DIMACS Technical Report 94-23
July 1995

Rook Placements and Cellular Decomposition of Partition Varieties

by

Kequan Ding[1,2]
DIMACS, Rutgers University
New Brunswick, New Jersey 08903
Institute for Advanced Study
Princeton, NJ 08540

[1]Partially supported by NSF Grant STC 91-19999, and New Jersey Commission on Science and Technology.
[2]Partially supported by Sloan Foundation Grant 93-6-6. The research is part of the author's Ph D thesis at University of Wisconsin-Madison under the direction of Professor Louis Solomon.

DIMACS is a cooperative project of Rutgers University, Princeton University, AT&T Bell Laboratories and Bellcore.

DIMACS is an NSF Science and Technology Center, funded under contract STC–91–19999; and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

In this paper, we study the geometric implication of the rook length polynomials introduced in the author's thesis. We introduce the idea of partition varieties. These are certain algebraic varieties which have CW-complex structures. We prove that the cell structure of a partition variety is in one to one correspondence with the rook placements on a Ferrers board. This correspondence enables us to characterize the geometric attachment between a cell and the closure of another cell combinatorially. The main result of this paper is that the Poincare polynomial of cohomology for a partition variety is given by the corresponding rook length polynomial.

This paper serves as the transition of our studies from combinatoriail aspects to the geometric aspects. To make it user friendly, we give three appendices on the known results on Grassmann manifolds and flag manifolds which are used frequently. One appendix is on a technical a lemma on the embedding of manifolds.

1 Introduction

Let $M^r_{m,n}(\mathbf{C})$ be the set of all m by n matrices of rank r over the complex field. Let M denote $M^m_{m,n}(\mathbf{C})$. Let $\lambda = (\lambda_1, \cdots, \lambda_m)$ be a partition of some positive integer. Let F_λ be the Ferrers board corresponding to λ. An r-rook placement on F_λ is a placement of r non-attacking rooks on F_λ. Let R^r_λ be the set of all r-rook placements on F_λ. Historically, people studied the relationship between rook placements and many other subjects. For example, Gouldman, Joichi, Reiner and White [9, 10, 11, 12, 13] studied the connection between rook placements and orthogonal polynomials, chromatic polynomials of graphs, Mobius inversion, the umbral calculus, Gould [14], Wachs and White [31] studied their connection with Stirling numbers and their q-analogues, Garsia and Remmel [6], Sagan [28] considered the relationship between rook placements and the major index, Garst [7], Björner, Lovász, Vrećica and Živaljević [2], and Ziegler [33] studied the topological structure of the set of rook placements as a simplicial complex, etc.. But the implication of rook placements in algebraic geometry has not yet been touched.

In this paper, we study the relationship between the combinatorics of rook placements on a Ferrers board and the geometry of certain algebraic varieties which have cellular decomposition analogous to the Schubert cells in a Grassmann manifold. We call these varieties *partition varieties*. We interpret the rook length polynomials introduced in [4] as Poincaré polynomials of partition varieties. The main theorem of this paper is Theorem 2.33.

In [4], we introduced the idea of invisible permutations, the length function for rook placements on a Ferrers board, the partially ordered set of rook placements and rook length polynomials. We proved a local formula for the length function of rook placements which evaluates the contribution of every rook to the length function individually. Then we gave a formula on the relationship between rook length polynomials and Garsia-Remmel polynomials. The major result in [4] is an explicit formula for rook length polynomials. As a consequence of this formula, we also obtained an explicit formula for Garsia-Remmel polynomials.

In this paper, we study the geometric aspect of rook placements. We introduce the partition variety as a quotient space $B\backslash M_\lambda$ where B is the Borel subgroup of upper triangular matrices of $GL_m(\mathbf{C})$ and

$$M_\lambda = \{a \in M \mid a_{i,j} = 0 \text{ if } (i,j) \notin F_\lambda\}.$$

We prove that $B\backslash M$ can be embedded into certain projective space $\mathbf{P}^N$. The image of $B\backslash M_\lambda$ is a projective subvariety. We transport the variety structure to $B\backslash M_\lambda$ itself and call $B\backslash M_\lambda$ a *partition variety*. We study the topological structure of partition varieties. We prove that every partition variety has a cellular decomposition. Further, we show that every partition variety has the structure of a CW-complex. The cells of the CW-complex and their closures can be described combinatorially by means of sequences of Ferrers boards. This description leads to a combinatorial criterion for one cell being a subset of the closure of another cell. By using the idea of invisible permutations introduce in [4], we prove that this inclusion relation can be characterized by the Bruhat order of certain symmetric group. The main result in this paper is that the Poincaré polynomials for cohomology of partition varieties with coefficients in real field $\mathbf{R}$ are given by our rook length polynomials introduced in [4] with $r = m$.

This paper serves as the transition from the combinatorics of rook placements in [4] to the geometry and topology in our study of partition varieties. To make this transition smooth for readers starting from both directions, we give basic definitions used in this paper in the next section. Also in order to give an appropriete preparation for this transition, we give four appendices at the end of this paper. The first one is an exposition on the CW-complex structure of Grassmannians over the complex field which is a complex analogue of that in Milnor and Stasheff [23]. The second describes the Schubert cells and Schubert varieties combinatorially. The third contains two different representations of the flag manifolds. The fourth is a lemma on the embedding of complex manifolds.

The main body of this paper consists of five sections. Section 2 gives the preliminaries and the statements of the main results. Section 3 proves the manifold structure for $B\backslash M$. Section 4 describes the embedding of $B\backslash M$ into $\mathbf{P}^N$. Section 5 shows that the partition varieties $B\backslash M_\lambda$ are indeed projective varieties which have cellular decomposition analogue to the cellular decomposition of Grassmannian into Schubert cells. Section 6 studies the cell structure of partition varieties. It is shown that partition varieties have CW-complex structure with the rook length polynomials obtained in [4] as their Poincaré polynomials of cohomology.

The author would like to thank a referee for many helpful suggestions including a simple proof to Jacobi's identity.

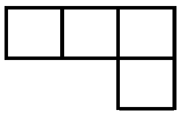
2 Preliminaries and the Statement of Main Theorems

In this section we give some preliminary definitions from combinatorics, geometry and topology, some known results which will be used in our argument, and the statements of our main results on partition varieties. The appendices contain an exposition of known facts about Grassmannians, Schubert cells, Schubert varieties, and the flag manifold. We will use the complex topology in this paper.

Let $[n]$ denote the set $\{1, \cdots, n\}$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of some integer where $\lambda_1 \geq \cdots \geq \lambda_m > 0$. Sometimes, we write $\lambda = (1^{\mu_1}\, 2^{\mu_2}\, \cdots\, n^{\mu_n})$ where μ_i is the number of λ_j's which are equal to i. We view a Ferrers board F_λ of shape λ as a subarray of an m by n matrix, where $n = \lambda_1$ and the k-th row has length λ_k for $1 \leq k \leq m$. For example, if $\lambda = (3, 1)$, then

$$F_\lambda = \begin{pmatrix} * & * & * \\ & & * \end{pmatrix}.$$

Sometimes we write

$F_\lambda =$  .

We assume, contrary to the usual convention, that F_λ is right justified; the reason for this is explained in [4]. If $r \geq 1$ let

$$\nu_r = \sum_{i=1}^{r} E_{i,n-r+i}. \tag{1}$$

where $E_{i,j}$ is the matrix with 1 at (i,j) and 0's elsewhere. Let W_n be the symmetric group on $[n] = \{1, \cdots, n\}$. Let $S(n)$ be the set of distinguished generators of W_n:

$$S(n) = \{(12), (23), \cdots, (n-1\ n)\}.$$

Definition 2.1 *Let λ be a partition such that $\nu_r \in F_\lambda$. When $r = 0$, let $\nu_0 = 0$. For $\sigma \in R^r_\lambda$, the length function $l(\sigma)$ is defined by*

$$l(\sigma) = min\{k + h \mid \sigma = s_k \cdots s_1 \nu_r s'_1 \cdots s'_h\}$$

where $s_i \in S(m)$ and $s'_j \in S(n)$ and

$$s_p \cdots s_1\, \nu_r\, s'_1 \cdots s'_q \in R^r_\lambda$$

for each $1 \leq p \leq k$ and $1 \leq q \leq h$.

Thus $l(\sigma)$ is the minimum number of adjacent row and/or column transpositions required to get σ from $\nu_r = \sum_{i=1}^{r} E_{i,n-r+i}$, such that all the intermediate rook placements are in the board F_λ. This length function was first used as the length of rook matrices by Solomon in his work on the Iwahori ring of $\mathbf{M}_n(\mathbf{F}_q)$ [29]. The length function l can be evaluated "locally" by a the following local formula which counts the contribution of each rook in σ, individually. In the following we often use a 1 to denote a rook and a 0 for an empty box in a Ferrers board.

Proposition 2.2 (Local Formula) *Let $\sigma \in R^r_\lambda$. Write $\sigma = \sum_{i=1}^{r} E_{c_i,b_i}$ where $c_1 < c_2 < \cdots < c_r$. Let α_i be the number of zero rows above the c_i-th row in σ, γ_i the number of zero columns to the right of the b_i-th column in σ, and β_i the number of 1's to the "northeast" of the i-th 1 (not including the i-th 1). Then,*

$$l(\sigma) = \sum_{i=1}^{r} (\alpha_i + \beta_i + \gamma_i). \tag{2}$$

Definition 2.3 *Let*

$$RL_r(\lambda, q) = \sum_{\sigma \in R^r_\lambda} q^{l(\sigma)}.$$

We call this a rook length polynomial.

In [4] we proved the following formula. One of the main results of this paper is to give a geometric interpretation of this formula when $r = m$.

Theorem 2.4 (Formula of Rook Length Polynomials) *If λ is a partition with m parts then*

$$RL_r(\lambda, q) = \sum_{1\le i_1<\cdots<i_r\le m} q^{\sum_{j=1}^r (i_j-j)} \prod_{j=1}^r (\lambda_{i_j} - r + j)_q \tag{3}$$

where $(k)_q = 1 + q + q^2 + \cdots + q^{k-1}$.

One of the main ideas in [4] is to extend a placement σ of r rooks on an m by n board to a placement $P(\sigma)$ of $m+n-r$ rooks on an $m+n-r$ by $m+n-r$ board. We identify $P(\sigma)$ with the corresponding permutation of $[m+n-r]$ and call $P(\sigma)$ the *invisible permutation* corresponding to σ. There are various ways to extend a rook placement σ to a permutation of $[m+n-r]$. The choice here is made so that $l(\sigma)$ is equal to the number of inversions in the permutation $P(\sigma)$.

In order to help the reader understand the definition of $P(\sigma)$ we give an example.

Example 2.5 *Let $r = 2$, $m = n = 4$. Let*

$$\sigma = E_{14} + E_{32} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the permutation matrix corresponding to σ is

$$\begin{aligned} P(\sigma) &= (E_{11} + E_{23}) + (E_{34} + E_{52}) + (E_{45} + E_{66}) \\ &= \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 \end{array}\right). \end{aligned} \tag{4}$$

The 4 by 4 submatrix of $P(\sigma)$ at the southwest corner is exactly the rook matrix σ.

The precise definition of the permutation $P(\sigma)$ is given in Definition 2.6. Since it is hard to see through the notation, we give an informal definition of the corresponding permutation matrix. The matrix $P(\sigma)$ is $m+n-r$ by $m+n-r$ and has σ in its southwest corner. We call this southwest corner of $P(\sigma)$ the *real part* of $P(\sigma)$. We call the columns $n+1, \ldots, n+m-r$ and the rows $1, \ldots, m-r$ the *imaginary part* of $P(\sigma)$. In the example these are the columns to the right of the vertical line and the rows above the horizontal line. The 1's are placed in the imaginary part of $P(\sigma)$ so that the column indices increase as the row indices increase. This describes their positions uniquely. From the method of placement of the 1's in the imaginary part we see that $l(\sigma)$ is equal to the number of inversions in the permutation $P(\sigma)$ which is the usual length of the permutation. So we have

$$l(\sigma) = l(P(\sigma)). \tag{5}$$

This mapping $\sigma \to P(\sigma)$ allows us to translate problems about rook placements into problems about inversions of permutations. It would also be possible to put the the real part σ of $P(\sigma)$ in the northeast corner. But, if σ is placed in either of the other two corners, the combinatorics becomes messy. In the following we think of $P(\sigma)$ as a permutation or a permutation matrix whichever is convenient. The formal definition of the permutation $P(\sigma)$ is as follows.

Definition 2.6 *Let $\sigma \in R^r_{nm}$. Let $\sigma = \sum_{i=1}^r E_{c_i,b_i}$ with $c_1 < c_2 < \cdots < c_r$. Define a permutation $P(\sigma) \in W_{m+n-r}$ by*

$$P(\sigma) = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & n-r & n-r+c_1 & \cdots & n-r+c_r & d_1 & \cdots & d_{m-r} \\ a_1 & \cdots & a_{n-r} & b_1 & \cdots & b_r & n+1 & \cdots & n+m-r \end{array} \right)$$

where $\{a_1, a_2, \ldots, a_{n-r}\}$ is the complement of $\{b_1, b_2, \ldots, b_r\}$ in $[n]$ with $a_1 < a_2 < \cdots < a_{n-r}$, and $\{d_1, d_2, \ldots, d_{m-r}\}$ is the complement of $\{n-r+c_1, \cdots, n-r+c_r\}$ in $\{n-r+1, \ldots, n-r+m\}$ with $d_1 < d_2 < \cdots < d_{m-r}$.

Remark. From this definition, we see that the a_i's index the zero columns of σ and the d_i's index the zero rows of σ.

In Example 2.5, since $\sigma = E_{14} + E_{32}$. we have $c_1 = 1$, $c_2 = 3$, $b_1 = 4$ and $b_2 = 2$. Thus $\{a_1, a_2\} = [n] - \{b_1, b_2\} = [4] - \{4, 2\} = \{1, 3\}$. So $a_1 = 1$ and $a_2 = 3$. Similarly,

$$\begin{aligned} \{d_1, d_2\} &= [n+m-r] - [n-r] - \{n-r+c_1, n-r+c_2\} \\ &= \{3, 4, 5, 6\} - \{2+c_1, 2+c_2\} = \{4, 6\} \end{aligned}$$

So $d_1 = 4$ and $d_2 = 6$. Thus

$$P(\sigma) = \left(\begin{array}{cc|cc|cc} 1 & 2 & 3 & 5 & 4 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{array} \right).$$

Then the permutation matrix $P(\sigma)$ is

$$P(\sigma) = (E_{11} + E_{23}) + (E_{34} + E_{52}) + (E_{45} + E_{66})$$

which is exactly the same as the one given in Example 2.5.

