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Rook Placements and Generalized Partition Varieties

by

Kequan Ding[1,2]
DIMACS, Rutgers University, New Brunswick, New Jersey 08903
Institute for Advanced Study, Princeton, NJ 08540

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ABSTRACT

This is a continuation on the studies of rook placements and partition varieties. In this paper, we generalize the paptition varieties to a quotient space of certain matrix space module a parabolic subgroup (vs a Borel subgroup) of general linear group. To study the topological structure of this variety, we introduce the ideas of Gamma-compatible partitions and Gamma-compatible rook placements. Then we introduce Gamma-compatible rook length polynomials. First we give an explicit formula for the Gamma-compatible rook length polynomials. Then we give a constructive correspondence between the CW-complex structure of partition varieties in this general setting and the rook placements on a Ferrers board of the shape defined by a Gamma-compatible partition. We prove that the Poincare polynomials of cohomology for such a partition variety is given by a Gamma-compatible rook length polynomial. The model of partition varieties in this general setting generalized Grassmann manifolds and flag manifolds. Thus we get a uniform treatment for the cohomology of Grassmannians and flag manifolds. This model suggests other combinatorial problems on restricted rook placements on a Ferrers board which are not yet explored.

1 Introduction

Let $M^r_{m,n}(\mathbf{C})$ be the set of all m by n matrices of rank r over the complex field. Let $M := M^m_{m,n}(\mathbf{C})$. Consider a partition $\lambda = (\lambda_1, \cdots, \lambda_m)$. Let F_λ be the right justified Ferrers board corresponding to λ, i.e., there are λ_i positions in the i-th row. Define

$$M_\lambda := \{a \in M \mid a_{i,j} = 0,\ (i,j) \notin F_\lambda\}.$$

An r-rook placement on F_λ is a placement of r non-attacking rooks on F_λ. Let R^r_λ be the set of all r-rook placements on F_λ. As a combinatorial subject rook placements have been studied for a long time [27]. In the last thirty years, there have been many interesting progresses in this area. For example, Foata and Schützenberger [6] classified Ferrers boards according to their rook polynomials. Gouldman, Joichi, Reiner and White [9, 10, 11, 12, 13] studied the connection between rook placements and orthogonal polynomials, chromatic polynomials of graphs, Mobius inversion, the umbral calculus etc.. Gould [14], Wachs and White [31] studied rook placements together with Stirling numbers and their q-analogues. Garsia and Remmel [7], Sagan [28] considered the relationship between rook placements and major indices, etc.. But the geometric aspect of the rook placements was only explored until recently [4]. In [3], we introduced a *length* function for rook placements on a Ferrers board and rook length polynomials. Then we gave a formula on the relationship between rook length polynomials and Garsia-Remmel polynomials. The major result in [3] is an explicit formula for rook length polynomials. As a consequence, we also obtained an explicit formula for Garsia-Remmel polynomials. In [4], we introduced the notion of a *partition variety* as a quotient space $B\backslash M_\lambda$ where B is the Borel subgroup of upper triangular matrices of $GL_m(\mathbf{C})$. We proved that $B\backslash M_\lambda$ is a projective subvariety which has the structure of a CW-complex. The Poincaré polynomials for cohomology of partition varieties with coefficients in real field $\mathbf{R}$ are proved to be rook length polynomials as introduced in [3] with $r = m$.

Definition 1.1 *Let m be a positive integer. A composition of m is a tuple $(m_1, \cdots, m_s)$ of positive integers such that $\sum_{i=1}^s m_i = m$.*

Definition 1.2 *Let $\gamma = (\gamma_1, \cdots, \gamma_k)$ be a composition of some positive integer. Let P_γ denote the parabolic subgroup of G_m of the shape*

$$P_\gamma = \begin{pmatrix} G_{\gamma_1} & * & \cdots & * \\ 0 & G_{\gamma_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & G_{\gamma_k} \end{pmatrix}$$

where $G_{\gamma_i} = GL_{\gamma_i}(\mathbf{C})$, and the $*$'s are arbitrary matrices of the appropriate sizes.

This present paper is a continuation of our study on rook placements and partition varieties in [4]. Here, we consider the partition varieties in the general setting $P_\gamma\backslash M_\lambda$ for certain partitions λ (called γ-compatible partitions), where P_γ is a parabolic subgroup of the general linear group $G_m = GL_m(\mathbf{C})$.

This paper consists of seven sections. Our main results are Theorem 3.4, 5.6, 7.2, 7.5 and 7.7.

§1 Introduction

§2 Preliminaries

§3 γ-compatible Rook Placements

§4 Complex Manifold $P_\gamma\backslash M$.

§5 Partition Variety $P_\gamma\backslash M_\lambda$

§6 Cellular Decomposition of $P_\gamma\backslash M_\lambda$

§7 CW-complex Structure of $P_\gamma\backslash M_\lambda$.

2 Preliminaries

Let $[n]$ denote the set $\{1, \cdots, n\}$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of some integer where $\lambda_1 \geq \cdots \geq \lambda_m > 0$. Sometimes, we write $\lambda = (1^{\mu_1}\, 2^{\mu_2}\, \cdots\, n^{\mu_n})$ where μ_i is the number of λ_j's which are equal to i. We view a Ferrers board F_λ of shape λ as a subarray of an m by n matrix, where $n = \lambda_1$ and the k-th row has length λ_k for $1 \leq k \leq m$. For example, if $\lambda = (3, 1)$, then

$$F_\lambda = \begin{pmatrix} * & * & * \\ & & * \end{pmatrix}.$$

We assume, contrary to the usual convention, that F_λ is right justified; the reason for this is explained in [3]. If $r \geq 1$ let

$$\nu_r = \sum_{i=1}^{r} E_{i,n-r+1}. \tag{1}$$

where $E_{i,j}$ is the matrix with 1 at (i,j) and 0's elsewhere. Let W_n be the symmetric group on $[n] = \{1, \cdots, n\}$. Let $S(n)$ be the set of distinguished generators of W_n: $S(n) = \{(12), (23), \cdots, (n-1\ n)\}$.

Definition 2.1 *Let λ be a partition such that $\nu_r \in F_\lambda$. When $r = 0$, let $\nu_0 = 0$. For $\sigma \in R^r_\lambda$, the length function $l(\sigma)$ is defined by*

$$l(\sigma) = min\{k + h \mid \sigma = s_k \cdots s_1 \nu_r s'_1 \cdots s'_h\}$$

where $s_i \in S(m)$ and $s'_j \in S(n)$ and

$$s_p \cdots s_1\, \nu_r\, s'_1 \cdots s'_q \in R^r_\lambda$$

for each $1 \leq p \leq k$ and $1 \leq q \leq h$.

Thus $l(\sigma)$ is the minimum number of adjacent row and/or column transpositions required to get σ from $\nu_r = \sum_{i=1}^{r} E_{i,n-r+i}$, such that all the intermediate rook placements are in the board F_λ. This length function was first used as the length of rook matrices by Solomon in his work on the Iwahori ring of $\mathbf{M}_n(\mathbf{F}_q)$ [29]. The length function l can be evaluated "locally" by the following formula which counts the contribution of each rook in σ, individually [3]. We often use a 1 to denote a rook and a 0 for an empty box in a Ferrers board.

Proposition 2.2 (Local Formula) *Let $\sigma \in R_\lambda^r$. Write $\sigma = \sum_{i=1}^{r} E_{c_i,b_i}$ where $c_1 < c_2 < \cdots < c_r$. Let α_i be the number of zero rows above the c_i-th row in σ, γ_i the number of zero columns to the right of the b_i-th column in σ, and β_i the number of 1's to the "northeast" of the i-th 1 (not including the i-th 1). Then,*

$$l(\sigma) = \sum_{i=1}^{r} (\alpha_i + \beta_i + \gamma_i). \tag{2}$$

Definition 2.3 *Let*

$$RL_r(\lambda, q) = \sum_{\sigma \in R_\lambda^r} q^{l(\sigma)}.$$

We call this a rook length polynomial.

Theorem 2.4 (Formula of Rook Length Polynomials [3]) *If λ is a partition with m parts then*

$$RL_r(\lambda, q) = \sum_{1 \le i_1 < \cdots < i_r \le m} q^{\sum_{j=1}^{r}(i_j - j)} \prod_{j=1}^{r} (\lambda_{i_j} - r + j)_q \tag{3}$$

where $(k)_q = 1 + q + q^2 + \cdots + q^{k-1}$.

One of the main ideas in [3] is to extend a placement σ of r rooks on an m by n board to a placement $P(\sigma)$ of $m+n-r$ rooks on an $m+n-r$ by $m+n-r$ board. We identify $P(\sigma)$ with the corresponding permutation of $[m+n-r]$ and call $P(\sigma)$ the *invisible permutation* corresponding to σ. There are various ways to extend a rook placement σ to a permutation of $[m+n-r]$. The choice here is made so that $l(\sigma)$ is equal to the number of inversions in the permutation $P(\sigma)$.

Example 2.5 *Let $r = 2$, $m = n = 4$. Let*

$$\sigma = E_{14} + E_{32} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the permutation matrix corresponding to σ is

$$
\begin{aligned}
P(\sigma) &= (E_{11}+E_{23})+(E_{34}+E_{52})+(E_{45}+E_{66}) \\
&= \left(\begin{array}{cccc|cc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 \\
\mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1
\end{array}\right).
\end{aligned} \tag{4}
$$

The 4 by 4 submatrix of $P(\sigma)$ at the southwest corner is exactly the rook matrix σ. The formal definition of the permutation $P(\sigma)$ is as follows.

Definition 2.6 *Let $\sigma \in R^r_{nm}$. Let $\sigma = \sum_{i=1}^r E_{c_i,b_i}$ with $c_1 < c_2 < \cdots < c_r$. Define a permutation $P(\sigma) \in W_{m+n-r}$ by*

$$
P(\sigma) = \left(\begin{array}{ccc|ccc|ccc}
1 & \cdots & n-r & n-r+c_1 & \cdots & n-r+c_r & d_1 & \cdots & d_{m-r} \\
a_1 & \cdots & a_{n-r} & b_1 & \cdots & b_r & n+1 & \cdots & n+m-r
\end{array}\right)
$$

where $\{a_1, a_2, \ldots, a_{n-r}\}$ is the complement of $\{b_1, b_2, \ldots, b_r\}$ in $[n]$ with $a_1 < a_2 < \cdots < a_{n-r}$, and $\{d_1, d_2, \ldots, d_{m-r}\}$ is the complement of $\{n-r+c_1, \cdots, n-r+c_r\}$ in $\{n-r+1, \ldots, n-r+m\}$ with $d_1 < d_2 < \cdots < d_{m-r}$.

Remark. From this definition, we see that the a_i's index the zero columns of σ and the d_i's index the zero rows of σ. In Example 2.5, since $\sigma = E_{14} + E_{32}$. we have $c_1 = 1$, $c_2 = 3$, $b_1 = 4$ and $b_2 = 2$. Thus $\{a_1, a_2\} = [n] - \{b_1, b_2\} = [4] - \{4, 2\} = \{1, 3\}$. So $a_1 = 1$ and $a_2 = 3$. Similarly,

$$
\begin{aligned}
\{d_1, d_2\} &= [n+m-r] - [n-r] - \{n-r+c_1, n-r+c_2\} \\
&= \{3,4,5,6\} - \{2+c_1, 2+c_2\} = \{4,6\}
\end{aligned}
$$

So $d_1 = 4$ and $d_2 = 6$. Thus

$$
P(\sigma) = \left(\begin{array}{cc|cc|cc}
1 & 2 & 3 & 5 & 4 & 6 \\
1 & 3 & 4 & 2 & 5 & 6
\end{array}\right).
$$

Then the permutation matrix $P(\sigma)$ is

$$
P(\sigma) = (E_{11}+E_{23})+(E_{34}+E_{52})+(E_{45}+E_{66})
$$

which is exactly the same as the one given in Example 2.5.

If we write $P(\sigma)$ as a two-row array as in Definition 2.6, the columns to the left of the first vertical line are called the *column imaginary part* of $P(\sigma)$. The columns to the right of the second vertical line are called the *row imaginary part* of $P(\sigma)$. The columns between the two vertical lines are called the *real part* of $P(\sigma)$. Note in Example 2.5 that the 4 by 4 submatrix of $P(\sigma)$ at the southwest corner, the real part of $P(\sigma)$, is $E_{34}+E_{52}$ which corresponds to $\sigma = E_{14}+E_{32}$. There will always be this shift of row indices. If $\sigma = \sum_{i=1}^{r} E_{c_i,b_i}$ then the corresponding block in the southwest corner of $P(\sigma)$ is $\sigma = \sum_{i=1}^{r} E_{n-r+c_i,b_i}$.

Let $\mathbf{C}^n$ be the n-dimensional affine space of row vectors. We use the usual complex topology in this paper.

Definition 2.7 (Affine Variety) *A subset of $\mathbf{C}^n$ is called an affine variety if it is the set of common zeroes of a collection of polynomials with coefficients in $\mathbf{C}$. These polynomials are called the defining polynomials of the affine variety.*

The n-dimensional projective space over $\mathbf{C}$ is defined by $\mathbf{P}^n = \mathbf{C}^{n+1} - \{\mathbf{0}\}/\sim$ where $(x_0, x_1, \cdots, x_n) \sim (y_0, y_1, \cdots, y_n)$ if there exists $c \in \mathbf{C} - \{0\}$ such that $y_i = cx_i$ for all i. Let $(x_0 : x_1 : \cdots : x_n)$ denote the equivalence class of $(x_0, x_1, \cdots, x_n)$. The x_i are called homogeneous coordinates. A polynomial $f(x_0, x_1, \cdots, x_n)$ is homogeneous of degree k if $f(cx_0, cx_1, \cdots, cx_n) = c^k f(x_0, x_1, \cdots, x_n)$.

