with C replaced by C_v). Then*

$$\int_P e^{\langle c,x\rangle} dx = \sum_{v\in\mathcal{F}_0(P)} \sigma_{C_v}(c),$$

whenever $c \in \mathbb{C}^n$ is nonsingular for all functions σ_{C_v}.

Note that in some sense Theorem 3.3.2 can be regarded as a generalization of the Gram-relation of LAWRENCE's [La91] approach.

The vector $c = 0$ that corresponds to volume computation is singular for all functionals σ_{C_v}; so we have to resort to computing the exponential integrals for nonzero vectors c with $0 < \|c\|_2 < \epsilon$ for some sufficiently small positive ϵ. Using such an approximation, BARVINOK [Ba93a] proves a theorem which, when combined with the fact that the volume of a given $\mathcal{V}$-polytope is polynomial in the size of the input, yields the following result.

3.3.3 *There is an algorithm which, for a given $\mathcal{V}$-polytope P, computes the volume of P in time $O(\pi(L)\beta(P))$, where π is a polynomial and*

$$\beta(P) = \sum_{v\in\mathcal{F}_0(P)} \binom{\hat{f}_1(v)}{n} \qquad \text{with } \hat{f}_1(v) = \operatorname{card}(\{e \in \mathcal{F}_1(P) : v \in e\}).$$

As a corollary to this theorem we see that for "near-simple" $\mathcal{V}$-polytopes P the volume of P can be computed in polynomial time. To be more precise, let τ be a nonnegative integer and define the class $\mathcal{P}_{\mathcal{V}}(\tau)$ by

$$\mathcal{P}_{\mathcal{V}}(\tau) = \bigcup_{n\in\mathbb{N}} \{P \in \mathcal{P}^n : P \text{ is a } \mathcal{V}\text{-polytope, and } \hat{f}_1(v) \leq n+\tau \text{ for each vertex } v \text{ of } P.\}$$

Then we obtain the following result, which is the "dual" counterpart of Theorem 3.1.2:

3.3.4 *Let τ be a nonnegative integer constant. Then there is a polynomial-time algorithm which, for a given $P \in \mathcal{P}_{\mathcal{V}}(\tau)$, computes the volume of P.*

Let us point out that 3.3.4 can also be derived from 3.2.2. Note that the validity of 3.3.4 is based on the fact that the number of facets of a simple polytope is bounded above by its number of vertices. Since, on the other hand, the number of vertices may be exponential in the number of facets, a similar result is not likely to be true for near-simple $\mathcal{H}$-polytopes. In fact, as we will see in Subsection 5.1, the problem of computing the volume of the intersection of a cube with a rational halfspace is already #$\mathbb{P}$-hard.

3.4 Numerical integration.

It may be fair to say that the modern study of volume computations began with Kepler [Ke1615] who derived the first *cubature formula* for measuring the capacities of wine barrels (see [St69, pp.192–197]), and that it was the task of volume computation that motivated the general field of integration. Many efforts have been made in *numerical analysis* to devise efficient algorithms for computing or approximating integrals, and it seems very natural to browse through the fund of numerical analysis to see what kind of approaches to numerical integration may, when suitably specialized, lead to efficient methods for volume computation.

Of course, we do not attempt to give a full account of the methods of numerical integration; for general treatments of this subject see any standard monograph, e.g. Stroud [St71] or Davis & Rabinovitz [DaR84]. Here we want to concentrate on two main approaches to numerical integration, the (*degree d*) *integration formulas*, and the (*quasi-*) *Monte Carlo methods.*

Many of the approximate methods for integration of a functional f over a compact region B of $\mathbb{R}^n$ have the form

$$\int_B f(x)dx \sim \sum_{i=1}^{r} \alpha_i f(y_i),$$

where the points $y_1, \ldots, y_r \in \mathbb{R}^n$ are the *nodes* and the numbers $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$ are the *coefficients* of the formula. Of course, the nodes and coefficients must not depend on f, and it is numerically desirable (to avoid annihilation) to have nonnegative coefficients. The integration formula is of *degree* d if it is exact for all multivariate polynomials f of degree at most d but inexact for some polynomial of degree $d+1$.

The theory of integration formulas for functions of one variable is well developed; subjects like the *Newton-Cotes formulas* or the *Gaussian quadrature formulas* are standard fare in every undergraduate course on this subject. However, already in dimension 2 the situation becomes significantly more complicated. One reason is that up to affine equivalence there is only one compact connected region in $\mathbb{R}^1$, while there are uncountably many such affinity classes in higher dimensions. Further, integration formulas for functions in one variable can be easily obtained by integration of interpolation polynomials, while for arbitrary point sets in higher dimensions, suitable interpolation is not always possible. Moreover, in contrast to the multivariate case, the theory of univariate orthogonal polynomials (which is of great use for constructing integration formulas in one variable) is simple and fairly well understood.

Suppose now that, given a region B in $\mathbb{R}^n$, we want to construct an integration formula of degree d. For $x = (\xi_1, \ldots, \xi_n)$ and $q = (\kappa_1, \ldots, \kappa_n) \in (\mathbb{N} \cup \{0\})^n$ let x^q denote the monomial

$$x^q = \xi_1^{\kappa_1} \cdot \xi_2^{\kappa_2} \cdot \ldots \cdot \xi_n^{\kappa_n}.$$

Further, let

$$S_{n,d} = \{q = (\kappa_1, \ldots, \kappa_n) \in (\mathbb{N} \cup \{0\})^n : \sum_{i=1}^{n} \kappa_i \leq d\}.$$

Note that there are

$$\binom{n+d}{d}$$

different multivariate monomials of degree at most d in $\mathbb{R}^n$. Thus in order to obtain an integration formula of degree d we have to solve the system

$$\int_B x^q dx = \sum_{i=1}^{r} \alpha_i y_i^q, \qquad q \in S_{n,d},$$

of $\binom{n+d}{d}$ nonlinear equations in $r(d+1)$ variables.

It is quite easy to see (e.g. [DaR84, p.366]) that the system cannot be solved with fewer than

$$\binom{n + \lfloor d/2 \rfloor}{\lfloor d/2 \rfloor}$$

nodes. The following theorem of TCHAKALOFF [Tc57] shows on the positive side that the system is always solvable with

$$r_0 = \binom{n+d}{d}$$

nodes even under the additional constraints that all coefficients be positive.

3.4.1 *Let $B \subset \mathbb{R}^n$ be compact with positive volume. Then there exist nodes $y_1, \ldots, y_{r_0} \in B$ and positive coefficients $\alpha_1, \ldots, \alpha_{r_0}$ such that*

$$\int_B f(x)dx = \sum_{i=1}^{r_0} \alpha_i f(y_i),$$

whenever f is a multivariate polynomial in x of degree at most d.

It may be worthwhile in our context to point out that the most elegant proof of this theorem makes fundamental use of the theory of convex cones.

Note that the equation in the above nonlinear system that corresponds to the monomial x^0 of degree 0 is just

$$V(B) = \int_B dx = \int_B x^0 dx = \sum_{i=1}^r \alpha_i.$$

Hence 3.4.1 is just tautological when applied to volume computation for a body K in its formulation $B = K$, $f \equiv 1$.

There are other ways of formulating (WEAK) VOLUME COMPUTATION as a problem in integration. For instance, it is equally natural to write

$$V(K) = \int_C \chi_K(x)dx,$$

where χ_K denotes the characteristic function of the body K and C is a body with $K \subset C$ whose volume can be computed easily. Then the formula would not be tautological. However, since the quality of the approximation

$$\int_C \chi_K(x)dx \sim \sum_{i=1}^r \alpha_i \chi_K(y_i)$$

would depend on the error in approximating the (noncontinuous) function χ_K by polynomials of (preferably) small degree, this formula would not be of great practical use.

If mere continuity were the issue, we could use yet another formulation. Suppose that $0 \in \operatorname{int} K$, and let γ_K denote the *gauge functional* of K; i.e. for $x \in \mathbb{R}^n$,

$$\gamma_K(x) = \min\{\lambda \geq 0 : x \in \lambda K\}.$$

Then

$$V(K) = \frac{1}{n!} \int_{\mathbb{R}^n} e^{-\gamma_K(x)} dx,$$

and hence we could get good approximations of $V(K)$ from the numerical value of the integral $\int_C e^{-\gamma_K(x)}dx$, where C is, say, a sufficiently large cube centered at the origin. By the *Stone-Weierstrass* theorem, any continuous function on C can be approximated uniformly on C by multivariate polynomials. However, in order to obtain sufficiently close approximation, the degrees of the polynomials must be very high.

There are many other ways in which one could try to utilize the rich fund of integration formulas (and their accompanying, sometimes very deep, theory of error bounds) for the apparently simpler task of volume computation. However, as we will see in Sections 5 and 6, there are some serious, apparently unavoidable obstacles to obtaining efficient deterministic algorithms for (WEAK) VOLUME COMPUTATION. With this in mind, it is natural to investigate techniques that use (or simulate) some kind of sampling. The general idea of the classical *Monte Carlo method* for numerical integration is to devise a *stochastic process* whose expected value is the integral under consideration, and then to estimate this expected value by sampling. More precisely, for approximating the integral $\int_B f(x)dx$ we choose, for a given integer r, random points $y_1, \dots, y_r$ independently uniformly distributed in B. Then the integral is estimated by

$$\int_B f(x)dx \sim \frac{1}{r}V(B)\sum_{i=1}^{r} f(y_i).$$

The expression $\frac{1}{r}V(B)\sum_{i=1}^{r} f(y_i)$ is a random variable with expected value $\int_B f(x)dx$ and standard deviation

$$\frac{1}{\sqrt{r}}\left(V(B)\int_B f^2(x)dx - \big(\int_B f(x)dx\big)^2\right).$$

Since the standard deviation does not decrease very rapidly in r, and since for most regions B it does not seem possible to actually perform random sampling, most practical applications resort to sequences of points that are specifically tailored for integration. They are in fact *quasi-Monte Carlo methods.*

(Note that the latter two of the three mentioned methods of formulating volume computation as a problem of integration do at least avoid the obvious drawback of the first formulation – that we would have to know in advance the volume of the body under consideration.)

A natural approach to deterministic sampling uses the points of B that belong to the point lattice $\delta\mathbb{Z}^n$ for some parameter $\delta \in]0,1]$. The corresponding formula is then

$$\int_B f(x)dx \sim \frac{V(B)}{G_\delta(B)} \sum_{y\in B\cap\delta\mathbb{Z}^n} f(y),$$

where $G_\delta(B) = \operatorname{card}(B\cap\delta\mathbb{Z}^n)$ is the *lattice-point enumerator* with respect to $\delta\mathbb{Z}^n$. Under assumptions on f that involve its variation it is possible to derive error estimates for such formulas; see [DaR84, p.352]. In the next subsection we will consider this lattice-point approach more closely in the context of (WEAK) VOLUME COMPUTATION.

Improved quasi-Monte Carlo methods can be obtained by "optimizing" the set of sampling points. The error estimates then rely explicitly on measures of equidistribution of the point set; see [St71, Sections 6.2 and 6.3], [DaR84, Section 5.5] and [SpM94].

Let us point out in passing that the lattice-point sampling corresponds to a dissection of space into cubes with centers at the lattice points. Rather than choosing these centers as sampling points one can choose one or more random points in each cube; this leads to the method of *stratified sampling*, see [St71, p.209].

As we will see in Section 7, the general idea of random sampling (when appropriately elaborated, utilizing special properties of convex bodies) does indeed lead to a randomized polynomial-time algorithm for volume computation (and hence also for some special integration problems; see Subsection 9.3). In fact, after suitable transformations, DYER, FRIEZE & KANNAN [DyFK89], [DyFK91] construct an ascending sequence of bodies

$$\mathbb{B}^n = K_0 \subset K_1 \subset \cdots \subset K_k = K$$

such that the corresponding volume ratios are small, and they then use random walks on the lattice points inside K_i to generate random points from the uniform distribution over K_i that lead to an estimate for $V(K_{i-1})/V(K_i)$.

3.5 Lattice point enumeration.

As was mentioned in the previous subsection, it is quite natural (though in general not optimal) to use the points of suitable lattices for sampling in a quasi-Monte Carlo approach to numerical integration. We want to consider the sampling with lattice points more closely now in the context of (Weak) Volume Computation. Let $R \in \mathbb{N}$, and suppose that the body K is contained in $R\mathbb{B}^n$. Set for $\delta \in]0,1[$

$$k_\delta = \left\lceil \frac{R}{\delta} - \frac{1}{2} \right\rceil \qquad \text{and} \qquad B_\delta = \left(k_\delta + \frac{1}{2}\right)[-\delta,\delta]^n.$$

Note that

$$K \subset B_\delta, \quad G_\delta(B_\delta) = (2k_\delta + 1)^n \qquad \text{and} \qquad V(B_\delta) = (2k_\delta+1)^n \delta^n.$$

Then, when applied to $B = B_\delta$ and $f = \chi_K$, the corresponding quasi-Monte Carlo integral formula of Subsection 3.4 becomes

$$V(K) = \int_{B_\delta} \chi_K(x)dx \sim \frac{V(B_\delta)}{G_\delta(B_\delta)} G_\delta(K) = \delta^n G_\delta(K),$$

and this relates (Weak) Volume Computation to the problem of counting lattice points. (See [GrW93] for a survey of lattice-point problems.) Now, we have the trivial upper bound

$$\delta^n G_\delta(K) \le V(K + \delta[-1,1]^n) \le \sum_{i=0}^{n} \binom{n}{i} V(\overbrace{K,\dots,K}^{n-i}, \overbrace{[-1,1]^n,\dots,[-1,1]^n}^{i})\delta^i,$$

and using the monotonicity of mixed volumes we obtain

$$\begin{aligned}\delta^n G_\delta(K) &\le V(K) + \delta \sum_{i=1}^{n} \binom{n}{i} V(\overbrace{R[-1,1]^n,\dots,R[-1,1]^n}^{n-i}, \overbrace{[-1,1]^n,\dots,[-1,1]^n}^{i}) \\ &\le V(K) + \delta(2(R+1))^n.\end{aligned}$$

On the other hand, the inequality of Bokowski, Hadwiger & Wills [BoHW72] yields

$$V(K) - \delta n(2R)^{n-1} \le V(K) - \delta V_{n-1}(K) \le \delta^n G_\delta(K),$$

whence

$$|V(K) - \delta^n G_\delta(K)| \le \delta(3R)^n.$$

Thus if λ is a positive rational, and we set $\delta = \lambda/(3R)^n$, the volume of K is approximated by $\delta^n G_\delta(K)$ up to the additive error λ.

By results of Dyer [Dy91] for $n \le 4$ (see also Zamanskii & Cherkasskii [ZaC83], [ZaC85]) and of Barvinok [Ba93b] in general (see also [DyK93]), the number of lattice points of an $\mathcal{H}$- or a $\mathcal{V}$-polytope can be computed in polynomial time when the dimension is *fixed*. Hence the above approach yields again Theorem 3.1.1 as a corollary.

Now, suppose there is an algorithm $\mathcal{A}$ which, accepting as input a pair of rationals $\epsilon, \delta \in]0,1[$ and a centered well-bounded body in $\mathbb{R}^n$ that is given by a weak membership oracle, produces a number g such that

$$|G_\delta(K) - g| \leq G_\delta(K(\epsilon)) - G_\delta(K(-\epsilon)).$$

Suppose further that for *fixed* n the running time of $\mathcal{A}$ is polynomial in size(K), size(ϵ) and size(δ). Then we can use the algorithm $\mathcal{A}$ to solve WEAK VOLUME COMPUTATION in fixed dimensions. In fact, let b, r, R denote the parameters of a centered well-bounded body K in $\mathbb{R}^n$, and suppose (without loss of generality) that $r = 1$. Note first that

$$K \subset K(-\epsilon) + \epsilon R\mathbb{B}^n,$$

and hence

$$\begin{aligned} V(K(-\epsilon)) &\geq V(K) - \epsilon \sum_{i=1}^{n} \binom{n}{i} V(\overbrace{R[-1,1]^n, \ldots, R[-1,1]^n}^{n-i}, \overbrace{[-1,1]^n, \ldots, [-1,1]^n}^{i}) R^i \\ &\geq V(K) - \epsilon(4R)^n. \end{aligned}$$

With $\hat{\delta} = \delta(3(R+1))^n$, this implies that

$$\begin{aligned} \delta^n |G_\delta(K) - g| &\leq \delta^n G_\delta(K(\epsilon)) - \delta^n G_\delta(K(-\epsilon)) \\ &\leq V(K(\epsilon)) + |\delta^n G_\delta(K(\epsilon)) - V(K(\epsilon))| - V(K(-\epsilon)) + \\ &\qquad + |\delta^n G_\delta(K(-\epsilon)) - V(K(-\epsilon))| \\ &\leq V(K(\epsilon)) - V(K(-\epsilon)) + \delta(3(R+\epsilon))^n + \delta(3(R-\epsilon))^n \\ &\leq 2\hat{\delta} + 2\epsilon(4R)^n \end{aligned}$$

and hence

$$\begin{aligned} |V(K) - \delta^n g| &\leq |V(K) - \delta^n G_\delta(K)| + \delta^n |G_\delta(K) - g| \\ &\leq 3\hat{\delta} + 2\epsilon(4R)^n. \end{aligned}$$

If λ is now the error parameter of the given instance of WEAK VOLUME COMPUTATION, we choose

$$\epsilon \leq \frac{\lambda}{4(4R)^n} \qquad \text{and} \qquad \delta = \frac{\hat{\delta}}{(3(R+1))^n)} \leq \frac{\lambda}{6(3(R+1))^n},$$

and run algorithm $\mathcal{A}$. This proves the following result.

3.5.1 *When the dimension is fixed, there is an algorithm for* WEAK VOLUME COMPUTATION *that uses a polynomial number of arithmetic operations on rationals of polynomially bounded sizes and a polynomial number of calls to the hypothetical algorithm $\mathcal{A}$.*

Note that we can of course check in oracle-polynomial time for each point y of $B_\delta \cap \delta\mathbb{Z}^n$ whether y is weakly contained in K. More precisely, given $y \in B_\delta \cap \delta\mathbb{Z}^n$, and a rational number $\epsilon > 0$, the oracle for K asserts that $y \in K(\epsilon)$ or that $y \notin K(-\epsilon)$. Further, the number g of input points $y \in B_\delta \cap \delta\mathbb{Z}^n$ for which the oracle asserts $y \in K(\epsilon)$ would satisfy the above requirements. Unfortunately, while the number of lattice points in $R[-1,1]^n$ that we have to check is polynomial in R and $1/\delta$, it is not bounded by a polynomial in size(K) and size(λ) and hence is not polynomial in the size of the input. Thus this simple checking procedure does not yield a suitable algorithm $\mathcal{A}$.

At present, we don't know whether such a polynomial algorithm $\mathcal{A}$ for WEAK LATTICE POINT ENUMERATION exists, nor do we know the precise status of WEAK VOLUME COMPUTATION in fixed dimensions. The latter question is of course closely related to the question of devising suitable algorithms for approximating bodies by polytopes.

3.6 Special convex bodies.

Since simplices are the most basic and elementary polytopes, formulas for volumes of simplices are of special interest. We begin this subsection with some formulas which supplement the basic determinantal formula given at the beginning of Subsection 3.1 and which are for some purposes more useful than that one. The following result expresses the volume of a j-dimensional simplex (short *j-simplex*) in $\mathbb{R}^n$ in terms of its edge-lengths.

3.6.1 *Suppose that S is a j-simplex in $\mathbb{R}^n$ with vertices $v_1, \dots, v_{j+1}$. Let $B = (\beta_{ik})$ denote the $(j+1) \times (j+1)$ matrix given by $\beta_{ik} = \|v_i - v_k\|_2^2$. Then*

$$2^j (j!)^2 V_j^2(S) = |\det(\hat{B})|,$$

where $\hat{B}$ is the $(j+2) \times (j+2)$ matrix obtained from B by bordering B with a top row $(0, 1, \dots, 1)$ and a left column $(0, 1, \dots, 1)^T$.