If we write $P(\sigma)$ as a two row array as in Definition 2.6, the columns to the left of the first vertical line are called the *column imaginary part* of $P(\sigma)$. The columns to the right of the second vertical line are called the *row imaginary part* of $P(\sigma)$. The columns between the two vertical lines are called the *real part* of $P(\sigma)$. Note in Example 2.5 that the 4 by 4 submatrix of $P(\sigma)$ at the southwest corner, the real part of $P(\sigma)$, is $E_{34} + E_{52}$ which corresponds to $\sigma = E_{14} + E_{32}$. There will always be this shift of row indices. If $\sigma = \sum_{i=1}^{r} E_{c_i,b_i}$ then the corresponding block in the southwest corner of $P(\sigma)$ is $\sigma = \sum_{i=1}^{r} E_{n-r+c_i,b_i}$.

Next we give with some definitions and basic theorems from geometry which we will use. Let $\mathbf{C}^n$ be the n-dimensional affine space of row vectors.

Definition 2.7 (Affine Variety) *A subset of $\mathbf{C}^n$ is called an affine variety if it is the set of common zeroes of a collection of polynomials with coefficients in $\mathbf{C}$. This collection of polynomials are called the defining polynomials of the affine variety.*

The n-dimensional projective space over $\mathbf{C}$ is defined by $\mathbf{P}^n = \mathbf{C}^{n+1} - \{\mathbf{0}\}/\sim$ where $(x_0, x_1, \cdots, x_n) \sim (y_0, y_1, \cdots, y_n)$ if there exists $c \in \mathbf{C} - \{0\}$ such that $y_i = cx_i$ for all i. We let $(x_0 : x_1 : \cdots : x_n)$ denote the equivalence class of $(x_0, x_1, \cdots, x_n)$. The x_i are called homogeneous coordinates. A polynomial $f(x_0, x_1, \cdots, x_n)$ is homogeneous of degree k if $f(cx_0, cx_1, \cdots, cx_n) = c^k f(x_0, x_1, \cdots, x_n)$.

Definition 2.8 (Projective Variety) *A subset of $\mathbf{P}^n$ is called a projective variety if it is the set of common zeros of a collection of homogeneous polynomials. This collection of homogeneous polynomials are called the defining polynomials of the projective variety.*

Definition 2.9 (Standard Affine Open Sets) *Let*

$$U_i = \{x = (x_0 : x_1 : \cdots : x_n) \in \mathbf{P}^n \mid x_i \neq 0\} \qquad 0 \leq i \leq n.$$

The U_i's are called the standard affine open sets. They form an open cover of $\mathbf{P}^n$ [17, page 4].

Lemma 2.10 *Each U_i is homeomorphic to the affine space $\mathbf{C}^n$ via $h : x = (x_0 : x_1 : \cdots : x_n) \mapsto (x_0/x_i, \cdots, x_{i-1}/x_i, x_{i+1}/x_i, \cdots, x_n/x_i)$.*

Let $\mathbf{P}^{n_1}, \ldots, \mathbf{P}^{n_m}$ be projective spaces. Let $N = \prod_{i=1}^{m}(n_i + 1) - 1$. The points in $\mathbf{P}^N$ will be written

$$(\ldots : x_{i_1,\ldots,i_m} : \ldots) \tag{6}$$

where $(i_1, \ldots, i_m) \in [n_1 + 1] \times \cdots \times [n_m + 1]$. Define a mapping $\Psi : \prod_{i=1}^{m} \mathbf{P}^{n_i} \to \mathbf{P}^N$ by

$$\Psi((\cdots : x_i : \cdots), (\cdots : y_j : \cdots), \cdots, (\cdots : z_k : \cdots)) = (\cdots : x_i y_j \cdots z_k : \cdots) \tag{7}$$

The mapping Ψ is injective [19, page 13].

Lemma 2.11 *Let A_i be a subvariety of $\mathbf{P}^{n_i}$ for $1 \leq i \leq m$. Then $\Psi(A_1 \times \cdots \times A_m)$ is a subvariety of P^N.*

Proof. Humphreys [19, page 13].

Remark. *We identify the Cartesian product of projective varieties with its Ψ image. Thus the Cartesian product of finitely many projective varieties is a projective variety.*

If X is a topological space and A is a subset of X then $\overline{A}$ denotes the closure of A.

Definition 2.12 (Finite CW-complex,[23]) *A Hausdorff space X is called a finite CW-complex if X has a partition $X = \bigsqcup_{i\in I} e_i$ into disjoint subsets $\{e_i\}_{i\in I}$ satisfying the following conditions:*

- *The index set I is finite.*
- *Each e_i is homeomorphic to an open ball in some vector space $\mathbf{R}^n$. (In this article we will view a complex field $\mathbf{C}$ as a two dimensional vector space over $\mathbf{R}$, topologically.)*
- *If $x \in \overline{e_i} - e_i$, then there is some e_j of lower dimension such that $x \in e_j$.*

We call the subsets e_i's cells. In this paper, we consider only finite CW-complexes, so whenever we say a CW-complex, we mean a finite CW-complex.

Definition 2.13 (Complex Manifold, [15]) *Let X be a Hausdorff space. The space X is a complex manifold of dimension n if there is an index set I and a set of pairs $\{(U_i, \phi_i) \mid i \in I\}$, called a local coordinate system, such that*

- *Each U_i is an open subset of X and $X = \cup_{i\in I} U_i$.*
- *Each $\phi_i : U_i \to \phi_i(U_i) \subset \mathbf{C}^n$ is a homeomorphism from U_i onto an open subset $\phi_i(U_i)$ of $\mathbf{C}^n$.*
- *Let $\phi_{i,j} := \phi_j \circ \phi_i^{-1}$. Then $\phi_{i,j}$ is a biholomorphic (the mapping and its inverse are both holomorphic) homeomorphism from $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$.*

Example 2.14 (Projective Space as Complex Manifold) *The projective space $\mathbf{P}^n$ is a complex manifold with the standard affine open sets as a local coordinate system.*

Definition 2.15 (Analytic Subvariety of a Complex Manifold) *Let X be a complex manifold. An analytic subvariety of X is a subset Z given locally as the zero locus of a finite number of holomorphic functions. Thus if $x \in Z$ there exists an open set U_i in the local coordinate system (2.13) such that (1) $x \in U_i$ and (2) there exist holomorphic functions $f_1, \ldots, f_r$ defined on U_i such that $Z \cap U_i$ is the set of common zeros of $f_1, \ldots, f_r$.*

Definition 2.16 (Holomorphic Map of Complex Manifold) *Let M and M' be complex manifolds with local coordinate systems $\{(U_I, \phi_I)\}$ and $\{(U_I', \phi_I')\}$ respectively. A map $\Phi : M \to M'$ is a holomorphic map of complex manifold if*

$$\phi_J' \circ \Phi \circ (\phi_I^{-1}|_{\phi_I(U_I \cap \Phi^{-1}(U_J'))}) \; : \; \phi_I(U_I \cap \Phi^{-1}(U_J')) \to \phi_I'(\Phi(U_I) \cap U_J')$$

is holomorphic for all I and J.

Definition 2.17 (Jacobian Matrix) *Let $D \subset \mathbf{C}^m$ and $D' \subset \mathbf{C}^n$ be connected open sets (called domains). Let $\Phi : D \to D'$ be a holomorphic map. The matrix*

$$J_{\mathbf{C}}(\Phi(z)) := (\frac{\partial \Phi_i(z)}{\partial z_j})_{1\le i\le n, 1\le j\le m}$$

is called the complex Jacobian matrix of f.

Definition 2.18 (Embedding of Complex Manifold) *Let M and M' be complex manifolds with $\dim M \leq \dim M'$. Let $\Phi : M \to M'$ be a holomorphic map of complex manifold. The map Φ is called an embedding of M into M' if Φ is injective and full ranked, i.e.,*

$$\text{rank } J_{\mathbf{C}}(\Phi)(x) = \dim M, \qquad \textit{for all} \quad x \in M \ . \tag{8}$$

Example 2.19 *The map $\Psi : \prod_{i=1}^{m} \mathbf{P}^{n_i} \to \mathbf{P}^N$ defined in (7) is an embedding of complex manifolds.*

Proposition 2.20 (Embedding Theorem) *Let X and Y be two complex manifolds. If X is compact and $\Theta : X \to Y$ is an embedding then $\Theta(X)$ is a complex analytic subvariety of Y.*

Proof. See Appendix D.

Theorem 2.21 (Chow's Theorem) *If $X \subseteq \mathbf{P}^n$ is an analytic subvariety then X is an projective subvariety [17, page 8].*

Corollary 2.22 (Embedding Theorem for Projective Space) *Let X be a complex manifold. If X is compact and $\Theta : X \to \mathbf{P}^n$ is an embedding, then $\Phi(X)$ is a projective subvariety of $\mathbf{P}^n$.*

Proof. Apply Chow's Theorem and the Embedding Theorem. Q.E.D.

Convention 2.23 *In the context of Corollary 2.22 we agree to transport the structure of projective variety from $\Theta(X)$ to X itself. In this sense X is a projective variety.*

This completes our preliminaries on affine varieties and projective varieties. Now we give some definitions which will allow us to state the main theorems. Let $G = GL_m(\mathbf{C})$ be the general linear group over the complex numbers. Let $B = B_m$ be the Borel subgroup of upper triangular matrices. Let M be the set of all m by n complex matrices of rank m. Thus $m \leq n$. Note that G, and hence B, acts on M by left multiplication.

Definition 2.24 *Let $I(m,n)$ be the set of all injective mappings from $[m] = \{1, \cdots, m\}$ into $[n] = \{1, \cdots, n\}$. Let*

$$J(m,n) = \{\sigma \in I(m,n) \,|\, \sigma(i) < \ \sigma(i+1), 1 \leq i \leq m-1\}.$$

Remark. We identify the mapping $\sigma \in I(m,n)$ with the m by n matrix $\Sigma_{i=1}^{m} E_{i,\sigma(i)}$. Thus we identify $I(m,n)$ with $R^m_{n^m}$ and we can talk about the rows and columns of $\sigma \in I(m,n)$.

Definition 2.25 *Let $I = \{i_1, \cdots, i_k\} \subseteq [m]$ and $J = \{j_1, \cdots, j_k\} \subseteq [n]$ with $k = |I| = |J|$, $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$. For $a \in M$, let $sub_{I,J}\, a$ denote the submatrix of a with row indices in I and column indices in J. (Note that when we change the order of row or column indices, the determinant of a submatrix may change its sign. If there is no special remark, in dealing with submatrices or determinants, we regard the subsets I and J as ordered set throughout this paper. For the same reason, if in certain situation the only useful information is whether a determinant is zero, we may ignore the order of the subindices.) Let*

$$a_{I,J} = det(sub_{I,J}\, a)$$

be the corresponding minor.

Thus, for $a \in M$ and $\sigma \in I(m,n)$, $a_{[m],\sigma([m])}$ is the minor of a with row indices $1, 2, \cdots, m$ and column indices $\sigma(1), \sigma(2), \cdots, \sigma(m)$.

Definition 2.26 *For each $\sigma \in I(m,n)$, there is a unique element $\sigma^* \in J(m,n)$ which can be obtained by rearranging the rows of σ. Let*

$$\lambda(\sigma) = (\lambda_1, \cdots, \lambda_m) = (n - \sigma^*(1) + 1, \cdots, n - \sigma^*(m) + 1). \tag{9}$$

Note that $n - \sigma^*(1) + 1 \geq \cdots \geq n - \sigma^*(m) + 1 > 0$. We call $\lambda(\sigma)$ the partition of σ.

For $\sigma \in I(m,n)$, let

$$J(\sigma) = \{\sigma(i) \mid 1 \leq i \leq m\}. \tag{10}$$

Let λ be a partition and let F_λ be its right justified Ferrers board.

Definition 2.27 *Let*

$$M_\lambda = \{a = (a_{i,j}) \in M \mid a_{i,j} = 0, \;\; if\; (i,j) \notin F_\lambda\}.$$

The space M_λ is a subset of the affine space M. We give it the subspace topology. Since $B\,M_\lambda \subseteq M_\lambda$, we may consider the geometry in the quotient space $B\backslash M_\lambda$. We give the quotient space $B\backslash M_\lambda$ the quotient topology. We will prove that $B\backslash M$ is a complex manifold and a projective variety. To state our main results we construct a transversal for the B-orbits on M. We construct a transversal for $B\backslash M$ of the form $\bigsqcup_{\sigma\in I(m,n)} Y(\sigma)$ where the sets $Y(\sigma)$ are defined as follows: Let

$$K(\sigma) = K_1(\sigma) \cup K_2(\sigma) \tag{11}$$

where

$$K_1(\sigma) = \{(i,j) \,|\, j \notin J(\sigma) \text{ and } \sigma(i) < j\} \tag{12}$$

$$K_2(\sigma) = \{(i,j) \,|\, j \in J(\sigma), \sigma(i) < j \text{ and } \sigma^{-1}(j) < i\}. \tag{13}$$

Note that σ is an injective map. The inverse σ^{-1} is only defined on $J(\sigma)$. Define

$$Y(\sigma) = \sigma + \sum_{(i,j)\in K(\sigma)} \mathbf{C} E_{i,j}. \tag{14}$$

Remark 2.28 Here is a pictorial explanation for this key definition. Think of σ as a placement of m rooks on an m by n board. Then the rooks are at positions $(i, \sigma(i))$ for $1 \leq i \leq m$. In particular, if $(i, j) \in K(\sigma)$ then no rook is in the position (i, j). The pictorial description of $K_1(\sigma)$ is as follows. For each fixed $(i, j) \in K_1(\sigma)$ there is no rook in the column j and there is a rook in the i-th row to the left of the j-th column. The pictorial description of $K_2(\sigma)$ is as follows. For $(i, j) \in K_2(\sigma)$ there is a rook in column j above the position (i, j) and there is a rook in the i-th row to the left of position (i, j).