Definition 2.8 (Projective Variety) *A subset of $\mathbf{P}^n$ is called a projective variety if it is the set of common zeros of a collection of homogeneous polynomials. These homogeneous polynomials are called the defining polynomials of the projective variety.*

Definition 2.9 (Standard Affine Open Sets) *Let*

$$U_i = \{x = (x_0 : x_1 : \cdots : x_n) \in \mathbf{P}^n \mid x_i \neq 0\} \qquad 0 \leq i \leq n.$$

The U_i's are called the standard affine open sets. They form an open cover of $\mathbf{P}^n$ [17, page 4].

Lemma 2.10 *Each U_i is homeomorphic to the affine space $\mathbf{C}^n$ via $h : x = (x_0 : x_1 : \cdots : x_n) \mapsto (x_0/x_i, \cdots, x_{i-1}/x_i, x_{i+1}/x_i, \cdots, x_n/x_i)$.*

Let $\mathbf{P}^{n_1}, \ldots, \mathbf{P}^{n_m}$ be projective spaces. Let $N = \prod_{i=1}^{m}(n_i+1) - 1$. The points in $\mathbf{P}^N$ will be written

$$(\ldots : x_{i_1,\ldots,i_m} : \ldots) \tag{5}$$

where $(i_1, \ldots, i_m) \in [n_1+1] \times \cdots \times [n_m+1]$. Define a mapping $\Psi : \prod_{i=1}^{m} \mathbf{P}^{n_i} \to \mathbf{P}^N$ by

$$\Psi((\cdots : x_i : \cdots), (\cdots : y_j : \cdots), \cdots, (\cdots : z_k : \cdots)) = (\cdots : x_i y_j \cdots z_k : \cdots) \tag{6}$$

Lemma 2.11 *Let A_i be a subvariety of $\mathbf{P}^{n_i}$ for $1 \leq i \leq m$. Then $\Psi(A_1 \times \cdots \times A_m)$ is a subvariety of P^N.*

Proof. Humphreys [19, page 13].

Remark. *We identify the Cartesian product of projective varieties with its Ψ image. Thus the Cartesian product of finitely many projective varieties is a projective variety.*

If X is a topological space and A is a subset of X then $\overline{A}$ denotes the closure of A.

Definition 2.12 (Finite CW-complex,[23]) *A Hausdorff space X is called a finite CW-complex if X has a partition $X = \bigsqcup_{i \in I} e_i$ into disjoint subsets $\{e_i\}_{i \in I}$ satisfying the following conditions:*

- *The index set I is finite.*
- *Each e_i is homeomorphic to an open ball in some $\mathbf{C}^n$.*
- *If $x \in \overline{e_i} - e_i$, then there is some e_j of lower dimension such that $x \in e_j$.*

We call the subsets e_i's cells. In this paper, whenever we say a CW-complex, we mean a finite CW-complex.

Definition 2.13 (Complex Manifold, [15]) *Let X be a Hausdorff space. The space X is a complex manifold of dimension n if there is an index set I and a set of pairs $\{(U_i, \phi_i) \mid i \in I\}$, called a local coordinate system, such that*

- *Each U_i is an open subset of X and $X = \cup_{i \in I} U_i$.*
- *Each $\phi_i : U_i \to \phi_i(U_i) \subset \mathbf{C}^n$ is a homeomorphism from U_i onto an open subset $\phi_i(U_i)$ of $\mathbf{C}^n$.*
- *Let $\phi_{i,j} := \phi_j \circ \phi_i^{-1}$. Then $\phi_{i,j}$ is a biholomorphic (the mapping and its inverse are both holomorphic) homeomorphism from $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$.*

Example 2.14 (Projective Space as Complex Manifold) *The projective space $\mathbf{P}^n$ is a complex manifold with the standard affine open sets as a local coordinate system.*

Definition 2.15 (Analytic Subvariety of a Complex Manifold) *Let X be a complex manifold. An analytic subvariety of X is a subset Z given locally as the zero locus of a finite number of holomorphic functions. Thus if $x \in Z$ there exists an open set U_i in the local coordinate system (2.13) such that (1) $x \in U_i$ and (2) there exist holomorphic functions $f_1, \ldots, f_r$ defined on U_i such that $Z \cap U_i$ is the set of common zeros of $f_1, \ldots, f_r$.*

Definition 2.16 (Holomorphic Map of Complex Manifold) *Let M and M' be complex manifolds with local coordinate systems $\{(U_I, \phi_I)\}$ and $\{(U'_I, \phi'_I)\}$ respectively. A map $\Phi : M \to M'$ is a holomorphic map of complex manifold if*

$$\phi'_J \circ \Phi \circ (\phi_I^{-1}|_{\phi_I(U_I \cap \Phi^{-1}(U'_J))}) \;:\; \phi_I(U_I \cap \Phi^{-1}(U'_J)) \to \phi'_I(\Phi(U_I) \cap U'_J)$$

is holomorphic for all I and J.

Definition 2.17 (Jacobian Matrix) *Let $D \subset \mathbf{C}^m$ and $D' \subset \mathbf{C}^n$ be connected open sets (called domains). Let $\Phi : D \to D'$ be a holomorphic map. The matrix*

$$J_{\mathbf{C}}(\Phi(z)) := (\frac{\partial \Phi_i(z)}{\partial z_j})_{1 \leq i \leq n, 1 \leq j \leq m}$$

is called the complex Jacobian matrix of f.

Definition 2.18 (Embedding of Complex Manifold) *Let M and M' be complex manifolds with $\dim M \leq \dim M'$. Let $\Phi : M \to M'$ be a holomorphic map of complex manifold. The map Φ is called an embedding of M into M' if Φ is injective and full ranked, i.e.,*

$$\text{rank } J_{\mathbf{C}}(\Phi)(x) = \dim M, \qquad \textit{for all} \quad x \in M \ . \tag{7}$$

Example 2.19 *The map $\Psi : \prod_{i=1}^{m} \mathbf{P}^{n_i} \to \mathbf{P}^N$ defined in (6) is an embedding of complex manifolds [19, page 13].*

I learned the following proposition from Professor Hiroaki Terao, which is crutial in our argument. See [4] or [5] for the proof.

Proposition 2.20 (Embedding Theorem) *Let X and Y be two complex manifolds. If X is compact and $\Theta : X \to Y$ is an embedding then $\Theta(X)$ is a complex analytic subvariety of Y.*

Theorem 2.21 (Chow's Theorem) *If $X \subseteq \mathbf{P}^n$ is an analytic subvariety then X is an projective subvariety [17, page 8].*

Corollary 2.22 (Embedding Theorem for Projective Space) *Let X be a complex manifold. If X is compact and $\Theta : X \to \mathbf{P}^n$ is an embedding, then $\Phi(X)$ is a projective subvariety of $\mathbf{P}^n$.*

Proof. Apply Chow's Theorem and the Embedding Theorem. Q.E.D.

Convention 2.23 *In the context of Corollary 2.22 we agree to transport the structure of projective variety from $\Theta(X)$ to X itself. In this sense X is a projective variety.*

Recall that M_λ is a subset of the affine space M. We give it the subspace topology. Since $P_\lambda M_\lambda \subseteq M_\lambda$, we may consider the geometry in the quotient space $P_\lambda \backslash M_\lambda$. We give the quotient space $P_\lambda \backslash M_\lambda$ the quotient topology. In this paper we use some well known results on Grassmannians. In the following we quote them without proof. There is an exposition in [5] on this which is a complex version of that in Milnor and Stasheff [23]. Let m and n be two positive integers such that $m \leq n$. As before let $G = GL_m(\mathbf{C})$. Let $G_m(\mathbf{C}^n)$ be the set of all m-dimensional subspaces of $\mathbf{C}^n$. Let V be an m-dimensional subspace of $\mathbf{C}^n$. Then V can be written as a linear span of m linearly independent row vectors $\{\alpha_i\}_{1 \leq i \leq m}$ in $\mathbf{C}^n$. Using α_i as the i-th row for $1 \leq i \leq m$ we get a m by n matrix a of rank m. Let $< a >$ denote the row space of a. Note that for $a, a' \in M$, $< a > = < a' >$ if and only if a and a' have the same G-orbit $[a] := G\,a = G\,a' = [a']$. Define a mapping $< a > \mapsto [a]$. This is a one to one correspondence between $G_m(\mathbf{C}^n)$ and the quotient space $G \backslash M = \{G\,a \mid a \in M\}$ where M is the set of all m by n matrices over $\mathbf{C}$ of rank m. The quotient space $G \backslash M$ has the quotient topology inherited from the natural topology of the space M. We transport this topology to $G_m(\mathbf{C}^n)$. Thus $G_m(\mathbf{C}^n)$ is a complex manifold, called the Grassmann manifold. In the following, for a fixed $a \in M$ we identity the row space $< a >$ with the G-orbit $[a] = G\,a$. Further, we identify the Grassmann manifold with the quotient space $G \backslash M = \{G\,a \,|\, a \in M\}$. Thus we consider an element of the Grassmann manifold $G_m(\mathbf{C}^n)$ either as a G-orbit of some matrix $a \in M$ or as an m-dimensional subspace of $\mathbf{C}^n$, whichever is convenient. Let

$$\mathbf{C}^k = \{(\underbrace{0, \cdots, 0}_{n-k}, z_k, \cdots, z_1) \in \mathbf{C}^n \,|\, z_i \in \mathbf{C}\}, \text{ for } 1 \leq k \leq n.$$

Remark. We call this special sequence of subspaces the standard flag. This definition is "backward"; one might prefer

$$\mathbf{C}^k = \{(z_1, \cdots, z_k, \underbrace{0, \cdots, 0}_{n-k}) \in \mathbf{C}^n \,|\, z_i \in \mathbf{C}\}.$$

The reason for us to use such a "backward" definition is that our Ferrers boards have to be right justified to include the parabolic boards as special cases. For the same reason, our row eliminations and Gram-Schmidt process are all from bottom to the top.

If $[a] \in G_m(\mathbf{C}^n)$, then,

$$0 = \dim([a] \cap \mathbf{C}^0) \leq \dim([a] \cap \mathbf{C}^1) \leq \cdots \leq \dim([a] \cap \mathbf{C}^n) = m.$$

Note that any two consecutive integers in this sequence differ by at most 1. Thus, there are exactly m jumps in the sequence. We collect all the $[a]$'s which jump at the same places.

Definition 2.24 (Schubert Cell) *For $\sigma \in J(m, n)$, let*

$$e(\sigma) = \{[a] \in G_m(\mathbf{C}^n) \,|\, \ dim([a] \cap \mathbf{C}^{\sigma(i)-1}) < \ dim([a] \cap \mathbf{C}^{\sigma(i)}), 1 \leq i \leq m\}.$$

We call $e(\sigma)$ the Schubert cell of σ [23, Section 6].

Let U be a subset of the Grassmann manifold $G_m(\mathbf{C}^n)$. We say that U has matrix representatives $S = \{x, y, \cdots\} \subset M$ if U is the set of row spaces of the elements in S. Note that a matrix determines its row space uniquely. On the other hand, for a fixed m dimensional subspace, there is a unique orthonomal m-frame $(v_1, \cdots, v_m)$ in $H^{\sigma(1)} \times \cdots \times H^{\sigma(m)}$. Use v_i as the i-th row vector for $1 \leq i \leq m$. We get a matrix a. This matrix is uniquely determined by the m dimensional subspace.

Lemma 2.25 *For $m \leq n$, let $e(\sigma)$ be a Schubert cell in a Grassmann manifold $G_m(\mathbf{C}^n)$. Then, $e(\sigma)$ has matrix representatives*

$$\begin{pmatrix} 0 & \cdots & 0 & p_1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * \\ & & & & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ & & & & & & * & 0 & * & \cdots & * & 0 & * & \cdots & * \\ & & & & & & & p_{m-1} & * & \cdots & * & 0 & * & \cdots & * \\ & & & & & & & & & & & p_m & * & \cdots & * \end{pmatrix} \tag{8}$$

with real positive p_k in the $\sigma(k)$-th place from the right, for $1 \leq k \leq m$ if and only if the corresponding Schubert variety $\overline{e(\sigma)}$ has matrix representatives

$$\begin{pmatrix} 0 & \cdots & 0 & * & * & \cdots & * & * & * & \cdots & * & * & * & \cdots & * \\ & & & & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ & & & & & & * & * & * & \cdots & * & * & * & \cdots & * \\ & & & & & & & * & * & \cdots & * & * & * & \cdots & * \\ & & & & & & & & & & & * & * & \cdots & * \end{pmatrix}. \tag{9}$$

with $\sigma(k)$ $*$'s in the k-th row, for $1 \leq k \leq m$ where the $*$'s are complex numbers.

Remark. The number of $*$'s in the matrix (8) equals the dimension of $e(\sigma)$. In the following, we write a Ferrers board consisting of all the $*$'s in the matrix (9) as $F_{\lambda(\sigma)}$.

For example, if $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, then $j_1 = 2$, and $j_2 = 3$, so $\overline{e(\sigma)}$ has a Ferrers board $\begin{pmatrix} * & * & * \\ & * & * \end{pmatrix}$ while $e(\sigma)$ has a matrix representative $\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \end{pmatrix}$.