The determinant appearing in 3.6.1 is often called the *Cayley-Menger determinant*. See [Dö65, p.285] and [BlG43] for references to low-dimensional cases of 3.6.1 associated with the names of Euler, Lagrange, Cayley, and Sylvester, and see SOMMERVILLE [So29, p.125] and BLUMENTHAL [Bl53, p.98] for proofs of 3.6.1.

The next formula, a close relative of 3.6.1, expresses the volume of a suitably located simplex in terms of the *Gram matrix* of inner products of its vertices; see [GrKL94] and [Bl53] for proofs of 3.6.2 and a variant of it.

3.6.2 *Suppose that S is a j-simplex in $\mathbb{R}^n$ with $0 \in \operatorname{aff} S$, and A is the $(j+1) \times n$ matrix whose rows list the coordinates of the vertices of S. Then*

$$(j!)^2 V_j^2(S) = \det(M + AA^T),$$

where M is the $(j+1) \times (j+1)$ matrix whose entries are all 1. If the origin is a vertex of S then

$$(j!)^2 V_j^2(S) = \det(A_0 A_0^T),$$

where A_0 is formed from A by discarding A's zero row.

For an n-simplex S in $\mathbb{R}^n$, the following formula expresses the volume of S in terms of the coefficients that appear in the affine functionals defining the facets of S. For general n the formula is due to KLEBANER, SUDBURY AND WATTERSON [KlSW89].

3.6.3 *Suppose that an n-simplex S in $\mathbb{R}^n$ is bounded by the $n+1$ hyperplanes whose equations are*

$$\alpha_{i0} + \sum_{j=1}^{n} \alpha_{ij} x_j = 0 \qquad (i = 0, 1, \ldots, n),$$

and let A denote the $(n+1) \times (n+1)$ matrix with elements α_{ij}, $0 \le i, j \le n$. Then the volume of S is given by

$$V(S) = \frac{|\det(A)|^n}{n! \prod_{i=0}^{n} A_{i0}},$$

where A_{i0} is the cofactor of α_{i0} in A.

The paper [KlSW89] also contains two formulas giving the volume of a j-simplex S in $\mathbb{R}^n$ when $j < n$. One formula is in terms of the coefficients that appear in the affine functionals defining the affine hull of S and the facets of S. The other formula is in terms of the coordinates of the vertices of the simplex. Like the volume formulas in [LiL90] and [Be92], it may be regarded as a higher-dimensional analogue of the Pythagorean theorem of plane geometry.

There are, of course, other classes of bodies or polytopes for which special volume formulas or special methods of computing volumes are known. Some of these can be found in the references listed at the start of Section 3. We do not discuss these here, but the case of zonotopes does seems worthy of special mention.

Let $(n, r; c; z_1, \ldots, z_r)$ be an $\mathcal{S}$-zonotope and set $Z = \sum_{i=1}^{r} [0,1] z_i$. Further, for $i = 1, \ldots, r$, let $K_i = [0,1] z_i$; i.e. the K_i's are all line segments. It follows from 2.4.1 that

$$V(Z) = V\left(\sum_{i=1}^{r} K_i\right) = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \cdots \sum_{i_n=1}^{r} V(K_{i_1}, \ldots, K_{i_n}).$$

If the indices $i_1, \ldots, i_n$ are not pairwise distinct, the zonotope $K_{i_1} + \cdots + K_{i_n}$ has volume 0, whence $V(K_{i_1}, \ldots, K_{i_n}) = 0$ and it follows with

$$V\left(K_{i_1} + \cdots + K_{i_n}\right) = V(K_{i_1}, \ldots, K_{i_n}) = n! \left|\det(z_{i_1}, \ldots, z_{i_n})\right|$$

that

$$V(Z) = \sum_{1 \le i_1 < i_2 < \cdots < i_n \le r} \left|\det(z_{i_1}, \ldots, z_{i_n})\right|;$$

see also [Mo89], [St91] and [Sh74].

Note that this formula for the volume of a zonotope leads to a polynomial-time algorithm for fixed dimensions, and also for varying dimensions if the input is restricted to the class of all "near-parallelotopal" $\mathcal{S}$-zonotopes, where $r - n$ is bounded by an a priori constant.

3.6.4 *When the dimension n is fixed or when $r - n$ is fixed,* Volume Computation *can be solved in polynomial time for $\mathcal{S}$-zonotopes.*

In general the above volume formula involves exponentially many summands, and this feature of zonotope volume computation cannot be avoided (unless $\mathbb{P} = \mathbb{NP}$); see 5.1.7.

For combining two convex sets J and K to form a third, the three most important ways are those of intersection, vector addition, and joining – forming the sets $J \cap K$, $J + K$, and

$\operatorname{conv}(J \cup K)$ respectively. When the sets J and K are sufficiently "independent," there are useful formulas relating $V(J+K)$ and $V(\operatorname{conv}(J \cup K))$ to $V(J)$ and $V(K)$.

For any two subsets X and Y of $\mathbb{E}^n$, we define the *distance*

$$\operatorname{dist}(X,Y) = \inf\{\|x-y\|_2 : x \in X, y \in Y\}.$$

Now suppose that X and Y are both flats (affine subspaces), that there is a unique pair of points $x_0 \in X$ and $y_0 \in Y$ for which $\|x_0 - y_0\|_2 = \operatorname{dist}(X,Y)$, and that the linear subspaces $X - x_0$ and $Y - y_0$ are mutually orthogonal. Then the flats X and Y are said to be *orthogonal* when $x_0 = y_0$ (so that $\operatorname{dist}(X,Y) = 0$) and *skew-orthogonal* when $x_0 \neq y_0$ (so that $\operatorname{dist}(X,Y) > 0$).

3.6.5 *Suppose that J is a j-dimensional body in $\mathbb{R}^n$ and K is a k-dimensional body in $\mathbb{R}^n$. Let*

$$P = J + K \qquad \text{and} \qquad Q = \operatorname{conv}(J \cup K).$$

If the flats aff J *and* aff K *are orthogonal, then P is a $(j+k)$-dimensional body with*

$$V_{j+k}(P) = V_j(J) \cdot V_k(K).$$

If the flats aff J *and* aff K *are skew-orthogonal, then Q is a $(j+k+1)$-dimensional body with*

$$V_{j+k+1}(Q) = \frac{j!k!}{(j+k+1)!} \operatorname{dist}\bigl(\operatorname{aff} F, \operatorname{aff} G\bigr) \cdot V_j(J) \cdot V_k(K).$$

The first formula in 3.6.5 is just the standard one for the volume of a cartesian product. Suppose, in particular, that $k, n_1, \dots, n_k$ are fixed positive integers. Then, for given ($\mathcal{H}$- or $\mathcal{V}$-) polytopes $P_1 \subset \mathbb{R}^{n_1}, \dots, P_k \subset \mathbb{R}^{n_k}$ and $P = P_1 \times \cdots \times P_k$ we have

$$V(P) = V(P_1) \cdot \ldots \cdot V(P_k),$$

and hence by 3.1.1, $V(P)$ can be computed in polynomial time. The computation of $V(P_1), \dots, V(P_k)$ is generally more efficient than direct computation of $V(P)$.

The second formula in 3.6.5 appears in [GrKL94] for the case in which J and K are both simplices, whence the general formula follows easily for polytopes by dissection and then for general bodies by approximation.

4. Deterministic methods for computing mixed volumes

4.1 Using a volume oracle.

In this first subsection we will outline the most natural approach for computing mixed volumes, a method directly suggested by Theorem 2.4.1, which is based on a procedure for volume computation.

Let us consider an arbitrary procedure $\mathcal{B}$ (efficient or not) for Weak Volume Computation; so, suppose that $\mathcal{B}$ is an algorithm which, for a body K given by a weak optimization oracle and for a given positive rational λ, produces a rational number μ such that

$|V(K) - \mu| \le \lambda$. In this subsection we show how such an algorithm can be used for the (weak) computation of mixed volumes.

Note first that when the number r of bodies $K_1, \ldots, K_r$ and the binary sizes of the positive rationals $\xi_1, \ldots, \xi_r$ are bounded by a polynomial in n, then the weak optimization oracles for $K_1, \ldots, K_r$ can be used to devise a weak optimization oracle for $\sum_{i=1}^r \xi_i K_i$. We will now try to gain information about mixed volumes by calling $\mathcal{B}$ for various such linear combinations of $K_1, \ldots, K_r$.

Let us begin with some remarks about the maximum number of different mixed volumes. By 2.4.1,

$$V\left(\sum_{i=1}^r \xi_i K_i\right) = \sum_{i_1=1}^r \sum_{i_2=1}^r \cdots \sum_{i_n=1}^r \xi_{i_1}\xi_{i_2}\cdots\xi_{i_n} V(K_{i_1}, K_{i_2}, \ldots, K_{i_n}).$$

Thus we have r^n coefficients $V(K_{i_1}, K_{i_2}, \ldots, K_{i_n})$. However, these coefficients are order-independent and hence only

$$\binom{n+r-1}{r-1}$$

of them can actually be distinct, for this is the number of different multivariate monomials of degree n in $\mathbb{R}^r$. It follows that *if r is fixed*, their number is polynomial in n, and *if n is fixed*, their number is polynomial in r. However, it also follows that in general, the task of computing *all* mixed volumes cannot be accomplished in polynomial time since the number of different mixed volumes may grow exponentially.

Now, for $x = (\xi_1, \ldots, \xi_r)$ and $q = (\kappa_1, \ldots, \kappa_r) \in (\mathbb{N} \cup \{0\})^r$, let x^q denote again the monomial

$$x^q = \xi_1^{\kappa_1} \cdot \xi_2^{\kappa_2} \cdot \ldots \cdot \xi_r^{\kappa_r}.$$

Further, let

$$Q_n = \Big\{q = (\kappa_1, \ldots, \kappa_r) \in (\mathbb{N} \cup \{0\})^r : \sum_{i=1}^r \kappa_i = n\Big\},$$

and for $q \in Q_n$ let

$$c_q = \binom{n}{\kappa_1, \ldots, \kappa_r} V(\overbrace{K_1, \ldots, K_1}^{\kappa_1}, \ldots, \overbrace{K_r, \ldots, K_r}^{\kappa_r}).$$

Here, as usual,

$$\binom{n}{\kappa_1, \ldots, \kappa_r} = n! \cdot \prod_{i=1}^r \frac{1}{\alpha_i} \qquad \text{where} \quad \alpha_i = \begin{cases} 1 & \text{if } \kappa_i = 0; \\ \kappa_i! & \text{else} \end{cases}$$

is a *multinomial coefficient.* Setting

$$\pi(x) = V\left(\sum_{i=1}^r \xi_i K_i\right),$$

the mixed volume expression 2.4.1 of π reads

$$\pi(x) = \sum_{q \in Q_n} c_q x^q.$$

Assume that the elements of Q_n are ordered (for instance lexicographically),

$$Q_n = \{q_1, \ldots, q_k\}, \qquad \text{with} \quad k = \binom{n+r-1}{r-1}.$$

Then, a choice of k nonnegative rational row vectors $y_1, \ldots, y_k$ of $\mathbb{R}^r$ and evaluation of $\pi(x)$ at these interpolation points leads to the matrix equation

$$p = \begin{pmatrix} \pi(y_1) \\ \vdots \\ \pi(y_k) \end{pmatrix} = \begin{pmatrix} y_1^{q_1} & y_1^{q_2} & \cdots & y_1^{q_k} \\ \vdots & \vdots & & \vdots \\ y_k^{q_1} & y_k^{q_2} & \cdots & y_k^{q_k} \end{pmatrix} \begin{pmatrix} c_{q_1} \\ \vdots \\ c_{q_k} \end{pmatrix} = Yz.$$

Note that for $r = 2$ and $y_1 = (1, \eta_1), \ldots, y_{n+1} = (1, \eta_{n+1})$, Y is just a *Vandermonde-matrix* and hence is nonsingular whenever $\eta_1, \ldots, \eta_{n+1}$ are pairwise distinct. In this case $\hat{\pi} = \pi|_{\{1\}\times\mathbb{R}}$ can be expressed in terms of the standard *Lagrange interpolation polynomials*, and there is a considerable literature on how to choose the interpolation nodes and do the computation in an efficient and numerically stable way; see e.g. [BeZ65], [Sa'74], [Ri75], [Ri90], or [MiM85]; see also 7.2 for a more explicit description of the case $r = 2$ in terms of Lagrange polynomials.

Now suppose we have chosen $y_j = (\eta_{j,1}, \ldots, \eta_{j,r})$ for $j = 1, \ldots, k$ such that Y is nonsingular. Further, let $\mu_1, \ldots, \mu_k$ be the rationals produced by $\mathcal{B}$ when applied to the bodies $\sum_{i=1}^{r} \eta_{j,i} K_i$, respectively, whence

$$|\pi(y_j) - \mu_j| \leq \lambda \qquad \text{for } j = 1, \ldots, k.$$

Now, let

$$A = Y^{-1}, \quad \hat{p} = (\mu_1, \ldots, \mu_k)^T, \quad \text{and} \quad \hat{z} = A\hat{p}.$$

Then

$$\|\hat{z} - z\|_\infty = \|A(\hat{p} - p)\|_\infty \leq \|A\| \, \|\hat{p} - p\|_\infty \leq \lambda \|A\|,$$

where $\|A\|$ is the matrix norm induced by $\| \quad \|_\infty$, i.e. the maximum of the ℓ_1 norms of the rows of A.

It can now be shown (see [St71, p.55], [ChY77], [Ol86]; cf. also 3.4.1) that the interpolation points $y_1, \ldots, y_k$ can be appropriately specified so as to yield the first case of the following result; the assertion for fixed n but varying r then follows in a standard way.

4.1.1 *Whenever r is fixed or n is fixed, there is an algorithm for (weakly) computing all mixed volumes of r bodies in $\mathbb{R}^n$ given by weak optimization oracles that uses a polynomial number of arithmetic operations on rationals of polynomially bounded sizes and a polynomial number of calls to the algorithm $\mathcal{B}$.*

Note that in order to compute one specific mixed volume by this method, we must essentially compute all of them. Further, 4.1.1 does not cover the case of varying n and r. In particular, it is unclear whether there is an efficient way of computing, say, $V(K_1, \dots, K_n)$. Theorem 5.2.2 below gives some indication that this might not be the case.

Finally note that we have used here a quite strong algorithm for volume approximation. We will use a weaker approximation routine in Subsection 7.2, and we will comment on the difference there. Different measures for approximation errors will be introduced in Subsection 6.1.

4.2 Polytopes.

Theorems 3.1.1 and 4.1.1 together show that there is a polynomial-time algorithm for computing all mixed volumes of polytopes in fixed dimensions.

4.2.1 *When the dimension n is fixed, there is a polynomial-time algorithm which, given $r \in \mathbb{N}$ and ($\mathcal{V}$- or $\mathcal{H}$-) polytopes $P_1, \dots, P_r$, computes all mixed volumes $V(P_{i_1}, \dots, P_{i_n})$.*

This algorithm makes use of the fact that the given bodies are actually polytopes only in the subroutine for VOLUME COMPUTATION. It is, however, possible to express the mixed volumes of polytopes as the sum of volumes of polytopes formed as sums of faces, and hence devise an algorithm that makes much stronger use of the facial structure of the polytopes, is more combinatorial and therefore possibly numerically more stable.

For $r \geq 2$ let $P_1, \dots, P_r$ be polytopes in $\mathbb{R}^n$. When applied to the mapping

$$\varphi : P_1 \times \cdots \times P_r \to P_1 + \cdots + P_r \qquad \text{defined by} \quad \varphi(x_1, \dots, x_r) = x_1 + \cdots + x_r,$$

the lifting theorem of WALKUP & WETS [WaW69] yields a dissection of $P_1 + \cdots + P_r$ into polytopes $F_1 + \cdots + F_r$, where $(F_1, \dots, F_r)$ varies over suitable r-tuples of faces F_i of P_i; see [McS83], [PeS92], [HuS93]. Using the expansion of $V(P_1 + \cdots + P_r)$ into mixed volumes and comparing coefficients, this dissection can be used to obtain a representation for mixed volumes in terms of the volumes $V(F_1 + \cdots + F_r)$. An explicit formula of this kind was given by BETKE [Be'92] for $r = 2$ and SCHNEIDER [Sc94] in the general case.

In order to state Schneider's result (in Theorem 4.2.2) precisely, we need to introduce some notation.

For a polytope P and a face F of P, let $N(P, F)$ denote the cone of outer normals of P at F. Further, let us call the vectors $v_1, \dots, v_r \in \mathbb{R}^n$ *admissible* for $P_1, \dots, P_r$ if

(i) there is an $i \in \{1, \dots, r\}$ such that $v_i \neq 0$;
(ii) $\sum_{i=1}^r v_i = 0$;
(iii) $\bigcap_{i=1}^r \big(\operatorname{relint} N(P_i, F_i) - v_i\big) = \emptyset$ whenever, for $i = 1, \dots, r$, F_i is a face of P_i and $\sum_{i=1}^r \dim F_i > n$.

Note that the third condition is invariant under a common translation of the vectors $v_1, \dots, v_r$, whence (iii) is the only relevant condition for admissibility. Now suppose that $v_1, \dots, v_r$ do not satisfy (iii). Then there exist faces $F_1, \dots, F_r$ of $P_1, \dots, P_r$, respectively, with $\sum_{i=1}^r \dim F_i > n$, and a vector $x_0 \in \mathbb{R}^n$ such that for each $i = 1, \dots, r$ the (appropriate hyperplane perpendicular to the) vector $x_0 + v_i$ supports P_i in the face F_i. Hence $z = ((x_0 + v_1)^T, \dots, (x_0 + v_r)^T)^T \in \mathbb{R}^{nr}$ supports the polytope $P = P_1 \times \cdots \times P_r$ of $\mathbb{R}^{nr}$ in its face $F = F_1 \times \cdots \times F_r$, whence $z \in \operatorname{relint} N(P, F)$. This implies in particular that $v + S \subset \operatorname{lin}(N(P, F))$, where $v = (v_1^T, \dots, v_r^T)^T$ and S is the n-dimensional subspace of $\mathbb{R}^{nr}$ of vectors of the form $(x^T, \dots, x^T)^T$ with $x \in \mathbb{R}^n$. Now consider the (linear) hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^{nr}$ that is formed by all hyperplanes that are orthogonal to an edge of P. The condition that $\sum_{i=1}^r \dim F_i > n$ implies that $\dim N(P,F) \le (r-1)n-1$, whence the n-dimensional affine subspace $v+S$ of $\mathbb{R}^{nr}$ meets a face of $\mathcal{H}$ of dimension $(r-1)n-1$; so, admissibility is just a general position condition that is "generically" satisfied.

In practice, to find vectors $v_1,\dots,v_r$ that are admissible for $P_1,\dots,P_r$ one would essentially choose $v_1,\dots,v_r$ at random. In a deterministic approach one might construct the face-lattice of $\mathcal{H}$ (using the algorithm of Edelsbrunner, O'Rourke & Seidel [EdOS86], [EdSS91]), then add S to each cell of dimension $(r-1)n-1$ (and if necessary add further lines to obtain hyperplanes), and then find an interior point of a full-dimensional cell of this new arrangement.

Let, as in the previous subsection,

$$Q_n = \Big\{q = (\kappa_1,\dots,\kappa_r) \in (\mathbb{N}\cup\{0\})^r : \sum_{i=1}^r \kappa_i = n\Big\},$$

and, for $q \in Q_n$, let $\mathcal{F}_q$ denote the set of all r-tuples $(F_1,\dots,F_r)$ of faces of $P_1,\dots,P_r$, respectively, for which

$\dim F_i = \kappa_i$, for $i = 1,\dots,r$;
$\dim(\sum_{i=1}^r F_i) = n$;
$\bigcap_{i=1}^r \big(N(P_i,F_i) - v_i\big) \ne \emptyset$.

Then Schneider [Sc94] proved the following representation theorem.