For example, if $\sigma = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$, then

$$Y(\sigma) = \{\begin{pmatrix} 0 & 1 & a_{1,3} \\ 1 & a_{2,2} & a_{2,3} \end{pmatrix} \mid a_{i,j} \in \mathbf{C}\}.$$

Remark. We write $a_{i,j}$ rather than a_{ij} because we think of an entry of a matrix as a 1 by 1 minor. Minors of size 2 will be written in the form $a_{ij,kl}$ and so forth.

Lemma 2.29 *$Y(\sigma)$ is an affine space of dimension $l(\sigma) = |K(\sigma)|$ where $l(\sigma)$ is the length function defined in Definition 2.1.*

Proof. Clearly dim $Y(\sigma) = |K(\sigma)|$. Thus we must prove $|K(\sigma)| = l(\sigma)$. By the local formula for the length function in Proposition 2.2, the contribution of the i-th rook to the length function equals the sum of the number of zero columns to the right of the rook and the number of rooks to the northeast of the rook (since the number of rooks equals the number of rows, there are no zero rows above the rook). From the pictorial description after the definition of the set $K(\sigma)$, we see that a pair (i, j) belongs to $K(\sigma)$ if and only if either the j-th column is a zero column or the rook in the j-th column is to the northeast of the i-th rook. Thus, $l(\sigma) = |K(\sigma)|$ and hence $Y(\sigma)$ is an affine space of dimension $l(\sigma)$. Q.E.D.

The following are the main results of this paper. We will prove them in the next several sections.

Theorem 2.30 *There exists a projective space $\mathbf{P}^N$ and an embedding $\Theta : B\backslash M \to \mathbf{P}^N$ of complex manifolds such that $\Theta(B\backslash M)$ is a projective variety and $\Theta(B\backslash M_\lambda)$ is a subvariety.*

Thus $B\backslash M_\lambda$ has the structure of a projective variety. We call $B\backslash M_\lambda$ a **partition variety**. For $1 \leq k \leq m$ let $M_{k,n}$ be the set of k by n matrices. Let

$$M^k_{k,n} = \{a \in M_{k,n} \mid \text{rank } a = k\} \tag{15}$$

Define a map $\rho_k : M \to M^k_{k,n}$ by

$$\rho_k(a) = a_k \tag{16}$$

where a_k is the submatrix of a consisting of its last k rows. For $\sigma \in I(m, n)$ define

$$X(\sigma) = \{B\, y \mid y \in Y(\sigma)\} \subseteq B\backslash M. \tag{17}$$

The space $X(\sigma)$ has the subspace topology from $B\backslash M$ and $Y(\sigma)$ has the subspace topology from M. There is a natural homeomorphism from $X(\sigma)$ onto $Y(\sigma)$ defined by $y \to By$. Let $\sigma_k = \rho_k(\sigma)$ denote the submatrix consisting of the last k rows in σ. Write $\sigma_k^* = (\sigma_k)^* \in J(k,n)$. By analogy with Definition 2.26 we define $\lambda(\sigma_k)$ to be the partition

$$\{n - \sigma_k^*(1) + 1, n - \sigma_k^*(2) + 1, \cdots, n - \sigma_k^*(k) + 1\}.$$

If $k = m$ then $\sigma_k = \sigma$. This agrees with Definition 2.26.

Proposition 2.31 *a) There is a one to one correspondence* Γ *between the set* $\mathcal{C} = \{\overline{X(\sigma)} \mid \sigma \in I(m,n)\}$ *and the set of sequences of Ferrers boards* $\mathcal{B} = \{F^{(1)} \subset \cdots \subset F^{(m)}\}$ *where* $F^{(k)}$ *is a Ferrers board of* k *rows which is obtained by deleting one row from* $F^{(k+1)}$ *for* $1 \le k \le m-1$. *The correspondence* Γ *is defined by*

$$\Gamma(\overline{X(\sigma)}) = (F_{\lambda(\sigma_1)}, \cdots, F_{\lambda(\sigma_m)}),$$

b) Let $\sigma \in I(m,n)$. *Then* $X(\sigma) \subseteq \overline{X(\sigma')}$ *if and only if* $F_{\lambda(\sigma_i)} \subseteq F_{\lambda(\sigma_i')}$, *for* $1 \le i \le m$.

For example, let $\sigma = \begin{pmatrix} 0&1&0 \\ 0&0&1 \end{pmatrix}$ and $\sigma' = \begin{pmatrix} 0&1&0 \\ 1&0&0 \end{pmatrix}$. Then,

$$F_{\lambda(\sigma_1)} = \begin{pmatrix} & & * \end{pmatrix} \qquad F_{\lambda(\sigma_2)} = \begin{pmatrix} & * & * \\ & & * \end{pmatrix}$$

and

$$F_{\lambda(\sigma_1')} = \begin{pmatrix} * & * & * \end{pmatrix} \qquad F_{\lambda(\sigma_2')} = \begin{pmatrix} * & * & * \\ & * & * \end{pmatrix}.$$

Thus, by Proposition 2.31, we know that $X(\sigma) \subseteq \overline{X(\sigma')}$.

Theorem 2.32 *a)* $X(\sigma) \subseteq \overline{X(\sigma')}$ *if and only if* $X(P(\sigma)) \subseteq \overline{X(P(\sigma'))}$.

b) $X(P(\sigma)) \subseteq \overline{X(P(\sigma'))}$ *if and only if* $P(\sigma) \le P(\sigma')$
where $P(\tau)$ *is the invisible permutation of* $\tau \in I(m,n)$ *and the order "$\le$" is the Bruhat order of the symmetric group* W_n, *i.e., if we write* $P(\sigma')$ *as a reduced word in terms of the distinguished transpositions* $\{(i,i+1)\}_{i=1}^{n-1}$, *then* $P(\sigma)$ *can be written as a subword of* $P(\sigma')$

Theorem 2.33 *The partition variety* $B\backslash M_\lambda$ *is a CW-complex which has a cellular decomposition*

$$B\backslash M_\lambda = \bigsqcup\nolimits_{\sigma \in R_\lambda^m} X(\sigma) \quad . \tag{18}$$

The Poincaré polynomial for cohomology of the partition variety $B\backslash M_\lambda$ *with real coefficients is*

$$\begin{aligned} Poin(B\backslash M_\lambda, q) &:= \sum dim\, H^i(B\backslash M_\lambda, \mathbf{R}) q^i \\ &= RL_m(\lambda, q^2). \end{aligned} \tag{19}$$

If $m = n$ and $\lambda = (n^m)$ then $M = G$. In this case (18) is the cellular decomposition

$$B\backslash G = \bigsqcup_{\sigma \in W_n} B\backslash B\sigma B \tag{20}$$

of the flag manifold $B\backslash G$ into Bruhat cells [19, page 173] and (19) gives the Poincaré polynomial

$$Poin(B\backslash G, q) = \prod_{i=1}^{n-1}(1 + q^2 + \cdots + q^{2i}) \tag{21}$$

of the flag manifold $B\backslash G$.

For example, let $\lambda = (3, 2)$, $n = 3$ and $r = m = 2$. Then there are four rook placements:

$$\begin{array}{|c|c|c|} \hline & 1 & \\ \hline \multicolumn{1}{c|}{} & & 1 \\ \cline{2-3} \end{array} \quad \begin{array}{|c|c|c|} \hline & & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & \\ \cline{2-3} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & & \\ \hline \multicolumn{1}{c|}{} & & 1 \\ \cline{2-3} \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & & \\ \hline \multicolumn{1}{c|}{} & 1 & \\ \cline{2-3} \end{array}\,.$$

Then $RL_2((3, 2), q) = 1 + 2q + q^2$. Therefore, $Poin(B_2\backslash M_{(3,2)}, q) = 1 + 2q^2 + q^4$.

3 The Complex Manifold $B\backslash M$

In this section we prove that $B\backslash M$ is a complex manifold. To begin we construct an open cover of M. If $\sigma \in I(m, n)$, let

$$L(\sigma) = \{(i, j) \mid j \notin J(\sigma)\} \cup \{(i, j) \mid j \in J(\sigma) \text{ and } i > \sigma^{-1}(j)\} . \tag{22}$$

(Although the shape of the formula for $L(\sigma)$ is very much like the formula for $K(\sigma)$ there is no connection between them!) Define

$$\mathcal{W}(\sigma) = \{\sigma + \sum_{(i,j)\in L(\sigma)} a_{ij}E_{ij} |\ a_{ij} \in \mathbf{C}\}. \tag{23}$$

Thus, if σ has no entry 1 in column j, the entries $a_{i,j}$ in this column are arbitrary in $\mathbf{C}$; if σ has an entry 1 in column j then the entries $a_{i,j}$ below the 1 are arbitrary in $\mathbf{C}$. We call the entries $a_{i,j}$ with $(i, j) \in L(\sigma)$ the *free entries* of $\mathcal{W}(\sigma)$. Note that $\mathcal{W}(\sigma)$ is an affine space of dimension

$$\dim\, \mathcal{W}(\sigma) = m(n - m) + \frac{m(m-1)}{2} \quad . \tag{24}$$

We will show in Corollary 3.7 that $\{B\mathcal{W}(\sigma) \mid \sigma \in I(m, n)\}$ is an open cover of M. Since the notation in the definition of $L(\sigma)$ is a bit complicated, we begin with an example.

Example 3.1 *If $m = 2$ and $n = 3$ the elements $\sigma \in I(2, 3)$ and the corresponding sets $\mathcal{W}(\sigma)$ are given by the following table.*

σ	$\mathcal{W}(\sigma)$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & a_{1,3} \\ a_{2,1} & 1 & a_{2,3} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a_{1,1} & 1 & 0 \\ a_{2,1} & a_{2,2} & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & a_{1,3} \\ 1 & a_{2,2} & a_{2,3} \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_{1,2} & 1 \\ 1 & a_{2,2} & a_{2,3} \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{1,1} & 0 & 1 \\ a_{2,1} & 1 & a_{2,3} \end{pmatrix}$

Here the $a_{i,j} \in \mathbf{C}$ are arbitrary. Therefore, we get an open cover $\{B\mathcal{W}(\sigma) \mid \sigma \in I(2,3)\}$ of M.

Let $[i,j] = \{i, i+1, \cdots, j\}$ for integers $i \leq j$. In particular, if $i = 1$, $[1,j] = [j]$ in our earlier notation. Let

$$\mathcal{U}_i(\sigma) = \{a \in M \mid a_{[i,m],\sigma([i,m])} \neq 0\}, \tag{25}$$

$$\mathcal{U}(\sigma) = \bigcap_{i=1}^{m} \mathcal{U}_i(\sigma). \tag{26}$$

where $a_{[i,m],\sigma([i,m])}$ is the minor of a defined in Definition 2.25

Lemma 3.2 *For $\sigma \in I(m,n)$, $a \in M$ and $b \in B$,*

$$(ba)_{[m-k+1,m],\sigma([n-k+1,n])} = b_{[m-k+1,m],[m-k+1,m]}\, a_{[m-k+1,m],\sigma([n-k+1,n])}\ .$$

Proof. Since all the matrices in B are upper triangular, the equality is obvious. Q.E.D.

Recall that each $a \in M$ has rank m. Thus, we can find m distinct integers $1 \leq j_1, j_2, \cdots, j_m \leq n$ such that the columns $j_1, \cdots, j_m$ of a are linearly independent.

Lemma 3.3 *a) Let $a \in M$. Choose a set $J = \{j_1, \cdots, j_m\}$ of m distinct integers such that the columns $j_1, \cdots, j_m$ of a are linearly independent. Then, there is some $\sigma \in I(m,n)$ such that $a \in \mathcal{U}(\sigma)$ and if we view $J(\sigma)$ as an unordered set, then $J(\sigma) = J$.*

b) Let $\sigma \in I(m,n)$. If $a \in \mathcal{U}(\sigma)$ then there is a unique $b \in B$ such that $ba \in \mathcal{W}(\sigma)$. Write this b as $b(\sigma, a)$.

Proof. Note that left multiplication by elements of B allows us to do elementary row operations "from bottom to top". The following construction will be used a few times, so we give it a name.

> *Construction OP*: Start with a matrix $a \in M$. Construct a sequence $a^{(0)}, a^{(1)}, \cdots, a^{(m)}$. Let $a^{(0)} = a$. If $a^{(k)}$ has been constructed, choose $a^{(k+1)}$ as follows. Since $a^{(k)}_{[m],J} \neq 0$, there is a non-zero entry $a^{(k)}_{m-k,j_{m-k}}$ with $j_{m-k} \in [n] - \{j_m, j_{m-1}, \cdots, j_{m-k+1}\}$. Divide the $(m-k)$-th row by $a^{(k)}_{m-k,j_{m-k}}$. Pivot at $(m-k, j_{m-k})$ to eliminate all the entries above. Let $a^{(k+1)}$ be the output. *End of Construction OP.*

Note by the construction that there exists $b \in B$ such that $a^{(m)} \in Ba$. Since the non-zero entries used in the construction are not in general unique, the sequence $j_1, \ldots, j_m$ and the output $a^{(m)}$ of the construction are not in general uniquely determined by a. However if the construction can be done with a preassigned sequence $j_1, \ldots, j_m$ then both b and $a^{(m)}$ are uniquely determined by a. Let $j_1, \ldots, j_m$ be one sequence of column indices obtained in this way. Define $\sigma(k) = j_k$, for $1 \leq k \leq m$. Then, $\sigma \in I(m,n)$ and the resulting matrix $a^{(m)}$ lies in $\mathcal{W}(\sigma)$. Thus $a \in Ba^{(m)} \subseteq B\mathcal{W}(\sigma) = \mathcal{U}(\sigma)$. Thus a) is proved.

Next, suppose $\sigma \in I(m,n)$ is fixed and $a \in \mathcal{U}(\sigma)$. Then the columns $\sigma(1), \cdots, \sigma(m)$ of a are linearly independent. By the definition of $\mathcal{U}(\sigma)$, for $k = m, m-1, \cdots, 2, 1$, we can choose the $j_k = \sigma(k)$ in *Construction OP.* By the definition of $\mathcal{W}(\sigma)$ to make sure that $b\,a \in \mathcal{W}(\sigma)$, we have to choose $j_k = \sigma(k)$ for $1 \leq k \leq m$. Thus, by our earlier remark on uniqueness, there is a unique $b \in B$ such that $b\,a \in \mathcal{W}(\sigma)$. Q.E.D.