Corollary 2.26 *For $\sigma, \sigma' \in J(m,n)$, $e(\sigma) \subseteq \overline{e(\sigma')}$ if and only if $F_{\lambda(\sigma)} \subseteq F_{\lambda(\sigma')}$.*

Theorem 2.27 *The Grassmann manifold $G\backslash M$ has a CW-complex structure.*

3 γ-Compatible Rook Placements

In this section, we give an explicit formula (see Theorem 3.4) of rook length polynomials for certain partition λ. These partitions are called γ-compatible partitions. The idea of γ-compatible partitions is to make sure that $P_\gamma M_\lambda \subset M_\lambda$.

Definition 3.1 *Let $\gamma = (\gamma_1, \cdots, \gamma_t)$ be a composition of m. A partition $\lambda = (\lambda_1, \cdots, \lambda_m)$ is called a γ-compatible partition if $\lambda = (k_1^{\gamma_1}, \cdots, k_t^{\gamma_t})$ where $k_1 \geq k_2 \geq \cdots \geq k_t > 0$.*

In the Ferrers board F_λ, we call the rows corresponding to $k_i^{\gamma_i}$ the i-th γ-block. If from the context the composition γ is clear, then we omit the γ and call these rows the i-th block.

Definition 3.2 *A rook placement σ is said to be γ-compatible if $\sigma(i) < \sigma(i+1)$ whenever $\sum_{j=1}^{j'} \gamma_j < i < i+1 \leq \sum_{j=1}^{j'+1} \gamma_j$, for some j'. Let $R_\lambda^r(\gamma)$ denote the set of all γ-compatible rook placements on the board F_λ with r rooks.*

This definition tells us that for a γ-compatible rook placements, in each block of the Ferrers board, the column indices of rooks increase as the row indices increase.

Definition 3.3 *Let λ be a γ-compatible partition. Define the γ-rook length polynomial*

$$RL_r(\lambda, \gamma, q) := \sum_{\sigma \in R_\lambda^r(\gamma)} q^{l(\sigma)}. \tag{10}$$

When $\gamma = (1^m)$, this is the rook length polynomial in Definition 2.3.

Theorem 3.4 *Let λ be a γ-compatible partition. Then*

$$RL_m(\lambda, \gamma, q) = \prod_{i=1}^{t} \begin{bmatrix} k_i - \gamma_{i+1} - \gamma_{i+2} - \cdots - \gamma_t \\ \gamma_i \end{bmatrix}_q \tag{11}$$

where $\left[{s \atop t}\right]_q = \frac{(s)!_q}{(t)!_q(s-t)!_q}$ *and* $(s)!_q = (s)_q(s-1)_q \cdots (1)_q$.

Proof. We use induction on the number t of blocks. If $t = 1$, then $\gamma_1 = m$, $k_1 = n$ and F_λ is an $m \times n$ rectangular board. Thus every γ-compatible rook placement σ satisfies the inequalities:

$$1 \leq \sigma(1) < \sigma(2) < \cdots < \sigma(m) \leq n \quad .$$

By the local formula of the length function, the i-th rook (the rook in the i-th row) contribute $n - m + i - \sigma(i) = n - \sigma(i) - (m - i)$ to the length $l(\sigma)$ for $1 \leq i \leq m$. Note that

$$0 \leq n - m + m - \sigma(m) < n - m + (m-1) - \sigma(2) < \cdots < n - m + 1 - \sigma(1) \leq n - m \quad .$$

So $RL_m(\lambda, \gamma, q) = \left[{n \atop m}\right]$ which is exactly our formula in the theorem. Suppose that the formula is true for $t < s$. If $t = s$, let σ be a fixed γ-compatible rook placement on F_λ. Consider the contribution of the rooks in the last block, i.e., the last γ_t rows. By definition of γ-compatible rook placements, if the rooks in the last block are in the columns of indices $k_1 - i_j + 1$ (the i_j-th column from the right) then we have $1 \leq i_1 < i_2 < \cdots < i_{\gamma_t} \leq k_t$. By the local formula in Corollary 2.2, these rooks contribute $\sum_{j=1}^{\gamma_t}(i_j - j)$ to the length function which is independent from the positions of the rooks in the first $t-1$ blocks. Now, cut off the last block and the columns containing the rooks in the last block from the Ferrers board. Then we get a rook placement σ' of $m - \gamma_t$ rooks on a Ferrers board of the shape $((k_1-\gamma_t)^{\gamma_1}, \cdots, (k_{t-1}-\gamma_t)^{\gamma_{t-1}})$. Clearly, σ' is $(\gamma_1, \cdots, \gamma_{t-1})$-compatible. By induction hypothesis,

$$\sum_{\sigma'} q^{l(\sigma')} = \prod_{i=1}^{t-1} \begin{bmatrix} (k_i - \gamma_t) - \gamma_{i+1} - \cdots - \gamma_{t-1} \\ \gamma_i \end{bmatrix}_q$$

where σ' goes through all the $(\gamma_1, \cdots, \gamma_{t-1})$-compatible rook placements with $m - \gamma_t$ rooks on a Ferrers board of the shape $((k_1-\gamma_t)^{\gamma_1}, \cdots, (k_{t-1}-\gamma_t)^{\gamma_{t-1}})$. Note that $l(\sigma) = l(\sigma') + \sum_{j=1}^{\gamma_t}(i_j - j)$. Let σ vary within $R_\lambda^m(\gamma)$. We have

$$\begin{aligned} &RL_m(\lambda, \gamma, q) = \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_{\gamma_t} \leq k_t} q^{\sum_{j=1}^{\gamma_t}(i_j - j)} \prod_{i=1}^{t-1} \begin{bmatrix} (k_i - \gamma_t) - \gamma_{i+1} - \cdots - \gamma_{t-1} \\ \gamma_i \end{bmatrix}_q \\ &= \prod_{i=1}^{t} \begin{bmatrix} k_i - \gamma_{i+1} - \gamma_{i+2} - \cdots - \gamma_t \\ \gamma_i \end{bmatrix}_q \end{aligned}$$

Hence the theorem is proved. Q.E.D.

Corollary 3.5 *If $t = 1$, then $\lambda_i = n$ for each i and*

$$RL_m(\lambda, \gamma, q) = \begin{bmatrix} n \\ m \end{bmatrix}_q . \tag{12}$$

Proof. This is the result in the first step of the induction in the proof of our theorem above. Q.E.D.

Corollary 3.6 *If $\gamma_i = 1$ for all i, then every partition is γ-compatible.*

$$\begin{aligned} RL_m(\lambda, \gamma, q) &= RL_m(\lambda, q) && (13) \\ &= \prod_{i=1}^{m} (\lambda_i - m + i)_q && (14) \end{aligned}$$

Proof. Take $k_i = \lambda_i$ for each i. Then, $\gamma_i = 1$ for each i. Q.E.D.

Example 3.7 Let $\gamma = (1, 2)$ and $\lambda = (4, 3, 3)$. Then, λ is clearly γ- compatible. Let

$$A = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline \multicolumn{1}{c|}{} & & 1 & \\ \cline{2-4} \multicolumn{1}{c|}{} & & & 1 \\ \cline{2-4}\end{array} \qquad B = \begin{array}{|c|c|c|c|}\hline & 1 & & \\ \hline \multicolumn{1}{c|}{} & & 1 & \\ \cline{2-4} \multicolumn{1}{c|}{} & & & 1 \\ \cline{2-4}\end{array}$$

$$C = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline \multicolumn{1}{c|}{} & 1 & & \\ \cline{2-4} \multicolumn{1}{c|}{} & & & 1 \\ \cline{2-4}\end{array} \qquad D = \begin{array}{|c|c|c|c|}\hline & & 1 & \\ \hline \multicolumn{1}{c|}{} & 1 & & \\ \cline{2-4} \multicolumn{1}{c|}{} & & & 1 \\ \cline{2-4}\end{array} \quad .$$

$$E = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline \multicolumn{1}{c|}{} & 1 & & \\ \cline{2-4} \multicolumn{1}{c|}{} & & 1 & \\ \cline{2-4}\end{array} \qquad F = \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & & \\ \cline{2-4} \multicolumn{1}{c|}{} & & 1 & \\ \cline{2-4}\end{array}$$

By the local formula of the length function, we know that

$$\begin{aligned} RL_3((4,3,3),(1,2),q) &= 1 + q + q + q^2 + q^2 + q^3 \\ &= (1+q)(1+q+q^2) \\ &= \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \begin{bmatrix} 4-2 \\ 1 \end{bmatrix}_q . \end{aligned}$$

4 Complex Manifold $P_\gamma \backslash M$

Definition 4.1 *Let $I(m, n)$ be the set of all injective mappings from $[m] = \{1, \cdots, m\}$ into $[n] = \{1, \cdots, n\}$. Let*

$$J(m, n) = \{\sigma \in I(m, n) \,|\, \sigma(i) < \sigma(i+1), 1 \leq i \leq m-1\}.$$

Remark. We identify the mapping $\sigma \in I(m,n)$ with the m by n matrix $\Sigma_{i=1}^m E_{i,\sigma(i)}$. Thus we identify $I(m,n)$ with $R_{n^m}^m$ and we can talk about the rows and columns of $\sigma \in I(m,n)$.

Definition 4.2 *For $a \in M$, $I \subseteq [m]$ and $J \subseteq [n]$ with $|I| = |J|$, let $sub_{I,J}\, a$ denote the submatrix of a with row indices in I and column indices in J. Let*

$$a_{I,J} = det(sub_{I,J}\, a)$$

be the corresponding minor.

Thus, for $a \in M$ and $\sigma \in J(m,n)$, $a_{[m],\sigma([m])}$ is the minor of a with row indices $1, 2, \cdots, m$ and column indices $\sigma(1), \sigma(2), \cdots, \sigma(m)$.

In this section we prove that $P_\gamma \backslash M$ is a complex manifold. For this reason, in this section our partition $\lambda = (n, \cdots, n) = (n^m)$. First, we introduce a family of subsets $\mathcal{W}_\gamma(\sigma)$, for $\sigma \in I(m,n)$. For a fixed composition γ of m, if the i-th row and the j-th row are in the same γ-block, then we write $i \sim_\gamma j$. Recall that $R_\lambda^m(\gamma)$ is the set of all γ-compatible rook placements on the board F_λ with m rooks. Define

$$\begin{aligned} L_\gamma(\sigma) &= \{(i,j) \,|\, j \notin J(\sigma)\} \\ &\cup \ \{(i,j) \,|\, j = \sigma^{-1}(i_1),\ i > i_1,\ i \not\sim_\gamma i_1\} && (15) \\ \mathcal{W}_\gamma(\sigma) &= \{\sigma + \sum_{(i,j)\in L_\gamma(\sigma)} a_{i,j} E_{ij} \,|\, a_{i,j} \in \mathbf{C}\} && (16) \\ \mathcal{U}(\sigma) &= \bigcup_{a \in \mathcal{W}_\gamma(\sigma)} P_\gamma\, a. && (17) \end{aligned}$$

In the following for a fixed σ, we will call a pair $(i,j) \in L_\gamma(\sigma)$ a free position of $\mathcal{W}_\gamma(\sigma)$ and we call the corresponding $a_{i,j}$ a free entry of $\mathcal{W}_\gamma(\sigma)$.

Lemma 4.3 $M = \cup_{\sigma \in R_\lambda^m(\gamma)}\ \mathcal{U}(\sigma)$.

Proof. We need only show that $M \subset \cup_{\sigma \in R_\lambda^m(\gamma)} \mathcal{U}(\sigma)$. Take $a \in M$, we show that there is some $\sigma \in R_\lambda^m(\gamma)$ such that $a \in \mathcal{U}(\sigma)$. Note that left multiplication by elements of P_γ corresponds to **a)** elementary row operations "from bottom to top" within the whole matrix, and **b)** elementary row operations "from top to bottom" within each block.

Let

$$\nu_j = \sum_{i=1}^{j} \gamma_i \qquad \text{for } 1 \le j \le t \tag{18}$$

Let c be a k by k matrix. Let $I_p \oplus c \oplus I_q$ denote the $p+k+q$ by $p+k+q$ diagonal block matrix

$$I_p \oplus c \oplus I_q := \begin{pmatrix} I_p & & \\ & c & \\ & & I_q \end{pmatrix}.$$

We need the following construction.

Construction OP*. Start with a matrix $a \in M$. Construct a sequence $a^{(0)}, a^{(1)}, \cdots, a^{(t)}$. Let $a^{(0)} = a$. If $a^{(k)}$ has been constructed, choose $a^{(k+1)}$ as follows.

Define $J_{t+1} = \emptyset$. Suppose $k \le t$. Since $\operatorname{rank} a^{(k)} = m$, there is a non-singular square submatrix which is in the rows from the $(\nu_{t-k-1}+1)$-st to the ν_{t-k}-th of a with column index set $J_{t-k} = \{j_{\nu_{t-k-1}+1}, \cdots, j_{\nu_{t-k}}\}$ such that $j_{\nu_{t-k-1}+1} < \cdots < j_{\nu_{t-k}}$ and $J_{t-k} \cap J_q = \emptyset$, for each $q > t-k$. Multiply a by $I_{\nu_{k-1}} \oplus (sub_{[\nu_{t-k-1}+1, \nu_{t-k}], J_t} a)^{-1} \oplus I_{m-\nu_k}$ on the left. Then pivot at $(m - \gamma_k - \cdots - \gamma_t + i, j_i)$ upwards to eliminate all the entries above for $\nu_{t-k-1} + 1 \le i \le \nu_{t-k}$. Then, we get a new matrix $a^{(k+1)}$. End of Construction OP*.

In the construction above the column index set J_{t-k} is in general not unique. Therefore, the output of Construction OP* is in general not unique. But if the J_k is fixed for $1 \le k \le t$, then the output of Construction OP* is uniquely determined by a and $\{J_k\}_{k=1}^t$.