4.2.2 *Let $P_1,\dots,P_r$ be polytopes of $\mathbb{R}^n$, let $v_1,\dots,v_r \in \mathbb{R}^n$ be admissible for $P_1,\dots,P_r$, and let $q = (\kappa_1,\dots,\kappa_r) \in Q_n$. Then*

$$\binom{n}{\kappa_1,\dots,\kappa_r} V(\overbrace{P_1,\dots,P_1}^{\kappa_1},\dots,\overbrace{P_r,\dots,P_r}^{\kappa_r}) = \sum_{\mathcal{F}_q} V(F_1+\cdots+F_r).$$

Let us point out that, when $P_1,\dots,P_r$ are $\mathcal{H}$-polytopes, for a given r-tuple $(F_1,\dots,F_r)$ of ($\mathcal{V}$- or $\mathcal{H}$-presented) faces of $P_1,\dots,P_r$, respectively, it can be checked in polynomial time (using Gaussian elimination and linear programming) whether $(F_1,\dots,F_r) \in \mathcal{F}_q$. Note, further, that 4.2.2 can be used to prove 4.2.1, that in fixed dimension, mixed volumes of polytopes can be computed in polynomial time.

We close this section with a tractability result of [DyGH94] that holds even when the dimension is part of the input.

4.2.3 *There is a polynomial time algorithm for checking whether, for given $n,r \in \mathbb{N}$, ($\mathcal{H}$- or $\mathcal{V}$-) polytpes $P_1,\dots,P_r$ of $\mathbb{R}^n$, and $q = (\kappa_1,\dots,\kappa_r) \in Q_n$,*

$$V(\overbrace{P_1,\dots,P_1}^{\kappa_1},\dots,\overbrace{P_r,\dots,P_r}^{\kappa_r}) = 0.$$

This result does not seem to be striking, but it is not trivial. In fact, suppose for notational simplicity that $r = n$, and that the origin belongs to the relative interior of all polytopes. Then select for each i a basis A_i of $\operatorname{lin} P_i$. It is easy to see that $V(P_1,\dots,P_n) \ne 0$ if and only if there is a choice of $(a_1,\dots,a_n) \in A_1 \times\cdots\times A_n$ such that $\det(a_1,\dots,a_n) \ne 0$. Let

$A = \bigcup_{i=1}^{n} A_i$, let $\mathcal{I}_L$ be the family of all linearly independent subsets of A, and let $\mathcal{I}_P$ denote the family of all subsets of A which meet each of the A_i in at most one element. Then the pairs $\mathcal{M}_L = (A, \mathcal{I}_L)$ and $\mathcal{M}_P = (A, \mathcal{I}_P)$ are *matroids*, called respectively a *linear matroid* and a *partition matroid*. Now $V(P_1, \ldots, P_n) \neq 0$ if and only if $\mathcal{M}_L$ and $\mathcal{M}_P$ have a common basis, and this can be detected in polynomial time by the *matroid intersection theorem* of EDMONDS [Ed70]; see also [GrLS88, Section 7.5].

4.3 Polytopes and bodies.

This subsection discusses a specific formula due to MINKOWSKI [Mi11] for the mixed volume of a convex body and $n-1$ copies of a polytope; see also [BoF34] and [Sc93].

With $h : \mathcal{K}^n \times \mathbb{S}^{n-1} \to \mathbb{R}$ denoting as before the support function, Minkowski proved the following result.

4.3.1 *Let K be a body and P a polytope in $\mathbb{R}^n$, let $F_1, \ldots, F_m$ denote P's facets, and let $u_1, \ldots, u_m$ be the corresponding outer unit normals. Then*

$$V(K, \overbrace{P, \ldots, P}^{n-1}) = \frac{1}{n} \sum_{i=1}^{m} h(K, u_i) V_{n-1}(F_i).$$

Let us mention, as a side remark, that 4.3.1 can be applied to a polytope P of the form $P = P_1 + \cdots + P_{n-1}$; it then yields a similar representation for $V(K, P_1, \ldots, P_{n-1})$ (see [BoF34, p.42]).

Suppose now that P is an $\mathcal{H}$-polytope and that the given presentation is irredundant. This means, in particular, that (not necessarily unit) normal vectors of all facets are given. Further suppose that the volumes of the facets of P are known, and that K is given by a weak optimization oracle. Then formula 4.3.1 allows us to approximate $V(K, P, \ldots, P)$ with the aid of m calls to the optimization oracle.

For general polytopes, and when the dimension is part of the input, this is not particularly encouraging since the problem is only polynomially reduced to VOLUME COMPUTATION for the facets of P. If, however, P belongs to a class of polytopes for which the facet volumes can be obtained efficiently, or if we just consider all computations that involve only P as "preprocessing" (since we may want to compute $V(K, P, \ldots, P)$ for many different bodies K but fixed P), then 4.3.1 may even be algorithmically useful.

It may be worthwhile to point out that some of the problems disappear when different *data structures* are used. This is particularly apparent in connection with the algorithmic significance of 4.3.1. Indeed, recall that by a theorem of MINKOWSKI [Mi1897], [Mi03], a polytope is uniquely determined (up to translation) by its facet volumes and its facet normals. Hence, the "tractability statement" related to 4.3.1 says essentially that if we choose, as our data structure for polytopes, a *Minkowski presentation* – i.e., a list of facet volumes and the associated facet normals – then $V(K, P, \ldots, P)$ can be approximated in polynomial time for arbitrary bodies given by a weak optimization oracle. However, the problem of passing from a given $\mathcal{V}$- or $\mathcal{H}$-presentation to a Minkowski-presentation is algorithmically difficult (see 5.2) unless the dimenson is fixed. The same is true for the reverse transformation; see [GrH94].

4.4 Special convex bodies.

There are other formulas and integral representations known for mixed volumes in general or for certain classes of bodies (see e.g. [BoF34], [Ha57], [BuZ88], [Sc93]) whose algorithmic significance seems, however, restricted to very particular cases. In the present subsection we will just mention two explicit formulas for the mixed volumes of a body and a ball or a parallelotope. We begin with the *intrinsic volumes of polytopes.*

As was noted already in Subsection 2.4, the expansion of $V(P+\xi\mathbb{B}^n)$ into mixed volumes leads to quermassintegrals or intrinsic volumes of a polytope P. For a face F of P let $\gamma(F,P)$ denote the *outer angle* of F at P (i.e. the fraction of space that is taken up by the cone of outer normals of P at some point that is relatively interior to F). Then McMullen [Mc75] gave the following representation of the intrinsic volumes.

4.4.1 *For* $i=0,\ldots,n$,

$$V_i(P)=\sum_{F\in\mathcal{F}_i(P)}\gamma(F,P)V_i(F).$$

Evaluation of this formula involves computing the volumes of all i-dimensional faces of P, and also of the $(n-i-1)$-dimensional (spherical) volumes of spherical polytopes that are obtained by intersecting the cones $N(P,F)$ of outer normals with $\mathbb{S}^{n-1}$. While the former is algorithmically easy only for small values of i (see 3.1.1), the latter is easy only for small values of $n-i$.

We mention in passing that Hadwiger [Ha75] has given the integral formula

$$\sum_{i=0}^{n}V_i(K)=\int_{\mathbb{R}^n}e^{-\pi\,\mathrm{dist}^2(K,x)}dx,$$

which is a useful tool for certain lattice-point problems. The same is true (see [GrW93]) for a formula that we are going to develop now; see [BuZ88, p.141] or [Sc93, p.294].

Let $a_1,\ldots,a_n\in\mathbb{R}^n$ such that $Z=\sum_{i=1}^{n}[0,1]a_i$ is a proper parallelotope, let $0<k<n$, and let $K_1,\ldots,K_{n-k}\in\mathcal{K}^n$. Further, $\Pi_S K$ denotes again the orthogonal projection of a body K onto a linear subspace S. Then the multilinearity of the mixed volume implies that

$$V(\overbrace{Z,\ldots,Z}^{k},K_1,\ldots,K_{n-k})=\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n}V([0,1]a_{i_1},\ldots,[0,1]a_{i_k},K_1,\ldots,K_{n-k}).$$

Now let $S_{i_1,\ldots,i_k}=\mathrm{lin}\{a_{i_1},\ldots,a_{i_k}\}$, then

$$\binom{n}{k}V([0,1]a_{i_1},\ldots,[0,1]a_{i_k},K_1,\ldots,K_{n-k})=\begin{cases}\alpha_{i_1,\ldots,i_k} & \text{if }\dim S_{i_1,\ldots,i_k}=k;\\ 0 & \text{otherwise,}\end{cases}$$

where

$$\alpha_{i_1,\ldots,i_k}=V_{S^\perp_{i_1,\ldots,i_k}}(\Pi_{S^\perp_{i_1,\ldots,i_k}}K_1,\ldots,\Pi_{S^\perp_{i_1,\ldots,i_k}}K_{n-k})\cdot V_{S_{i_1,\ldots,i_k}}([0,1]a_{i_1},\ldots,[0,1]a_{i_k});$$

the subscripts $S^\perp_{i_1,\ldots,i_k}$ and $S_{i_1,\ldots,i_k}$ indicate that the corresponding mixed volumes are taken with respect to the spaces $S^\perp_{i_1,\ldots,i_k}$ and $S_{i_1,\ldots,i_k}$, respectively. If we specialize this formula to the case $Z=C_n=[0,1]^n=\sum_{i=1}^{n}[0,1]e_i$ and $K_1=\cdots=K_{n-k}=K$ we obtain the following result.

4.4.2 *For $k = 0, \dots, n$,*

$$\binom{n}{k} V(\overbrace{C_n, \dots, C_n}^{k}, \overbrace{K, \dots, K}^{n-k}) = \sum V_{n-k}(\Pi_S K),$$

where S ranges over all $(n-k)$-dimensional coordinate subspaces of $\mathbb{R}^n$.

Another useful specialization is obtained for $K_1 = \cdots = K_{n-k} = \mathbb{B}^n$; it leads to a simple formula for the intrinsic volumes of a parallelotope.

4.4.3 *Let $Z = \sum_{i=1}^n [0,1]a_i$ be a proper parallelotope. Then, for $k = 0, \dots, n$,*

$$2^{n-k} V_k(Z) = \sum_{F \in \mathcal{F}_k(Z)} V_k(F).$$

Hence, in order to compute the intrinsic volumes of Z, we need only compute the k-volume of its *k-skeleton* $\bigcup_{F \in \mathcal{F}_k(Z)} F$. This can be done inductively. In fact, if for $j = 0, \dots, k$ and $m = 1, \dots, n$

$$S(j,m) = \sum_{F \in \mathcal{F}_j(\sum_{i=1}^m [0,1]a_i)} V_j(F),$$

then we have (with appropriate conventions in the "boundary cases")

$$S(j, m+1) = 2S(j,m) + S(j-1,m) \cdot \|a_{m+1} - A(A^T A)^{-1} A^T a_{m+1}\|_2,$$

where A is the $n \times m$ matrix with column vectors $a_1, \dots, a_m$. Thus we obtain the following result as a corollary of 4.4.3.

4.4.4 *The mixed volumes of an $\mathcal{S}$-parallelotope Z in $\mathbb{R}^n$ can be approximated up to an additive rational error $\epsilon > 0$ in time that is polynomial in n, in the size of Z's presentation, and in size(ϵ).*

5. Intractability results

5.1 Volume computations.

In striking contrast to the "positive" results 3.1.1, 3.1.2, 3.3.4, 3.6.4, there are several strong intractability results for Volume Computation. Theorem 5.1.1 summarizes the former, and the latter appear in Theorems 5.1.3–5.1.5 and 5.1.7.

5.1.1 *The volume of a polytope P can be computed in polynomial time in the following cases:*

(i) *if the dimension is fixed and P is a $\mathcal{V}$- or $\mathcal{H}$-polytope or an $\mathcal{S}$-zonotope;*
(ii) *if the dimension is part of the input and P is a near-simple $\mathcal{V}$- or a near-simplicial $\mathcal{H}$-polytope or a near-parallelotopal $\mathcal{S}$-zonotope.*

By 5.1.1 (i) there is a polynomial-time algorithm for VOLUME COMPUTATION when the dimension is fixed. However, the methods of volume computation that we described in Section 3 all require exponential time when n is part of the input. Hence it is natural to wonder whether there is a more robustly polynomial-time procedure for volume computation.

To set the stage, let us begin with a negative result that was mentioned in a different setting in Theorem 3.2.3.

5.1.2 *There does not exist a polynomial-space algorithm for exact computation of the volume of $\mathcal{H}$-polytopes.*

Since each polynomial-time algorithm uses only a polynomial amount of space, Theorem 5.1.2 implies that there is no polynomial-time algorithm which, given a dimension n and an n-dimensional $\mathcal{H}$-polytope P, computes $V(P)$. Doesn't this result already show that volume computation is actually much harder than such $\mathbb{NP}$-complete problems as the TRAVELING SALESMAN PROBLEM? The answer is "Not really!" and we take a few sentences to explain why (to a reader who is less familiar with the relevant concepts of complexity theory).

In the realm of $\mathbb{P}$ and $\mathbb{NP}$, complexity theory usually deals with problems whose answer is "yes" or "no" since this corresponds to the results of a halting Turing machine computation. (When dealing with the class $\#\mathbb{P}$, the Turing machine is augmented by a device that counts accepting computations.) This means that when dealing with related complexity results, the proper formulation of VOLUME COMPUTATION is as follows.

VOLUME.

Instance:	*A positive integer n, an $\mathcal{H}$-polytope $\langle$or a $\mathcal{V}$-polytope, or an $\mathcal{S}$-zonotope$\rangle$ P, a nonnegative rational ν.*
Question:	*Is the volume of P bounded above by ν, i.e. is $V(P) \leq \nu$?*

In order to distinguish the different classes of input polytopes we will sometimes speak of the problems $\mathcal{H}$-VOLUME, $\mathcal{V}$-VOLUME and $\mathcal{S}$-VOLUME, respectively.

Note that the above problems could have equally well been phrased in terms of lower bounding $V(P)$. We use upper-bounding only to associate a "yes" answer with instances in which P is lower-dimensional, a special and easy case.

Suppose now that we had a polynomial-time routine for solving $\mathcal{H}$-VOLUME. Then, using binary search (with appropriately specified values of ν) we could approximate $V(P)$ with any polynomial-size accuracy in polynomial time. Hence, Theorem 5.1.2 does not rule out the possibility that $\mathcal{H}$-VOLUME is in $\mathbb{P}$ and that computing any number of polynomially many digits of $V(P)$ for $\mathcal{H}$-polytopes P is actually easy.

DYER & FRIEZE [DyF88] showed, however, that both $\mathcal{H}$-VOLUME and $\mathcal{V}$-VOLUME are $\#\mathbb{P}$-hard, and we are going to describe various hardness proofs that are all geometric in nature; (see also KHACHIYAN [Kh88], [Kh89], [Kh93]). We begin with $\mathcal{H}$-VOLUME.

Let us point out that, in the following, we are going to deal with hardness results which involve classes of $\mathcal{H}$-polytopes for which the volume is of polynomial size. Hence, a polynomial time method for VOLUME would, in fact, result in a polynomial-time algorithm for VOLUME COMPUTATION.

The first proof stated here for the $\mathbb{NP}$-hardness of the problem of computing the volume of certain simple $\mathcal{H}$-polytopes utilizes the sweeping-plane formula 3.2.1; it is due to KHACHIYAN [Kh88], [Kh93].

As in 3.2.1, let P be a simple $\mathcal{H}$-polytope with corresponding irredundant $\mathcal{H}$-presentation $(n, m; A, b)$. Further, let $c \in \mathbb{R}^n$ such that $\langle c, \ \rangle$ is not constant on any edge of P, and let $H(\tau) = \{x \in \mathbb{R}^n : \langle c, x\rangle \leq \tau\}$ for $\tau \in \mathbb{R}$. Then, by 3.2.1,

$$V(P \cap H(\tau)) = \frac{(-1)^n}{n!} \sum_{v \in \mathcal{F}_0(P)} \frac{(\max\{0, \tau - \langle c, v\rangle\})^n}{\prod_{i=1}^n e_i^T A_{I_v}^{-1} c |\det(A_{I_v})|}.$$

This implies that

$$\varphi(\tau) = V(P \cap H(\tau))$$

is a piecewise polynomial of degree at most n, and is $(n-1)$-times continuously differentiable. (This result has also been proved in the theory of splines (cf. DE BOOR & HÖLLIG [BoH82]), and it is also relevant to some problems in geometric tomography; see [GaG94].) Further, if $\tau_0 < \cdots < \tau_k$ are the (ordered) values of τ for which $\operatorname{bd} H(\tau)$ contains a vertex of P, then the nth derivative of φ is discontinuous at most at $\tau_0, \ldots, \tau_r$, and at these points the one-sided derivatives satisfy the equation

$$\frac{d^n\varphi(\tau_j+)}{d\tau^n} - \frac{d^n\varphi(\tau_j-)}{d\tau^n} = \sum_{v \in \mathcal{F}_0(P), \langle c,v\rangle = \tau} \frac{(-1)^{\delta(v)}}{|\prod_{i=1}^n e_i^T A_{I_v}^{-1} c \det(A_{I_v})|},$$

where $\delta(v) = \operatorname{card}\{w \in \mathcal{F}_0(P) : \langle c, w\rangle < \langle c, v\rangle\}$.

Specifically, if $C_n = [0,1]^n$ and $c > 0$, then for each vertex v of C_n we have $\delta(v) = \|v\|_1$. Let us assume that c has the following "constant-one" property that whenever two vertices v, w of C_n are such that $\langle c, v\rangle = \langle c, w\rangle$, then $\|v\|_1 = \|w\|_1$. Then,

$$\text{there is a vector } v \in \{0,1\}^n \text{ with } \langle c, v\rangle = \kappa$$
$$\text{if and only if}$$
$$\frac{d^n\varphi(\kappa+)}{d\tau^n} - \frac{d^n\varphi(\kappa-)}{d\tau^n} \neq 0.$$

Now suppose that we could compute the volume of the intersection $\varphi(\tau) = C_n \cap H(\tau)$ in polynomial time. Since, unless $\operatorname{bd} H(\tau)$ meets a vertex of C_n, $\varphi(\tau)$ is a polynomial in τ of degree at most n, we can check the differentiability condition in polynomial time. Hence the problem,

> Given $c \in \mathbb{N}^n$ with the constant-one property and $\kappa \in \mathbb{N}$, is there a 0-1-vector v with $\langle c, v\rangle = \kappa$?,

is transformed in polynomial time into the problem of computing the volume of an $\mathcal{H}$-polytope. But except for the additional property of c this is equivalent to SUBSET-SUM:

> Given positive integers $n, \gamma_1, \ldots, \gamma_n$ and a positive integer κ, is there a subset I of $\{1, \ldots, n\}$ such that $\sum_{i \in I} \gamma_i = \kappa$?

SUBSET-SUM is known to be $\mathbb{NP}$-complete, [Sa74] (see also [GaJ79]). On the other hand, any instance of SUBSET-SUM can easily be transformed to the n instances of the required restricted version that are obtained by replacing c by the vector $c + (\|c\|_1 + 1, \ldots, \|c\|_1 + 1)^T$, and κ successively for $\theta = 1, \ldots, n$ by $\kappa + (\|c\|_1 + 1)\theta$. This shows that computing the volume of an $\mathcal{H}$-polytope is $\mathbb{NP}$-hard even for polytopes that are intersections of C_n with one additional (rational) halfspace.

DYER & FRIEZE [DyF88] actually proved $\#\mathbb{P}$-hardness of $\mathcal{H}$-VOLUME; see also [Kh89], [DyF91], [Kh93].

5.1.3 *The problem of computing the volume of the intersection of the unit cube with a rational halfspace is* $\#\mathbb{P}$*-hard.*

To prove 5.1.3, [DyF88] use a reduction of the following counting version of 0-1 KNAPSACK, a problem that is known to be $\#\mathbb{P}$-hard; see [GaJ79].

#(0-1 KNAPSACK).

Instance: *A positive integer n, a positive integer n-vector a, a positive integer β.*
Task: *Determine the cardinality of $\{v \in \{0,1\}^n : \langle a, v\rangle \le \beta\}$.*

The polynomial-time reduction of #(0-1 KNAPSACK) to $\mathcal{H}$-VOLUME uses some ideas that are similar in spirit to the ideas exploited in the above $\mathbb{NP}$-hardness proof. In particular, an inclusion-exclusion formula is used, and the volumes that will be computed are again values of a certain polynomial. There are, however, important differences, and it may be useful to sketch the explicit geometric construction underlying the reduction of 5.1.3.