Definition 3.4 *Let $u_\sigma : \mathcal{U}(\sigma) \to \mathcal{W}(\sigma)$ be the map defined by $u_\sigma(a) = b(\sigma, a)\,a$.*

Remark 3.5 *Since $\mathcal{W}(\sigma) \subseteq \mathcal{U}(\sigma)$, the map u_σ is surjective.*

Corollary 3.6 *$\mathcal{U}(\sigma) = B\,\mathcal{W}(\sigma)$, for all $\sigma \in I(m,n)$.*

Proof. By the definition of $\mathcal{W}(\sigma)$, we know that $a_{[i,m],\sigma([i,m])} = \pm 1 \neq 0$, for $1 \leq i \leq m$ and $a \in \mathcal{W}(\sigma)$. So $\mathcal{W}(\sigma) \subseteq \mathcal{U}(\sigma)$. By Lemma 3.2 $(b\,a)_{[i,m],\sigma([i,m])} = b_{[i,m],[i,m]}\,a_{[i,m],\sigma([i,m])} \neq 0$ so $b\,a \in \mathcal{U}(\sigma)$. Hence $B\mathcal{W}(\sigma) \subseteq \mathcal{U}(\sigma)$. On the other hand, if $a \in \mathcal{U}(\sigma)$, let $j_k = \sigma(k)$, for $1 \leq k \leq m$. The conclusion b) in Lemma 3.3 tells us that $a \in B\,\mathcal{W}(\sigma)$. Therefore, $\mathcal{U}(\sigma) = B\,\mathcal{W}(\sigma)$. Q.E.D.

Corollary 3.7 *Each $\mathcal{U}(\sigma)$ is open and*

$$M = \bigcup_{\sigma \in I(m,n)} \mathcal{U}(\sigma).$$

Proof. For each $a \in M$, by Lemma 3.3, there is some $\sigma \in I(m,n)$ such that $a \in \mathcal{U}(\sigma)$. So, $M \subseteq \bigcup_{\sigma \in I(m,n)} \mathcal{U}(\sigma)$. The reverse inclusion is clear. Since $\mathcal{U}(\sigma)$ is defined as the set of solutions of the system of polynomial inequalities

$$\det(sub_{\{m-k+1,\cdots,m\},\{\sigma(m-k+1),\cdots,\sigma(m)\}}a) \neq 0, \tag{27}$$

for $1 \leq k \leq m$, each $\mathcal{U}(\sigma)$ is an open set. Q.E.D.

Remark on Subindices. *In the rest of this section,* we will consider equalities involving minors of a matrix. The order of subindices become crutial. When the subindices are given by a set, we mean an ordered set. In particular, let I be an ordered set $I = \{i_1, \cdots, i_k\}$, then $\{j\} \cup I$ means the ordered set $\{j, i_1, \cdots, i_k\}$. Similarly, $[j, k]$ means the ordered set $\{j, j+1, \cdots, k\}$ and $\sigma([j,k])$ means the ordered set $\{\sigma(j), \sigma(j+1), \cdots, \sigma(k)\}$.

Lemma 3.8 *Let a be an m by n matrix of arbitrary rank. Let $I \subseteq [m]$, $J \subseteq [n]$ and $|I| = |J| = k$. Let $i, p \in [m] - I$ with $i \neq p$, and $j, q \in [n] - J$ with $j \neq q$. Then*

$$\begin{aligned} a_{I,J}\ a_{\{i,p\}\cup I,\{j,q\}\cup J} &= a_{\{p\}\cup I,\{q\}\cup J}\ a_{\{i\}\cup I,\{j\}\cup J} \\ &- a_{\{i\}\cup I,\{q\}\cup J}\ a_{\{p\}\cup I,\{j\}\cup J}. \end{aligned} \tag{28}$$

This is the well-known Jacobi's identity. For completeness, we give a proof which is suggested by a referee, we thank him for it.

Proof. Let $I = \{i_1, \cdots, i_k\}$ and $J = \{j_1, \cdots, j_k\}$. Without loss of generality, we assume that $i < p < i_1 < \cdots < i_k$ and $j < q < j_1 < \cdots < j_k$ since other cases can be transformed into this case by row or column transpositions. Since both sides of the identity are polynomials of entries of the matrix a, it is enough to prove the case that $a_{I,J} \neq 0$. In such a case, let c be a matrix

$$c = \begin{pmatrix} 1 & 0 & O_{2,k} \\ 0 & 1 & \\ O_{k,2} & & c' \end{pmatrix}$$

where $c' = (sub_{I,J} a)^{-1}$ and $O_{\lambda,\mu}$ denotes the λ by μ zero matrix. Replace a by ca, both sides of the identity change by a factor $(\det c')^2$. Thus we may replace a by ca and assume $a_{I,J} = identity$. Thus we need only show

$$\begin{aligned} a_{\{i,p\}\cup I,\{j,q\}\cup J} &= a_{\{p\}\cup I,\{q\}\cup J}\ a_{\{i\}\cup I,\{j\}\cup J} \\ &- a_{\{i\}\cup I,\{q\}\cup J}\ a_{\{p\}\cup I,\{j\}\cup J}. \end{aligned}$$

This can be verify by calculation as follows: let

$$sub_{\{i,p\}\cup I,\{j,q\}\cup J}\, a = \begin{pmatrix} \alpha & \beta & u_1 & \cdots & u_k \\ \gamma & \delta & v_1 & \cdots & v_k \\ x_1 & y_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & y_k & 0 & \cdots & 1 \end{pmatrix}.$$

Pivot at $(l+2, l+2)$ to eliminate the $(1, l+2)$ entry and $(2, l+2)$ entry for $1 \leq l \leq k$. Then the matrix is transformed to

$$\begin{pmatrix} \alpha - \sum_{l=1}^k u_l x_l & \beta - \sum_{l=1}^k u_l y_l & 0 & \cdots & 0 \\ \gamma - \sum_{l=1}^k v_l x_l & \delta - \sum_{l=1}^k v_l y_l & 0 & \cdots & 0 \\ x_1 & y_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_k & y_k & 0 & \cdots & 1 \end{pmatrix}. \tag{29}$$

Thus

$$a_{\{i,p\}\cup I,\{j,q\}\cup J} = (\alpha - \sum_{l=1}^{k} u_l x_l)(\delta - \sum_{l=1}^{k} v_l y_l) - (\beta - \sum_{l=1}^{k} u_l y_l)(\gamma - \sum_{l=1}^{k} v_l x_l).$$

On the other hand, the matrix (29) gives

$$\begin{aligned} a_{\{i,p\}\cup I,\{j,q\}\cup J} &= \alpha - \sum_{l=1}^{k} u_l x_l \\ a_{\{i\}\cup I,\{q\}\cup J} &= \beta - \sum_{l=1}^{k} u_l y_l \\ a_{\{p\}\cup I,\{j\}\cup J} &= \gamma - \sum_{l=1}^{k} v_l x_l \\ a_{\{p\}\cup I,\{q\}\cup J} &= \delta - \sum_{l=1}^{k} v_l y_l \end{aligned}$$

This proves the Jacobi's identity. Q.E.D.

Recall that for a fixed σ the sequence $a = a^{(0)}, a^{(1)}, \cdots, a^{(m)}$ in Construction OP is uniquely determined by a.

Lemma 3.9 *Fix σ and preassign $j_1, \ldots, j_m$ such that $\sigma(k) = j_k$ for $1 \leq k \leq m$. Then the matrix $a^{(m-k)}$ obtained in Construction OP is*

$$C_k \, (\sum_{\substack{1\leq i\leq k \\ 1\leq j\leq n}} a_{\{i\}\cup\sigma([k+1,m]),\{j\}\cup\sigma([k+1,m])} E_{ij} + \sum_{\substack{k+1\leq i\leq m \\ 1\leq j\leq n}} a_{[i,m],\{j\}\cup\sigma([k+1,m])} E_{ij}) \tag{30}$$

where $C_k = diag(1/a_{[k,m],\sigma([k,m])}, \cdots, 1/a_{[k,m],\sigma([k,m])}, \underbrace{1, \cdots, 1}_{m-k})$.

Proof. Use induction on $m - k$. If $m - k = 1$, then the first step of Construction OP implies that all the entries in the $\sigma(m)$-th column are zero except the last entry, which is 1. By the elimination procedure, the (i,j) entry of $a^{(1)}$ is $a_{i,j} - a_{i,\sigma(m)} a_{m,j}/a_{m,\sigma(m)} = a_{\{i,m\},\{j,\sigma(m)\}}/a_{m,\sigma(m)}$, for $j \neq \sigma(m)$. So the lemma is true for $m - k = 1$. Assume the lemma is true for any positive integer less than $m - k$. By induction hypothesis,

$$\begin{aligned} a^{(m-k-1)} &= C_{k+1} \, (\sum_{\substack{1\leq i\leq k+1 \\ 1\leq j\leq n}} a_{\{i\}\cup[k+2,m],\{j\}\cup\sigma([k+2,m])} E_{ij} \\ &+ \sum_{\substack{k+2\leq i\leq m \\ 1\leq j\leq n}} a_{[i,m],\{j\}\cup\sigma([k+2,m])} E_{ij}) \end{aligned} \tag{31}$$

where $C_{k+1} = diag(1/a_{[k+1,m],\sigma([k+1,m])}, \cdots, 1/a_{[k+1,m],\sigma([k+1,m])}, \underbrace{1, \cdots, 1}_{m-k-1})$. By the elimination procedure, the (i,j) entry of $a^{(m-k)}$ is

$$\frac{1}{a_{[k,m],\sigma([k,m])}} \left(a_{\{i\}\cup[k+1,m],\{j\}\cup\sigma([k+1,m])} - \frac{a_{\{i\}\cup[k+1,m],\sigma([k,m])} a_{[k,m],\{j\}\cup\sigma([k+1,m])}}{a_{[k,m],\sigma([k,m])}} \right). \tag{32}$$

By the previous lemma, we know this is

$$a_{\{i\}\cup[k,m],\{j\}\cup\sigma([k,m])}/a_{[k,m],\sigma([k,m])} \tag{33}$$

Hence, the lemma is proved. Q.E.D.

Corollary 3.10

$$u_\sigma(a) = \sigma + \sum_{(i,j)\in L(\sigma)} \frac{a_{[i,m],\{j\}\cup\sigma([i+1,m])}}{a_{[i,m],\sigma([i,m])}} E_{ij} \tag{34}$$

Proof. By the definition of $L(\sigma)$, $(i,j) \in L(\sigma)$ if and only if either $j \notin J(\sigma)$ or $j \in J(\sigma)$ and $i > \sigma^{-1}(j)$. Note that if $j \in J(\sigma)$ and $i < \sigma^{-1}(j)$, then $j \in \sigma([i+1,m])$. Thus $a_{[i,m],\{j\}\cup\sigma([i+1,m])} = 0$. Moreover if $j = \sigma(i)$, then $a_{[i,m],\{j\}\cup\sigma([i+1,m])}/a_{[i,m],\sigma([i,m])} = 1$. Take $k = 0$ in (30). Then the first term of (30) does not exist and the second term of (30) is exactly the formula in this corollary. Q.E.D.

Example 3.11 *Let $m = 2$. If $\sigma = \begin{pmatrix} 0&0&1 \\ 0&1&0 \end{pmatrix}$, then*

$$a = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix} \mapsto u_\sigma(a) = b(\sigma,a)\,a = \begin{pmatrix} \frac{a_{12,12}}{a_{12,23}} & 0 & 1 \\ \frac{a_{2,1}}{a_{2,2}} & 1 & \frac{a_{2,3}}{a_{2,2}} \end{pmatrix}$$

Proposition 3.12 *If $\sigma \in I(m,n)$, then $V(\sigma) := B\backslash\mathcal{U}(\sigma)$ is homeomorphic to $\mathcal{W}(\sigma)$.*

Proof. It follows from Formula (34) that

$$u_\sigma(b\,a) = \sigma + \sum_{(i,j)\in L(\sigma)} \frac{b_{[i,m],[i,m]}a_{[i,m],\{j\}\cup\sigma([i+1,m])}}{b_{[i,m],[i,m]}a_{[i,m],\sigma([i,m])}} E_{ij} = u_\sigma(a). \tag{35}$$

Thus we may define $v_\sigma : V(\sigma) \to \mathcal{W}(\sigma)$ by $v_\sigma(B\,a) = u_\sigma(a)$. We will show that v_σ is the desired homeomorphism. Let $\pi : \mathcal{U}(\sigma) \to V(\sigma)$ be the projection defined by $\pi : a \mapsto B\,a$. Clearly, π is an onto map. Since $B\backslash M$ has the quotient topology, a subset of $V(\sigma)$ is open in $V(\sigma)$ if and only if its pre-image under π is open in $\mathcal{U}(\sigma)$. Thus we have a commutative diagram

$$\begin{tikzcd} \mathcal{U}(\sigma) \arrow[rr, "\mathbf{u}_\sigma"] \arrow[dr, "\pi"'] & & \mathcal{W}(\sigma) \\ & V(\sigma) \arrow[ur, "\mathbf{v}_\sigma"'] & \end{tikzcd}$$

By Remark 3.5 the map u_σ is surjective, so $\mathcal{W}(\sigma) = \{u_\sigma(a) |\, a \in \mathcal{U}(\sigma)\}$. To prove that v_σ is a homeomorphism we must show a) v_σ is continuous, b) v_σ is bijective, and c) v_σ^{-1} is continuous.

a) Since $V(\sigma)$ has the quotient topology, to show that v_σ is continuous we need only show that u_σ is continuous. But u_σ is obviously continuous because each entry of u_σ is a rational function of the entries of a with nonzero denominators.

b) Since u_σ is onto, so is v_σ. To show that v_σ is one to one, assume $v_\sigma(B\,a) = v_\sigma(B\,a')$ for some $a, a' \in M$. Then, by Definition 3.4 we have

$$b(\sigma, a)a = u_\sigma(a) = v_\sigma(B\,a) = v_\sigma(B\,a') = u_\sigma(a') = b(\sigma, a')a'.$$

Thus $B\,a' = B\,a$.

c) v_σ^{-1} is clearly continuous since the topology on $\mathcal{W}(\sigma)$ is the subspace topology. Q.E.D.