Let $J_t, J_{t-1}, \cdots, J_1$ be the sequence of column index sets obtained in this way. Define $\sigma(k) = j_k$, for $1 \le k \le m$. Then, $\sigma \in I(m,n)$ and the resulting matrix in the process above lies in $\mathcal{W}_\gamma(\sigma)$. In particular, for $i \sim_\gamma i'$, if $i < i'$ then $\sigma(i) < \sigma(i')$. Thus we know that $\sigma \in R_\lambda^m(\gamma)$. The definition of Construction OP* implies that there is some $p \in P_\gamma$ such that $a^{(m)} = p\,a \in \mathcal{W}_\gamma(\sigma) \subset \mathcal{U}(\sigma)$. By the definition of $\mathcal{U}(\sigma)$, we have $a = p^{-1} a^{(m)} \in \mathcal{U}(\sigma)$. Q.E.D.

Lemma 4.4 (Uniqueness) *Let $\sigma \in R_{n^m}^m(\gamma)$. If $a \in \mathcal{U}(\sigma)$, then there is a unique $p \in P_\gamma$ such that $p\,a \in \mathcal{W}_\gamma(\sigma)$. Write this p as $p(\sigma, a)$.*

Proof. Suppose $\sigma \in R_{n^m}^m(\gamma)$ is fixed and $a \in \mathcal{U}(\sigma)$. Then the columns $\sigma(1), \cdots, \sigma(m)$ of a are linearly independent. By the definition of $\mathcal{U}(\sigma)$, for $k = 0, 1, \cdots, t-1$, we can choose the

$$J_{t-k} = \{\sigma(\nu_{t-k-1}+1), \cdots, \sigma(\nu_{t-k})\}.$$

in Construction OP*. On the other hand, by the definition of $\mathcal{W}_\gamma(\sigma)$, to make sure that $p\,a \in \mathcal{W}_\gamma(\sigma)$ we have to choose $\sigma(k) = j_k$ for all k. Thus, by our earlier remark on uniqueness, there is a unique $p \in P_\gamma$ such that $p\,a \in \mathcal{W}_\gamma(\sigma)$. Q.E.D.
Let $u_\sigma : \mathcal{U}(\sigma) \to \mathcal{W}_\gamma(\sigma)$ be the map defined by $a \mapsto p(\sigma, a)\,a$. then Lemma 4.4 tells us that this is a well defined map. Since $\mathcal{W}_\gamma(\sigma) \subset \mathcal{U}(\sigma)$, this map is onto.

Corollary 4.5
a)

$$\mathcal{U}(\sigma) = \{a \in M \mid a_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]} \neq 0, \quad 1 \le k \le t\ \} \tag{19}$$

b)

$$P_\gamma \mathcal{U}(\sigma) \subset \mathcal{U}(\sigma) \textit{ for all } \ \sigma \in R_{n^m}^m(\gamma). \tag{20}$$

c)

$$P_\gamma \backslash M = \bigcup_{\sigma \in R_\lambda^m(\gamma)} V(\sigma) \tag{21}$$

where $V(\sigma) := P_\gamma \backslash \mathcal{U}(\sigma)$ is open in $P_\gamma \backslash M$.

Proof. **a)** By Lemma 4.4 for each $a \in \mathcal{U}(\sigma)$, there is some $p \in P_\gamma$ and $w \in \mathcal{W}_\gamma(\sigma)$ such that $a = pw$. By the definition of $\mathcal{W}_\gamma(\sigma)$, we have

$$w_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]} = \pm 1 \neq 0,$$

Since for each $p \in P_\gamma$,

$$\begin{aligned} & sub_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]}a = \\ = \;& sub_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]}(pw) \\ = \;& sub_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]}p \; sub_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]}w \end{aligned}$$

Take determinant on both sides, we have

$$a_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]} = p_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]} \; w_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]} \neq 0,$$

for $1 \leq k \leq t$. Therefore, the left hand of (19) is a subset of the right hand side of (19). On the other hand, if $a \in M$ such that $a_{[m-\nu_k+1,m],\sigma[m-\nu_k+1,m]} \neq 0$, for $1 \leq k \leq t$, then by Construction OP*, we can find $p \in P_\gamma$ and $w \in \mathcal{W}_\gamma(\sigma)$ such that $a = p\,w \in P_\gamma \mathcal{W}_\gamma(\sigma) = \mathcal{U}(\sigma)$, i.e., the right hand side of (19) is a subset of the left the left hand side of (19). Hence, (19) is proved. **b)** By the formula in **a)** this is obvious as all the matrices in P_γ are non-singular with all the non-zero entries in the parabolic board determined by γ. **c)** By the definition of quotient topology, we need to show that $\mathcal{U}(\sigma)$ is open in M. Note that the inequalities in (19) are strict polynomial inequalities. Thus, $\mathcal{U}(\sigma)$ is open in M. Q.E.D.

Lemma 4.6 *$V(\sigma)$ is an affine space of dimension $m(n-m) + \frac{m(m-1)}{2} + \sum_{i=1}^{t} \binom{\gamma_i}{2}$ for every $\sigma \in R_\lambda^m(\gamma)$.*

Proof. Define a map $v_\sigma : V(\sigma) \to \mathcal{W}_\gamma(\sigma)$ as the following: Let $\pi : M \to P_\gamma \backslash M$ be the canonical mapping defined by $a \mapsto P_\gamma\, a$. Define

$$v_\sigma(P_\gamma\, a) := u_\sigma(a).$$

Thus we have the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{U}(\sigma) & & \stackrel{u_\sigma}{\to} & & \mathcal{W}_\gamma(\sigma) \\ & \pi \searrow & & \nearrow v_\sigma & \\ & & V(\sigma) & & \end{array}$$

By the definitions of the quotient topology on $P_\gamma \backslash M$ and the subspace topology on $\mathcal{W}_\gamma(\sigma) \subset M$ we know that v_σ is a homeomorphism from $V(\sigma)$ onto $\mathcal{W}_\gamma(\sigma)$. By the definition of $L_\gamma(\sigma)$, dim $\mathcal{W}_\gamma(\sigma) = m(n-m) + \frac{m(m-1)}{2} + \sum_{i=1}^{t} \binom{\gamma_i}{2}$. Q.E.D.

Lemma 4.7 *Suppose $V(\sigma) \cap V(\sigma') \neq \emptyset$, for some $\sigma, \sigma' \in R_\lambda^m(\gamma)$. Then $v_\sigma \circ v_\sigma^{-1}$ is a biholomorphic homeomorphism from $v_\sigma(V(\sigma) \cap V(\sigma'))$ onto $v_{\sigma'}(V(\sigma) \cap V(\sigma'))$.*

Proof. Since $u_\sigma = \pi \circ v_\sigma$, by the definition of quotient topology, we need only show that $u_\sigma \circ u_\sigma^{-1}$ is a biholomorphic homeomorphism. Recall that for a fixed $a \in M$ and a fixed $\sigma \in R_\lambda^m(\gamma)$, Construction OP* gives

$$u_\sigma(a) = p_s \cdots p_1\, a$$

where each $p_i \in P_\gamma$. Moreover, by Construction PO*, we know that the p_i's are uniquely determined by the fixed $a \in M$ and a fixed $\sigma \in R_\lambda^m(\gamma)$. Since each p_i is either of the form of $I_{k_1} \oplus g \oplus I_{k_2}$ or $I + cE_{i,j}$ where g is the inverse of a k_3 by k_3 non-singular submatrix of a such that $k_1 + k_2 + k_3 = m$, and c is a rational function of the entries of a such that all the denominators are nonzero, for all $a \in \mathcal{U}(\sigma)$. Thus, u_σ is holomorphic from $\mathcal{U}(\sigma)$ onto $u_\sigma(\mathcal{U}(\sigma)) = \mathcal{W}_\gamma(\sigma)$. Note that the matrices $I_{k_1} \oplus g \oplus I_{k_2}$ or $I + cE_{i,j}$ are invertible. We know that u_σ is one to one and onto. Therefore, u_σ^{-1} exists. By the similar argument as above we know that u_σ^{-1} is holomorphic. Q.E.D.

Combine the three lemmas above, we have

Proposition 4.8 *$P_\gamma \backslash M$ is a complex manifold.*

The following is an example for an open cover of the quotient space $P_\gamma \backslash M$.

Example 4.9 *Let $\gamma = (1,2)$ and $\lambda = (4,4,4)$. Then, λ is clearly γ-compatible. Let*

$$\sigma^{(1)} = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array} \qquad \sigma^{(2)} = \begin{array}{|c|c|c|c|}\hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array}$$

$$\sigma^{(3)} = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & 1 & & \\ \hline & & & 1 \\ \hline \end{array} \qquad \sigma^{(4)} = \begin{array}{|c|c|c|c|}\hline & & 1 & \\ \hline & 1 & & \\ \hline & & & 1 \\ \hline \end{array}$$

$$\sigma^{(5)} = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline \end{array} \qquad \sigma^{(6)} = \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline \end{array}$$

$$\sigma^{(7)} = \begin{array}{|c|c|c|c|}\hline & 1 & & \\ \hline 1 & & & \\ \hline & & & 1 \\ \hline \end{array} \qquad \sigma^{(8)} = \begin{array}{|c|c|c|c|}\hline & & 1 & \\ \hline 1 & & & \\ \hline & & & 1 \\ \hline \end{array}$$

$$\sigma^{(9)} = \begin{array}{|c|c|c|c|}\hline & 1 & & \\ \hline 1 & & & \\ \hline & & 1 & \\ \hline \end{array} \qquad \sigma^{(10)} = \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 1 & & & \\ \hline & & 1 & \\ \hline \end{array}$$

$$\sigma^{(11)} = \begin{array}{|c|c|c|c|}\hline & & 1 & \\ \hline 1 & & & \\ \hline & 1 & & \\ \hline \end{array} \qquad \sigma^{(12)} = \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 1 & & & \\ \hline & 1 & & \\ \hline \end{array}$$

$$a(\sigma^{(1)}) = \begin{pmatrix} 1 & a_{1,2} & 0 & 0 \\ a_{2,1} & a_{2,2} & 1 & 0 \\ a_{3,1} & a_{3,2} & 0 & 1 \end{pmatrix} \qquad a(\sigma^{(2)}) = \begin{pmatrix} a_{1,1} & 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & 1 & 0 \\ a_{3,1} & a_{3,2} & 0 & 1 \end{pmatrix}$$

$$a(\sigma^{(3)}) = \begin{pmatrix} 1 & 0 & a_{1,3} & 0 \\ a_{2,1} & 1 & a_{2,3} & 0 \\ a_{3,1} & 0 & a_{3,3} & 1 \end{pmatrix} \qquad a(\sigma^{(4)}) = \begin{pmatrix} a_{1,1} & 0 & 1 & 0 \\ a_{2,1} & 1 & a_{2,3} & 0 \\ a_{3,1} & 0 & a_{3,3} & 1 \end{pmatrix}$$

$$a(\sigma^{(5)}) = \begin{pmatrix} 1 & 0 & 0 & a_{1,4} \\ a_{2,1} & 1 & 0 & a_{2,4} \\ a_{3,1} & 0 & 1 & a_{3,4} \end{pmatrix} \qquad a(\sigma^{(6)}) = \begin{pmatrix} a_{1,1} & 0 & 0 & 1 \\ a_{2,1} & 1 & 0 & a_{2,4} \\ a_{3,1} & 0 & 1 & a_{3,4} \end{pmatrix}$$

$$a(\sigma^{(7)}) = \begin{pmatrix} 0 & 1 & a_{1,3} & 0 \\ 1 & a_{2,2} & a_{2,3} & 0 \\ 0 & a_{3,2} & a_{3,3} & 1 \end{pmatrix} \qquad a(\sigma^{(8)}) = \begin{pmatrix} 0 & a_{1,2} & 1 & 0 \\ 1 & a_{2,2} & a_{2,3} & 0 \\ 0 & a_{3,2} & a_{3,3} & 1 \end{pmatrix}$$

$$a(\sigma^{(9)}) = \begin{pmatrix} 0 & 1 & 0 & a_{1,4} \\ 1 & a_{2,2} & 0 & a_{2,4} \\ 0 & a_{3,2} & 1 & a_{3,4} \end{pmatrix} \qquad a(\sigma^{(10)}) = \begin{pmatrix} 0 & a_{1,2} & 0 & 1 \\ 1 & a_{2,2} & 0 & a_{2,4} \\ 0 & a_{3,2} & 1 & a_{3,4} \end{pmatrix}$$

$$a(\sigma^{(11)}) = \begin{pmatrix} 0 & 0 & 1 & a_{1,4} \\ 1 & 0 & a_{2,3} & a_{2,4} \\ 0 & 1 & a_{3,3} & a_{3,4} \end{pmatrix} \qquad a(\sigma^{(12)}) = \begin{pmatrix} 0 & 0 & a_{1,3} & 1 \\ 1 & 0 & a_{2,3} & a_{2,4} \\ 0 & 1 & a_{3,3} & a_{3,4} \end{pmatrix}$$

For $1 \leq k \leq 12$

$$\begin{aligned} \mathcal{W}_\gamma(\sigma^{(k)}) &= \{a(\sigma^{(k)}) \mid a_{i,j} \in \mathbf{C}\}, \\ \mathcal{U}(\sigma^{(k)}) &= \bigcup_{a(\sigma^{(k)}) \in \mathcal{W}_\gamma(\sigma^{(k)})} P_\gamma\, a(\sigma^{(k)}) \\ V(\sigma^{(k)}) &= \{P_\gamma\, a(\sigma^{(k)}) \mid a(\sigma^{(k)}) \in \mathcal{W}_\gamma(\sigma^{(k)})\} \end{aligned}$$

Thus we have a) each $V(\sigma^{(k)})$ *is open and b)* $P_\gamma \backslash M = \cup_{k=1}^{12} V(\sigma^{(k)})$.