Let $(n; a_0, \beta_0)$ be an instance of #(0-1 KNAPSACK). We may assume (by considering the instance $(n; 2a_0, 2\beta_0 + 1)$, if necessary) that $\{v \in \{0,1\}^n : \langle a_0, v\rangle = \beta_0\} = \emptyset$.

Now, let us define for each $v \in \{0,1\}^n$ the polytope

$$S_v = \{x \in \mathbb{R}^n : x \ge v,\ \langle a_0, x\rangle \le \beta_0\}.$$

If S_v is full-dimensional, it is the simplex with vertices v and $(\beta_0 - \langle a_0, v\rangle)e_i/\alpha_i$, where α_i is the ith coordinate of a_0. Hence, by the standard determinant formula for the volume of a simplex,

$$V(S_v) = \frac{1}{n!}\prod_{i=1}^{n} \frac{\max\{0, \beta_0 - \langle a_0, v\rangle\}}{\alpha_i}.$$

Now, let $P = \{x \in C_n : \langle a_0, x\rangle \le \beta_0\}$, and let $\mathbb{1} = (1, \dots, 1)^T \in \mathbb{R}^n$. Then the inclusion-exclusion principle yields

$$V(P) = \sum_{v\in\{0,1\}^n} (-1)^{\langle v,\mathbb{1}\rangle} V(S_v) = \frac{1}{n!}\left(\prod_{i=1}^{n}\frac{1}{\alpha_i}\right) \sum_{v\in\{0,1\}^n} (-1)^{\langle v,\mathbb{1}\rangle}(\max\{0, \beta_0 - \langle a_0, v\rangle\})^n.$$

In the neighborhood $]\beta_0 - 1, \beta_0 + 1[$ of β_0, the function π_{a_0,β_0} defined by

$$\pi_{a_0,\beta_0}(\beta) = \sum_{v\in\{0,1\}^n} (-1)^{\langle v,\mathbb{1}\rangle}(\max\{0, \beta - \langle a_0, v\rangle\})^n$$

is a polynomial in β, and a procedure for volume computation would allow us to compute all coefficients of π_{a_0,β_0}. Note that the coefficient of β^n is just

$$\sum_{\substack{v\in\{0,1\}^n \\ \langle a_0,v\rangle\le\beta_0}} (-1)^{\langle v,\mathbb{1}\rangle}.$$

Let us now compute the leading coefficients for various choices of a_0 and β_0; set for $k = 1, \dots, n$

$$\mu = \langle a_0, \mathbb{1}\rangle + 1, \quad a_k = a_0 + \mu\mathbb{1}, \quad \text{and} \quad \beta_k = \beta_0 + \mu k.$$

We may assume in the following that $\langle a_0, \mathbb{1}\rangle > \beta_0$, since otherwise the original instance of #(0-1 KNAPSACK) is trivial.

Now let $v \in \{0,1\}^n$. Then $\langle a_k, v\rangle \le \beta_k$ if and only if v satisfies one of the following two conditions:

(i) $\langle v, \mathbb{1}\rangle < k$;

(ii) $\langle v, \mathbb{1}\rangle = k$ and $\langle a_0, v\rangle \le \beta_0$.

Hence the leading coefficient of π_{a_k,β_k} is

$$(-1)^k \operatorname{card}\{v \in \{0,1\}^n : \langle a_0, v\rangle \le \beta_0,\ \langle v, \mathbb{1}\rangle = k\} + \sum_{i=0}^{k-1}(-1)^i \binom{n}{i}.$$

Since

$$\sum_{k=0}^{n} \operatorname{card}\{v \in \{0,1\}^n : \langle a_0, v\rangle \le \beta_0,\ \langle v, \mathbb{1}\rangle = k\}$$

is actually the solution for the given instance $(n; a_0, \beta_0)$ of #(0-1-KNAPSACK), we see that a polynomial-time algorithm for volume computation would yield a polynomial-time algorithm for #(0-1-KNAPSACK).

Note that this and the previous hardness result involve, as part of the input, integers whose absolute values are not bounded by a polynomial in n. In fact, a result of KOZLOV [Ko86] shows that the volume of the intersection of the unit cube with a constant number of rational halfspaces can be computed in *pseudopolynomial time*. Thus it is natural to wonder whether the problem retains its hardness if we restrict all input data to numbers whose absolute values are bounded by a polynomial in n. It turns out that the problem of computing the volume of $\mathcal{H}$-polytopes is #$\mathbb{P}$-hard even in this strong sense. This follows from the two facts that the problem of computing the number of linear extensions of a given partially ordered set $\mathcal{O} = (\{1,\dots,n\}, \prec)$ is #$\mathbb{P}$-complete, BRIGHTWELL & WINKLER [BrW91], and that this number is equal to $n!V(P_{\mathcal{O}})$, where the set

$$P_{\mathcal{O}} = \{x = (\xi_1,\dots,\xi_n)^T \in [0,1]^n : \xi_i \le \xi_j \iff i \prec j\}$$

is the *order polytope* of $\mathcal{O}$; STANLEY [St86a]. In the following we will indicate the geometric essence of the latter result.

Let $N = \{1,\dots,n\}$, and let $\mathcal{O} = (N, \prec)$ be an arbitrary poset. A *linear extension* of $\mathcal{O}$ is a total ordering of N that is compatible with $\prec$. A linear extension of $\mathcal{O}$ can be regarded as a permutation π of N (or, equally, as a vector $(\pi(1), \pi(2), \dots, \pi(n))$) which has the property

$$i, j \in N \ \wedge\ i \prec j \quad \Longrightarrow \quad \pi^{-1}(i) < \pi^{-1}(j).$$

Let $E(\mathcal{O})$ denote the set of linear extensions of $\mathcal{O}$. Now consider for a given linear extension $\pi \in E(\mathcal{O})$ the polytope

$$T_\pi = \{x \in [0,1]^n : \xi_{\pi(1)} \le \xi_{\pi(2)} \le \dots \le \xi_{\pi(n)}\}.$$

Observe that T_π is a simplex, and that all the constraints that define the order polytope $P_{\mathcal{O}}$ are also constraints of T_π; hence $T_\pi \subset P_{\mathcal{O}}$. Further,

if π_1 and π_2 are different linear extensions of $\mathcal{O}$ then $\operatorname{int}(T_{\pi_1}) \cap \operatorname{int}(T_{\pi_2}) = \emptyset$, and also

$$\bigcup_{\pi\in E(\mathcal{O})} T_\pi = P_{\mathcal{O}}.$$

Hence the simplices T_π, $\pi \in E(\mathcal{O})$ form a dissection of the order polytope $P_{\mathcal{O}}$. Finally note that all these simplices are congruent, and hence

$$V(T_\pi) = \frac{1}{n!}.$$

But this shows that each linear extension π of $\mathcal{O}$ contributes $1/(n!)$ to the volume of $P_{\mathcal{O}}$, and therefore

$$\operatorname{card}(E(\mathcal{O})) = n!V(P_{\mathcal{O}}).$$

Observe that the number of inequalities defining P is $\mathrm{O}(n^2)$.

5.1.4 *$\mathcal{H}$-*VOLUME *is $\#\mathbb{P}$-hard in the strong sense.*

Let us now turn to $\mathcal{V}$-VOLUME, a problem that, in general, is slightly easier since the volume of $\mathcal{V}$-polytopes is of polynomial size; see 3.1.3. However, as DYER & FRIEZE [DyF88] show, it is not much easier; see also [Kh89], [Kh93].

5.1.5 *The problem of computing the volume of the convex hull of the regular $\mathcal{V}$-cross-polytope and an additional integer vector is $\#\mathbb{P}$-hard.*

The following proof is due to KHACHIYAN [Kh89]. Let $Q_n = \operatorname{conv}\{\pm e_1, \dots, \pm e_n\}$, the regular cross-polytope, and for each $a \in \mathbb{Z}^n$ let $P_a = \operatorname{conv}(\{a\} \cup Q_n)$. Then P_a can be dissected into Q_n and the set $\mathcal{S}$ of all simplices $S_F = \operatorname{conv}(F \cup \{a\})$, where F is a facet of Q_n that is visible from a. Now, let $S_F \in \mathcal{S}$, and let $z \in \{-1,1\}^n$ be an outer normal to F. Then

$$V(S_F) = V(F)\cdot \operatorname{dist}(a,F) = V(F)\cdot \frac{\langle a,z\rangle - 1}{\sqrt{n}} = \frac{1}{n!}(\langle a,z\rangle - 1).$$

Therefore

$$n!V(P_a) = n!V(Q_n) + n!\sum_{S\in\mathcal{S}} V(S) = 2^n + \sum_{z\in\{-1,1\}^n} \max\{0, \langle a,z\rangle - 1\},$$

whence

$$\begin{aligned} n!\big(V(P_{a+e_1}) - 2V(P_a) + V(P_{a-e_1})\big) &= \\ = \sum_{z\in\{-1,1\}^n} &\big(\max\{0,\langle a,z\rangle - 2\} - 2\max\{0,\langle a,z\rangle - 1\} + \max\{0,\langle a,z\rangle\}\big) \\ = \sum_{\substack{z\in\{-1,1\}^n\\ \langle a,z\rangle = 1}} &1. \end{aligned}$$

This implies that if we could compute the volume of a $\mathcal{V}$-polytope in polynomial time, then we could also solve the following counting problem in polynomial time:

> Given $n \in \mathbb{N}$ and $a \in \mathbb{Z}^n$, determine the number of solutions $z \in \{-1,1\}^n$ of $\langle a,z\rangle = 1$.

However, this problem is closely related to #(0-1 KNAPSACK) and is in fact #$\mathbb{P}$-complete.

It is not known whether the problem of computing the volume of a $\mathcal{V}$-polytope is #$\mathbb{P}$-hard in the strong sense.

DYER & FRIEZE [DyF88] also show that the problem of computing the volume of a $\mathcal{V}$-polytope is #$\mathbb{P}$-easy in the following sense.

5.1.6 *Let Π be any #$\mathbb{P}$-complete problem. Then any oracle $\mathcal{O}_\Pi$ for solving Π can be used to produce an algorithm that runs in time that is oracle-polynomial in L and size(ϵ) for solving the following problem:*

> *Given $n \in \mathbb{N}$, a $\mathcal{V}$- or an $\mathcal{H}$-polytope P and a positive rational ϵ, compute a rational number μ such that $V(P) - \epsilon \leq \mu \leq V(P) + \epsilon$.*

It follows from 5.1.6 that for $\mathcal{V}$-polytopes, $\mathcal{O}_\Pi$ can be used to actually *compute* $V(P)$, while (due to Theorem 5.1.2) for $\mathcal{H}$-polytopes, $V(P)$ can only be approximated (yet in a very strong sense). Note, however, that (as remarked in [DyF88]) the question remains open as to whether there exist a *fixed constant* λ and a polynomial-time algorithm which, given $n \in \mathbb{N}$ and a $\mathcal{V}$- or an $\mathcal{H}$-polytope P, computes a rational number μ such that

$$(1-\lambda)V(P) \leq \mu \leq (1+\lambda)V(P).$$

See Subsection 6.3 for some related "negative" results in a different model of computation.

The final subject of this subsection is the complexity of volume computations for zonotopes.

The fact that $\mathcal{V}$- and $\mathcal{H}$-VOLUME is #$\mathbb{P}$-hard does not necessarily mean that the same is true for $\mathcal{S}$-zonotopes since, typically, zonotopes have a number of vertices and a number of facets that grow exponentially in the number of generating segments. Recall from 3.6 that we can express the volume of the zonotope $Z = \sum_{i=1}^{r}[0,1]z_i$ as a sum of determinants

$$V(Z) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq r} \left|\det(z_{i_1}, \ldots, z_{i_n})\right|.$$

Hence $\mathcal{S}$-VOLUME is equivalent to the following problem, SUM-OF-DETERMINANTS:

> Given positive integers n, r with $r \geq n$, and an integer $n \times r$ matrix A, compute $\sum |\det(B)|$, where the sum extends over all $n \times n$ submatrices of A.

Clearly, in fixed dimension, this problem can be solved in polynomial time (see 3.6.4), and, even when the dimension is part of the input, each summand can be computed in polynomial time. There are, however, exponentially many summands, and this fact accounts for the hardness of the problem. (See BEN ISRAEL [Be92] for a related but different notion of "volume" associated with the determinants of the $n \times n$ submatrices of an $n \times r$ matrix, and for the relevance of his notion to Moore-Penrose inverses of rectangular matrices.)

5.1.7 $\mathcal{S}$-VOLUME *is #$\mathbb{P}$-hard and also #$\mathbb{P}$-easy.*

Theorem 5.1.7 is due to [DyGH94]. Its hardness result is obtained by a reduction of #PARTITION, the #$\mathbb{P}$-complete task to

> determine for given $m \in \mathbb{N}$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$, the number of different subsets $I \subset M = \{1, \ldots, m\}$ such that $\sum_{i \in I} \alpha_i = \sum_{i \in M \setminus I} \alpha_i$.

It is not known whether $\mathcal{S}$-VOLUME is #$\mathbb{P}$-hard in the strong sense.

Let us mention in passing that the problem

> given positive integers n, r with $r \geq n$, an integer $n \times r$ matrix A, a positive integer λ, determine whether there exists an $n \times n$ submatrix of A such that $|\det(B)| \geq \lambda$,

is $\mathbb{NP}$-complete. This follows from the $\mathbb{NP}$-completeness of HAMILTONIAN CYCLE for directed graphs by a construction of PAPADIMITRIOU & YANNAKAKIS [PaY90]; see [GrKL94] for applications of this result to the problem of finding j-simplices of maximum volume in n-polytopes.

5.2 Computing mixed volumes.

Since volume computation is just a special case of computing mixed volumes, the hardness results of the previous subsection carry over:

5.2.1 *For each fixed $k \in \mathbb{N}$, and for each fixed sequence $(q_n)_{n\in\mathbb{N}}$, where each q_n is a k-tuple $(\kappa_1, \dots, \kappa_k)$ of nonnegative integers with $\sum_{i=1}^{k} \kappa_i = n$, the following problem is #$\mathbb{P}$-hard:*

Instance: *A positive integer n, $\mathcal{H}$- (or $\mathcal{V}$-) polytopes (or $\mathcal{S}$-zonotopes) $P_1, \dots, P_k$ of $\mathbb{R}^n$.*

Task: *Determine the mixed volume $V(\overbrace{P_1, \dots, P_1}^{\kappa_1}, \dots, \overbrace{P_k, \dots, P_k}^{\kappa_k})$.*

In the remainder of this subsection we will give some additional hardness results for mixed volumes that do not trivially depend on the hardness of volume computations. Let us start with the following extreme example of such a result, the hardness of computing mixed volumes of *boxes*, by which we mean rectangular parallelotopes with axis-aligned edges.

5.2.2 *The following problem is #$\mathbb{P}$-hard.*

Instance: *A positive integer n, for $i, j = 1, \dots, n$ positive integers $\alpha_{i,j}$.*

Task: *Determine the mixed volume $V(Z_1, \dots, Z_n)$, where $Z_i = \sum_{j=1}^{n}[0, \alpha_{i,j}]e_j$ for $i = 1, \dots, n$.*

Note that this result, which is due to [DyGH94], is indeed of a different nature than 5.2.1. In fact, each of the Z_i is just a rectangular box, and so is $Z = \sum_{i=1}^{n} Z_i$. Hence the volume $V(Z) = \prod_{j=1}^{n}(\sum_{i=1}^{n} \alpha_{i,j})$ can be computed very easily. Nevertheless, the mixed volume $V(Z_1, \dots, Z_n)$ is hard to compute. This is in interesting contrast to the hardness result of 5.1.7, where the volume of a sum of segments is hard to compute even though each of their mixed volumes can be computed in polynomial time.

As was shown in [DyGH94], 5.2.2 can be extended to show that the #$\mathbb{P}$-hardness persists even if the integers $\alpha_{i,j}$ have only two different values α and β.

To sketch the reasoning for this result, let us compute $V(Z_1, \dots, Z_n)$, where $Z_i = \sum_{j=1}^{n}[0, \alpha_{i,j}]e_j$.

Let $\xi_1, \dots, \xi_n \geq 0$. Then

$$V\left(\sum_{i=1}^{n} \xi_i Z_i\right) = V\left(\sum_{j=1}^{n}\left[0, \sum_{i=1}^{n} \xi_i \alpha_{i,j}\right] e_j\right) = \prod_{j=1}^{n}\left(\sum_{i=1}^{n} \xi_i \alpha_{i,j}\right),$$

and a comparison of the coefficients of $\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_n$ yields

$$V(Z_1, \ldots, Z_n) = \frac{1}{n!} \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} \delta_{j_1,\ldots,j_n} \alpha_{1,j_1} \cdot \ldots \cdot \alpha_{n,j_n},$$

where

$$\delta_{j_1,\ldots,j_n} = \begin{cases} 1 & \text{if } \{j_1, \ldots, j_n\} \text{ is a permutation of } \{1, 2, \ldots, n\}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$n! V(Z_1, \ldots, Z_n) = \mathrm{per}(A)$$

is the *permanent* of the matrix $A = (\alpha_{i,j})_{i,j=1,\ldots,n}$.

Now, VALIANT [Va77] has established the #ℙ-hardness of the problem of computing the permanent even for 0-1-matrices. (In fact, this problem is equivalent to counting the number of *perfect matchings* in a bipartite graph.) This gives already the hardness result 5.2.2. The sharpening, however, relies on an extension of Valiant's result since it requires α and β to be positive or, equivalently, the parallelotopes to be full-dimensional.

Note that by 4.1.1 (in conjunction with 1.2.2), the mixed volumes of boxes $Z_1, \ldots, Z_r$ can be computed in polynomial time if the number r of boxes is constant. (Recall that in 5.2.2 we had $r = n$.) However, this result relies in an essential way on the fact that each of the rectangular parallelotopes has axis-parallel edges. When this restriction is lifted, even the case $r = 2$ becomes hard [DyGH94].

5.2.3 *The following problem is #ℙ-hard.*

Instance: *Positive integers n and k with $k < n$, two n-tuples $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of integer vectors which each form an orthogonal basis of $\mathbb{R}^n$.*

Task: *Compute the mixed volume*

$$V(\overbrace{Z_1, \ldots, Z_1}^{k}, \overbrace{Z_2, \ldots, Z_2}^{n-k}),$$

where $Z_1 = \sum_{j=1}^{n} [0,1] v_j$ and $Z_2 = \sum_{j=1}^{n} [0,1] w_j$.

6. DETERMINISTIC APPROXIMATION OF VOLUMES AND MIXED VOLUMES

6.1 Measures for approximation errors.

Since it is algorithmically difficult to compute the volume of a given body (or polytope) K, it is of interest to approximate $V(K)$ from above or below. The same is true for mixed volumes.

In general, the approximation of a (nonnegative) functional ρ defined on a class of bodies involves, first, an a priori measure for the closeness of approximation.

Typical measures of the closeness of a number μ and the function value $\rho(K)$ for a given convex body K include the *absolute error*

$$|\mu - \rho(K)|$$

and the *relative error*

$$\left|\frac{\mu - \rho(K)}{\rho(K)}\right|.$$

Obviously, the results of Section 5 and the fact that the absolute error changes after scaling K indicate that the absolute error is not an adequate measure for our purposes. The relative error introduced above is adequate for "positive" results that involve a small positive rational error bound λ. However, the relative error is biased toward underestimation in the sense that $\mu = 0$ always produces the error 1. Since we are interested in a *symmetric relative error* measure we define for an arbitrary positive rational λ a (rational) λ-*approximation* of $\rho(K)$ to be a positive rational number μ such that

$$\frac{\rho(K)}{\mu} \leq 1 + \lambda \qquad \text{and} \qquad \frac{\mu}{\rho(K)} \leq 1 + \lambda.$$

Note that this criterion can also be stated as follows

$$-\frac{\lambda}{1+\lambda} \leq \frac{\mu - \rho(K)}{\rho(K)} \leq \lambda.$$

In the remainder of the section we will deal mainly with the following problem for a positive functional $\lambda : \mathbb{N} \to \mathbb{R}$ and with ρ representing the volume or some mixed volume.