Proposition 3.13 *Let $\sigma, \sigma' \in I(m,n)$. If $V(\sigma) \cap V(\sigma') \neq \emptyset$, then $v_{\sigma'} \circ v_\sigma^{-1} : v_\sigma(V(\sigma) \cap V(\sigma')) \to v_{\sigma'}(V(\sigma) \cap V(\sigma'))$ is a biholomorphic map.*

Proof. Note that $\mathcal{W}(\sigma) \subseteq \mathcal{U}(\sigma)$. If $a \in v_\sigma(V(\sigma) \cap V(\sigma'))$ then $a \in \mathcal{U}(\sigma) \cap \mathcal{U}(\sigma')$. Therefore $(v_{\sigma'} \circ v_\sigma^{-1})(a) = u_{\sigma'}(a)$. Thus, by the formula (34) for $u_{\sigma'}(a)$, $v_{\sigma'} \circ v_\sigma^{-1} : v_\sigma(V(\sigma) \cap V(\sigma')) \to v_{\sigma'}(V(\sigma) \cap V(\sigma'))$ is a holomorphic map. The same argument applies to the opposite direction. So $v_{\sigma'} \circ v_\sigma^{-1}$ is biholomorphic. Q.E.D.

It follows from Corollary 3.7, Proposition 3.12 and Proposition 3.13 that

Theorem 3.14 *$B\backslash M$ is a complex manifold with $\{(V(\sigma), v_\sigma) |\ \sigma \in I(m,n)\}$ as a local coordinate system.*

The dimension of $B\backslash M$ is equal to the dimension of the each of the affine spaces $\mathcal{W}(\sigma)$. It follows from (24) that dim $B\backslash M = m(n-m) + \frac{m(m-1)}{2}$.

4 Embedding of $B\backslash M$ into $\prod_{i=1}^m \mathbf{P}^{\binom{n}{i}-1}$

In this section we prove the following special case of Theorem 2.30 in which $\lambda = (n^m)$ and thus $B\backslash M_\lambda = B\backslash M$.

Theorem 4.1 *There exist a projective space $\mathbf{P}^N$ and an embedding $\Theta : B\backslash M \to \mathbf{P}^N$ of complex manifolds such that $\Theta(B\backslash M)$ is a projective subvariety of $\mathbf{P}^N$.*

The same embedding Θ will be used in the proof of Theorem 2.30 in the next section. We will prove Theorem 4.1 by showing (1) that there is an embedding $\Phi : B\backslash M \to \mathbf{P}^{\binom{n}{1}-1} \times \cdots \times \mathbf{P}^{\binom{n}{m}-1}$ of complex manifolds and (2) that $B\backslash M$ is compact. It follows from Corollary 2.22 that $\Phi(B\backslash M)$ is a projective subvariety of $\mathbf{P}^{\binom{n}{1}-1} \times \cdots \times \mathbf{P}^{\binom{n}{m}-1}$. Let $N = \prod_{i=1}^m \binom{n}{i} - 1$ and let

$$\Psi : \mathbf{P}^{\binom{n}{1}-1} \times \cdots \times \mathbf{P}^{\binom{n}{m}-1} \to \mathbf{P}^N \tag{36}$$

be the mapping defined in (7) with $n_i = \binom{n}{i} - 1$ for $1 \leq i \leq m$. Then $\Theta = \Psi \circ \Phi$ is the desired embedding.

Recall that for a positive integer s, a point $p \in \mathbf{P}^s$ is written as $p = (p_0 : \ldots : p_s)$. If $s = \binom{n}{k} - 1$ we agree to index the components of p by k-subsets J of $[n]$ listed in lexicographic order. Thus a point of $\mathbf{P}^{\binom{n}{k}-1}$ has the form $p = (\ldots : p_J : \ldots)$. Recall from (15) and (16) that we have map $\rho_k : M \to M^k_{k,n}$ where $\rho_k(a) = a_k$ is the matrix which consists of the last k rows of a. Define

$$\phi_k : M^k_{k,n} \to \mathbf{P}^{\binom{n}{k}-1} \tag{37}$$

by

$$\phi_k(x) = (\ldots : x_{[k],J} : \ldots) \quad , \text{ for } x \in M^k_{k,n}$$

where $x_{[k],J}$ is the k by k minor of x corresponding to $J \subseteq [n]$. We write

$$(\phi_k(x))_J = x_{[k],J} \quad , \text{ for } x \in M^k_{k,n} \tag{38}$$

Define a map $\phi : M \to \prod_{k=1}^m \mathbf{P}^{\binom{n}{k}-1}$ by

$$\phi(a) = (\phi_1(a_1) : \ldots : \phi_m(a_m)), \qquad \text{for } a \in M \tag{39}$$

where $a_k \in M^k_{k,n}$ consists of the last k rows of a.

Proposition 4.2 *$\phi(b\,a) = \phi(a)$, for all $b \in B$.*

Proof. Since the matrices in B are upper triangular, for any k subset of $[n]$ and $b \in B$,

$$\begin{aligned}
(\phi_k((b\,a)_k))_J &= (b\,a)_{[m-k+1,m],J} \\
&= b_{[m-k+1,m],[m-k+1,m]}\, a_{[m-k+1,m],J} \\
&= b_{[m-k+1,m],[m-k+1,m]}\, (\phi_k((a)_k))_J \ .
\end{aligned}$$

Thus $\phi(b\,a) = \phi(a)$, for all $b \in B$. Q.E.D.

It follows from Proposition 4.2 that we can define a map $\Phi : B\backslash M \to \prod_{i=1}^m \mathbf{P}^{\binom{n}{i}-1}$ by

$$\Phi(Ba) := \phi(a). \tag{40}$$

Proposition 4.3 *The mapping Φ defined by (40) is an embedding of complex manifolds.*

Proof. By Definition 2.18, we need to show that Φ is a holomorphic map of complex manifold, Φ is full ranked and Φ is injective. The holomorphic part is obvious as the components of each ϕ_i are polynomials in the entries of the matrix a. We prove that Φ is injective and full ranked. First we show that the restriction of Φ to $V(\sigma)$ is injective for each $\sigma \in I(m,n)$ and then, as a by-product, we prove that Φ is full ranked. In Proposition 3.12, we proved that $v_\sigma : V(\sigma) \to \mathcal{W}(\sigma)$ is a homeomorphism. Thus it will suffice to show that $\Phi \circ v_\sigma^{-1} : \mathcal{W}(\sigma) \to \mathbf{P}^{\binom{n}{1}-1} \times \cdots \times \mathbf{P}^{\binom{n}{m}-1}$ is injective. For every $j \in [n] - \sigma([m-k+2,m])$. if $a \in \mathcal{W}(\sigma)$ then, from (40) and (39), the component of $\phi_k(a_k)$ with set of column indices equal to $\{j\} \cup \sigma([m-k+2,m])$ is $\pm a_{m-k+1,j}$, i.e.,

$$(\phi_k(a_k))_{\{j\}\cup\sigma([m-k+2,m])} = \pm a_{m-k+1,j} \tag{41}$$

where $j \notin \sigma([m-k+2, m])$. If $(i,j) \in L(\sigma)$, choose k so that $i = m-k+1$. Thus $a_{i,j}$ is determined by $\phi_k(a_k)$ and hence by $\Phi(Ba)$. Thus the restriction of Φ to $V(\sigma)$ is injective.

Let $c = \dim \mathcal{W}(\sigma)$. Note that the equality in (41) implies that every free entry (or its negative multiple) of the matrices in $\mathcal{W}(\sigma)$ is a component of some ϕ_k. Thus at every $a \in \mathcal{W}(\sigma)$, the Jacobian of ϕ is of rank c. So ϕ is full ranked. By the definition of Φ in (40) we know that Φ is full ranked.

To illustrate the idea here we give an example. Suppose

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If $a \in \mathcal{W}(\sigma)$, then

$$a = \begin{pmatrix} 1 & 0 & 0 & a_{1,4} \\ a_{2,1} & 1 & 0 & a_{2,4} \\ a_{3,1} & a_{3,2} & 1 & a_{3,4} \end{pmatrix}.$$

Thus

$$\Phi(Ba) = (\phi_1(a_1), \phi_2(a_2), \phi_3(a_3))$$

where

$$\begin{aligned} \phi_1(a_1) &= (a_{3,1} : a_{3,2} : a_{3,3} = 1 : a_{3,4}) \\ \phi_2(a_2) &= (a_{23,12} : a_{23,13} = a_{2,1} : a_{23,14} : a_{23,23} = 1 : a_{23,24} : a_{23,34} = -a_{2,4}) \\ \phi_3(a_3) &= (a_{123,123} = 1 : a_{123,124} : a_{123,134} : a_{123,234} = a_{1,4}) \quad . \end{aligned}$$

Since

$$L(\sigma) = \{(1,4), (2,1), (2,4), (3,1), (3,2), (3,4)\}$$

the preceding equations show that $\Phi(Ba)$ determines a. Thus the restriction of Φ to $V(\sigma)$ is injective.

Next we show that Φ is injective. Suppose $a, a' \in M$ and $\Phi(Ba) = \Phi(Ba')$ and $\sigma \neq \sigma'$. We want to show that $Ba = Ba'$. If there exists some $\tau \in I(m,n)$ such that $a, a' \in \mathcal{U}(\tau)$, then we are done since the restriction of Φ to $V(\tau)$ is injective. So assume that $a' \in \mathcal{U}(\sigma)$. Then from the definition of $\mathcal{U}(\sigma)$, there must be some i such that $a'_{[i,m],\sigma([i,m])} = 0$. On the other hand, $a_{[i,m],\sigma([i,m])} \neq 0$. This shows that $\phi_i(a_i) \neq \phi_i(a'_i)$, and this is a contradiction. Q.E.D.

Example 4.4 *Let $\sigma = E_{12} + E_{23} + E_{31}$. Then*

$$u_\sigma(a) = \begin{pmatrix} 0 & 1 & 0 & a_{1,4} & a_{1,5} \\ 0 & a_{2,2} & 1 & a_{2,4} & a_{2,5} \\ 1 & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \end{pmatrix}.$$

Since $\sigma(\{3\}) = \{1\}$, *we have* $\phi_1(a_1) = (1 : a_{3,2} : a_{3,3} : a_{3,4} : a_{3,5})$. *Since* $\sigma(\{2,3\}) = \{3,1\}$, *we have*

$$\begin{aligned}\phi_2(a_2) &= ((\phi_2)_{\{1,2\}} : (\phi_2)_{\{1,4\}} : (\phi_2)_{\{1,5\}} : \cdots) \\ &= (a_{2,2} : a_{2,4} : a_{2,5} : \cdots).\end{aligned}$$

Since $\sigma(\{1,2,3\}) = \{2,3,1\}$, *we have*

$$\begin{aligned}\phi_3(a_3) &= ((\phi_3)_{\{1,3,4\}} : (\phi_3)_{\{1,3,5\}} : \cdots) \\ &= (a_{1,4} : a_{1,5} : \cdots) \quad .\end{aligned}$$

So all the entries $a_{i,j}$ *in* $u_\sigma(a)$ *are listed in* $\Phi(Ba)$.

Proposition 4.3 tells us that our mapping Φ is an embedding of $B\backslash M$ into $\mathbf{P}^{\binom{n}{1}-1} \times \cdots \times \mathbf{P}^{\binom{n}{m}-1}$. Let $N = \prod_{i=1}^{m} \binom{n}{i} - 1$. By Lemma 2.11 and Example 2.19 there is an embedding $\mathbf{P}^{\binom{n}{1}-1} \times \cdots \times \mathbf{P}^{\binom{n}{m}-1} \to \mathbf{P}^N$. The composite of these two embeddings gives us an embedding of $B\backslash M$ into $\mathbf{P}^N$. Thus, in view of Corollary 2.22, to complete the proof of Theorem 4.1 it suffices to show that $B\backslash M$ is compact.

Lemma 4.5 $B\backslash M$ *is compact.*

Proof. Here and in the rest of this article we use the standard inner product

$$< u, v > := \sum_{i=1}^{n} u_i \overline{v_i}, \text{ for } u, v \in \mathbf{C}^n. \tag{42}$$

and define

$$||\mathbf{u}||^2 = < \mathbf{u}, \mathbf{u} > \quad . \tag{43}$$

Let

$$A := \{a \in M \mid \text{ the row vectors of } a \text{ are orthonormal } \}.$$

Then, for each $a \in A$,

$$\sum_{k=1}^{n} a_{i,k} \overline{a_{j,k}} = \delta_{i,j} \quad .$$

Thus, A is a compact subset of M. If $a \in M$ let $\gamma(a) \in A$ be the matrix produced by the Gram-Schmidt process. By the formulas in the Gram-Schmidt process, $\gamma(a) \in Ba$. Thus $Ba = B\gamma(a)$. It follows that the restriction $\pi|_A : A \to B\backslash M$ is surjective. Thus $B\backslash M$ is compact. Q. E. D.

This completes the proof of Theorem 4.1.

5 Partition Variety $B\backslash M_\lambda$ and its Cellular Decomposition

So far, we have proved that $B\backslash M$ is a complex manifold and a projective variety. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition with $\lambda_1 = n$. Recall that

$$M_\lambda := \{a \in M |\, a_{ij} = 0 \text{ if } (i,j) \notin F_\lambda\}$$

in Definition 2.27. Note that for $r = m$, M_λ is non-empty if and only if $\lambda_i \geq m - i + 1$ for $1 \leq i \leq m$. We assume this condition throughout. The purpose of this section is to show

1) $B\backslash M_\lambda$ is a projective variety.
2) $B\backslash M_\lambda$ has a cellular decomposition.

First we show that $B\backslash M_\lambda$ is a projective variety.

Lemma 5.1 $B\,M_\lambda \subseteq M_\lambda$.

Proof. If $a \in M_\lambda$ and $b \in B$, then $a_{i,j} = 0$ if $(i,j) \notin F_\lambda$. Thus, $(ba)_{i,j} = \sum_{k=i}^m b_{i,k}a_{k,j}$. In particular, if $(i,j) \notin F_\lambda$, then $a_{k,j} = 0$ for all $k \geq i$. Hence, $(ba)_{i,j} = \sum_{k=i}^m b_{ik}a_{k,j} = 0$. Thus $ba \in M_\lambda$. Q.E.D.