5 Partition Variety $P_\gamma \backslash M_\lambda$

In this section, we prove that the quotient space $P_\gamma \backslash M_\lambda$ is a projective variety which justifies the name *partition variety*. We first embed the quotient space $P_\gamma \backslash M$ into a direct product of projective spaces. Then we show that the image of the quotient space is an algebraic subvariety of the direct product of projective spaces. At the end, we prove that $P_\gamma \backslash M_\lambda$ is an algebraic subvariety of $P_\gamma \backslash M$.

For $a \in M$, let a_k denote the submatrix of a consisting of the last k rows in a.

Definition 5.1 *Let* $\Phi_\gamma : M \to \prod_{i=1}^t \mathbf{P}^{\binom{n}{\mu_i}-1}$ *be the mapping defined by*

$$\Phi_\gamma(a) := (\phi_{\mu_t}(a_{\mu_t}), \cdots, \phi_{\mu_1}(a_{\mu_1})) \tag{22}$$

where

$$\phi_k(a_k) = (\cdots : a_{[m-k+1.m],J} : \cdots) \tag{23}$$

with J *running throughout all the* k *subsets of* $[n]$ *and* $\mu_i := \sum_{j=i}^{t} \gamma_j$ *for* $1 \le i \le t$.

For a matrix $a \in M$, let $\triangle_k(a)$ denote the submatrix in the southeast corner of a.

Proposition 5.2 $\Phi_\gamma(a) = \Phi_\gamma(p\,a)$, *for all* $p \in P_\gamma$.

Proof. Take $a \in M$. By the definition of P_γ, we have

$$\triangle_{\mu_i}(p\,a) = \triangle_{\mu_i}(p)\,\triangle_{\mu_i}(a), \text{ for } 1 \le i \le t.$$

In general, for an arbitrary μ_i by μ_i submatrix in the last μ_i rows of $p\,a$, it can be viewed as $\triangle_{\mu_i}(p\,a)$ multiplied by an n by n permutation matrix on the right. Since the action of P_γ on the left and the action of S_n on the right commute with each other, each μ_i by μ_i submatrix in the last μ_i rows of $p\,a$ is equal to the product of the μ_i by μ_i submatrix in the last μ_i rows of a with the same column index set and the μ_i by μ_i submatrix in the south east corner of p. Therefore, $\phi_{\mu_i}((p\,a)_{\mu_i}) = \triangle_{\mu_i}(p)\phi_{\mu_i}(a_{\mu_i}) = \phi_{\mu_i}(a_{\mu_i})$ for $1 \le i \le t$. Q.E.D.

So, we can define $\mathbf{\Phi}(P_\gamma\,a) := \Phi_\gamma(a)$. By proposition 5.2 the map $\mathbf{\Phi}$ is well defined from $P_\gamma \backslash M$ into $\prod_{i=1}^{t} \mathbf{P}^{\binom{n}{\mu_i}-1}$.

Proposition 5.3 *The mapping* $\mathbf{\Phi}$ *is an embedding of complex manifolds.*

Proof. By definition of embeddings of complex manifolds, we need to show three things: **a)** $\mathbf{\Phi}$ is holomorphic. **b)** $\mathbf{\Phi}$ is of full rank. **c)** $\mathbf{\Phi}$ is injective.

a) This is obvious as the components of each ϕ_i are polynomials in the entries of the matrix a.

b) Let $j_i = \sigma(i)$. By definition of Φ_γ, the component of $\phi_k(a_k)$ with the set of column indices equal to $\{j\} \cup (\sigma[m-k+2, m] - \{j_i\})$ is $\pm a_{m-k+1,j}$, i.e.,

$$(\phi_k(a_k))_{\{j\}\cup(\sigma[m-k+1,m]-\{j_i\})} = \pm a_{i,j}. \tag{24}$$

where $j \notin \sigma([m-k+1, m]) - \{j_i\}$, $k \in \{\mu_i\}_{1\le i\le t}$ and k is the minimal number in this set such that $m-k+1 \le i$. Thus every free entry of $\mathcal{W}_\gamma(\sigma)$ or its negative multiple appears as a component in the image of $\mathbf{\Phi}$. Thus the map $\mathbf{\Phi}$ is of full rank.

c) The equation (24) also tells us that $a_{i,j}$ is determined by $\mathbf{\Phi}(P_\gamma\,a)$. Thus $\mathbf{\Phi}$ is injective on each open set $V(\sigma)$. Next, we show that Φ_γ is injective on the whole $P_\gamma \backslash M$. Suppose $a, a' \in M$ and $\mathbf{\Phi}(P_\gamma\,a) = \mathbf{\Phi}(P_\gamma\,a')$. We want to show that $P_\gamma\,a = P_\gamma\,a'$. If there exists some $\tau \in R^m_{(n^m)}(\gamma)$ such that $a, a' \in \mathcal{U}(\tau)$, then we are done since the restriction of $\mathbf{\Phi}$ to $V(\tau)$ is injective. Suppose such τ does not exist. We prove that $\mathbf{\Phi}(P_\gamma\,a) \ne \mathbf{\Phi}(P_\gamma\,a')$, which is a contradiction.

Choose $\sigma, \sigma' \in R^m_{(n^m)}(\gamma)$ such that $a \in \mathcal{U}(\sigma)$ and $a' \in \mathcal{U}(\sigma')$. Choose $w \in W_n$ so that $(\sigma w)(i) = i$. Since $\mathcal{U}(\sigma w) = \mathcal{U}(\sigma)w$ and the components of $\phi_k((aw)_k)$ are a permutation of the components of $\phi_k(a_k)$ we may replace σ by σw and assume, without loss of generality, that $\sigma(i) = i$. Thus $a_{[m],[m]} \neq 0$.

Case 1: The first m columns of a' are linearly dependent. In this case, consider the component with $J = [m]$ in (23), we get $(\phi_m(a))_{[m]} = a_{[m],[m]} \neq 0$ and $(\phi_m(a'))_{[m]} = a'_{[m],[m]} = 0$. Thus $\phi_m(a'_m) \neq \phi_m(a_m)$. So $\Phi_\gamma(a') \neq \Phi_\gamma(a)$. Therefore $\mathbf{\Phi}(P_\gamma\, a) \neq \mathbf{\Phi}(P_\gamma\, a')$

Case 2: The first m columns of a' are linearly independent. Thus we can choose σ' such that $a' \in \mathcal{U}(\sigma')$ and $\sigma'([m]) = [m]$. Since there does not exist $\tau \in R^m_{(n^m)}(\gamma)$ such that $a, a' \in \mathcal{U}(\tau)$, we have $\sigma \neq \sigma'$.

For the fixed orbit Ba', there may be several τ's in $R^m_\lambda(\gamma)$ such that $P_\gamma\, a' \subseteq \mathcal{U}(\tau)$ and $\tau([m]) = [m]$. Define

$$\Omega(P_\gamma\, a') = \{\tau \in P_\gamma \,|\, Ba' \subseteq \mathcal{U}(\tau) \text{ and } \tau([m]) = [m]\}.$$

Choose $\sigma' \in \Omega(P_\gamma\, a')$ such that σ and σ' agree in the last $m-k$ rows and $m-k$ is maximal. By definition of u_σ,

$$u_\sigma(a) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & a_{1,m+1} & \cdots & a_{1,n} \\ a_{2,1} & 1 & 0 & 0 & \cdots & 0 & a_{2,m+1} & \cdots & a_{2,n} \\ \vdots & \ddots & \ddots & \ddots & & \vdots & & & \\ a_{k-1,1} & \cdots & a_{k-1,k-2} & 1 & 0 & 0 & \vdots & & \vdots \\ \vdots & & & \ddots & \ddots & 0 & & & \\ a_{m,1} & & \cdots & & a_{m,m-1} & 1 & a_{m,m+1} & \cdots & a_{m,n} \end{pmatrix}.$$

Note that

$$\begin{aligned} L_\gamma(\sigma) \cap L_\gamma(\sigma') \quad \supset \quad & \{(i,j) \in L_\gamma(\sigma) \,|\, k \leq i \leq m\} \\ & \cup \{(i,j) \in L_\gamma(\sigma) \,|\, n-m+1 \leq j \leq n\} \end{aligned}$$

In other words, the free entries of $\mathcal{W}(\sigma)$ and $\mathcal{W}(\sigma')$ agree in the last $m-k+1$ rows and the last $n-m$ columns. But, in the $(k-1)$-st row, since $\sigma'(k-1) \neq \sigma(k-1)$, we have $j := \sigma'(k-1) < \sigma(k-1) = k-1$. This follows from the facts that $\sigma'([m]) = [m]$, $\sigma'(i) = \sigma(i)$ for $i = k, \cdots, m$ and σ' is injective.

Suppose $\mu_i \leq k-1 \leq \mu_{i-1}$. Note that $\phi_{\mu_i}(a_{\mu_i}) = \phi_{\mu_i}(a'_{\mu_i})$ in $\mathbf{P}^{\binom{n}{\mu_i}-1}$. Consider the $[m-\mu_i+1, m]$-th and the $(\{j\} \cup ([m-\mu_i+1, m] - \{k-1\}))$-th components, we have

$$\begin{aligned} & (\cdots : a_{k-1,j} : \cdots : 1 : \cdots) = \\ = \;\; & (\cdots : (\phi_{\mu_i}(a_{\mu_i}))_{(\{j\}\cup([m-\mu_i+1,m]-\{k-1\}))} : \cdots : (\phi_{\mu_i}(a_{\mu_i}))_{[m-\mu_i+1,m]:\cdots}) \\ = \;\; & (\cdots : (\phi_{\mu_i}(a'_{\mu_i}))_{(\{j\}\cup([m-\mu_i+1,m]-\{k-1\}))} : \cdots : (\phi_{\mu_i}(a'_{\mu_i}))_{[m-\mu_i+1,m]:\cdots}) \\ = \;\; & (\cdots : 1 : \cdots : a'_{k-1,k-1} : \cdots). \end{aligned}$$

Therefore, $a_{k-1,j}/1 = 1/a'_{k-1,k-1}$ which implies that $a'_{k-1,k-1} \neq 0$. Let $(a')^{(0)}, \ldots, (a')^{(m)}$ be the output of Construction OP*. By Construction OP*, we know that we can pivot at the $(k-1, k-1)$ position on $(a')^{(m-k)}$. This implies that $P_\gamma\, a' \subset \mathcal{U}(\tau)$ for some τ such that $\tau([m]) = [m]$ and $\sigma(i) = \tau(i)$ for $i = k-1, \cdots, m$. This contradicts the maximality of $m-k$. So we know that $\mathbf{\Phi}$ is injective. Q.E.D.

Lemma 5.4 *$P_\gamma\backslash M$ is compact.*

Proof. Let $N = \{a \in M \mid$ the row vectors of a are orthonormal $\}$. Then, for each $a \in N$, $\sum_{k=1}^n |a_{i,k}|^2 = 1$, for $1 \le i \le m$, $\sum_{k=1}^n a_{i,k}\overline{a_{j,k}} = 0$, if $i \ne j$. Thus, we know that N is a bounded and closed subset of M which is compact. Since the matrix corresponding to Gram-Schmidt process is upper triangular, in each P_γ-orbit, there is some matrix in N. let $\pi : M \to P_\gamma\backslash M$ be the canonical mapping. Then, $\pi(N) = P_\gamma\backslash M$. Note that the left multiplication by a matrix corresponding to Gram-Schmidt process is a continuous mapping. $P_\gamma\backslash M$ is compact. Q. E. D.

Recall the embedding mapping defined in (6)

$$\Psi : \mathbf{P}^{\binom{n}{\mu_1}-1} \times \mathbf{P}^{\binom{n}{\mu_2}-1} \times \cdots \times \mathbf{P}^{\binom{n}{\mu_t}-1} \to \mathbf{P}^N$$

where $N = \sum_{i=1}^m \binom{n}{\mu_i} - 1$. Combine all the results proved in this section, by the Embedding Theorem —refpro:locemb and Chow's Theorem we have

Proposition 5.5 *Let $(\Psi \circ \mathbf{\Phi})(P_\gamma\backslash M)$ is an algebraic subvariety of $\mathbf{P}^N$ and hence $P_\gamma\backslash M$ itself is a projective variety.*

Theorem 5.6 *$P_\gamma\backslash M_\lambda$ is a projective variety if $\lambda_i \ge m - i + 1$, $1 \le i \le m$ and λ is γ-compatible.*

Proof. Choose any $\sigma \in R_\lambda^m(\gamma)$ and fix it temporarily. By (24) every free entry of $a \in \mathcal{W}_\gamma(\sigma)$ is a component of some ϕ_k. Consider equations

$$a_{i,j} = 0, \qquad \text{for} \quad (i,j) \notin F_\lambda. \tag{25}$$

Note that $\lambda_i \ge m - i + 1$, $1 \le i \le m$ and λ is γ-compatible. These imply that the parabolic board $F_\gamma \subset F_\lambda$. Thus, if $(i,j) \notin F_\lambda$, then $(i,j) \notin F_\gamma$. Hence, $p_{i,k} = 0$ for all $k < j$. In particular, $p_{i,k} = 0$ for all $k < i$.