λ-APPROXIMATION for ρ

Instance: *A positive integer n, a well-bounded body K given by a (strong or weak) separation oracle.*

Task: *Determine a positive rational μ such that*

$$-\frac{\lambda}{1+\lambda} \leq \frac{\mu - \rho(K)}{\rho(K)} \leq \lambda.$$

For abbreviation we will sometimes use the terms VOLUME APPROXIMATION and MIXED VOLUME APPROXIMATION for the task of solving λ-APPROXIMATION for the volume or for some mixed volumes, respectively.

6.2 Upper bounds.

A quite general tool for obtaining estimates of functionals, even for arbitrary convex bodies, is suggested by a theorem of JOHN [Jo48]. (A strengthening of this result for symmetric bodies appeared in [Jo42].)

6.2.1 *For a body K in $\mathbb{R}^n$, let $a_0 \in \mathbb{R}^n$ and let A_0 be a linear transformation such that $E_0 = a_0 + A(\mathbb{B}^n)$ is the ellipsoid of maximum volume inscribed in K. Then*

$$a_0 + A_0(\mathbb{B}^n) \subset K \subset a_0 + nA_0(\mathbb{B}^n).$$

Any ellipsoid $E = a + A(\mathbb{B}^n)$ that satisfies the inclusion relation $a + A(\mathbb{B}^n) \subset K \subset a + nA(\mathbb{B}^n)$ is called a *Löwner-John ellipsoid* for K. Observe that the dilatation factor n in John's theorem is best possible for the simplex (and only for the simplex [Pa92]). See the book [Pi89] for additional results on contained and containing ellipsoids.

In order to obtain approximative algorithms, one needs of course an algorithmic version of Theorem 6.2.1, or at least a polynomial-time method for approximating the ellipsoid E_0 in 6.2.1 (and in this way obtaining *weak Löwner-John ellipsoids*). Such an algorithm was devised by GRÖTSCHEL, LOVÁSZ & SCHRIJVER [GrLS88], using the ellipsoid method of linear programming.

6.2.2 *There exists an oracle-polynomial-time algorithm which, for any well-bounded body K of $\mathbb{R}^n$ given by a weak separation oracle, finds a point a and a linear transformation A such that*

$$a + A(\mathbb{B}^n) \subset K \subset a + (n+1)\sqrt{n}A(\mathbb{B}^n).$$

Further, the dilatation factor $(n+1)\sqrt{n}$ can be replaced by $\sqrt{n(n+1)}$ when K is symmetric, by $(n+1)$ when K is an $\mathcal{H}$-polytope, and by $\sqrt{n+1}$ when K is a symmetric ($\mathcal{V}$- or $\mathcal{H}$-) polytope.

Since the volume of the ellipsoid $a + A(\mathbb{B}^n)$ can be easily computed, taking the geometric mean of the upper and lower bound in 6.2.2 gives a polynomial-time $(n+1)^{3n/4}$-approximation μ to $V(K)$.

Tarasov, Khachiyan & Erlich [TaKE88] and Khachiyan & Todd [KhT93] give polynomial-time algorithms for approximating the ellipsoid of maximum volume that is contained in an $\mathcal{H}$-polytope. In particular, the following appears in [KhT93].

6.2.3 *For each rational $\gamma \in]0,1[$ there exists a polynomial-time algorithm which, given $n, m \in \mathbb{N}$, and $a_1, \dots, a_m \in \mathbb{Q}^n$, produces an ellipsoid $E = a + A(\mathbb{B}^n)$ such that*

$$E \subset P = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq 1, \text{ for } i = 1, \dots, m\} \quad \text{and} \quad V(E) \geq \gamma \cdot V(E_0),$$

where E_0 is the ellipsoid of maximum volume contained in P. The running time of the algorithm is

$$O\left(m^{3.5} \log(mR)/(r\log(1/\gamma)) \log(nR)/(r\log(1/\gamma))\right),$$

where r and R are respectively a lower bound on P's inradius and an upper bound on P's circumradius.

Note that it can be determined in polynomial time whether a given $\mathcal{H}$-polytope has interior points, and, if it does, such a point b can be found in polynomial time. Then, if necessary, a translation about $-b$ (and a suitable scaling) will transform the given $\mathcal{H}$-polytope into one of the kind used in 6.2.3. Hence, the condition on the right-hand side of P's $\mathcal{H}$-presentation does not impose any severe restrictions. It is not known whether a result similar to 6.2.3 can also be obtained for $\mathcal{V}$-polytopes; see [KhT93, p.158].

Now note that, as shown in [TaKE88], an approximation of E_0 of the kind given in Theorem 6.2.3 leads to the following inclusion:

$$a + A(\mathbb{B}^n) \subset K \subset a + \frac{n(1+3\sqrt{1-\gamma})}{\gamma}A(\mathbb{B}^n),$$

and hence leads, for every $\alpha > 1$, to an $(\alpha n)^{n/2}$-approximation of $V(P)$ for $\mathcal{H}$-polytopes P.

A similar bound can also be derived for convex bodies that are given by an appropriate oracle. In particular, Applegate & Kannan [ApK90] give the following algorithmic Löwner-John-type result for parallelotopes.

6.2.4 *There exists an oracle-polynomial-time algorithm which, for any well-bounded body K of $\mathbb{R}^n$ given by a weak separation oracle, finds a point a and a linear transformation A such that*

$$a + A([-1,1]^n) \subset K \subset a + 2(n+1)A([-1,1]^n).$$

While this result has direct applications in the design of improved randomized algorithms for volume computation (see Subsection 8.1), the following result of Betke & Henk [BeH93] gives a slightly better approximation error.

6.2.5 *There exists an oracle-polynomial-time algorithm which, for any body K of $\mathbb{R}^n$ given by a weak optimization oracle, and for every $\epsilon > 0$, finds rationals μ_1 and μ_2 such that*

$$\mu_1 \leq V(K) \leq \mu_2 \quad \text{and} \quad \mu_2 \leq n!(1+\epsilon)^n \mu_1.$$

In fact, two calls to a strong optimization oracle for directions c_1 and $-c_1$ give two supporting halfspaces H_1^+, H_1^- and two optimizers z_1^+, z_1^-. This procedure is now repeated for the directions $\pm c_2$, with a c_2 orthogonal to $\operatorname{aff}\{z_1^+, z_1^-\}$, etc. After n steps one gets the parallelotope $P = \bigcap_{i=1}^n (H_i^+ \cap H_i^-) \supset K$ and the polytope $Q = \operatorname{conv}\{z_i^+, z_i^- : i = 1, \ldots, n\} \subset K$, with $V(P)/V(Q) \leq n!$. The use of a weak rather than a strong optimization oracle brings in an additional factor $(1+\epsilon)^n$.

Let us now turn to the case of mixed volumes for some bodies $K_1, \ldots, K_r$. There are two natural general approaches to this problem, namely to approximate the bodies $K_1, \ldots, K_r$ by bodies $C_1, \ldots, C_r$, respectively, and then to use the corresponding mixed volumes of $C_1, \ldots, C_r$ as approximations, or to approximate $V(\sum_{i=1}^r \xi_i K_i)$ for appropriately chosen nonnegative parameters $\xi_1, \ldots, \xi_r$, and then to use the techniques outlined in Subsection 4.1 to derive estimates for the mixed volumes of $K_1, \ldots, K_r$. The remainder of this subsection will address both possibilities.

Note, first, that the Minkowski sum of two ellipsoids is, in general, no longer an ellipsoid. Hence a straightforward extension of the Löwner-John approach to mixed volumes fails because of the lack of an efficient algorithmic procedure for computing mixed volumes of ellipsoids. Also the approach of 6.2.4 is bound to fail for mixed volumes, for we have seen in Theorem 5.2.3 that computing mixed volumes of parallelotopes is #$\mathbb{P}$-hard. The general problem that we are facing here is that there don't seem to be rich enough classes of bodies (which could be used for approximating the given bodies $K_1, \ldots, K_r$) for which mixed volumes can actually be computed, and this is closely related to the obvious lack of rich enough classes of bodies for which the volume of their Minkowski sums can actually be computed.

There is however one case where the mixed volumes can be (weakly) computed, and this is the case $r = 2$ where $C_1 = \mathbb{B}^n$ and C_2 is a parallelotope. Recall, in fact, that by Theorem 4.4.4 the intrinsic volumes of an $\mathcal{S}$-parallelotope can be approximated (with respect to arbitrarily small additive error) in polynomial time. Hence we can combine Theorems 6.2.2 and 6.2.4 as follows. First we construct an ellipsoid $E = a_1 + A_1(\mathbb{B}^n)$ and a parallelotope $Z = a_2 + A_2([-1,1]^n)$ such that

$$\begin{aligned} &a_1 + A_1(\mathbb{B}^n) \subset K_1 \subset a_1 + (n+1)\sqrt{n}A_1(\mathbb{B}^n) \qquad \text{and} \\ &a_2 + A_2([-1,1]^n) \subset K_2 \subset a_2 + 2(n+1)A_2([-1,1]^n). \end{aligned}$$

Then, with

$$\begin{aligned} C_1' &= a_1 + A_1(\mathbb{B}^n), & C_1'' &= a_1 + (n+1)\sqrt{n}A_1(\mathbb{B}^n), \\ C_2' &= a_2 + A_2([-1,1]^n), & C_2'' &= a_2 + 2(n+1)A_2([-1,1]^n), \end{aligned}$$

we have

$$V(\overbrace{C_1', \ldots, C_1'}^{k}, \overbrace{C_2', \ldots, C_2'}^{n-k}) \leq V(\overbrace{K_1, \ldots, K_1}^{k}, \overbrace{K_2, \ldots, K_2}^{n-k}) \leq V(\overbrace{C_1'', \ldots, C_1''}^{k}, \overbrace{C_2'', \ldots, C_2''}^{n-k}).$$

By 2.4.2 (iii), application of the affine transformation $x \mapsto A_1^{-1}(x - a_1)$ changes the mixed volume only by the common factor $|\det(A_1)|^{-1}$, and this is irrelevant for relative approximation. But now we have arrived at an approximation by means of the intrinsic volumes of the parallelotope $Z = A_1^{-1}(a_2 + A_2([-1,1]^n) - a_1)$ which can, in fact, be (weakly) computed. Hence we can compute a lower bound μ $(= V(\overbrace{C_1', \dots, C_1'}^{k}, \overbrace{C_2', \dots, C_2'}^{n-k}))$ such that

$$\mu \le V(\overbrace{K_1, \dots, K_1}^{k}, \overbrace{K_2, \dots, K_2}^{n-k}) \le \big((n+1)\sqrt{n}\big)^k \big(2(n+1)\big)^{n-k} \mu.$$

Taking the geometric mean of the lower and upper bound and, if necessary, interchanging the roles of K_1 and K_2, we obtain the following result; see [DyGH94].

6.2.6 *There is a polynomial-time algorithm for* $\mathrm{O}(2^{n/4} n^{5n/8})$-APPROXIMATION *of all mixed volumes of any two well-bounded bodies K_1 and K_2 given by a weak separation oracle.*

The approximation error in 6.2.6 is only an upper bound for the precise value that we get from 6.2.2 and 6.2.4 with the outlined method; it is in fact

$$\lambda(n) = (n+1)^{n/2} \min\{n^{\frac{k}{4}} 2^{\frac{n-k}{2}}, n^{\frac{n-k}{4}} 2^{\frac{k}{2}}\}.$$

Note further that for $\mathcal{H}$-polytopes, 6.2.6 can be improved by using 6.2.3 rather than 6.2.2. However, we don't know of any result that extends 6.2.6 to the general case of n bodies. It is easy to obtain some approximation results that depend on auxiliary parameters such as the inradius or the circumradius of the specific bodies, but such results are much weaker than 6.2.6 which depends only on the dimension.

Another way of attempting to obtain, for some functional $\lambda : \mathbb{N} \to \mathbb{R}$, a λ-APPROXIMATION of certain mixed volumes, is to try to extend 4.1.1 to λ-APPROXIMATION.

Recall that 4.1.1 utilized the fact that an algorithm for approximating a polynomial with respect to the absolute error can be used to obtain approximations of the coefficients (again with respect to the absolute error). It turns out, however, that such a procedure does not exist with respect to the (symmetric) relative error. In fact, let us consider the following simple univariate example. Suppose that we want to estimate the middle coefficient α of a quadratic polynomial π with constant 1 and leading coefficient 1. In other words, we know that $\pi = \pi_\alpha = x^2 + \alpha x + 1$ for some α, and we want to find or approximate α. Now let $\epsilon > 0$, and suppose that $\eta_0, \dots, \eta_k$ are nodes at which we want to approximately evaluate π in order to estimate α. We may further suppose that $\eta_0, \dots, \eta_k > 0$ (for this is the only situation that is relevant in the context of MIXED VOLUME APPROXIMATION, and also, the construction can be easily adapted to the general case if desired). Now assume that the approximation oracle uses the exact values of $\pi_0 = 1 + x^2$ at $\eta_0, \dots, \eta_k$ to produce estimates for $\pi(\eta_0), \dots, \pi(\eta_k)$.

For $j = 0, \dots, k$ and each α with $0 \le \alpha \le \epsilon/(\max_{i=0,\dots,k} \eta_i)$, we have

$$\frac{\pi_0(\eta_j)}{\pi(\eta_j)} \le 1 \quad \text{and} \quad \frac{\pi(\eta_j)}{\pi_0(\eta_j)} \le 1 + \alpha\eta_j \le 1 + \epsilon.$$

Hence the approximation oracle produces estimates for the values of the polynomial with symmetric relative error bounded by ϵ. On the other hand, since α may be (at least) any coefficient between 0 and $\epsilon/(\max_{i=0,\ldots,k}\eta_i)$, we cannot use the approximations of the function values to obtain any symmetric relative approximation for this coefficient with finite error bound.

The obstruction here is the lack of some kind of correlation between the various coefficients of π. However, with mixed volumes we are here in a special situation since we can use the *Aleksandrov-Fenchel inequality.* For two bodies K_1 and K_2, 2.4.3 reads as follows:

$$V(\overbrace{K_1,\ldots,K_1}^{n-i},\overbrace{K_2,\ldots,K_2}^{i})^2 \geq V(\overbrace{K_1,\ldots,K_1}^{n-i+1},\overbrace{K_2,\ldots,K_2}^{i-1})V(\overbrace{K_1,\ldots,K_1}^{n-i-1},\overbrace{K_2,\ldots,K_2}^{i+1}),$$

This implies that the sequence of coefficients $\tau_0,\ldots,\tau_n$ is unimodal. Furthermore, in the special case of mixed volumes of two bodies an appropriate "scaling" can be utilized, [DyGH94].

6.2.7 *For any pair K_1, K_2 of well-bounded bodies given by a weak separation oracle and for any $k=1,\ldots,n$ one can construct in polynomial time an affine transformation α and a positive rational scaling factor λ such that the mixed volumes*

$$\tau_i = V(\overbrace{K_1',\ldots,K_1'}^{n-i},\overbrace{K_2',\ldots,K_2',}^{i}) \qquad i=k-1,k,$$

of the transformed bodies $K_1'=\alpha(K_1)$ and $K_2'=\lambda\alpha(K_2)$ satisfy the inequality

$$1 \leq \frac{\tau_{k-1}}{\tau_k} \leq (n+1)^{8k}.$$

Note that the right-hand bound does depend only on n and k, and not on special properties or measures of the bodies K_1 and K_2.

These special properties of mixed volumes can be used to obtain approximation results, and they are crucial for the randomized algorithm described in Subsection 7.2. There are, however, still major obstacles to extending Theorem 4.1.1 to relative volume approximation, and we will deal with these problems in Subsection 7.2.

6.3 Lower bounds in the oracle model.

It turns out that the above bounds for VOLUME APPROXIMATION are not too far away from the best one can achieve. ELEKES [El86] showed that even if our bodies K are given by a strong separation oracle, a subexponential number of calls to the oracle does not suffice to obtain a polynomial approximation. His argument is based on the following observation. Suppose that $K\subset\mathbb{B}^n$, that for some $k\in\mathbb{N}$ the inputs to the oracle are points $y_1,\ldots,y_k\in\mathbb{B}^n$, and suppose further that all membership tests are affirmative (and hence we never get a separating hyperplane). Then, with $P=\operatorname{conv}\{y_1,\ldots,y_k\}$ we know that $P\subset K\subset\mathbb{B}^n$, but this is all the information that is available, and based on this information an approximation μ of $V(K)$ is determined by our approximation algorithm. This implies that

$$\max_{\{K:P\subset K\subset\mathbb{B}^n\}}\left\{\frac{\mu}{V(K)},\frac{V(K)}{\mu}\right\} \geq \sqrt{\frac{V(\mathbb{B}^n)}{V(P)}}.$$

Now, ELEKES [El86] shows that

$$P \subset \frac{1}{2}\bigcup_{i=1}^{k}(y_i + \mathbb{B}^n),$$

and this yields

$$\frac{V(\mathbb{B}^n)}{V(P)} \geq \frac{k}{2^n}.$$

BÁRÁNY & FÜREDI [BaF86] improve this result by proving the following theorem.

6.3.1 *Suppose that*

$$\lambda(n) < \left(\frac{n}{\log n}\right)^{n/2} - 1 \qquad \textit{for all } n \in \mathbb{N}.$$

*Then there is no deterministic oracle-polynomial-time algorithm for λ-*APPROXIMATION *of the volume.*

Now, it is clear by Theorem 2.4.2 (ii) that Theorem 6.3.1 carries over to MIXED VOLUME APPROXIMATION simply because it includes the case where all bodies are the same. It is very likely, though, that in more general situations the bound of 6.3.1 can be improved. In particular, the worst-case approximation error for $V(K_1, \dots, K_n)$ (where the worst case is taken over all possible choices of $K_1, \dots, K_n$) should be much worse than $(n/\log n)^{n/2} - 1$.

7. RANDOMIZED ALGORITHMS

7.1 Approximating the volume.

As we have seen, volume computation and even volume approximation is hard when we restrict our algorithms to deterministic ones. The situation changes drastically if we allow randomized algorithms. In fact, DYER, FRIEZE & KANNAN [DyFK89] give a polynomial-time randomized algorithm for relative approximation of the volume of convex bodies that are given by appropriate oracles. The algorithm is a *random walk*, and its analysis is based on the notion of *rapidly mixing Markov chains*. We are going to describe the basic ideas of this approach, skipping however a lot of technical details, particularly those related to the stochastic analysis. For further details, background information, a sketch of the corresponding history and more references see the papers by DYER & FRIEZE [DyF91], KHACHIYAN [Kh93], LOVÁSZ [Lo92], [Lo94] and LOVÁSZ & SIMONOVITZ [LoS93].

EXPECTED VOLUME COMPUTATION.

Instance: *A positive integer n, a centered well-bounded body K in $\mathbb{R}^n$ given by a weak membership oracle, positive rationals β and ϵ.*

Task: *Determine a positive rational random variable μ such that*

$$\text{prob}\left\{\left|\frac{\mu}{V(K)} - 1\right| \leq \epsilon\right\} \geq 1 - \beta.$$

Note that in the above problem, the relative error measure is employed; see Subsection 6.1. This indicates already that we are aiming at "close approximation," and in fact, the main theorem of this section due to DYER, FRIEZE & KANNAN [DyFK89] is as follows.

7.1.1 *There is a randomized algorithm for* EXPECTED VOLUME COMPUTATION *which runs in time that is oracle-polynomial in* n, $1/\epsilon$ *and* $\log(1/\beta)$.

Before giving an (informal) description of the algorithm let us clarify that the existence of a polynomial-time randomized algorithm for volume computations does not contradict the negative results of Subsections 5.1 and 6.3. In fact, for a deterministic algorithm all that counts is what it produces as output, while for a nondeterministic algorithm what can *potentially* be produced is relevant. In fact, the results depend on the distribution of these potential outcomes rather than on the outcomes themselves. As will become clear, the randomized algorithm described below does have the potential to reach *exponentially* many points, and this is crucial for the polynomial running time.