Proof of Theorem 2.30. Choose any $\sigma \in I(m,n)$ and fix it temporarily. By (41) every free entry of $a \in \mathcal{W}(\sigma)$ is a component of some ϕ_k. Thus the equations

$$a_{i,j} = 0, \qquad \text{for} \quad (i,j) \notin F_\lambda$$

give a set of homogeneous equations in $\Phi(V(\sigma)) \subset \Phi(B\backslash M)$. These homogeneous equations define the image $\Phi(V(\sigma) \cap (B\backslash M_\lambda)) \subset \Phi(B\backslash M_\lambda)$ as a subvariety of the projective variety $\Phi(B\backslash M)$. Recall that we have the embedding $\Psi : \prod_{i=1}^m \mathbf{P}^{\binom{n}{i}-1} \to \mathbf{P}^N$ defined as in (36) where $N = \prod_{i=1}^m \binom{n}{i} - 1$. Thus $(\Psi \circ \Phi)(B\backslash M_\lambda)$ is an analytic subvariety of $\mathbf{P}^N$. By Chow's Theorem 2.21 $(\Psi \circ \Phi)(B\backslash M_\lambda)$ is a projective subvariety of $\mathbf{P}^N$. Let $\Theta := \Psi \circ \Phi$. We get the conclusion of our theorem. Q.E.D.

Next we show that the partition variety $B\backslash M_\lambda$ has a cellular decomposition. Recall from Lemma 2.29 that each $Y(\sigma)$ is an affine space of dimension $l(\sigma)$. If $1 \leq s \leq n$, we list all the s-subsets of $[n]$ lexicographically. If J_1 and J_2 are two s-subsets of $[n]$ and J_1 precedes J_2 in the lexicographic order then we write $J_1 \prec J_2$. Recall that for $\sigma \in I(m,n)$, σ_k denotes the submatrix consisting of the last k rows of σ for $1 \leq k \leq m$. To illustrate the idea of the next lemma we give an example first.

Example 5.2 Let $m = 2$ and $n = 4$. Let $\sigma = \left(\begin{smallmatrix} 0&0&1&0 \\ 0&1&0&0 \end{smallmatrix}\right)$. $J(\sigma_1) = 2$ and $J(\sigma_2) = J(\sigma) = \{2,3\}$. Then $K(\sigma) = \{(1,4),(2,3),(2,4)\}$ and $Y(\sigma) = \sigma + \mathbf{C}E_{1,4} + \mathbf{C}E_{2,3} + \mathbf{C}E_{2,4}$. Take $a \in Y(\sigma)$. Then a is of the form

$$a = \begin{pmatrix} 0 & 0 & 1 & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \end{pmatrix}.$$

Thus the 1 by 1 minors in the last row of a are $a_{2,1} = 0,\ \ a_{2,2} = 1,\ \ a_{2,3},$ and $a_{2,4}$. The 2 by 2 minors in the last 2 rows are $a_{12,12} = 0,\ \ a_{12,13} = 0,\ \ a_{12,14} = 0,\ \ a_{12,23} = -1,\ \ a_{12,24} = -a_{1,4}$, $a_{12,34} = a_{2,4} - a_{1,4}a_{2,3}$. Clearly in the lexicographic order of the 2-subsets of column indices in [4], $a_{[2,2],J(\sigma_1)} = a_{2,2} = 1$ is the first nonzero 1 by 1 minor. Similarly in the lexicographic order of the 2-subsets of column indices in [4], $a_{[1,2],J(\sigma_2)} = a_{12,23} = -1$ is the first nonzero 2 by 2 minor. The next lemma shows that this is not a coincidence.

Lemma 5.3 *If* $a \in Y(\sigma)$ *then* $a_{[m-k+1,m],J(\sigma_k)} = \pm 1$ *and* $a_{[m-k+1,m],J} = 0$ *for any* $J \prec J(\sigma_k)$ *in the lexicographic order of* k*-subsets of* $[n]$.

Remark In this lemma, we view the sets of indices of the minors as unordered (ordinary) sets instead of ordered sets.

Proof. We use induction on k. If $k = 1$, then $a_{[m-k+1,m],J(\sigma_k)} = a_{m,j_m} = 1$ and $a_{[m-k+1,m],J} = a_{m,j} = 0$ for $j < j_m$ by the definition of $K(\sigma)$ in (11). Suppose the lemma is true for $k < t$. Then by the definition of $Y(\sigma)$ in (14) and the induction hypothesis

$$a_{[m-t+1,m],J(\sigma_t)} = \pm a_{m-t+1,j_{m-t+1}} a_{[m-(t-1)+1,m],J(\sigma_{t-1})} = \pm 1.$$

Suppose $J \prec J(\sigma_t)$. There exists a $j \in J - J(\sigma_t)$ such that for all $j' \in J$ with $j' < j$ we have $j' \in J(\sigma_t)$. Let $J' = \{j' \in J \mid j' < j\}$. Thus the definition of $K(\sigma)$ in (11) implies that $a_{i,j} = 0$ for $i \in [m-t+1,m] \backslash \sigma_t^{-1}(J')$. Therefore, $a_{[m-t+1,m],J} = 0$. Q.E.D.

Proposition 5.4 (Cellular Decomposition of M)

$$M = \bigsqcup_{\sigma \in I(m,n)} BY(\sigma).$$

Proof. The right hand side is clearly included in the left. In our proof we use the following construction. For $a \in M$ if $a_{i,j} \neq 0$, we define :

> *Construction* (i,j): Pivot at (i,j) to eliminate all the entries above $a_{i,j}$. Then divide the i-th row by $a_{i,j}$. Thus, we get a new matrix a'. *End of Construction* (i,j).

We will construct, inductively, a sequence of matrices $a = a^{(0)}, a^{(1)}, \ldots, a^{(m)}$ and a sequence $(i_1,j_1), \ldots, (i_m,j_m)$ of pairs of integers with $1 \le i_k \le m$ and $1 \le j_k \le n$ such that $i_1, \ldots, i_m$ are distinct and $j_1 < \ldots < j_m$. Let $a^{(0)} = a$. Suppose the first nonzero column of $a^{(0)}$ is the j_1-th column. Choose i_1 so that $a_{i_1,j_1} \neq 0$ and i_1 is maximal with this property. Then, do Construction (i_1,j_1). Let $a^{(1)}$ be the output matrix. Suppose $a^{(k)}$ and (i_k,j_k) have been constructed with $k < m$. Since $a^{(k)}$ has rank m, there exists a pair (i,j) such that $a_{i,j} \neq 0$ and $j_k < j \le n$, and i is different from $i_1, \ldots, i_k$. Let S be the set of all such pairs. Choose $(i,j) \in S$ so that j is minimal. Let $j_{k+1} = j$. Now choose $(i, j_{k+1}) \in S$ so that i is maximal. Let $i_{k+1} = i$. Now let $a^{(k+1)}$ be the output of Construction (i_{k+1}, j_{k+1}).

This completes the induction. Since $i_1,\dots,i_m$ are distinct and $j_1,\dots,j_m$ are distinct we may define $\sigma \in I(m,n)$ by $\sigma(i_k) = j_k$ for $1 \le k \le m$. If we can show that $a^{(m)} \in Y(\sigma)$ then $M = \bigcup_{\sigma\in I(m,n)} BY(\sigma)$. Clearly the entries at the position $(k,\sigma(k))$ in $a^{(m)}$ are all 1's for $1 \le k \le m$. By (14) and the pictorial descriptions of $K_1(\sigma)$ and $K_2(\sigma)$ in Remark 2.28, we need only show that all the non-zero entries of $a^{(m)}$ which are not at the position $(k,\sigma(k))$ (for $1 \le k \le m$) are at the position (i,j) which belongs to either $K_1(\sigma)$ or $K_2(\sigma)$. The minimality of j_{k+1} implies that if an entry $a^{(m)}_{i,j}$ is non-zero and $\sigma(i) \ne j$ then $j > j_{k+1}$. Thus there are two posibilities: either a) $j \notin J(\sigma)$ or b) by the choice of i_{k+1} $j \in J(\sigma)$ and $i < \sigma^{-1}(j)$. In case a) we have $(i,j) \in K_1(\sigma)$. In case b) we have $(i,j) \in K_2(\sigma)$. Therefore $a^{(m)} \in Y(\sigma)$.

Next we prove that the union in a) is disjoint. Suppose $\sigma \ne \sigma'$. We assume there exists some $x \in BY(\sigma) \cap BY(\sigma')$ and arrive at a contradiction. Write $x = ba = b'a'$ where $b, b' \in B$, $a \in Y(\sigma)$, and $a' \in Y(\sigma')$. Scan the rows of σ and σ' from bottom to top, and look for the first place where they differ. Recall that σ_k denotes the matrix which consists of the last k rows of σ. Since $\sigma \ne \sigma'$, there exists some $s \in [m]$ such that $\sigma_k = \sigma'_k$ for $1 \le k < s$ and $\sigma_s \ne \sigma'_s$. Thus $J(\sigma_s) \ne J(\sigma'_s)$. We may assume without loss of generality that $J(\sigma_s) \prec J(\sigma'_s)$. Since $a \in Y(\sigma)$ we may apply the first equation in Lemma 5.3 with $k = s$ to conclude that $a_{[m-s+1,m],J(\sigma_s)} \ne 0$. Since b is non-singular and upper triangular, we conclude that $x_{[m-s+1,m],J(\sigma_s)} \ne 0$. On the other hand, since $J(\sigma_s) \prec J(\sigma'_s)$, we may apply the second equation in Lemma 5.3 with $k = s$ and σ' in place of σ to conclude that $a'_{[m-s+1,m],J(\sigma_s)} = 0$. Since b is non-singular and upper triangular, we conclude that $x_{[m-s+1,m],J(\sigma_s)} = 0$. This is a contradiction. Q.E.D.

Theorem 5.5 (Cellular Decomposition of $B\backslash M_\lambda$)

$$B\backslash M_\lambda = \bigsqcup_{\substack{\sigma\in I(m,n)\\ (i,\sigma(i))\in F_\lambda, 1\le i\le m}} X(\sigma).$$

Proof. By the definition of $X(\sigma)$ in (17), Proposition 5.4 implies that

$$B\backslash M = \bigsqcup_{\sigma\in I(m,n)} X(\sigma).$$

Take $\sigma \in I(m,n)$. Suppose $(i,\sigma(i)) \in F_\lambda$ for $1 \le i \le m$. By the definition of $Y(\sigma)$, $Y(\sigma) \subset M_\lambda$. By Lemma 5.1 this implies that $BY(\sigma) \subset M_\lambda$. Thus $X(\sigma) \subset B\backslash M_\lambda$. On the other hand if for some $\sigma' \in I(m,n)$, there exists some i such that $(i,\sigma'(i)) \notin F_\lambda$ then $Y(\sigma') \cap M_\lambda = \emptyset$. Thus $X(\sigma') \cap B\backslash M_\lambda = \emptyset$. Therefore our theorem is proved. Q.E.D.

Remark. Note that the conditions $\sigma \in I(m,n)$ and $(i,\sigma(i)) \in F_\lambda$, for $1 \le i \le m$ are the same as $\sigma \in R^m_\lambda$. The formula in Theorem 5.5 is exactly the formula (18) in theorem 2.33.

6 Cell Structure of $B\backslash M_\lambda$

In this section, we show that $B\backslash M_\lambda$ is a CW-complex. We will use the terminology and results on Grassmann manifolds, Schubert cells, Schubert variety, flag manifolds and their combinatorial descriptions. To make this article more accessible we write some expositions on these topics in the appendices.

Recall that for $a \in M$ we let a_k denote the submatrix consisting of the last k rows of a. Let B_k be the k by k Borel subgroup of general linear group $G_k = GL_k(\mathbf{C})$. Then $B_k a_k$ denotes the B_k-orbit of a_k in $M^k_{k,n}$ (see (15) for definition).

Define the set of partial flags by

$$\mathcal{F}_m(\mathbf{C}^n) := \{(V_1, \cdots, V_m) \,|\, \dim V_i = i \text{ and } V_i \subset V_{i+1}, \text{ for } 1 \leq i \leq m\}$$

where each V_i is a subspace of $\mathbf{C}^n$.

Lemma 6.1 *Define a map $h : B\backslash M \to \mathcal{F}_m(\mathbf{C}^n)$ by*

$$\begin{aligned} h_k(a_k) &:= <\alpha_m(a), \alpha_{m-1}(a), \cdots, \alpha_{m-k+1}(a)> \\ h(Ba) &:= (h_1(a_1), h_2(a_2), \cdots, h_m(a_m)) \end{aligned}$$

where $\alpha_i(a)$ denotes the i-th row of a and $< v_1, \cdots, v_k >$ denotes the subspace spanned by $v_1, \cdots, v_k$. Then h is a homeomorphism.

Proof. We need to show a) h is well defined, b) h is a one to one correspondence, and c) h and its inverse are both continuous.

a) If $b \in B$, then since b is upper triangular and nonsingular,

$$<\alpha_m(b\,a), \cdots, \alpha_k(b\,a)>=<\alpha_m(a), \cdots, \alpha_k(a)>, \quad \text{for } 1 \leq k \leq m.$$

So h is well defined.

b) Let $(V_1, \cdots, V_m) \in \mathcal{F}_m(\mathbf{C}^n)$, choose $v_i \in V_{i+1} - V_i$, for $0 \leq i \leq m-1$ where we write $V_0 = (0)$. Take

$$a := \begin{pmatrix} v_m \\ \vdots \\ v_1 \end{pmatrix}.$$

Then, by definition, $h(B_m a) = (V_1, \cdots, V_m)$. So, h is onto.

On the other hand, suppose $h(B_m a) = h(B_m a')$. Use induction on $m-k$ to show that $a' = b\,a$ for some $b \in B$. First, $< \alpha_m(a) > \; = \; < \alpha_m(a') >$ implies that $\alpha_m(a') = b_{m,m}\,\alpha_m(a)$ for some $b_{m,m} \in \mathbf{C} - \{0\}$. Assume that

$$\alpha_{m-k}(a') = \sum_{j=m-k}^{m} b_{m-k,j}\,\alpha_j(a) \;\text{ for some } b_{m-k,j} \in \mathbf{C}$$

where $b_{m-k,m-k} \neq 0$. Since $h_{m-k-1}(a_{m-k-1}) = h_{m-k-1}(a'_{m-k-1})$, we have

$$<\alpha_{m-k-1}(a), \alpha_{m-k}(a), \cdots, \alpha_m(a)>=<\alpha_{m-k-1}(a'), \alpha_{m-k}(a'), \cdots, \alpha_m(a')>. \tag{44}$$

Therefore,

$$\alpha_{m-k-1}(a') = \sum_{j=m-k-1}^{m} b_{m-k-1,j}\,\alpha_j(a).$$

for some $b_{m-k-1,j} \in \mathbf{C}$ where $b_{m-k-1,m-k-1} \neq 0$. Take $b = (b_{i,j})$ with $b_{i,j} = 0$, for all $i > j$. Then $b \in B$. Hence, $a' = b\,a$, i.e., $B_m a = B_m a'$. So, h is one to one.

c) Here we show both h and h^{-1} are continuous. First we show that h is continuous. Let O be any open set in $\mathcal{F}_m(\mathbf{C}^n)$. We show that $h^{-1}(O)$ is open in $B\backslash M$. Our idea is illustrated in the following diagram.

$$\begin{array}{ccccccc} \mathbf{C}^{mn} & \supset & M & \xrightarrow{\rho_i} & M^i_{i,n} & \xrightarrow{\pi_i} & G_i(\mathbf{C}^n) \\ & & \cup & & \cup & & \cup \\ & & O'_i := \rho_i^{-1}\pi_i^{-1}(O_i) & \rightarrow & \pi_i^{-1}(O_i) & \rightarrow & O_i \end{array} \tag{45}$$

where $G_i(\mathbf{C}^n)$ denotes the Grassmannian consisting of all i-dimensional subspaces of $\mathbf{C}^n$.