In fact, for each $a \in M$ if $a \in M_\lambda$, i.e., a satisfies the equations in (25), then for any $p \in P_\gamma$, $pa = (\sum_{k=i}^m p_{i,k}a_{k,j})$. In particular, if $(i,j) \notin F_\lambda$, then $a_{k,j} = 0$ for all $k \ge i$. Hence,

$$\begin{aligned} (pa)_{i,j} &= \sum_{k=1}^m p_{i,k}a_{k,j} \\ &= \sum_{k=1}^{i-1} p_{i,k}a_{k,j} \\ &= 0. \end{aligned}$$

This tells us that $(pa) \in M_\lambda$. Since p is arbitary in P_γ, we have $P_\gamma\, a \subset M_\lambda$ which implies that equations in (25) give a set of homogeneous equations in $\mathbf{\Phi}(V(\sigma) \subset \mathbf{\Phi}(P_\gamma\backslash M)$. These homogeneous equations define the image $\mathbf{\Phi}(V(\sigma) \cap (P_\gamma\backslash M_\lambda)) \subset \mathbf{\Phi}(P_\gamma\backslash M_\lambda)$ as a subvariety of the projective variety $\mathbf{\Phi}(P_\gamma\backslash M)$. Recall that we have the embedding $\Psi : \prod_{i=1}^m \mathbf{P}^{\binom{n}{\mu_i}-1} \to \mathbf{P}^N$ defined as in (6) where $N = \binom{n}{\mu_1} + \cdots + \binom{n}{\mu_m} - 1$. Thus $(\Psi \circ \mathbf{\Phi})(P_\gamma\backslash M_\lambda)$ is an analytic subvariety of $\mathbf{P}^N$. By Chow's Theorem 2.21 $(\Psi \circ \mathbf{\Phi})(P_\gamma\backslash M_\lambda)$ is a projective subvariety of $\mathbf{P}^N$. Q.E.D.

6 Cellular Decomposition of $P_\gamma\backslash M_\lambda$

In this section we give a cellular decomposition of the partition variety $P_\gamma\backslash M_\lambda$. The topological implication of this cellular decomposition is the topic of the next section. To state our main results we construct a transversal for the P_γ-orbits on M. We construct a transversal for $P_\gamma\backslash M$ of the form $\bigsqcup_{\sigma\in I(m,n)} Y_\gamma(\sigma)$ where the sets $Y_\gamma(\sigma)$ are defined as follows: Let

$$H(\sigma) = H_1(\sigma)\cup H_2(\sigma) \tag{26}$$

where

$$\begin{aligned} H_1(\sigma) &= \{(i,j)\,|\,\sigma(i)<j, \text{ for } j\notin J(\sigma)\} &(27)\\ H_2(\sigma) &= \{(i,j)\,|\,\sigma(i)<j, \text{ for } j\in J(\sigma) \text{ and } \sigma^{-1}(j)<i\}. &(28)\end{aligned}$$

Define

$$Y(\sigma) = \sigma + \sum_{(i,j)\in H(\sigma)} \mathbf{C}E_{i,j}. \tag{29}$$

Lemma 6.1 *$Y_\gamma(\sigma)$ is an affine space of dimension $l(\sigma) = |H(\sigma)|$ where $l(\sigma)$ is the length function defined in Definition 2.1.*

Proof. Clearly dim $Y_\gamma(\sigma) = |H(\sigma)|$. Thus we must prove $|H(\sigma)| = l(\sigma)$. By the local formula for the length function in Proposition 2.2, the contribution of the i-th rook to the length function equals the sum of the number of zero columns to the right of the rook and the number of rooks to the northeast of the rook (since the number of rooks equals the number of rows, there are no zero rows above the rook). By the definition of the set $H(\sigma)$, a pair (i,j) belongs to $H(\sigma)$ if and only if either the j-th column is a zero column or the rook in the j-th column is to the northeast of the i-th rook. Thus, $l(\sigma) = |H(\sigma)|$ and hence $Y_\gamma(\sigma)$ is an affine space of dimension $l(\sigma)$. Q.E.D.

For $\sigma\in I(m,n)$ define

$$X_\gamma(\sigma) = \{P_\gamma\, y\,|\, y\in Y_\gamma(\sigma)\}\subseteq P_\gamma\backslash M. \tag{30}$$

The space $X_\gamma(\sigma)$ has the subspace topology from $P_\gamma\backslash M$ and $Y(\sigma)$ has the subspace topology from M. There is a natural homeomorphism from $X_\gamma(\sigma)$ onto $Y_\gamma(\sigma)$ defined by $P_\gamma\, y\to y$. Here we give an example to illustrate the definition of $Y_\gamma(\sigma)$.

Example 6.2 *Recall that in Example 4.9, $m=3$, $n=4$, $\lambda=(4^3)$ and $\gamma=(1,2)$. Then $\{Y_\gamma(\sigma)\}_{\sigma\in R^m_\lambda(\gamma)}$ is given as follows.*

$$Y_\gamma(\sigma^{(1)}) = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Y_\gamma(\sigma^{(2)}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Y_\gamma(\sigma^{(3)}) = \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Y_\gamma(\sigma^{(4)}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Y_\gamma(\sigma^{(5)}) = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \end{pmatrix} \qquad Y_\gamma(\sigma^{(6)}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \end{pmatrix}$$

$$Y_\gamma(\sigma^{(7)}) = \begin{pmatrix} 0 & 1 & x & 0 \\ 1 & x & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Y_\gamma(\sigma^{(8)}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & x & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Y_\gamma(\sigma^{(9)}) = \begin{pmatrix} 0 & 1 & 0 & x \\ 1 & x & 0 & x \\ 0 & 0 & 1 & x \end{pmatrix} \qquad Y_\gamma(\sigma^{(10)}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & x & 0 & x \\ 0 & 0 & 1 & x \end{pmatrix}$$

$$Y_\gamma(\sigma^{(11)}) = \begin{pmatrix} 0 & 0 & 1 & x \\ 1 & 0 & x & x \\ 0 & 1 & x & x \end{pmatrix} \qquad Y_\gamma(\sigma^{(12)}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & x & x \\ 0 & 1 & x & x \end{pmatrix}.$$

where we use an "x" for an entry which is arbitrary in $\mathbf{C}$.

Proposition 6.3 (Cellular Decomposition of M) *The manifold M has the following cellular decomposition:*

$$M = \bigsqcup_{\sigma \in R^m_{\binom{n}{m}}(\gamma)} \mathcal{X}_\gamma(\sigma)$$

where $\bigsqcup$ denotes the disjoint union and $\mathcal{X}_\gamma(\sigma) := P_\gamma Y_\gamma(\sigma)$.

Proof. For each fixed $a \in M$, list all the γ_t by γ_t submatrices of a in its last block accoording to the column indices, lexicographically. Let $c_t := sub_{[m-\gamma_t+1,m],J_t} a$ the first non-singular such submatrix in this order. Write $J_t = \{j_{m-\gamma_t+1}, \cdots, j_m\}$ such that $j_{m-\gamma_t+1} < \cdots < j_m$. Multiply a by

$$d_t(a) := \begin{pmatrix} I_{\gamma_1} & & & & \\ & I_{\gamma_2} & & & \\ & & \ddots & & \\ & & & I_{\gamma_{t-1}} & \\ & & & & (c_t)^{-1} \end{pmatrix}$$

on the left. Then do row eliminations upwards, using the identity submatrix with row index set $[m-\gamma_t+1, m]$ and column index set J_t to eliminate all the entries above in these columns. Then, we get a new matrix $a^{(t)}$. Clearly,

$$a^{(t)} = u_t(a)\, d_t(a)\, a$$

where $u_t(a) = I - \sum_{i<m-\gamma_t+1, j\in J_t} a_{i,j} E_{i,j}$ and I is the m by m identity matrix.

Construction Block k

For $a^{(k+1)}$ where $k \geq 1$, list all the γ_k by γ_k submatrices of $a^{(k+1)}$ in its k-th block accoording to the column indices, lexicographically. Recall that $\mu_k = \sum_{j=k}^{t} \gamma_j$. Let $c_k := sub_{[m-\mu_k+1, m-\mu_{k+1}], J_k} a$ the first non-singular such submatrix in this order with

$$J_t = \{j_{m-\mu_k+1}, \cdots, j_{m-\mu_{k+1}}\}.$$

Multiply a by

$$d_k(a) := \begin{pmatrix} I_{\gamma_1} & & & & \\ & \ddots & & & \\ & & I_{\gamma_{k-1}} & & \\ & & & (c_k)^{-1} & \\ & & & & I_{\mu_{k+1}} \end{pmatrix}$$

on the left. Then do row eliminations upwards, using the identity submatrix with row index set $[m-\mu_k+1, m-\mu_{k+1}]$ and column index set J_k to eliminate all the entries above in these columns. Then, we get a new matrix $a^{(k)}$. **End of Construction Block k.**

Thus

$$\begin{aligned} a^{(k)} &= u_k(a)\, d_k(a)\, a^{(k+1)} \\ &= u_k(a)\, d_k(a)\, u_{k+1}(a)\, d_{k+1}(a) \cdots u_t(a)\, d_t(a)\, a \quad . \end{aligned}$$

Repeat this process until we get $a^{(1)}$. The definition of **Construction Block** k implies that $a^{(1)} \in Y_\gamma(\sigma)$ where $\sigma \in R^m_{(n^m)}(\gamma)$ is defined by $\sigma(i) = j_i$. Note that

$$\begin{aligned} a^{(1)} &= u_1(a)\, d_1(a)\, a^{(2)} \\ &= \cdots = \\ &= u_1(a)\, d_1(a)\, u_2(a)\, d_2(a) \cdots u_t(a)\, d_t(a)\, a \quad . \end{aligned}$$

Let

$$p' := u_1(a)\, d_1(a)\, u_2(a)\, d_2(a) \cdots u_t(a)\, d_t(a).$$

and $p := (p')^{-1}$. Then $p \in P_\gamma$ such that $a = p\, a^{(1)}$. So $M \subset \bigcup_{\sigma\in R^m_{(n^m)}(\gamma)} \mathcal{X}_\gamma(\sigma)$ where $\mathcal{X}_\gamma(\sigma) = P_\gamma Y_\gamma(\sigma)$.

Next, we need to show that $P_\gamma Y_\gamma(\sigma) \cap P_\gamma Y_\gamma(\sigma') = \emptyset$ if $\sigma \neq \sigma'$. Without loss of generality, assume $\sigma = \sum_{i=1}^m E_{i,i}$. Let i_0 be the smallest row index such that $j_0 = i_0 = \sigma(i_0) \neq \sigma'(i_0) = j_0'$. Assume the i_0-th row is in the k-th block. Assume there were some $a' \in P_\gamma Y_\gamma(\sigma') \cap P_\gamma Y_\gamma(\sigma)$. Then, there are some $p_1, p_2 \in P_\gamma$, $y_1 \in Y_\gamma(\sigma)$ and $y_2 \in Y_\gamma(\sigma')$ such that $p_1 y_1 = a' = p_2 y_2$. Hence, $y_1 = p_1^{-1} p_2 y_2$. Let $p = p_1^{-1} p_2$. By Construction Block k, after using Construction i for $k+1 \leq i \leq t$, since $p\, y_2 = y_1$, the first non-singular submatrix in the k-th block of y_2 in the lexicogrraphic order is of the same column index set as that in y_1. Thus, the definitions of $Y_\gamma(\sigma)$ and $Y_\gamma(\sigma')$ imply that $\sigma(i_0) = \sigma'(i_0)$. This contradicts the choice of i_0. Therefore, such an a' does not exist. So $P_\gamma Y_\gamma(\sigma') \cap P_\gamma Y_\gamma(\sigma) = \emptyset$. Q.E.D.

Theorem 6.4 (Cellular Decomposition of $P_\gamma\backslash M_\lambda$)
Let λ be a γ-compatible partition. Then the partition variety $P_\gamma\backslash M_\lambda$ has the following cellular decomposition:

$$P_\gamma\backslash M_\lambda = \bigsqcup_{\sigma\in R^m_\lambda(\gamma)} X_\gamma(\sigma)$$

where $\bigsqcup$ denotes the disjoint union and $X_\gamma(\sigma) := P_\gamma\backslash\mathcal{X}_\gamma(\sigma)$.

Proof. By the definition of $\mathcal{X}_\gamma(\sigma)$ and $X_\gamma(\sigma)$, Proposition 6.3 implies that

$$P_\gamma\backslash M = \bigsqcup_{\sigma\in R^m_{(n^m)}(\gamma)} X_\gamma(\sigma).$$

Take $\sigma \in R^m_\lambda(\gamma)$. Suppose $(i,\sigma(i)) \in F_\lambda$ for $1 \le i \le m$. By *Construction OP^** in Section 4, $Y_\gamma(\sigma) \subset M_\lambda$. Since λ is γ-compatible, we have $P_\gamma\, M_\lambda \subseteq M_\lambda$. In particular this implies that $P_\gamma\, Y_\gamma(\sigma) \subset M_\lambda$. Thus $X_\gamma(\sigma) \subset P_\gamma\backslash M_\lambda$. On the other hand if for some $\sigma' \in R^m_{(n^m)}(\gamma)$, there exists some i such that $(i,\sigma'(i)) \notin F_\lambda$ then $Y_\gamma(\sigma') \cap M_\lambda = \emptyset$. Thus $X_\gamma(\sigma') \cap P_\gamma\backslash M_\lambda = \emptyset$. Therefore our theorem is proved. Q.E.D.