Let us now describe the original algorithm for 7.1.1; some improvements will be outlined later in this subsection. The first step is a rounding procedure that utilizes (in conjunction with 1.2.1) the algorithmic version 6.1.2 [GrLS88] of JOHN's [Jo48] result. According to this version, there exists an oracle-polynomial-time algorithm which, for any well-bounded body K of $\mathbb{R}^n$ given by a weak separation oracle, finds a point a and a linear transformation A such that

$$a + A(\mathbb{B}^n) \subset K \subset a + (n+1)\sqrt{n}A(\mathbb{B}^n).$$

Hence,

$$\mathbb{B}^n \subset A^{-1}a - A^{-1}(K) \subset (n+1)\sqrt{n}\mathbb{B}^n.$$

This rounding procedure is a deterministic algorithm that uses $\mathrm{O}(n^4(\mathrm{size}(r) + \mathrm{size}(R)))$ operations on numbers of size $\mathrm{O}(n^2(\mathrm{size}(r)+\mathrm{size}(R)))$, where r, R are (as usual) the a priori bounds for K's inradius and circumradius; see [GrLS88, p.122]. Since

$$V(K) = |\det(A)|V\big(A^{-1}a + A^{-1}(K)\big),$$

we may, for the second step of the randomized algorithm, assume that

$$\mathbb{B}^n \subset K \subset (n+1)\sqrt{n}\mathbb{B}^n.$$

One could now try to estimate the ratio $V((n+1)\sqrt{n}\mathbb{B}^n)/V(K)$ by means of a randomized procedure. However, this ratio may be exponential, and this leads to a blowup of the complexity of the randomized approach outlined below. For this reason, the next step reduces the problem to a series of problems with suitably bounded volume ratios. Let

$$k = \left\lceil \frac{3}{2}(n+1)\log(n+1) \right\rceil, \qquad \text{and} \qquad K_i = K \cap \left(1 + \frac{1}{n}\right)^i \mathbb{B}^n \quad \text{for } i = 0, \dots, k.$$

Then

$$\mathbb{B}^n = K_0 \subset K_1 \subset \cdots \subset K_{k-1} \subset K_k = K \subset (n+1)\sqrt{n}\mathbb{B}^n,$$

and, more importantly, for $i = 1, \dots, k$,

$$1 \le \frac{V(K_i)}{V(K_{i-1})} \le \left(1 + \frac{1}{n}\right)^n < e.$$

Clearly,

$$V(K) = V(\mathbb{B}^n) \prod_{i=1}^{k} \frac{V(K_i)}{V(K_{i-1})},$$

whence it suffices to estimate each ratio $V(K_i)/V(K_{i-1})$ up to a relative error of order $\epsilon/(n \log n)$ with error probability of order $\beta/(n \log n)$.

Now, the main step of the algorithm of Dyer, Frieze & Kannan [DyFK89] is based on a method for sampling nearly uniformly from within K_i. It superimposes a chess-board grid of small cubes (say of edge length δ) on K_i (compare 3.4 and 3.5) and performs a random walk over the set $\mathcal{C}_i$ of cubes in this grid that intersect a suitable parallel body $K + \alpha\mathbb{B}^n$ where α is small. This walk is performed by moving through a facet with probability $1/f_{n-1}(C_n) = (2n)^{-1}$ if this move ends up in a cube of $\mathcal{C}_i$, and staying at the current cube if the move would lead outside of $\mathcal{C}_i$. The random walk gives a *Markov chain* which is irreducible (since the moves are connected), aperiodic and hence ergodic. But this implies that there is a unique stationary distribution, the limit distribution of the chain, which is easily seen to be a *uniform distribution*. Thus after a sufficiently large number of steps we can use the current cube in the random walk to sample nearly uniformly from $\mathcal{C}_i$. Having obtained such a uniformly sampled cube, it is determined whether it belongs to $\mathcal{C}_{i-1}$ or to $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$.

Now note that if ν_i is the number of cubes in $\mathcal{C}_i$, then the number $\mu_i = \nu_i/\nu_{i-1}$ is an estimate for the volume ratio $V(K_i)/V(K_{i-1})$. It is this number μ_i that can now be "randomly approximated" using the above constructed approximation of a uniform sampling over $\mathcal{C}_i$. In fact, a cube C that is reached after sufficiently many steps in the random walk will lie in $\mathcal{C}_{i-1}$ with probability approximately $1/\mu_i$; hence by repeated sampling we can approximate this number closely.

This informal description of the randomized algorithm must of course be rigorously analyzed to determine its complexity. A main question is just how quickly the random walk approximates a "reasonably uniform" distribution. In their analysis, Dyer, Frieze & Kannan [DyFK89] use a result of Sinclair & Jerrum [SiJ89] that relates the speed of convergence to the *conductance* of the chain. With the aid of a geometric interpretation of this quantity and an isoperimetric inequality of Bérad, Besson & Gallot [BeBG85] on the volume of subsets of smooth Riemannian manifolds with positive curvature, it is shown in [DyFK89] that the Markov chain is, indeed, mixing rapidly enough to yield polynomiality. The following inequality (which is stronger than what was needed in [DyFK89]'s original proof) is taken from [DyF91]; see also [LoS90], [ApK90] and [LoS93].

7.1.2 *Let K be a convex body in $\mathbb{R}^n$, and let f be a real-valued log-concave function on* $\mathrm{int}(K)$. *Further, let $S_1, S_2 \subset K$ be measurable, $S = K \setminus (S_1 \cup S_2)$, and suppose that* $\mathrm{dist}(S_1, S_2) > 0$. *Then*

$$\min\left\{\int_{S_1} f(x)dx, \int_{S_2} f(x)dx\right\} \le \frac{R_1(K)}{\mathrm{dist}(S_1,S_2)} \int_S f(x)dx,$$

where $R_1(K)$ is half of K's diameter.

A corollary which conveys the flavor of this inequality (and which is sufficient for the proof of polynomiality of the randomized volume-algorithm) says that if K is a convex body in $\mathbb{R}^n$, and S is a minimal surface that partitions K into two sets S_1, S_2, then

$$\min\{V(S_1), V(S_2)\} \leq R_1(K)A(S),$$

where $A(S)$ denotes the surface area of S. This formulation shows that 7.1.2 is an extension of the result that a body K is contained in any cylinder whose base is the projection of K on the hyperplane orthogonal to some direction u, and whose height in direction u is K's breadth in this direction.

[DyFK89]'s polynomial-time randomized algorithm for Expected Volume Computation was subsequently improved in various papers, including [LoS90], [ApK90], [DyF91], [LoS93] and [KaLS94].

One key ingredient for improvements is 7.1.2, while another major improvement can be obtained by replacing the "rounding"

$$\mathbb{B}^n \subset K \subset (n+1)\sqrt{n}\mathbb{B}^n$$

by the "normalization"

$$[-1,1]^n \subset K \subset 2(n+1)[-1,1]^n;$$

Applegate & Kannan [ApK90], see 6.2.4. Another idea of [ApK90] that avoids difficulties caused by inherent "nonsmoothness" is to approximate the characteristic function of K by a smooth function; cf. 3.4.

Lovász & Simonovits [LoS93] improve on these ideas extend the theory of conductance and rapid mixing from the finite case to arbitrary Markov chains (so that now steps can be chosen uniformly from a ball with fixed radius about the current point), and replace the rounding phase by an "approximate sandwiching:" an affine transformation α is produced such that 2/3 of the volume of $\mathbb{B}^n$ is contained in $\alpha(K)$ and 2/3 of the volume of $\alpha(K)$ is contained in $n\mathbb{B}^n$. In their extensive study, they achieve the following complexity bound for the second step (after the normalization) of

$$\mathrm{O}\left(\frac{1}{\epsilon^2} n^7 \log^2(n) \log^3(\frac{1}{\epsilon}) \log(\frac{1}{\beta})\right).$$

Very recently, Kannan, Lovász & Simonovits [KaLS94] gave a further substantial improvement; see [Lo94]. They achieve the currently best known bound where now n enters only in fifth power.

Let us close this subsection with a few remarks.

Sometimes it is possible to devise random walks not over a superimposed grid of cubes but over objects that are more closely related to the specific bodies. One natural example is the class of *order polytopes*. As we have seen in Subsection 5.1 (the discussion preceding Theorem 5.1.4), an order polytope can be dissected into simplices of the same volume which correspond to the linear extensions of the given order $\mathcal{O}$. This approach gives rise to a random walk over the linear extensions of $\mathcal{O}$ which, itself, has interesting applications; see Karzanov & Khachiyan [KaK90], Khachiyan [Kh93] and Lovász [Lo94].

A second class of bodies that come with a natural dissection are the zonotopes. Zonotopes can be dissected into parallelotopes, and it is intriguing to try to use these parallelotopes instead of the cubes. Unfortunately, the volumes of the parallelotopes may in general vary exponentially, and hence a direct extension of the above approach will work only in very special cases. Thus it is unknown whether, for general zonotopes, there is a randomized algorithm for volume computation that is more efficient than randomized algorithms that work for arbitrary convex bodies.

The key step of the randomized volume-algorithms is to compute a *nearly uniform distribution* on a body K. DYER & FRIEZE [DyF91] show that the converse is also true: A polynomial number of calls to a volume approximator suffice to generate with high probability uniformly distributed points in K.

7.2 Approximating mixed volumes.

Now that we have a randomized polynomial-time algorithm at hand for solving EXPECTED VOLUME COMPUTATION, it is natural to try to use it for devising a similar procedure for mixed volumes. This subsection will outline such an approach of [DyGH94].

Let us begin with the case of two centered well-bounded bodies K_1 and K_2 that are given by weak membership oracles. Let us consider the polynomial π given by

$$\pi(\xi) = V(K_1 + \xi K_2) = \sum_{i=1}^{n} \binom{n}{i} V(\overbrace{K_1, \ldots, K_1}^{n-i}, \overbrace{K_2, \ldots, K_2}^{i})\xi^i.$$

We will sometimes use the abbreviation

$$\tau_i = V(\overbrace{K_1, \ldots, K_1}^{n-i}, \overbrace{K_2, \ldots, K_2}^{i}) \quad \text{and} \quad \zeta_i = \binom{n}{i}\tau_i.$$

Then, following the approach of Subsection 4.1 our goal is to use the randomized volume algorithm to evaluate the polynomial $\pi(\xi) = \sum_{i=1}^{n} \zeta_i \xi^i$ at suitable nodes in order to obtain estimates for its coefficients ζ_i.

As we have already seen in Subsection 6.2, there is no general way to derive *relative* estimates for the coefficients of a polynomial from *relative* estimates of certain function values of π. However, we are here in a special situation in which we can use both 6.2.7 and the specialization to two bodies of the *Aleksandrov-Fenchel inequality.* As we have already mentioned at the end of Subsection 6.2, it turns out, though, that there are still major obstacles to extending Theorem 4.1.1 to randomized relative volume approximation, and before we state the results of [DyGH94] we want to point out what the additional problems are.

As we have seen in Subsection 4.1, computing the coefficients of a polynomial from some of its values can in principle be done by numerical differentiation. Let $\eta_0, \ldots, \eta_n$ be pairwise different *interpolation points*, and let for $j = 0, \ldots, n$

$$\ell_j(\xi) = \sum_{i=0}^{n} \beta_{ij}\xi^i$$

denote the jth *Lagrange interpolation polynomial* on the node set $Y = \{\eta_0, \ldots, \eta_n\}$. Recall that for $j, k = 0, \ldots, n$,

$$\ell_j(\xi_k) = \sum_{i=0}^{n} \beta_{ij}\xi_k^i = \delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\pi(\xi) = \sum_{j=0}^{n} \pi(\xi_j)\ell_j(\xi) = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \beta_{ij}\pi(\xi_j) \right) \xi^i,$$

whence for each $i = 0, \dots, n$,

$$\zeta_i = \sum_{j=0}^{n} \beta_{ij}\pi(\xi_j).$$

Now, suppose we have approximations $\mu_0, \dots, \mu_n$ of the values $\pi(\xi_0), \dots, \pi(\xi_n)$, respectively, with relative error bounded by some $\epsilon > 0$, and for $i = 0, \dots, n$ we use

$$\hat{\zeta}_i = \sum_{j=0}^{n} \beta_{ij}\mu_j$$

as an estimate for ζ_i. Then it is easy to see that

$$|\hat{\zeta}_i - \zeta_i| \le \epsilon \max_{k=0,\dots,n} \pi(\xi_k) \sum_{j=0}^{n} |\beta_{ij}|,$$

and this bound is tight. This means, in order to bound the relative error of the approximation $\hat{\zeta}_k$ of ζ_k we need to be able to bound the right-hand side in terms of ζ_k. Unfortunately, as is pointed out in [DyGH94], $\max_{k=0,\dots,n} \pi(\xi_k) \sum_{j=0}^{n} |\beta_{ij}|$ grows exponentially, and that is why only a certain portion of the coefficients may become approximable by such an approach. (Recall that the randomized volume algorithm is polynomial in $1/\epsilon$ but exponential in $\log(1/\epsilon)$.) Hence we introduce a version of the problem that depends on an additional function $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ with $\psi(n) \le n$ for every $n \in \mathbb{N}_0$.

Expected ψ-Mixed Volume Computation.

Instance: *A positive integer n, centered well-bounded bodies K_1 and K_2 in $\mathbb{R}^n$ given by weak membership oracles, positive rationals β and ϵ.*

Task: *Determine for each nonnegative integer i with $i \le \psi(n)$ a positive rational random variable $\hat{\tau}_i$ such that*

$$\text{prob}\left\{ \left| \frac{\hat{\tau}_i}{\tau_i} - 1 \right| \le \epsilon \right\} \ge 1 - \beta.$$

Then [DyGH94] prove the following theorem.

7.2.1 *Let $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ with $\psi(n) \le n$ for every $n \in \mathbb{N}_0$, and suppose that*

$$\psi(n) \log \psi(n) = o(\log n).$$

Then there is a randomized algorithm for Expected ψ-Mixed Volume Computation *which runs in time that is oracle-polynomial in n, $1/\epsilon$ and $\log(1/\beta)$.*

Observe that $\psi(n) = \lceil \log(n)/\log^2 \log(n) \rceil$ is a choice that satisfies the assumptions of Theorem 7.2.1.

The algorithm underlying 7.2.1 proceeds inductively, beginning with τ_0 which can be approximated by the volume procedure 6.1.1. Suppose that for some k all mixed volumes $\tau_0, \ldots, \tau_{k-1}$ have already been approximated. As was mentioned earlier, the algorithm now uses the scaling described in 6.2.7 as preprocessing for the next step. This yields a rough estimate for τ_k. Then, using the volume algorithm again and choosing the nodes appropriately, approximations of $\pi(\xi)$ are computed. Next, a binary search procedure is used to improve the initial relative estimate of τ_k to within a constant error, and finally the last step achieves an approximation of τ_k to within a relative error ϵ, as desired. Of course, the interpolation points now depend on ϵ, and they are chosen in such a way that the higher order terms of π can be bounded appropriately so as to allow the use of only a small part of the coefficient matrix $B = (\beta_{ij})$. This makes it possible to keep the error small.

It may be worth mentioning that as compared to algorithms for EXPECTED VOLUME COMPUTATION, the complexity of the above algorithm is only marginally worse.

Let us point out explicitly that it is not known whether EXPECTED ψ-MIXED VOLUME COMPUTATION can be solved in polynomial time under assumptions on ψ that are less restrictive than those stated in Theorem 7.2.1. In particular, it is not known how to efficiently approximate $V(\overbrace{K_1, \ldots, K_1}^{n}, \overbrace{K_2, \ldots, K_2}^{n})$ for bodies in $\mathbb{R}^{2n}$.

On the positive side, it is possible to extend 7.2.1 to the case of more than two bodies and to show that there is a randomized polynomial time algorithm for computing

$$V(\overbrace{K_1, \ldots, K_1}^{i_1}, \overbrace{K_2, \ldots, K_2}^{i_2}, \ldots, \overbrace{K_{r-1}, \ldots, K_{r-1}}^{i_{r-1}}, \overbrace{K_r, \ldots, K_r}^{i_r}),$$

where $\sum_{j=1}^{r} i_j = n$ and $\sum_{j=1}^{r-1} i_j = \psi(n)$ with a function ψ as in 7.2.1.

In fact, suppose we have a procedure for r sets. Then, utilizing the multilinearity of the mixed volume, we consider

$$\pi(\xi) = V(\overbrace{K_1, \ldots, K_1}^{i_1}, \ldots, \overbrace{K_{r-1}, \ldots, K_{r-1}}^{i_r - 1}, \overbrace{K_r + \xi K_{r+1}, \ldots, K_r + \xi K_{r+1}}^{i_r + i_{r+1}}),$$

which can be estimated recursively for fixed ξ. On the other hand, π is a polynomial of degree $i_r + i_{r+1}$ in ξ, for which we wish to estimate the coefficient of $\xi^{i_{r+1}}$. The coefficients of π are themselves mixed volumes, and consequently satisfy the Alexandrov-Fenchel inequalities. Thus the approach above for two sets can be used with very little change. There is, however, one difficulty. We do not have a polynomial-time procedure for producing a "good" initial scaling of the sets, as we had with 6.2.7 for two bodies; and we leave as an open question whether such a procedure exists. Without such a scaling, one has to resort to the "well-boundedness" parameters r_k, R_k that come as bounds for the inradius and the circumradius of the bodies. Unfortunately, these parameters may be exponentially large, and this feeds into the recursion. However, [DyGH94] show that one can approximate the mixed volumes for any *fixed* r in polynomial time, where each of the first $r - 1$ sets may be repeated up to $o(\log n / \log\log n)$ times. Further, if the ratios R_k/r_k are "quasi-polynomial" in n, i.e. of the form $O(2^{\gamma(\log(n))})$, where γ is a polynomial, we can approximate mixed volumes for any $r = o(\log n / \log\log n)$ in polynomial time. For larger ratio $\rho = \max_{k=1,\ldots,r} R_k/r_k$ we can approximate up to $r = o(\log n / \log\log \rho)$ in similar time.

Let us finally point out that, particularly in view of the applications stated in Subsections 9.6 – 9.9, it would be desirable to be able to extend the above results to the general case. Specifically, it would be useful to be able to compute $V(K_1, \ldots, K_n)$ by means of a randomized polynomial time algorithm. It is not known whether such a procedure exists.

8. Miscellaneous

In the present section, we will mention some results that are closely related to volume computation.

8.1 Projections and sections.

The problem of maximizing or minimizing the volumes of orthogonal projections of polytopes onto hyperplanes has received some attention in geometry because it is related to various illumination and optimization problems, see e.g. Martini [Ma85]. It has been treated from a computational viewpoint by McKenna & Seidel [McS85], whose algorithm finds a direction in which the orthogonal projection has maximum (or minimum) volume. Their algorithm is asymptotically optimal when the dimension is fixed.

The more general case of projections onto subspaces of arbitrary intermediate dimension is studied (for fixed and for variable dimensions) in [BuGK94a]. Let $\gamma : \mathbb{N} \to \mathbb{N}$ denote a functional with the property that $1 \leq \gamma(n) \leq n-1$ for each n. Then we have the following decision problems.

Maximum γ-Projection ⟨Minimum γ-Projection⟩.

Instance: *A positive integer n, an $\mathcal{H}$-polytope ⟨a $\mathcal{V}$-polytope, or an $\mathcal{S}$-zonotope⟩ P, a nonnegative rational μ.*

Question: *Is there a $\gamma(n)$-dimensional subspace S of $\mathbb{R}^n$ such that $V^2_{\gamma(n)}(\Pi_S P) \geq \mu$ ⟨$V^2_{\gamma(n)}(\Pi_S P) \leq \mu$⟩?*

Here, as before, $V_{\gamma(n)}(\Pi_S P)$ denotes the $\gamma(n)$-dimensional volume of the orthogonal projection $\Pi_S P$ of P on S.

Note that with the special choice $\gamma \equiv 1$, Maximum γ-Projection is the problem of lower bounding (the square of) a polytope P's *diameter*. This problem is easy for $\mathcal{V}$-polytopes; however, it is already $\mathbb{NP}$-complete for $\mathcal{H}$- (or $\mathcal{S}$-) parallelotopes, [BoGKL90], [GrK93a]; see also [GrK94a].