Since the topology in $\mathcal{F}_m(\mathbf{C}^n)$ is inherited from the product topology in $G_1(\mathbf{C}^n) \times \cdots \times G_m(\mathbf{C}^n)$ we may assume that $O = (\prod_{i=1}^m O_i) \cap \mathcal{F}_m(\mathbf{C}^n)$ where O_i is an open set in $G_m(\mathbf{C}^n)$. Recall that $M^i_{i,n}$ is the space of $i \times n$ matrices of rank i and let $\pi_i : M^i_{i,n} \to G_i(\mathbf{C}^n)$ be the natural map which takes a matrix to its row space. Then $\pi_i^{-1}(O_i)$ is open by definition of the quotient topology in $G_i(\mathbf{C}^n)$. Recall that $\rho_i : M \to M^i_{i,n}$ is defined by $\rho_i(a) = a_i$ where, as before, a_i consists of the last i rows of a. Since ρ_i is continuous, $O'_i := \rho_i^{-1}\pi_i^{-1}(O_i)$ is open in M.

Since M consists of the matrices in $M_{m,n}(\mathbf{C})$ of maximum rank m, it is open in $M_{m,n}(\mathbf{C})$. Thus O'_i is open in $M_{m,n}(\mathbf{C})$. Write $O'_i = \cup_j O'_{ij}$ where the O'_{ij} are basic open sets. Each O'_{ij} is a direct product of mn open intervals in $\mathbf{C}$. Clearly every such open interval is the solution set of a strict inequality of the form $a < x < b$. Define $O' = \cap_{i=1}^m O'_i$. So O' is the solution set of a system of strict inequalities. In the following we call those strict inequalities with the $a_{i,j}$ involved (as the variables) the *open restriction on* $a_{i,j}$. Then O' is an open subset of $M_{m,n}(\mathbf{C})$. By definition of ρ_i,

$$\pi_i\rho_i(a) = < a_i > \subset < a_{i+1} > = \pi_{i+1}\rho_{i+1}(a) \text{ for } 1 \leq i \leq m-1. \tag{46}$$

Thus O' is a union of B-orbits in M. By definition of the quotient topology on $B\backslash M$, $B\backslash O'$ is open in $B\backslash M$.

Now we want to show that $B\backslash O' = h^{-1}(O)$ which implies that h is continuous. Recall that $h(B_m a) = (h_1(a_1), \cdots, h_m(a_m))$ and $h_k(a_k)$ is the linear span of the vectors $\alpha_m(a)$, $\alpha_{m-1}(a)$, $\cdots$, $\alpha_{m-k+1}(a)$. If $B_m x \in B\backslash O'$, then the fact that $x \in O'$ implies that $x \in O'_i$ for $1 \leq i \leq m$. then

$$h(B_m x) = (h_1(x_1), \cdots, h_m(x_m)) \in \prod_{i=1}^m O_i = O.$$

So $B\backslash O' \subset h^{-1}(O)$. Next, take $B_m x \in h^{-1}(O)$. Then $h(B_m x) \in O$. Thus, $h_i(x_i) \in O_i$, for all i. Therefore, $x \in O'$. Because of the nested condition $x_i \subset x_{i+1}$, $1 \leq i \leq m-1$, $B_m x \in B\backslash O'$. Therefore, $B\backslash O' = h^{-1}(O)$. Hence, h is continuous.

Next we show that h^{-1} is continuous. Let U be an open subset of $B\backslash M$. Then, there is an open set $U' \subset M$ such that U' is the preimage of U in M. We show that $h(U)$ is open in $\mathcal{F}_m(\mathbf{C}^n)$. Since U' is open in M, without loss of generality we may assume that U' is a basic open subset of M. That is to say that U' is a direct product of mn open intervals:

one interval for each entry of the matrix. Collect all the open restrictions on the entries in the last i rows of the matrices in U'. The collection of these open restrictions define an open set $U_i' \subset M_i$ for $1 \leq i \leq m$. For $(x_1, \cdots, x_k) \in h(U)$, there exists some $a \in U'$ such that $x_i = h_i(a_i) =< a_i >$ for $1 \leq i \leq k$. Thus $x_1 \subset \cdots \subset x_k$. By the definition of quotient topology on $G_i(\mathbf{C}^n)$, $U_i = \pi_i(U_i')$ is open for $1 \leq i \leq m$. Since $\mathcal{F}_m(\mathbf{C}^n)$ has the subspace topology inherited from $\prod_{i=1}^m G_i(\mathbf{C}^n)$, we know that $h(U) = (\prod_{i=1}^m U_i) \cap \mathcal{F}_m(\mathbf{C}^n)$ is open. Therefore, h^{-1} is continuous. Hence h is a homeomorphism. Q.E.D.

Proof of Proposition 2.31. As in Definition 2.26 , for $\sigma \in I(k,n)$, let $\sigma^* \in J(k,n)$ be obtained by rearranging the rows of σ. By Lemma 6.1 we can view $X(\sigma)$ as a subset of the direct product of Grassmannians $\prod_{i=1}^m G_i(\mathbf{C}^n)$. For each $B_m a \in X(\sigma)$, let $< a_k >$ be the k-dimensional subspace spanned by the last k rows of a, for $1 \leq k \leq m$. Then, $< a_k > \in e(\sigma_k^*)$, for $1 \leq k \leq m$. Conversely, if we have a partial flag $\{< a_i >\}_{1 \leq i \leq m}$ such that $< a_k > \in e(\sigma_k^*)$, write $< a_0 >= \{0\}$ and take α_k from $< a_k > - < a_{k-1} >$, for $1 \leq k \leq m$. Use α_k as the k-th row of a matrix, for $1 \leq k \leq m$, we get a matrix a. Then $B_m a \in X(\sigma)$. By Lemma B.1, we have a one to one correspondence Γ between the sequence of Ferrers boards of $\{\sigma_k\}_{1 \leq k \leq m}$ and the set $X(\sigma)$. The second statement follows from Corollary B.2 that in a Grassmannian, $e(\sigma^*) \subset \overline{e((\sigma')^*)}$ if and only if $F_{\lambda(\sigma^*)} \subset F_{\lambda((\sigma')^*)}$. Q.E.D.

Definition 6.2 *Let $m_1, \cdots, m_s$ and $n_1, \cdots, n_t$ be two decreasing subsequences of $[m]$ such that as two subsets of $[m]$, $[m] = \{m_1, \cdots, m_s\} \sqcup \{n_1, \cdots, n_t\}$ disjoint union. We call such a pair of subsequences a (s,t)-pair.*

In this definition, either s or t may be 0.

Lemma 6.3 *Let $(\{m_i\}_{i=1}^s, \{n_i\}_{i=1}^t)$ and $(\{m_i'\}_{i=1}^s, \{n_i'\}_{i=1}^t)$ be two (s,t)-pairs of $[m]$ such that $n_i \leq n_i'$. If we insert m_s into the proper position in $n_1, \cdots, n_t$ to make a $(s-1, t+1)$-pair $(\{m_i\}_{i=1}^{s-1}, \{n_i^*\}_{i=1}^{t+1})$ and at the same time we insert m_s' into the proper position in $n_1', \cdots, n_t'$ to make another $(s-1, t+1)$-pair $(\{m_i'\}_{i=1}^{s-1}, \{(n')_i^*\}_{i=1}^{t+1})$, then $n_i^* \leq (n')_i^*$, for $1 \leq i \leq t+1$.*

Proof. Clearly, the only nontrivial case is when $m_s > m_s'$. Suppose m_s is inserted at the i-th place so that $n_i^* = m_s$. Then,

$$m_s = m - (i-1) - (s-1) = n - i - s + 2.$$

For the same i, since $m_s > m_s'$, we have $(n')_i^* = n_i' \geq m - (s-1) - (i-1) = m_s$. Assume $m_s' = (n')_k^*$, $k > i$. Then, for each l such that $i < l < k$, $n_l = n_{l+1}^*$. Therefore, $n_{l+1}^* = m - (s-1) - l \leq (n')_{l+1}^*$, as $(n')_{l+1}^* > m_s'$. The case when $l \geq k$ is trivial. Hence the lemma is proved. Q.E.D.

Proof of Theorem 2.32. The second statement is exactly the type A case of Chevalley's theorem on Chevalley-Bruhat order. There is a combinatorial proof in Macdonald [22, page 7-8]

Now we prove a). Assume $X(P(\sigma)) \subseteq \overline{X(P(\sigma'))}$. By Lemma 2.31, $F_{\lambda(P(\sigma)_i)} \subseteq F_{\lambda(P(\sigma')_i)}$, for $1 \leq i \leq n$. In particular, we have $F_{\lambda(\sigma_i)} = F_{\lambda(P(\sigma)_i)} \subseteq F_{\lambda(P(\sigma')_i)} = F_{\lambda(\sigma_i')}$, for $1 \leq i \leq m$. By Lemma 2.31 we have $X(\sigma) \subseteq \overline{X(\sigma')}$.

Assume $X(\sigma) \subseteq \overline{X(\sigma')}$. We use induction on $n-m$ to show $X(P(\sigma)) \subseteq \overline{X(P(\sigma'))}$. If $n-m=0$, there is nothing to prove. If $n-m=1$, then by Lemma 2.31,

$$F_\lambda(\sigma_k) \subseteq F_\lambda(\sigma'_k), \text{ for } 1 \leq k \leq n-1.$$

That means $\sigma_k^*(i) \geq (\sigma')_k^*(i)$, for each i. By Definition 2.6,

$$(\{P(\sigma(1))\}, \{\sigma_k^*(i)\}_{i=1}^{n-1}) \text{ and } (\{P(\sigma'(1))\}, \{(\sigma')_k^*(i)\}_{i=1}^{n-1})$$

are two $(i, n-1)$-pairs such that $\sigma_k^*(i) \geq (\sigma')_k^*(i)$, for each i. By Lemma 6.3, we know that $(P(\sigma))_k^*(i) \geq (P(\sigma'))_k^*(i)$ for $1 \leq i \leq n$. Thus $X(P(\sigma)) \subseteq \overline{X(P(\sigma'))}$. Assume that $X(\sigma) \subseteq X(\sigma')$ implies $X(P(\sigma)) \subseteq \overline{X(P(\sigma'))}$ for $n-m=j$. When $n-m=j+1$, repeat the argument above word by word, and use Lemma 6.3 again. We get $X(P(\sigma)) \subseteq \overline{X(P(\sigma'))}$ from $X(\sigma) \subseteq X(\sigma')$ for the $n-m=j+1$ case. Q.E.D.

Corollary 6.4 *If $X(\sigma) \cap \overline{X(\sigma')} \neq \emptyset$, then $X(\sigma) \subseteq \overline{X(\sigma')}$.*

Proof. Take $B_m a \in X(\sigma) \cap \overline{X(\sigma')}$. Then $B_i a_i$ has a matrix representative as the matrix (59) in Lemma B.1 with respect to $(\sigma_i)^*$ and a matrix representative as the matrix (60) in Lemma B.1 with respect to $(\sigma'_i)^*$ for $1 \leq i \leq m$. Therefore, $F_\lambda(\sigma_k) \subseteq F_\lambda(\sigma'_k)$ for $1 \leq k \leq m$. So $X(\sigma) \subseteq \overline{X(\sigma')}$. Q.E.D.

Now we are ready to prove the main theorem 2.33 of this chapter. We quote the statement as follows.

Main Theorem

a) $B\backslash M_\lambda$ is a CW-complex consisting of all the cells of $B\backslash M$ which fit into the board F_λ.

b)The Poincaré polynomial homology (and the cohomology) of the partition variety $B\backslash M_\lambda$ with real coefficients is given by

$$Poin(B\backslash M_\lambda, \mathbf{R}) = Poin(B\backslash M_\lambda, \mathbf{R}, q) = RL_m(\lambda, q^2). \tag{47}$$

Proof. By Lemma 2.32 the formula in Theorem 5.5 can be rewritten as

$$B\backslash M_\lambda = \bigsqcup_{\sigma \in I(m,n), F_{\lambda(\sigma)} \subset F_\lambda} X(\sigma).$$

To show that this is a CW-complex, we need only to prove that for each $B_m a \in \overline{X(\sigma)} - X(\sigma) \subseteq B\backslash M_\lambda$, there exists some cell $X(\sigma') \subseteq B\backslash M_\lambda$ of lower dimension such that $B_m a \in X(\sigma')$. Since $B_m a \in \overline{X(\sigma)} \subseteq B\backslash M_\lambda$, there is a unique $X(\sigma')$ containing $B_m a$. By the last corollary, $X(\sigma') \subset \overline{X(\sigma)}$. So $F_{\lambda(\sigma'_i)} \subseteq F_{\lambda(\sigma_i)}$ for all i. But $B_m a \notin X(\sigma)$ implies that there is at least one i such that the inclusion is strict, i.e. $F_{\lambda(\sigma'_i)} \subset F_{\lambda(\sigma_i)}$ but $F_{\lambda(\sigma'_i)} \neq F_{\lambda(\sigma_i)}$. This means that $X(\sigma') \subset B\backslash M_\lambda$ and $P(\sigma') < P(\sigma)$ in the Bruhat order of W_n. Therefore,

$$dim_{\mathbf{C}} X(\sigma') = l(P(\sigma')) < l(P(\sigma)) = dim_{\mathbf{C}} X(\sigma).$$

So we have proved that $B\backslash M_\lambda$ is a CW-complex.