7 CW-complex Structure of $P_\gamma\backslash M_\lambda$

In this section, we will show that the partition variety

$$P_\gamma\backslash M_\lambda = \bigsqcup_{\sigma\in R^m_\lambda(\gamma), F_\lambda(\sigma)\subset F_\lambda} X_\gamma(\sigma)$$

is a CW-complex where $X_\gamma(\sigma) = P_\gamma\backslash\mathcal{X}_\gamma(\sigma)$. For $1 \le k \le m$ let $M_{k,n}$ be the set of k by n matrices. Let

$$M^k_{k,n} = \{a \in M_{k,n} \mid \text{rank } a = k\} \tag{31}$$

Define a map $\rho_k : M \to M^k_{k,n}$ by

$$\rho_k(a) = a_k \tag{32}$$

where a_k is the submatrix of a consisting of its last k rows. Let $\sigma_k = \rho_k(\sigma)$ denote the submatrix consisting of the last k rows in σ. Define $\lambda(\sigma_k)$ to be the partition

$$\{n - \sigma^*(m-k+1) + 1, n - \sigma^*(m-k+2) + 1, \cdots, n - \sigma^*(m) + 1\}.$$

If $k = m$ then $\sigma_k = \sigma$.

Recall that for a fixed $\gamma = (\gamma_1, \cdots, \gamma_t)$, $\mu_i = \sum_{j=i}^t \gamma_j$. Let $\gamma_0 = 0$. Write $\nu_i = \sum_{j=0}^i \gamma_j$. Clearly, for each i, $\mu_{i+1} + \nu_i = m$. Let a be a matrix in M. Write the k-th row of a as $\alpha_k(a)$ for $1 \le k \le m$. Let a_k denote the submatrix of a consisting of the last k rows of a. Now consider the following set of partial flags

$$\mathcal{F}_\gamma(\mathbf{C}^n) = \{(V_1, \cdots, V_t) \mid \dim V_i = \mu_{t-i+1} \text{ and } V_i \subset V_{i+1}, \forall i\}..$$

We have the following lemma

Lemma 7.1 *$h_\gamma : P_\gamma \backslash M \to \mathcal{F}_\gamma(\mathbf{C}^n)$ is a homeomorphism defined by*

$$h_\gamma([a]) = (h_{\mu_t}(a_{\mu_t}), \cdots, h_{\mu_1}(a_{\mu_1})) \tag{33}$$

where for $1 \le k \le m$

$$a_k = \begin{pmatrix} \alpha_{m-k+1}(a) \\ \vdots \\ \alpha_m(a) \end{pmatrix}$$

and

$$h_k(a_k) := < \alpha_m(a), \alpha_{m-1}(a), \cdots, \alpha_{m-k+1}(a) >,$$

i.e., $h_k(a_k)$ is the linear span of the row vectors $\alpha_m(a), \alpha_{m-1}(a), \cdots, \alpha_{m-k+1}(a)$ of a.

Proof. We need to show a) h_γ is well defined, b) h_γ is a one to one correspondence, and c) h_γ and its inverse are both continuous.

a) For any given $p \in P_\gamma$, since p has all its nonzero elements in the parabolic board corresponding to γ,

$$< \alpha_m(p\,a), \cdots, \alpha_{\mu_i}(p\,a) >=< \alpha_m(a), \cdots, \alpha_{\mu_i}(a) >, \quad \text{for } 1 \le i \le t.$$

So the image of h_γ is independent of the choice of representatives in $[a]$, i.e., h_γ is well defined.

b) For each $(V_1, \cdots, V_t) \in \mathcal{F}_\gamma(\mathbf{C}^n)$, let $V_0 = (0)$ and $\mu_{t+1} = 0$, take a maximal linearly independent set $\{v_j\}_{j=\nu_{i-1}+1}^{\nu_i} \subset V_{i+1} - V_i, 0 \le i \le t-1$. Take

$$a := \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}.$$

By the definition of h_γ, we have $h_\gamma([a]) = (V_1, \cdots, V_t)$. So, h_γ is onto.

Next, we prove that h_γ is one to one. Suppose $h_\gamma([a]) = h_\gamma([a'])$. Use induction on t to show that $a' = p\,a$ for some $p \in P_\gamma$. If $t = 1$, then $\gamma = (n^m)$. Thus, the equation $h_\gamma([a]) = h_\gamma([a'])$ implies that $[a] = [a']$ as m dimensional subspaces of $\mathbf{C}^n$. Therefore, there is some $p \in P_\gamma = G_m$ such that $a' = p\,a$. Assume that h_γ is one to one when γ has less than t blocks. When γ has t blocks, $h_\gamma([a]) = h_\gamma([a'])$ implies that

$$\begin{aligned} & < \alpha_m(a), \alpha_{m-1}(a), \cdots, \alpha_{m-\mu_i+1}(a) > \\ = \;& h_{\mu_i}(a_{\mu_i}) = h_{\mu_i}(a'_{\mu_i}) \\ = \;& < \alpha_m(a'), \alpha_{m-1}(a'), \cdots, \alpha_{m-\mu_i+1}(a') > \end{aligned}$$

for $1 \le i \le t$. By induction hypothesis, there is a $p' \in P_{(\gamma_2, \cdots, \gamma_t)}$ such that $a'_{\mu_2} = p'\,a_{\mu_2}$. Thus, for each $i \ge 1$, $\alpha_{\gamma_1+i}(a')$ is a linear combination

$$\alpha_{\gamma_1+i}(a') = \sum_j p'_{i,j} \alpha_{\gamma_1+j}(a')$$

where j goes throughout all the indices such that $p'_{i,j} \neq 0$. Since

$$\begin{aligned} & < \alpha_m(a), \alpha_{m-1}(a), \cdots, \alpha_1(a) > \\ = \ & h_{\mu_1}(a_{\mu_1}) = h_{\mu_1}(a'_{\mu_1}) \\ = \ & < \alpha_m(a'), \alpha_{m-1}(a'), \cdots, \alpha_1(a') >, \end{aligned}$$

where $\mu_1 = m$, $\alpha_k(a')$ is a linear combination of $\{\alpha_i(a')\}_{i=1}^m$, for $1 \leq k \leq \gamma_1$, there exists some $p \in P_\gamma$ such that $a' = p\,a$. Therefore h_γ is one to one.

c) Here we show both h and h^{-1} are continuous. First we show that h is continuous. Let O be any open set in $\mathcal{F}_\gamma(\mathbf{C}^n)$. We show that $h^{-1}(O)$ is open in $B\backslash M$. Our idea is illustrated in the following diagram.

$$\begin{array}{ccccccc} \mathbf{C}^{mn} & \supset & M & \xrightarrow{\rho_i} & M_i & \xrightarrow{\pi_i} & G_i(\mathbf{C}^n) \\ & & \cup & & \cup & & \cup \\ & & O'_i := \rho_i^{-1}\pi_i^{-1}(O_i) & \rightarrow & \pi_i^{-1}(O_i) & \rightarrow & O_i \end{array} \tag{34}$$

Since the topology in $\mathcal{F}_\gamma(\mathbf{C}^n)$ is the subspace topology inherited from the product topology in $G_{\mu_t}(\mathbf{C}^n) \times \cdots \times G_{\mu_1}(\mathbf{C}^n)$ we may assume that $O = (\prod_{i=1}^t O_i) \cap \mathcal{F}_\gamma(\mathbf{C}^n)$ where O_i is an open set in $G_{\mu_{t-i+1}}(\mathbf{C}^n)$ for $1 \leq i \leq t$. Recall that $M^i_{i,n}$ is the space of $i \times n$ matrices of rank i and let $\pi_i : M^i_{i,n} \rightarrow G_i(\mathbf{C}^n)$ be the natural map which takes a matrix to its row space. Then $\pi^{-1}_{\mu_{t-i+1}}(O_i)$ is open by definition of the quotient topology in $G_i(\mathbf{C}^n)$. Recall that $\rho_i : M \rightarrow M^i_{i,n}$ is defined by by $\rho_i(a) = a_i$ where, as before, a_i consists of the last i rows of a. Since ρ_i is continuous, $O'_i := \rho^{-1}_{\mu_{t-i+1}}\pi_i^{-1}(O_i)$ is open in M.

Since the set M consists of all the $m \times n$ matrices of rank m, it is open in $M_{m,n}(\mathbf{C})$. Thus O'_i is open in $M_{m,n}(\mathbf{C})$. Write $O'_i = \cup_j O'_{ij}$ where the O'_{ij} are basic open sets. Each O'_{ij} is a direct product of mn open intervals in $\mathbf{C}$. Clearly every such open interval is the solution set of a strict inequality of the form $a < x < b$. Define $O' = \cap_{i=1}^t O'_i$. So O' is the solution set of a system of strict inequalities. In the following we call those strict inequalities with the $a_{i,j}$ involved (as the variables) the *open restriction on* $a_{i,j}$. Then O' is an open subset of $M_{m,n}(\mathbf{C})$. By definition of $\rho_{\mu_{t-i+1}}$,

$$\pi_{\mu_{t-i+1}}\rho_{\mu_{t-i+1}}(a) =< a_{\mu_{t-i+1}} > \subset < a_{\mu_{t-i}} >= \pi_{\mu_{t-i}}\rho_{\mu_{t-i}}(a) \text{ for } 1 \leq i \leq m-1. \tag{35}$$

Thus O' is a union of P_γ-orbits in M. By definition of the quotient topology on $P_\gamma\backslash M$, $P_\gamma\backslash O'$ is open in $P_\gamma\backslash M$.

Now we want to show that $B\backslash O' = h_\gamma^{-1}(O)$ which implies that h_γ is continuous. Recall that $h_\gamma(P_\gamma\, a) = (h_{\mu_t}(a_{\mu_t}), \cdots, h_{\mu_1}(a_{\mu_t}))$ and $h_k(a_k)$ is the linear span of the vectors $\alpha_m(a), \alpha_{m-1}(a), \cdots, \alpha_{m-k+1}(a)$. If $P_\gamma x \in P_\gamma\backslash O'$, then the fact that $x \in O'$ implies that $x \in O'_i$ for $1 \leq i \leq t$. then

$$h_\gamma(P_\gamma x) = (h_{\mu_t}(x_{\mu_t}), \cdots, h_{\mu_1}(x_{\mu_1})) \in \prod_{i=1}^t O_i = O.$$

So $P_\gamma\backslash O' \subset h_\gamma^{-1}(O)$. Next, take $P_\gamma x \in h_\gamma^{-1}(O)$. Then $h(B_m x) \in O$. Thus, $h_{\mu_{t-i+1}}(x_{\mu_{t-i+1}}) \in O_i$, for all i. Therefore, $x \in O'$. Because of the nested condition $x_i \subset x_{i+1}$, $1 \leq i \leq m-1$, $P_\gamma x \in P_\gamma\backslash O'$. Therefore, $P_\gamma\backslash O' = h_\gamma^{-1}(O)$. Hence, h_γ is continuous.

Next we show that h_γ^{-1} is continuous. Let U be an open subset of $P_\gamma\backslash M$. Then, there is an open set $U' \subset M$ such that U' is the preimage of U in M. We show that $h_\gamma(U)$ is open in $\mathcal{F}_\gamma(\mathbf{C}^n)$. Since U' is open in M, without loss of generality we may assume that U' is a basic open subset of M. That is to say that U' is a direct product of mn open intervals: one interval for each entry of the matrix. Collect all the open restrictions on the entries in the last μ_{t-i+1} rows of the matrices in U'. The collection of these open restrictions define an open set $U_i' \subset M_{\mu_{t-i+1}}$ for $1 \leq i \leq m$. For $(x_1, \cdots, x_t) \in h_\gamma(U)$, there exists some $a \in U'$ such that $x_i = h_{\mu_{t-i+1}}(a_{\mu_{t-i+1}}) =< a_{\mu_{t-i+1}} >$ for $1 \leq i \leq t$. Thus $x_1 \subset \cdots \subset x_t$. By the definition of quotient topology on $G_{\mu_{t-i+1}}(\mathbf{C}^n)$, $U_i = \pi_{\mu_{t-i+1}}(U_i')$ is open for $1 \leq i \leq t$. Since $\mathcal{F}_\gamma(\mathbf{C}^n)$ has the subspace topology inherited from $\prod_{i=1}^m G_{\mu_{t-i+1}}(\mathbf{C}^n)$, we know that $h_\gamma(U) = (\prod_{i=1}^t U_i) \cap \mathcal{F}_\gamma(\mathbf{C}^n)$ is open. Therefore, h_γ^{-1} is continuous. Hence h_γ is a homeomorphism. Q.E.D.

Theorem 7.2 (Ehresmann Argument) *a) There is a one to one correspondence Γ between the set $\mathcal{C}_\gamma = \{\overline{X_\gamma(\sigma)} \mid \sigma \in R^m_{n^m}(\gamma)\}$ and the set of sequences of Ferrers boards $\mathcal{B}_\gamma = \{F^{(\mu_t)} \subset \cdots \subset F^{(\mu_1)}\}$ where $F^{(\mu_k)}$ is a Ferrers board of μ_k rows which is obtained by deleting γ_{k-1} rows from $F^{\mu_(k-1)}$ for $2 \leq k \leq t$. The correspondence Γ is defined by*

$$\Gamma(\overline{X_\gamma(\sigma)}) = (F_{\lambda(\sigma_{\mu_t})}, \cdots, F_{\lambda(\sigma_{\mu_1})}),$$

b) Let $\sigma \in R^m_{n^m}(\gamma)$. Then $X(\sigma) \subseteq \overline{X(\sigma')}$ if and only if $F_{\lambda(\sigma_{\mu_i})} \subseteq F_{\lambda(\sigma'_{\mu_i})}$, for $1 \leq i \leq t$.

As before, for any $\sigma \in I(k,n)$, let $\sigma^* \in J(k,n)$ be such that σ^* is obtained by rearranging the rows of σ.