In view of the results of Subsection 5.1, it is not surprising that the variants of Maximum γ-Projection and Minimum γ-Projection that ask for the actual volumes of optimal projections are #$\mathbb{P}$-hard. However, it turns out that Maximum γ-Projection is hard for other reasons as well. In fact, even for $\gamma(n) = n-1$ (the case of projections onto hyperplanes), the problem Maximum γ-Projection is $\mathbb{NP}$-complete even for the class of all ($\mathcal{V}$- or $\mathcal{H}$-) simplices ([BuGK94a]), even though the $(n-1)$-dimensional volume of any projection of a (rational) simplex on a (rationally presented) hyperplane can be computed in polynomial time. On the other hand, minimizing projections of simplices on hyperplanes is easy, but Minimum γ-Projection is $\mathbb{NP}$-hard for many classes of functionals γ and polytopes P (see [BuGK94a]).

Recalling from Subsection 4.4 that for any $z \in \mathbb{S}^{n-1}$,

$$nV([0,1]z, \overbrace{K,\ldots,K}^{n-1}) = V(\Pi_{\mathrm{lin}\{z\}^\perp}(K)),$$

these results imply that the problem of maximizing $V([0,1]z, \overbrace{K,\ldots,K}^{n-1})$ is already $\mathbb{NP}$-hard for K being a simplex, while the problem of minimizing $V([0,1]z, \overbrace{K,\ldots,K}^{n-1})$ is $\mathbb{NP}$-hard for arbitrary $\mathcal{H}$-polytopes (but easy for simplices). Extensions of these and other results can be found in [BuGK94a].

For some interesting theoretical results on projections see FILLIMAN [Fi88], [Fi90] and [Fi92]. The problem of estimating the intrinsic volume $V_i(K)$ of a body K from the intrinsic volumes $V_i(\Pi_{S_j} K)$ of K's projections onto certain j-dimensional subspaces $S_1, \ldots, S_m$ (with $1 \le i \le j \le n-1$) has been studied by BETKE & MCMULLEN [BeM83].

Problems similar to those for projections can also be investigated for sections (with some additional constraints in the case of minimizing sections). In fact, for $\gamma \equiv 1$, the problem MAXIMIMUM γ-PROJECTION is the same as the (appropriately defined) problem MAXIMIMUM γ-SECTION, and the latter is hence again $\mathbb{NP}$-hard. Additional algorithmic results can be found in [BuGK94b].

The general problem of finding the maximum of the volumes of the j-dimensional sections of P (i.e., of the j-dimensional convex sets formed by intersecting P with a j-flat) is discussed by FILLIMAN [Fi92], who finds geometric conditions that must be satisfied by critical sections. For results related to extremal j-sections of simplices and cubes, see [Wa68], [Fi92].

Finally, we mention the survey article of MARTINI [Ma94], which discusses a variety of questions related to sections and projections.

8.2 Expected volumes.

For a proper body K in $\mathbb{R}^n$ and an integer $m > n$, let $\varphi_m(K) = \mu_m(K)/V(K)$, where $\mu_m(K)$ is the expected volume of the convex hull of m points chosen independently and at random from the uniform distribution over K. For each m this is an affine invariant of K, because volume ratios are invariant under nonsingular affine transformations. The literature contains many results concerning the functions φ_m, and a good short survey with many references was given by CROFT, FALCONER & GUY [CrFG91, pp.54–57]. See BÁRÁNY & BUCHTA [BaB93] for later results and references.

Despite the extensive literature, we are not aware of any general algorithmic approach to the computation of $\varphi_m(P)$ when P is a given n-polytope. Indeed, even the numbers $s(n) = \varphi_{n+1}(S)$ for an n-simplex S have proved to be resistant. (These numbers are of interest for a comparison of the efficiency of two algorithms for the analysis of multicomponent phase diagrams; see [Kl65].) Although it is easy to see that $s(1) = 1/3$ and has long been known that $s(2) = 1/6$, for many years the best that could be done with $s(3)$ was to approximate it by means of Monte Carlo experiments (see [BuR92] and its references). Recently, however, BUCHTA & REITZINGER [BuR92] showed in a tour de force that

$$s(3) = \frac{13}{720} - \frac{\pi^2}{15015}.$$

For $n > 3$, $s(n)$ is still not known precisely.

It seems reasonable to conjecture that for each fixed n, $\varphi_{n+1}(K)$ is a maximum when K is a simplex, but the conjecture is open for all $n \geq 3$. GROEMER [Gr73] showed for all n that $\varphi_{n+1}(K)$ is a minimum when K is an ellipsoid, and the value in this case has been computed by KINGMAN [Ki69].

8.3 Volumes of unions and intersections of special bodies.

The special bodies to be discussed in this subsection are boxes, balls, and simplices. For $a = (\alpha_1, \ldots, \alpha_n)^T, b = (\beta_1, \ldots, \beta_n)^T \in \mathbb{R}^n$ with $\alpha_i \leq \beta_i$ for all i, let $B(a, b)$ denote the box

$$B(a, b) = \{x = (\xi_1, \ldots, \xi_n)^T : \alpha_i \leq \xi_i \leq \beta_i \quad \text{for } 1 \leq i \leq n\}.$$

Now suppose that $B(a_1, b_1), \ldots, B(a_k, b_k)$ are boxes with $a_j = (\alpha_{j,1}, \ldots, \alpha_{j,n})$ and $b_j = (\beta_{j,1}, \ldots, \beta_{j,n})$ for $1 \leq j \leq k$, and for $1 \leq i \leq n$ let

$$\overline{\alpha}_i = \max\{\alpha_{1,i}, \ldots, \alpha_{k,i}\},$$

and

$$\underline{\beta}_i = \min\{\beta_{1,i}, \ldots, \beta_{k,i}\}.$$

Then the intersection $\bigcap_{i=1}^k B(a_i, b_i)$ is empty if $\overline{\alpha}_i > \underline{\beta}_i$ for some i, and otherwise the intersection is the box $B(\overline{a}, \underline{b})$. This representation yields a fast algorithm for computing volumes of intersections of boxes.

Note that any algorithm for computing volumes of intersections of bodies of a special sort yields also an algorithm for computing volumes of unions of the same sort of bodies. That is true because the volume function is a valuation and hence

$$\begin{aligned} V\left(\bigcup_{i=1}^m K_i\right) &= \sum_i V(K_i) - \sum_{i<j} V(K_i \cap K_j) + \\ &\quad + \sum_{i<j<k} V(K_i \cap K_j \cap K_k) - \cdots + (-1)^{m-1} V\left(\bigcap_{i=1}^m K_i\right). \end{aligned}$$

However, this direct use of the principle of inclusion and exclusion is often not the best way to compute volumes of unions. For better ways to compute the volume of a union of k boxes in $\mathbb{R}^n$, see FREDMAN & WEIDE [FrW78] for an optimal $O(k \log k)$ algorithm when $n = 1$; see VAN LEEUWEN & WOOD [VaW81] for an $O(k \log k)$ algorithm (doe to J.L. Bentley) when $n = 2$ and for an $O(k^{n-1})$ algorithm when $n \geq 3$.

Among the papers that contain algorithms for computing volumes of unions and intersections of balls, we mention [Au86], [AvBI88], [Sp85], and especially EDELSBRUNNER [Ed93b] and EDELSBRUNNER & FU [EdF93]. See Subsection 9.11 for a suggested use of such algorithms in experimental computation on a famous unsolved problem.

¿From the viewpoint of computational complexity, the most interesting problem to be mentioned in this subsection is that of computing the volume of a union of n d-simplices in $\mathbb{R}^d$. (The change in notation – d rather than n for the dimension – is necessary in order to conform to the notation in the term *n^2-hard* below that is standard in the relevant part of the literature.) When $d = 2$, this problem belongs to the class of so-called n^2-*hard* problems introduced by GAJENTAAN & OVERMARS [GaO93] with the aid of quadratic transformations from a "base problem" that is linearly equivalent to the following:

> Given three sets of integers A, B, and C with $|A| + |B| + |C| = n$, decide whether there exist $a \in A$, $b \in B$, and $c \in C$ such that $a + b = c$?

For the problem of computing the area of a union of triangles, as for other problems in the class, there are no known subquadratic algorithms. (In addition to the original paper of [GaO93], see [Or94b], [EfLS93], [ErS93], and their references for more details. The present account is taken from [Or94b].) From the viewpoint of computational convexity, it would be interesting to know what can be said for $d > 2$ about the complexity of computing the volume of the union of n simplices in $\mathbb{R}^d$, and what happens when the dimension d is part of the input.

8.4 More about dissections and triangulations.

As we have seen, any sort of deterministic computation of the volume of a polytope P is apt to be time-consuming. However, since the volume of a simplex is so easy to compute, and since dissecting P into simplices is easy to understand and not hard to program (see Subsection 3.1), the use of such a dissection is the most convenient method in many practical cases. When faced with a polytope P whose volume is to be computed by means of dissection into simplices, it is natural to wonder what is the *minimum* number of simplices possible. That suggests the following decision problems.

8.4.1 Given a positive integer n, a proper $\mathcal{H}$-polytope $\langle$or $\mathcal{V}$-polytope$\rangle$ $P \subset \mathbb{R}^n$, a positive integer k.

- (A) Can P be dissected into k or fewer n-simplices, each having all of its vertices among those of P?
- (B) Can P be dissected into k or fewer n-simplices, each having all of its vertices at points of P that have rational coordinates?
- (C) Can P be dissected into k or fewer n-simplices?

Note that in 8.4.1 (A), only the vertices of P can be used in forming the n-simplices of the dissection. It seems plausible that the minimum number of such simplices cannot be reduced by the use of additional vertices. However, we are not aware of any proof of this even for the case in which P is an n-cube. In 8.4.1 (B), additional vertices are permitted but are required to have all rational coordinates (as do the vertices of P), while in 8.4.1 (C) there is no restriction on the position of additional vertices. Hence it is conceivable that these three similar-sounding problems are of different computational complexity. It can be shown that 8.4.1 (A) belongs to the class $\mathbb{NP}$. However, for 8.4.1 (B) it does not seem obvious even that there exists a finite decision algorithm. (This is vaguely reminiscent of the fact that the problem of deciding whether a given polytope is combinatorially equivalent to one with exclusively rational vertices is algorithmically equivalent to the problem of deciding whether a diophantine equation is solvable in rationals (see STURMFELS [St87]) – and it is not known whether there is an algorithm for the latter problem (see [KlW91, p.95]).) For 8.4.1 (C), the existence of a finite decision algorithm follows from the decision theory of TARSKI [Ta61] (see also [ChK73] and RENEGAR [Re92a], [Re92b], [Re92c]) because the existence of a dissection of the desired sort can be expressed in terms of the consistency (now over $\mathbb{R}^n$ rather than $\mathbb{Q}^n$) of a system of polynomial equalities and inequalities involving the vertex-coordinates.

When the above problems are posed for *triangulations* (as opposed to dissections), the above statements about computational complexity still apply to 8.4.1 (A) and 8.4.1 (C). However, in the case of triangulations, the same decision algorithm that works for 8.4.1 (C) applies also to 8.4.1 (B). That is a consequence of the following fact: If P is a polytope whose vertices are all at rational points, and $\mathcal{T}$ is a triangulation of P, then the nonrational vertices of $\mathcal{T}$ can be moved, one at a time, to nearby rational points so as to produce a triangulation $\mathcal{T}'$ of P whose vertices are all at rational points.

Though it has nothing to do with computing the volume of a polytope, we want to mention the following fact, simply for its intrinsic interest: An n-cube can be dissected into k n-simplices of equal volume if and only if k is a multiple of $n!$. This was first proved for $n = 2$ in [Mo70], then extended to arbitrary n in [Me79]. The proof depends in an essential way on valuation theory. For further results on dissecting polygons into triangles of equal area, see [KaS90] and [Mo90].

There is an extensive literature concerning the following questions.

8.4.2 What is the minimum number $T(n)$ $\langle S(n)\rangle$ of n-simplices into which an n-cube can be triangulated $\langle$dissected$\rangle$?

The number $S(n)$ has been fully determined only for $n \leq 5$, $T(n)$ only for $n \leq 7$ (see Hughes [Hu93] and Hughes & Anderson [HuA94]), and as $n \to \infty$ the best asymptotic lower and upper bounds are far apart. The best asymptotic lower bounds result from volume considerations (see [Ha91] for references), and the best asymptotic upper bounds come from the construction of specific triangulations in low dimensions together with a simple but elegant method of Haiman [Ha91] for extending these to higher dimensions. For $n \leq 11$, the best lower bounds for both $S(n)$ and $T(n)$ come from a linear programming approach proposed by Sallee [Sa82] and developed further in [Hu93] and [HuA94].

Triangulations of n-cubes are of interest for their role in complementary pivoting algorithms used to find approximately fixed points of continuous mappings [To76]. In this connection, Todd [To76] proposes the number $(\mathrm{card}(\mathcal{T})/n!)^{1/n}$ as a measure of the efficiency of a triangulation $\mathcal{T}$ of the n-cube. The construction of [Ha91] shows that any value of this measure that is attained for some fixed n is also attainable asymptotically. However, this measure does not tell the whole story of efficiency, for it often happens that triangulations into fewer simplices require more complicated pivoting rules. See [ToT93] and [HuAH94] for the details of some recent triangulations and for references to earlier work.

To end this subsection, we mention that Ong [On91], [On94], has analyzed a triangulation of the 3-cube that is notable for a number of geometric properties that make it especially convenient for use in the finite-element method for approximating solutions to partial differential equations. It would be worthwhile to produce and study higher-dimensional analogues of her triangulation.

9. Applications

This last section collects some of the more recent and probably less known applications of volume and mixed volume computation. The applications 9.3–9.5 and part of 9.9 were explored in more detail in [DyF91], the applications of 9.1, 9.2, 9.6–9.8 and part of 9.9 are dealt with in [DyGH94].

9.1 Counting integer points in lattice polytopes.

As we have seen already in Subsection 3.5, there is a close connection between Volume Computation and counting lattice points. Here we show how a "mixed-volume-like" approach to lattice-point enumeration can be used to deduce some complexity results.

Let $\mathbb{L}$ denote an integer lattice of $\mathbb{R}^n$ whose vectors span $\mathbb{R}^n$, denote by $\mathcal{P}^n(\mathbb{L})$ the set of all polyopes in $\mathbb{R}^n$ whose vertex set belongs to $\mathbb{L}$, and let $G_{\mathbb{L}}$ denote the *lattice-point enumerator*, i.e. the functional $G_{\mathbb{L}} : \mathcal{P}^n(\mathbb{L}) \to \mathbb{N}_0$ defined by $G_{\mathbb{L}}(P) = \mathrm{card}(P \cap \mathbb{L})$. By Ehrhart [Eh67], [Eh68], [Eh69], [Eh77] there are functionals $G_{\mathbb{L},i} : \mathcal{P}^n(\mathbb{L}) \to \mathbb{N}_0$ such that for every $P \in \mathcal{P}^n(\mathbb{L})$ and $k \in \mathbb{N}$,

$$G_{\mathbb{L}}(kP) = \sum_{i=0}^{n} k^i G_{\mathbb{L},i}(P).$$

The polynomial on the right-hand side is often referred to as *Ehrhart-polynomial*; see Stanley [St86b] for basic facts on this polynomial, and see [GrW93] for a survey on lattice-point problems. For simplicity we restrict the further considerations to the case where $\mathbb{L}$ is the standard integer lattice $\mathbb{Z}^n$ and omit the subscript $\mathbb{Z}^n$.

Note that $G_n(P)$ is just the volume of P; see 3.5. Suppose now, we could determine in polynomial time the number $G(P)$ of lattice points of a polytope $P \in \mathcal{P}(\mathbb{Z}^n)$. We could run this algorithm for the polytopes $1 \cdot P, \ldots, n \cdot P$, and obtain $V(P) = G_n(P)$ by solving the system

$$Mx = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1^1 & \ldots & 1^n \\ 1 & 2^1 & \ldots & 2^n \\ \vdots & \vdots & & \vdots \\ 1 & n^1 & \ldots & n^n \end{pmatrix} \begin{pmatrix} G_0(P) \\ G_1(P) \\ \vdots \\ G_n(P) \end{pmatrix} = \begin{pmatrix} G(0 \cdot P) \\ G(1 \cdot P) \\ \vdots \\ G(n \cdot P) \end{pmatrix} = b$$

of linear equations. Application of Theorems 5.1.4, 5.1.5 and 5.1.7 shows that the problem of evaluating $G(P)$ for integer $\mathcal{H}$-polytopes, integer $\mathcal{V}$-polytopes, or integer $\mathcal{S}$-zonotopes is $\#\mathbb{P}$-complete.

Recall from Section 3.5 that a recent result of Barvinok [Ba93b] shows that in fixed dimension, $G(P)$ can be computed in polynomial time, see also [DyK93].

9.2 Zonotopes and Mixture-Management.

The typical approach to standard problems in mixture-management models the problem as a linear program by assigning costs to each basic mixture. However, Girad & Valentin [GiV89] remark that for many applications in the petroleum industry the cost of these basic mixtures are essentially identical, and this accounts for the fact that the linear programming model may not always be the best one. They propose an approach that involves zonotopes.

Suppose that a seller has m containers, each of which contains a mixture of n basic chemicals. For $i \in \{1, \ldots, m\}$ let $z_i = (\zeta_{i1}, \ldots, \zeta_{in})^T$ represent the mixture in container i, where ζ_{ij} is the quantity of chemical j in container i. (The ζ_{ij} are assumed to be nonnegative rationals.)

Suppose, further, that a customer demands a certain mixture $b = (\beta_1, \ldots, \beta_n)^T$ that consists of the quantities $\beta_1, \ldots, \beta_n$ of chemical $1, \ldots, n$, respectively.

In order to satisfy this demand the seller takes, for $i = 1, \ldots, m$, a proportion λ_i of container i's content such that

$$\beta_j = \sum_{i=1}^{m} \lambda_i \zeta_{ij} \qquad \text{for } j \in \{1, \ldots, n\}.$$

Hence, the zonotope

$$Z = \sum_{i=1}^{m} [0,1] z_i$$

is the set of all possible demands that the seller can satisfy. In general, there is more than one way to satisfy the demand; thus the seller will have a choice of vectors

$$l \in \Lambda(b) = \left\{ l = (\lambda_1, \ldots, \lambda_m)^T \in [0,1]^m : \sum_{i=1}^{m} \lambda_i z_i = b \right\},$$

and the question is: What is a good strategy for the choice of l so as to be able to satisfy the widest possible variety of possible future demands. If the seller has no information on the distribution of these demands it might be reasonable to assume that they are uniformly distributed.

Then the objective for the seller is to maximize the volume of the zonotope $Z(l)$ that is the set of all mixtures that are still possible after the current demand b has been satisfied by the choice l. This maximization criterion was suggested in [GiV89]. Of course, the volume of $Z(l) = \sum_{i=1}^{m} [0,1](1-\lambda_i) z_i$ is a homogenous polynomial in the $(1-\lambda_i)$'s, and its nth root is concave by the Brunn-Minkowski theorem 2.4.4. Hence the maximization problem is algorithmically tractable if the computation of function values is easy. However, Theorem 5.1.7 shows that the problem of computing $V(Z(l))$ is $\#\mathbb{P}$-hard. Thus the algorithm suggested by [GiV89] is not efficient unless the number n of basic chemicals is small. Note, however, that the randomized algorithm of Theorem 7.1.1 could be used.

9.3 Integration over bodies.

Suppose that K is a convex body of $\mathbb{R}^n$ and that the function $f : K \to \mathbb{R}$ is nonnegative and concave. Then

$$\int_K f(x)dx = V(K_f), \qquad \text{where} \quad K_f = \left\{ \begin{pmatrix} x \\ \xi_{n+1} \end{pmatrix} : x \in K,\, 0 \le \xi_{n+1} \le f(x) \right\}.$$

Since K_f is a convex body in $\mathbb{R}^{n+1}$, we can use the algorithm of Theorem 7.1.1 to approximate $\int_K f(x)dx$.

In order to bound the running time of the corresponding randomized algorithm, we need to make some assumptions about the *a priori* guarantees for K and f. Naturally, we will assume again that K is given by a centered well-bounded membership oracle with parameters r, R and b. Dyer & Frieze [DyF91] suggest, as measures for the size of f, the size L_1 of an upper bound of f on K and the size L_2 of a positive lower bound on f's average

$\int_K f(x)dx/V(K)$ on K. Using 7.7.1, the integral can then be approximated in time that is polynomial in size(K) and L_1 and L_2.