Proof. a) View $X(\sigma)$ as a subset of the direct product of Grassmannians $\prod_{i=1}^t G_{\mu_{t-i+1}}(\mathbf{C}^n)$. Then, for each $P_\gamma a \in X(\sigma)$, let $V_{\mu_{t-i+1}}$ be the μ_{t-i+1}-dimensional subspace spanned by the last μ_{t-i+1} rows of a, for $1 \leq i \leq t$. Then, $V_{\mu_{t-i+1}} \in e(\sigma^*_{\mu_{t-i+1}})$, for $1 \leq i \leq t$.

Conversely, for any partial flag $V_{\mu_t} \subset \cdots \subset V_{\mu_1}$ such that dim $V_{\mu_i} = \mu_i$, the nested condition implies that there is some $\sigma \in R^m_\lambda$ such that $V_{\mu_k} \in e(\sigma^*_{\mu_k})$. Write $V_0 = \{0\}$. Recall that $\nu_j = \sum_{i=1}^j \gamma_i$. Take any maximal linearly independent set $\{v_j\}_{j=\nu_{i-1}+1}^{\nu_i} \subset V_{\mu_i} - V_{\mu_{i+1}}, 0 \leq i \leq t-1$. Use v_i as the i-th row, for $1 \leq k \leq m$, we get a matrix a. If necessary, rearrange the vectors in the set $\{v_j\}_{j=\nu_{i-1}+1}^{\nu_i}$ such that the σ can be chosen from $R^m_\lambda(\gamma)$ and $P_\gamma a \in X(\sigma)$ (note that such a rearrangement does not change the partial flag $V_{\mu_t} \subset \cdots \subset V_{\mu_1}$ which is a P_γ-orbit). For this fixed partial flag, the σ is unique in $R^m_\lambda(\gamma)$. Thus, we have a one to one correspondence between the sequence of Ferrers boards of $\{F_\lambda((\sigma_k)^*)\}_{1 \leq k \leq m}$ and the set $X(\sigma)$.

The second statement follows from the fact that in a Grassmannian, $e(\sigma^*) \subset \overline{e((\sigma')^*)}$ if and only if $F_{\lambda(\sigma^*)} \subset F_{\lambda((\sigma')^*)}$. Q.E.D.

Definition 7.3 *Let $m_1, \cdots, m_s$ and $n_1, \cdots, n_t$ be two decreasing subsequences of $[m]$ such that as two subsets of $[m]$, $[m] = \{m_1, \cdots, m_s\} \sqcup \{n_1, \cdots, n_t\}$ disjoint union. We call such a pair of subsequences a (s,t)-pair.*

In this definition, either s or t may be 0.

Lemma 7.4 *Let $(\{m_i\}_{i=1}^s, \{n_i\}_{i=1}^t)$ and $(\{m_i'\}_{i=1}^s, \{n_i'\}_{i=1}^t)$ be two (s,t)-pairs of $[m]$ such that $n_i \leq n_i'$. If we insert m_s into the proper position in $n_1, \cdots, n_t$ to make a $(s-1, t+1)$-pair $(\{m_i\}_{i=1}^{s-1}, \{n_i^*\}_{i=1}^{t+1})$ and at the same time we insert m_s' into the proper position in $n_1', \cdots, n_t'$ to make another $(s-1, t+1)$-pair $(\{m_i'\}_{i=1}^{s-1}, \{(n')_i^*\}_{i=1}^{t+1})$, then $n_i^* \leq (n')_i^*$, for $1 \leq i \leq t+1$.*

Proof. Clearly, the only nontrivial case is when $m_s > m_s'$. Suppose m_s is inserted at the i-th place so that $n_i^* = m_s$. Then,

$$m_s = m - (i-1) - (s-1) = n - i - s + 2.$$

For the same i, since $m_s > m_s'$, we have $(n')_i^* = n_i' \geq m - (s-1) - (i-1) = m_s$. Assume $m_s' = (n')_k^*$, $k > i$. Then, for each l such that $i < l < k$, $n_l = n_{l+1}^*$. Therefore, $n_{l+1}^* = m - (s-1) - l \leq (n')_{l+1}^*$, as $(n')_{l+1}^* > m_s'$. The case when $l \geq k$ is trivial. Hence the lemma is proved. Q.E.D.

Theorem 7.5 *For any given composition γ of m, let λ be a γ-compatible partition. For σ and $\sigma' \in R_\lambda^m(\gamma)$, let $P(\sigma)$ and $P(\sigma')$ be the invisible permutations of σ and σ', respectively.*

a) $X(\sigma) \subset \overline{X(\sigma')}$ if and only if $X*(P(\sigma)) \subset \overline{X*(P(\sigma'))}$.

b) $X*(P(\sigma)) \subset \overline{X*(P(\sigma'))}$ if and only if $P(\sigma) \leq P(\sigma')$

where

- *the order "$\leq$" is the Bruhat order of the symmetric group S_n, i.e., write $P(\sigma')$ as a reduced word in terms of the distinguished transpositions $\{(i, i+1)\}_{i=1}^{n-1}$, $P(\sigma)$ can be written as a subword of $P(\sigma')$.*

- *we use $X*(P(\sigma))$ to denote the Schubert cell of $P(\sigma)$ in the homogeneous space $P_{(1^{n-m},\gamma)} \backslash G_n$ where*
$$(1^{n-m}, \gamma) := (\underbrace{1, \cdots, 1}_{n-m}, \gamma_1, \cdots, \gamma_t).$$

Proof. a) Let $\theta = (1^{n-m}, \gamma)$. Then $\theta_i = 1$ for $1 \leq i \leq n-m$ and $\theta_j = \gamma_j$ for $n-m \leq j \leq n-m+t$. Assume $X(P(\sigma)) \subseteq \overline{X(P(\sigma'))}$. By Theorem 7.2, $F_{\lambda(P(\sigma)_{\theta_i})} \subseteq F_{\lambda(P(\sigma')_{\theta_i})}$, for $1 \leq i \leq n-m+t$. In particular, we have

$$F_{\lambda(\sigma_{\mu_i})} = F_{\lambda(P(\sigma)_{n-m+\mu_i})} \subseteq F_{\lambda(P(\sigma')_{n-m+\mu_i})} = F_{\lambda(\sigma'_{\mu_i})},$$

for $1 \leq i \leq t$. By Theorem 7.2 we have $X(\sigma) \subseteq \overline{X(\sigma')}$.

Assume $X(\sigma) \subset X(\sigma')$. We use induction on $n-m$ to show $X(P(\sigma)) \subset \overline{X(P(\sigma'))}$. If $n-m=0$, there is nothing to prove. If $n-m=1$, then by Theorem 7.2,

$$F_\lambda(\sigma_k) \subset F_\lambda(\sigma'_k), \text{ for } 1 \leq k \leq n-1.$$

That means $\sigma^*_k(i) \geq (\sigma')^*_k(i)$, for each i. By the definition of invisible permutations, $(\{P(\sigma(1))\}, \{\sigma^*_k(i)\}_{i=1}^{n-1})$ and $(\{P(\sigma'(1))\}, \{(\sigma')^*_k(i)\}_{i=1}^{n-1})$ are two $(i, n-1)$- pairs such that $\sigma^*_k(i) \geq (\sigma')^*_k(i)$, for each i. By Lemma 7.4, we know that $(P(\sigma))^*_k(i) \geq (P(\sigma'))^*_k(i)$, for $1 \leq i \leq n$ which imply that $X(P(\sigma)) \subset \overline{X(P(\sigma'))}$. Assume the inclusion above is true for $n-m=j$, when $n-m=j+1$, repeat the argument above, and use Lemma 7.4 again. We get the inclusion for the $n-m=j+1$ case. Q.E.D.

Corollary 7.6 *For $\sigma \in R^m_\lambda(\gamma)$, if $X(\sigma) \cap \overline{X(\sigma')} \neq \emptyset$, then $X(\sigma) \subset \overline{X(\sigma')}$.*

Proof. Take $P_\gamma \in X(\sigma) \cap \overline{X(\sigma')}$, then $[a_i]$ has a matrix representative of the form as the first one in Lemma 2.25 according to σ_i and a matrix representative of the form as the second one in Lemma 2.25 according to σ'_i for $1 \leq i \leq m$. Therefore,

$$F_\lambda(\sigma_{\mu_i}) \subset F_\lambda(\sigma'_{\mu_i}), \text{ for } 1 \leq i \leq t.$$

So $X(\sigma) \subset \overline{X(\sigma')}$. Q.E.D.

Theorem 7.7 (Main Theorem) *Let λ be a γ-compatible partition. Then we have*

a) the partition variety $P_\gamma \backslash M_\lambda$ is a CW-complex consisting of all the cells $X(\sigma)$ of $P_\gamma \backslash M$ which fit into the board F_λ

$$P_\gamma \backslash M_\lambda = \bigsqcup{}_{\sigma \in R^m_\lambda(\gamma)} X_\gamma(\sigma) \quad . \tag{36}$$

b) the Poincaré polynomial for cohomology of the partition variety $P_\gamma \backslash M_\lambda$ with real coefficients is

$$\begin{aligned} Poin(P_\gamma \backslash M_\lambda, \mathbf{R}) &= \sum \dim H^i(P_\gamma \backslash M_\lambda, \mathbf{R}) q^i \\ &= RL_m(\lambda, \gamma, q^2). \end{aligned} \tag{37}$$

Proof. The previous lemma tells us that

$$P_\gamma \backslash M_\lambda = \sqcup_{\sigma \in R^m_\lambda(\gamma), F_{\lambda(\sigma)} \subset F_\lambda} X_\gamma(\sigma).$$

To show that this is a CW-complex, we need only to prove that for each $P_\gamma a \in (\overline{X_\gamma(\sigma)} - X_\gamma(\sigma)) \subset P_\gamma \backslash M_\lambda$, there must be some cell $X_\gamma(\sigma') \subset P_\gamma \backslash M_\lambda$ of lower dimension such that $P_\gamma a \in X_\gamma(\sigma')$. Since $P_\gamma a \in (\overline{X_\gamma(\sigma)} \subset P_\gamma \backslash M_\lambda$, there is a unique $X_\gamma(\sigma')$ containing $P_\gamma a$. By the last corollary, $X_\gamma(\sigma') \subset \overline{X_\gamma(\sigma)}$. So, $F_{\lambda(\sigma'_i)} \subseteq F_{\lambda(\sigma_i)}$, for all i. But, $P_\gamma a \notin X_\gamma(\sigma)$ impies that there is at least one i such that the inclusion is strict, i.e., $F_{\lambda(\sigma'_i)} \subset F_{\lambda(\sigma_i)}$. This means that $X_\gamma(\sigma') \subset P_\gamma \backslash M_\lambda$ and $P(\sigma') < P(\sigma)$ in the Bruhat order of S_n. Therefore,

$$dim_{\mathbf{C}} X_\gamma(\sigma') = l(P(\sigma')) < l(P(\sigma)) = dim_{\mathbf{C}} X_\gamma(\sigma).$$

So we have proved that $P_\gamma\backslash M_\lambda$ is a CW-complex. Since this CW-complex structure is over the complex field which is viewed as two dimensional real space, all the cells are of even dimensions. So the boundary operator ∂ is zero. Thus, the homology (cohomology) groups of $P_\gamma\backslash M_\lambda$ are freely generated by its cells. Hence, we have $Poin_{\mathbf{R}}(P_\gamma\backslash M_\lambda, \mathbf{R}) = RL_m(\lambda,\, \gamma,\, q^2)$. Q.E.D.

We conclude this paper with three well-known examples which are special cases of the partition varieties.

Corollary 7.8 (Homogeneous Space $P_\gamma\backslash G_m$)
Let γ be any decomposition of m.

a) The homogeneous space $P_\gamma\backslash G_m$ is a CW-complex.

b) The homology (and the cohomology) groups of the homogenous space $P_\gamma\backslash G_m$ are given by

$$\begin{aligned} Poin(P_\gamma\backslash G_m, \mathbf{R}) &= RL_m((m^m),\, \gamma,\, q^2) \\ &= \prod_{i=1}^{t} \begin{bmatrix} k_i - \gamma_{i+1} - \gamma_{i+1} - \cdots - \gamma_t \\ \gamma_i \end{bmatrix}_{q^2} \end{aligned}$$

Proof. Take $n = m$ and $\lambda = (m^m)$. Q.E.D.

Corollary 7.9 (Flag Manifold)

a) The flag manifold $B_m\backslash G_m$ is a CW-complex consisting of Bruhat cells.

b)The homology (and the cohomology) groups of the flag manifold $B_m\backslash G_m$ are given by

$$\begin{aligned} Poin(B_m\backslash G_m, \mathbf{R}) &= RL_m((m^m),\, (1^m),\, q^2) \\ &= [m]_{q^2}! \end{aligned}$$

Proof. Take $n = m$, $\gamma = (1^m)$ and $\lambda = (m^m)$. Q.E.D.

Corollary 7.10 (Grassmann Manifold)

a) The Grassmann Manifold $G_m\backslash M_{n^m}$ is a CW-complex consisting of Schubert cells.

b)The homology (and the cohomology) groups of the Grassmann manifold are given by

$$\begin{aligned} Poin(G_m\backslash M_{n^m}, \mathbf{R}) &= RL_m((n^m),\, (m^m),\, q^2) \\ &= \begin{bmatrix} n \\ m \end{bmatrix}_{q^2}. \end{aligned}$$

Remark. Here, we have a unified treatment to the homology (and the cohomology) for both the flag manifold and the Grassmann manifold. It is interesting that this model, partition variety, in general has singularities while the commonly used model , homogeneous space $P_\gamma\backslash G_m$, has to be a smooth manifold. The structure of the singularities in a partition variety is an interesting topic to study further.

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Address from fall of 1995:
Kequan Ding
Department of Mathematics
University of Illinois
Urbana, IL 61801
e-mail: k-ding@orion.math.uiuc.edu