Dyer & Frieze [DyF91] further show that this approach can be extended to quasi-concave functions satisfying a Lipschitz-condition, and they derive a *pseudopolynomial* randomized algorithm for general integrable functions.

9.4 Stochastic programming.

As is pointed out by Dyer & Frieze [DyF91], the randomized algorithm for computing the volume of convex bodies can, in certain cases, be used to approximate the expected value of certain stochastic programming problems. Examples discussed in [DyF91] include the problem of computing the expected value of the functional $\varphi(b)$ that is defined by

$$\varphi(b) = \max f(x)$$
$$g_i(x) \leq \beta_i, \quad i = 1, \ldots, m,$$

where f is a concave functional, $g_1, \ldots, g_m$ are convex, and $b = (\beta_1, \ldots, \beta_m)^T$ is chosen uniformly from a convex body $K \in \mathcal{K}^m$.

Another example of [DyF91] deals with a question that comes up in the sensitivity analysis for linear programs. When, in the linear program

$$\begin{aligned} \min\langle c, x\rangle \\ Ax = b \\ x \geq 0 \end{aligned}$$

the parameters $(b^T, c^T)^T \in \mathbb{R}^m \times \mathbb{R}^n$ are chosen uniformly from a convex body K in $\mathbb{R}^{m+n}$, sensitivity analysis may ask for the probability that a specific nonsingular $(m \times m)$ submatrix B of A gives an optimal basic solution. This can be expressed in terms of volumes as follows. Since B is nonsingular, the condition $Ax = b$ is equivalent to $B^{-1}Ax = B^{-1}b$. Now, let $x_B = B^{-1}b$, let $\hat{x}_B$ denote the corresponding n-vector that is obtained from x_B by augmenting components 0 whenever the corresponding column of A does not belong to B, and let c_B denote the m-vector obtained from c by deleting all components that do not correspond to a column in B. Then it is well-known from duality theory of linear programming (and quite easy to derive, see e.g. [GoT89] or [GrK93b]) that $\hat{x}_B$ is an *optimal basic feasible solution* if and only if $\hat{x}_B$ is primal feasible and $\hat{y}_B = (c_B^T B^{-1})^T$ is dual feasible. Since, by definition, $A\hat{x}_B = b$, this is equivalent to

$$x_B \geq 0 \quad \text{and} \quad c_B^T B^{-1} A \leq c^T.$$

Hence the choices $(b^T, c^T)^T$ of K for which B is an optimal basis are those which belong to the subset

$$K_B = K \cap \left\{ \begin{pmatrix} b \\ c \end{pmatrix} : B^{-1}b \geq 0,\, c_B^T B^{-1} A \leq c^T \right\}$$

of K, and the probability of B being an optimal basis is given as the volume ratio

$$V(K_B)/V(K).$$

Again, under some reasonable assumptions, the randomized algorithm 7.1.1 for volume computation can be used.

9.5 Learning a halfspace.

Another application for volume computation, due to DYER, FRIEZE & KANNAN (see [DyF91]) is related to certain questions in "learning theory." Suppose, an algorithm $\mathcal{A}$ wanted to "learn" an unknown inequality $\langle a, x\rangle \geq \alpha$ where $a \in [-1,1]^n$ and $\alpha \in [-1,1]$. Suppose, further, that there is a sequence $(x_i)_{i\in\mathbb{N}}$ of points provided, and at step i, a guess is made by $\mathcal{A}$ as to whether x_i satisfies the inequality. It is then revealed to $\mathcal{A}$ whether the guess is correct. The goal for $\mathcal{A}$ is to devise a strategy which minimizes the proportion of errors made.

Now, each query point x_i leads to two halfspaces

$$H_i^+ = \left\{\begin{pmatrix} a \\ \alpha \end{pmatrix} : \langle a, x_i\rangle \geq \alpha\right\} \quad \text{and} \quad H_i^- = \left\{\begin{pmatrix} a \\ \alpha \end{pmatrix} : \langle a, x_i\rangle \leq \alpha\right\},$$

and each "verification" if the guessed answer is correct or not rules out one of the halfspaces. Hence after step i, $\mathcal{A}$ knows a polytope P_i, and a good strategy for deciding, whether for point x_{i+1} the guess should be "yes" or "no" may be guided by the volumes of the two parts

$$P_i \cap H_{i+1}^+, \quad P_i \cap H_{i+1}^-.$$

An analysis of this approach and a comparison with a method of MAASS & TURÁN [MaT89] can be found in DYER & FRIEZE [DF91]. See BLUM & KANNAN [BlK93] for a polynomial time method for learning an intersection of a constant number of halfspaces over a uniform distribution of query points.

9.6 Permanents.

For $i, j \in \{1, \ldots, n\}$ let α_{ij} be a nonnegative integer, and set $Z_i = \sum_{j=1}^n [0, \alpha_{ij}]e_j$. As we have seen in Subsection 5.2,

$$n!V(Z_1, \ldots, Z_n) = \mathrm{per}(A)$$

is the permanent of the matrix $A = (\alpha_{ij})_{i,j=1,\ldots,n}$.

We have used the $\#\mathbb{P}$-hardness of the problem of computing the permanent of a matrix to show in 5.2.2 that the problem of computing the mixed volume $V(Z_1, \ldots, Z_n)$ of the rectangular parallelotopes $Z_1, \ldots, Z_n$ is $\#\mathbb{P}$-hard. However, the correspondence goes both ways, and any progress for mixed volume computation leads to new results on the "positive side" for permanent computation. It follows, for instance, from the results stated in Subsection 7.2 that one can approximate the permanent of matrices with positive integer entries of quasi-polynomial size, at least if they have the property that all but $n - o(\log n/\log\log n)$ of the rows are identical, see [DyGH94].

Note that the fastest deterministic algorithm known for computing the permanent of a square 0-1 matrix with n rows runs in $n2^{n-1}$ time (see RYSER [Ry63] and the improvement by NIJENHUIS & WILF [NiW78]), while the best known randomized algorithms for producing a relative approximation p with

$$\mathrm{prob}\left\{\left|\frac{p}{\mathrm{per}(A)} - 1\right| \leq \epsilon\right\} \geq 1 - \beta.$$

still use time of order $2^{O(\sqrt{n}\log^2(n))}\epsilon^{-2}\log(1/\beta)$, JERRUM & VAZIRANI [JeV91], see also KARMARKAR, KARP, LIPTON, LOVÁSZ & LUBY [KaKLLL93].

Let us point out that besides the well-known applications in mathematical programming and combinatorics, there is need for computing or approximating the permanents of certain matrices that arise in particle physics (see ZHU [Zh93]). The lack of efficient procedures for this task leads to difficulties in the study of the Bose-Einstein correlation between particles.

It seems appropriate to end this subsection by mentioning van der Waerden's 1926 conjecture that on the $(n-1)^2$-dimensional polytope formed by all $n \times n$ doubly stochastic matrices, the permanent attains its minimum at the matrix $\frac{1}{n}J_n$ whose entries are all equal to $1/n$. This was finally proved by EGORYCHEV [Eg81], who showed that the minimum is in fact attained *uniquely* at $\frac{1}{n}J_n$. An essential tool in his proof was the Aleksandrov-Fenchel inequality 2.4.3 for mixed volumes. KNUTH [Kn81] later gave a relatively elementary, self-contained proof of Egorychev's result, and it turned out that FALIKMAN [Fa81] had independently proved the conjecture (but not the uniqueness) by different methods. Nevertheless, we believe that mixed volumes will continue to be a useful tool in dealing with specific problems that may at first not appear to have any connection with mixed volumes. In the words of Egorychev [Eg81, p.299]: "The method of mixed volumes is ideally suited to solving extremal problems and problems of uniqueness, and obtaining deep new inequalities. It is reasonable to assume that in the future the method of mixed volumes will stand with that of generating functions as one of the basic analytical tools of combinatorial analysis."

9.7 Polynomial equations.

Mixed volumes play an important role in algebraic geometry. Let us here discuss the relation of mixed volumes and the number of solutions of a system of equations involving *Laurent polynomials.*

We use a notation similar to that introduced in Subsection 3.4: when $x = (\xi_1, \ldots, \xi_n)$, and $q = (\kappa_1, \ldots, \kappa_n) \in \mathbb{Z}^n$, then

$$x^q = \xi_1^{\kappa_1} \cdot \xi_2^{\kappa_2} \cdot \ldots \cdot \xi_n^{\kappa_n}.$$

Now, let $S_1, S_2, \ldots, S_n \subset \mathbb{Z}^n$, and let for $i = 1, \ldots, n$,

$$f_i(x) = \sum_{q \in S_i} c_q^{(i)} x^q,$$

where the coefficients $c_q^{(i)}$ are fixed complex numbers. Hence $f_i \in \mathbb{C}\,[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, and we are interested in determining the number $L(F)$ of distinct common roots of the system $F = \{f_1, \ldots, f_n\}$. By a result of BERNSTHEIN [Be75] (see also [BuZ88, Chapter 27]) this number depends, if the coefficients $c_q^{(i)}$ $(q \in S_i)$ are chosen generically (i.e. in sufficiently general position), only on the *Newton polytopes*

$$P_i = \mathrm{conv}(S_i)$$

of the polynomials, and, more precisely,

$$L(F) \;=\; n! \cdot V(P_1, P_2, \ldots, P_n).$$

HUBER & STURMFELS [HuS93] use this result in conjunction with an approach similar to those of Subsection 4.2 to devise a numerical continuation algorithm for computing all isolated common roots of a system of polynomial equations. See GELFAND, KAPRANOV & ZELEVINSKY [GeKZ90] and also [GrS93] for further information about Newton polytopes and related concepts, and VERSCHELDE & COOL [VeC92], PETERSEN & STURMFELS [PeS93], CANNY & ROJAS [CaR93], and VERSCHELDE, VERLINDEN & COOL [VeVC94] for further results along these lines.

9.8 Bases of unimodular matroids.

Let $\mathcal{M}$ be a unimodular matroid of rank n with representation $v_1, v_2, \ldots, v_m$ over the reals. Let $S_1, \ldots, S_r$ be a partition of $\{1, 2, \ldots, m\}$ and let $i_1, \ldots, i_r$ be nonnegative integers such that $\sum_{j=1}^{r} i_j = n$. STANLEY [St81] shows that the number of bases of $\mathcal{M}$ with i_j elements in S_i for $i = 1, \ldots, r$ is

$$\binom{n}{i_1, \ldots, i_r} V(\overbrace{Z_1, \ldots, Z_1}^{i_1}, \ldots, \overbrace{Z_r, \ldots, Z_r}^{i_r}),$$

where Z_i is the zonotope

$$Z_i = \sum_{j \in S_i} [0, 1] v_j.$$

Hence the results of Subsection 5.2 can be applied to yield $\#\mathbb{P}$-hardness results for this counting problem. See BJÖRNER, LAS VERGNAS, STURMFELS, WHITE & ZIEGLER [BjLSWZ93] for a state-of-the-art account of oriented matroids.

9.9 Partial orders and linear extensions.

When dealing with the strong $\#\mathbb{P}$-hardness of volume computation for $\mathcal{H}$-polytopes in Subsection 5.1, we showed that the number of linear extensions of a given ordering $\mathcal{O}$ is equal to the volume of the order polytope $P_{\mathcal{O}}$. Now we outline an extension of this result, also due to STANLEY [St81], that involves mixed volumes.

Let $N = \{p_1, \ldots, p_r, q_1, \ldots, q_{n-r}\}$ be a poset, let $N_p = \{p_1, \ldots, p_r\}$, $N_q = \{q_1, \ldots, q_{n-r}\}$, and suppose that $p_1 < p_2 < \cdots < p_r$. For $i_1, \ldots, i_r \in \{1, \ldots, n\}$ let $e(i_1, \ldots, i_r)$ be the number of linear extensions π of N such that $\pi(p_j) = i_j$ for $j = 1, 2, \ldots, r$.

Now, define for $j = 0, \ldots, r$ the order polytopes P_j of $\mathbb{R}^{n-r}$ as the sets of points $x = (\xi_1, \ldots, \xi_{n-r})^T$ that satisfy the following constraints:

$$\begin{aligned} 0 \leq \xi_i &\leq 1 && \text{for } i = 1, \ldots, n-r; \\ \xi_i &\leq \xi_k && \text{if } q_i < q_k;\ i, k = 1, \ldots, n-r; \\ \xi_i &= 0 && \text{if } j > 0 \text{ and } q_i < p_j;\ i = 1, \ldots, n-r; \\ \xi_i &= 1 && \text{if } j < r \text{ and } q_i > p_{j+1};\ i = 1, \ldots, n-r. \end{aligned}$$

Note, that the polytopes P_j reflect the restriction of N to N_q that lies "between" p_j and p_{j+1}.

STANLEY [St81] showed that (with $i_0 = 0, i_{r+1} = n + 1$),

$$e(i_1, i_2, \ldots, i_r) = (n-r)!\, V(\overbrace{P_0, \ldots, P_0}^{i_1 - i_0 - 1}, \ldots, \overbrace{P_r, \ldots, P_r}^{i_{r+1} - i_r - 1}).$$

As a side remark, observe that, when $r = 1$, the Aleksandrov-Fenchel inequalities 2.4.3 imply that for $i = 1, \ldots, n-1$

$$e(i)^2 \geq e(i-1)e(i+1),$$

and hence the sequence $e(1), \ldots, e(n)$ is unimodal.

9.10 Experimental computation: Points on spheres.

There is a large collection of unsolved mathematical problems involving volumes or mixed volumes of polytopes or other bodies. In many cases, an important source of difficulty is a lack of intuition or computational experience that might guide one to a solution. That is especially true of extremum problems, and especially true in higher dimensions. We expect that improved methods of volume computation, in conjunction with heuristic optimization methods, will facilitate computational experiments to provide increased insight concerning these problems. Here we describe two problems that are not completely solved even in low dimensions and for which the range of ignorance increases rapidly as the dimension grows.

The sort of development that we have in mind is well illustrated by the history of the following problem:

9.10.1 How should k points be arranged on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ so as to maximize the volume of their convex hull?

For $n = 2$, the solution is the obvious one: place the points at the vertices of a regular k-gon inscribed in the unit circle $\mathbb{S}^1$ [Fe53]. As far as we know, the only case of 9.10.1 that has been settled for all n is that in which $k = n+1$, where the regular arrangements (placing the points at the vertices of a regular n-simplex) are the only ones that maximize the volume (Fejes Tóth [Fe64, p.313], Slepian [Sl69]). Fejes Tóth [Fe64] discusses the difficulties in the case $n = 4$, $k = 120$, saying that "It may be taken as certain that of the 4-dimensional polytopes with 120 vertices and unit circumradius $\{3, 3, 5\}$ (the regular one) has the greatest possible volume ... But so far we have no methods for proving these conjectures ... ." Of course, computer experimentation cannot provide proofs of such conjectures. However, algorithms combining volume computation with optimization methods may provide useful clues in cases where there is no obvious candidate for the optimizing shape or when the "obvious" candidate turns out not to be the optimum. Consider, for example, the case $n = 3$, $k = 8$, where the 8 vertices of a cube do not yield even a relative maximum. The volume-maximizing arrangement of 8 points was first discovered (as a relative maximum) by computer experimentation (Grace [Gr63]), and was later proved by Berman & Hanes [BeH70] to be the maximum. We expect that, as algorithms for volume computation are improved, higher-dimensional analogues of this sort of compute-conjecture-prove development will occur.

The problem of the preceding paragraph is closely related to the problem of finding, for a given n and k, the n-polytopes of minimum volume among those that have k facets and are circumscribed about a given sphere in $\mathbb{R}^n$. Again, the regular solution is known to be optimum for all k when $n = 2$ [Fe53] and for all n when $k = n+1$. Beyond that, the solution is known when $n = 3$ for a few values of $k > 4$, but ignorance is almost total in the higher-dimensional cases. For the case $n = 3$, see Goldberg [Go35] for the history of this problem and Schoen [Sc86] for an algorithmic approach.

The following problem turns out to involve a specific mixed volume.

9.10.2 How should k points be arranged on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ so as to maximize the mean width of their convex hull?

As pointed out at the end of Section 2, the mean width is just a multiple of the first intrinsic volume V_1. Let us add that for k-polytopes P in $\mathbb{R}^n$, there is a formula of SHEPHARD [Sh68]:

$$(1+(-1)^k)V_1(P) = \sum_{i=1}^{k-1} \sum_{F\in\mathcal{F}_i(P)} (-1)^{i+1}V_1(F).$$

The extent of ignorance concerning 9.10.2 is even greater than that for 9.10.1. For the important case in which $k = n+1$, several authors ([Gi52], [Ba63], [Ba65], [We68]) have assumed the existence of a proof that the regular arrangement maximizes the mean width. However, we are not aware of any such proof. (It is known, ALEXANDER [Al77], that the regular arrangement of $n+1$ points maximizes the *width* of the inscribed simplex.)

The problem 9.10.2 is of interest because of its connection with a problem in communication theory. Suppose that Z denotes the Gaussian distribution in $\mathbb{R}^n$ that has zero mean and whose covariance matrix is the $n \times n$ identity matrix. For a fixed $k > n$, a fixed $\lambda > 0$, and a fixed set U consisting of k points of the unit sphere $\mathbb{S}^{n-1}$, let Y be the vector random variable of the form $Y = (\lambda U) + Z$. Upon receiving Y, we are asked to determine which point of U has been transmitted, all points of U being equally likely a priori. The problem is to arrange the points of U so as to maximize the probability of this detection. The *simplex code conjecture* asserts that when $k = n+1$, the optimum arrangement is the regular one. A claimed proof [LaS66] of the conjecture was shown by FARBER [Fa68] and TANNER [Ta70] to be invalid, but the conjecture itself is still open. See [Fa68], [Ta74] for stronger forms of this conjecture, BALAKRISHNAN [Ba61], [Ba65], and TANNER [Ta70] for the relationship of the conjecture to mean widths of simplices, and CHAKERIAN & KLAMKIN [ChK73] for other conjectures on mean widths of simplices.

9.11 Experimental computation: Pushing balls together.

For each point $c \in \mathbb{R}^n$, let $B(c) = \{x \in \mathbb{R}^n : \|x-c\|_2 \le 1\}$, the Euclidean ball of unit radius centered at the point c. Now suppose that $p_1, \dots, p_k$ and $q_1, \dots, q_k$ are points of $\mathbb{R}^n$ such that for all i, j, $\|q_i - q_j\|_2 \le \|p_i - p_j\|_2$ – in other words, the q_i are pairwise at least as close together as the p_i. It has been conjectured that under these circumstances, the greater extent of overlapping of the balls $B(q_i)$ insures that

$$V\left(\bigcap_{i=1}^{k} B(q_i)\right) \ge V\left(\bigcap_{i=1}^{k} B(p_i)\right)$$

and

$$V\left(\bigcup_{i=1}^{k} B(q_i)\right) \le V\left(\bigcup_{i=1}^{k} B(p_i)\right).$$

(see [Th54] and [Kn55] for the latter conjecture). Both conjectures have been proved for the case in which $k \le n+1$ and for unrestricted k when $n = 1$, but for unrestricted k both are open for all $n \ge 2$. Concerning the intersections, it follows from a theorem of Kirszbraun

[Ki34] that if $V(\bigcap_{i=1}^k B(p_i)) > 0$ then $V(\bigcap_{i=1}^k B(q_i)) > 0$, and concerning the unions, it was proved by M. KNESER [Kn55] that

$$V\left(\bigcup_{i=1}^{k} B(q_i)\right) \leq 3^n V\left(\bigcup_{i=1}^{k} B(p_i)\right).$$

See [KlW91] for a detailed discussion of the above conjectures and some of their relatives, including a stronger conjecture of Kneser that implies the above conjectures. Kneser's interest in these questions arose from his study [Kn55] of a measure of surface area proposed by Minkowski.

Despite the plausibility of the above conjectures, it would not surprise us if they fail even when $n = 2$. It seems that for small n and for k not too much larger than $n+1$, it should be possible to design a computer experiment that would greatly improve the multiplier 3^n and would at the same time have a good chance of discovering a counterexample to the original conjecture (if one exists). Such an experiment would require fast algorithms for computing the volumes of intersections and unions of balls. See subsection 8.3 for references to such algorithms.

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