Since this CW-complex structure is over the complex field, viewed as two dimensional real space, all the cells are of even dimensions. So the boundary operator ∂ is zero. (This is a standard way to compute cohomology of complex manifold with cellular decomposition. See for example [24, page 232] for the simplest case of $\mathbf{P}^n$.) Thus, the homology and cohomology groups of $B\backslash M_\lambda$ are freely generated by its cells. Hence, we have $Poin_{\mathbf{R}}(B\backslash M_\lambda, q) = RL_m(\lambda, q^2)$. Q.E.D.

APPENDIX

A Schubert Cells in Grassmann Manifold

In this section we give an exposition on the Schubert cells and Schubert varieties in a Grassmann manifold. The results are all known. The structure is an complex analogue to that in [23, Section 6] which is over the real field.

Let m and n be two positive integers such that $m \leq n$. As before let $G = GL_m(\mathbf{C})$. Let $G_m(\mathbf{C}^n)$ be the set of all m-dimensional subspaces of $\mathbf{C}^n$. Let V be an m-dimensional subspace of $\mathbf{C}^n$. Then V can be written as a linear span of m linearly independent row vectors $\{\alpha_i\}_{1\leq i\leq m}$ in $\mathbf{C}^n$. Using α_i as the i-th row for $1 \leq i \leq m$ we get an m by n matrix a of rank m. Let $<a>$ denote the row space of a. Note that for $a, a' \in M$, $<a>=<a'>$ if and only if a and a' have the same G-orbit $[a] := G\,a = G\,a' = [a']$. Define a mapping $<a>\mapsto [a]$. This is a one to one correspondence between $G_m(\mathbf{C}^n)$ and the quotient space $G\backslash M = \{G\,a \,|\, a \in M\}$ where M is the set of all m by n matrices over $\mathbf{C}$ of rank m. The quotient space $G\backslash M$ has the quotient topology inherited from the natural topology of the space M. We transport this topology to $G_m(\mathbf{C}^n)$. Thus $G_m(\mathbf{C}^n)$ is a complex manifold, called the Grassmann manifold. In the following, for a fixed $a \in M$ we identify the row space $<a>$ with the G-orbit $[a] = G\,a$. With this identification in our mind we will use these two notations in the rest of this paper whichever is convenient. Further, we identify the Grassmann manifold with the quotient space $G\backslash M = \{G\,a \,|\, a \in M\}$. Thus we consider an element of the Grassmann manifold $G_m(\mathbf{C}^n)$ either as a G-orbit of some matrix $a \in M$ or as an m-dimensional subspace of $\mathbf{C}^n$, whichever is convenient. Let

$$\mathbf{C}^k = \{(\underbrace{0, \cdots, 0}_{n-k}, z_k, \cdots, z_1) \in \mathbf{C}^n \,|\, z_i \in \mathbf{C}\}, \text{ for } 1 \leq k \leq n.$$

Remark. We call this special sequence of subspaces the standard flag. This definition is "backward"; one might prefer

$$\mathbf{C}^k = \{(z_1, \cdots, z_k, \underbrace{0, \cdots, 0}_{n-k}) \in \mathbf{C}^n \,|\, z_i \in \mathbf{C}\}.$$

The reason for us to use such a "backward" definition is that our Ferrers boards have to be right justified to include the parabolic boards as special cases. For the same reason, our row eliminations and Gram-Schmidt process are all from bottom to the top.

If $[a] \in G_m(\mathbf{C}^n)$, then,

$$0 = \dim([a] \cap \mathbf{C}^0) \leq \dim([a] \cap \mathbf{C}^1) \leq \cdots \leq \dim([a] \cap \mathbf{C}^n) = m.$$

Note that any two consecutive integers in this sequence differ by at most 1. Thus, there are exactly m jumps in the sequence. We collect all the $[a]$'s which jump at the same places.

Definition A.1 (Schubert Cell) *For $\sigma \in J(m,n)$, let*

$$e(\sigma) = \{[a] \in G_m(\mathbf{C}^n) \mid \; dim([a] \cap \mathbf{C}^{n-\sigma(i)}) < \; dim([a] \cap \mathbf{C}^{n-\sigma(i)+1}), 1 \le i \le m\}.$$

We call $e(\sigma)$ the Schubert cell of σ [23, Section 6].

For example, if $\sigma = \binom{100}{001}$, then, $e(\sigma)$ is the set of all $[a] \in G_2(\mathbf{C}^3)$ such that $\dim([a] \cap \mathbf{C}^1) = \dim([a] \cap \mathbf{C}^2) = 1$, and $\dim([a] \cap \mathbf{C}^3) = 2$. In general, we have

$$\dim([a] \cap \mathbf{C}^{n-\sigma(m-i+1)+1}) = i$$

and

$$\dim([a] \cap \mathbf{C}^{n-\sigma(m-i+1)}) = i - 1$$

for $1 \le i \le m$. Clearly, we have a partition

$$G_m(\mathbf{C}^n) = \bigsqcup_{\sigma \in J(m,n)} e(\sigma).$$

Next, we give a description of the Grassmannian $G_m(\mathbf{C}^n)$ in terms of orthonormal bases and use it to describe the closure $\overline{e(\sigma)}$ of the Schubert cell $e(\sigma)$. This description was given in [23] over the field $\mathbf{R}$. In the following we call m-tuple of orthonormal vectors in $\mathbf{C}^n$ an orthonormal m-frame (see [20]). Let Σ_m be the set of orthonormal m-frames in $\mathbf{C}^n$ [20, page 13]. Let

$$H^k = \{(\underbrace{0, \cdots, 0}_{n-k} z_k, \cdots, z_1) \mid z_i \in \mathbf{C}, \; \text{for } 1 \le i < k, z_k \in \mathbf{R}, z_k > 0\} \subset \mathbf{C}^k. \tag{48}$$

For $\sigma \in J(m,n)$, give

$$H^\sigma := H^{n-\sigma(1)+1} \times \cdots \times H^{n-\sigma(m)+1} \subset \underbrace{\mathbf{C}^n \times \cdots \times \mathbf{C}^n}_{m} \tag{49}$$

the subspace topology obtained from the product topology on $\mathbf{C}^n \times \cdots \times \mathbf{C}^n$.

Lemma A.2 *For any $x \in e(\sigma)$ with $\sigma \in J(m,n)$, there is a unique orthonormal m-frame $(v_1, \cdots, v_m)$ for x which is in $H^{n-\sigma(1)+1} \times \cdots \times H^{n-\sigma(m)+1}$. Let Ψ denote the mapping defined by $x \mapsto (v_1, \cdots, v_m)$.*

Proof. Since $x \in e(\sigma)$, there is an ordered basis $(x_1, \cdots, x_m)$ of x such that the first $\sigma(i) - 1$ components (from the left)of x_i are all zeros and the $\sigma(i)$-th component (from the left) is non-zero, for $1 \le i \le m$. Let a be the matrix with x_i as its i-th row. Starting from x_m, do the Gram-Schmidt orthonormalization to the rows of a, from bottom to the top as follows: Divide the m-th row by its norm and then multiply it by $e^{-i\theta}$ where θ $(0 \le \theta < 2\pi)$ is the argument of its $\sigma(m)$-th component (from the left). Then, we get a vector v_m which is a unit vector with its $\sigma(m)$-th component (from the left) being a positive number, i.e., $v_m \in H^{n-\sigma(m)+1}$. Then, let

$$v'_{m-1} = x_{m-1} - < x_{m-1}, v_m > v_m.$$

Let

$$v_{m-1} = \frac{v'_{m-1}}{||v'_{m-1}||} exp(-i\, Arg((v'_{m-1})_{n-\sigma(m-1)+1}))$$

where $(v'_{m-1})_{n-\sigma(m-1)+1}$ is the $\sigma(m-1)$-th component (from the left) of v'_{m-1}. Then $v_{m-1} \in H^{n-\sigma(m-1)+1}$ and $v_{m-1} \perp v_m$. In general, for $k \geq 2$, let

$$\begin{aligned} v'_{m-k} &:= x_{m-k} - \sum_{j=0}^{k-1} < x_{m-k}, v_{m-j} > v_{m-j} && (50) \\ v_{m-k} &:= \frac{v'_{m-k}}{||v'_{m-k}||} exp(-i\, Arg((v'_{m-k})_{n-\sigma(m-k)+1})). && (51) \end{aligned}$$

Therefore, $v_{m-k} \in H^{n-\sigma(m-k)+1}$ and $v_{m-k} \perp v_{m-j}$, for $0 \leq j \leq k-1$. The construction determines the uniqueness, clearly. Q.E.D.

Remark A.3 Note that the Gram-Schmidt orthonormalization applied to the rows of a is done by multiplication of a on the left by an upper triangular matrix whose entries are rational functions of the entries of a with all the denominators nonzero. Therefore these rational functions are continuous.

Let

$$e'(\sigma) = \Sigma_m \cap H^{n-\sigma(1)+1} \times \cdots \times H^{n-\sigma(m)+1}. \tag{52}$$

In Lemma A.5, we will prove that

$$\begin{aligned} \overline{e'(\sigma)} &:= \Sigma_m \cap \overline{H^{n-\sigma(1)+1} \times \cdots \times H^{n-\sigma(m)+1}} \\ &= \Sigma_m \cap \overline{H^{n-\sigma(1)+1}} \times \cdots \times \overline{H^{n-\sigma(m)+1}} && (53) \end{aligned}$$

where

$$\overline{H^k} = \{(\underbrace{0, \cdots, 0}_{n-k} z_k, \cdots, z_1) \,|\, z_i \in \mathbf{C},\ i < k,\ z_k \in \mathbf{R},\ z_k \geq 0\}.$$

Note that z_k may be zero. We will prove that this is another description of Schubert cells. The reason for us to use this description of Schubert cells is that the closure of $e'(\sigma)$ has an interesting combinatorial background(see Appendix B).

In the following we will prove a lemma on Schubert cells (Lemma A.5) which is a complex analogue of Lemma 6.3 (page. 76) in [23]. To do this we need to introduce some notation and a technical lemma. Let

$$\begin{aligned} \Delta^i &= \{x \in \overline{H^i} \mid ||x|| = 1\} && (54) \\ b_i &= (0, \cdots, 0, 1, \underbrace{0, \cdots, 0}_{n-\sigma(i)}) && (55) \end{aligned}$$

where H^i is defined in Formula (48) and $||x|| =< x, x >$. For $\sigma \in I(m,n)$ and $i > 1$, define

$$D(x_i, \cdots, x_m) = \{x \in \Delta^{n-\sigma(i-1)+1} | \, x \perp x_j, i \leq j \leq m\}. \tag{56}$$

Then Δ^k is a closed unit disk of complex dimension $k-1$ with center at the origin corresponding to $z_k = 1$ and boundary corresponding to $z_k = 0$. In particular, if $k = 1$, then Δ^k is a point. If $u, v \in \mathbf{C}^n$ are two unit vectors and $< u, v > \neq -1$, define a $\mathbf{C}$ -linear map $T(u,v) : \mathbf{C}^n \to \mathbf{C}^n$ by

$$T(u,v)x = x - \frac{< x, \, u+v >}{1+ < u, v >} (u+v) \, + 2 < x, u > v.$$

Then $T(u,v)$ has the following properties:

Lemma A.4

P1: The mapping $(u,v,x) \mapsto T(u,v)x$ *is continuous in the three variables* $u, v, \, x$.

P2: If $u, \, v \in \mathbf{C}^k$ *then* $T(u,v)x \equiv x \; mod \, \mathbf{C}^k$.

P3: $T(u,u) = identity$.

P4: If $x \perp u$ *and* $x \perp v$ *then* $T(u,v)x = x$.

P5: $T(u,v)u = v$.

P6: $T(u,v)$ *is a unitary operator, i.e.,* $< T(u,v)x, T(u,v)y > \, = \, < x, y >$, *for all* $x, y \in \mathbf{C}^n$.

P7: $T(u,v)^{-1} = T(v,u)$ *if* $< u, v > \in \mathbf{R}$.

Proof. P1 to P5 are immediate from the definition of T. We prove P6. For any $x, y \in \mathbf{C}^n$,

$$\begin{aligned}
<T(u,v)x, T(u,v)y> \; &= \; <x,y> + \frac{<x, u+v> \overline{<y, u+v>}}{(1+ <u,v>)(1+\overline{<u,v>})} <u+v, u+v> \\
&+ \; 4 <x,u> \overline{<y,u>} <v,v> - \frac{\overline{<y, u+v>}}{1+\overline{<u,v>}} <x, u+v> \\
&- \; \frac{<x, u+v>}{1+<u,v>} <u+v, y> + 2<x,u><v,y> + 2\overline{<y,u>} <x,v> \\
&- \; \frac{<x, u+v>}{1+ <u,v>} 2\overline{<y,u>} <u+v, v> \\
&- \; \frac{\overline{<y, u+v>}}{1+\overline{<u,v>}} 2 <x,u><v, u+v> .
\end{aligned}$$

Since $< a, b >= \overline{< b, a >}$ and $< u, u >=< v, v >= 1$, we have

$$\begin{aligned}
< T(u,v)x, T(u,v)y > \; &= \; < x, y > + \frac{< x, u+v >< u+v, y >}{(1+ < u, v >)(1+ < v, u >)} (2+ < u, v > + < v, u >) \\
&+ \; 4 < x, u >< u, y > - \frac{< u+v, y >< x, u+v >}{1+ < v, u >} \\
&- \; \frac{< x, u+v >< u+v, y >}{1+ < u, v >} \\
&+ \; 2 < x, u >< v, y > + 2 < u, y >< x, v > \\
&- \; < x, u+v > 2 < u, y > - < u+v, y > 2 < x, u >
\end{aligned}$$

Clearly, the 4-th and the 5-th term cancel the 2nd term, and the last two terms cancel the 3rd, the 6-th and the 7-th term. Hence,

$$< T(u,v)x, T(u,v)y > \; = \; < x, y > .$$

This proves $P\,6$.

We show P7. Let u, v be unit vectors in $\mathbf{C}^n$ which are linearly independent. Let V_1 be the linear span of u and v. Since $\mathbf{C}^n = V_1 \oplus V_1^{\perp}$ and both $T(u,v)$ and $T(v,u)$ fix $V_1^{\perp}$, it is enough to show that

$$T(u,v)\,T(v,u)\,w = w, \text{ for } w \in \{u, v\}$$

This is clear for $w = v$ by P5. Suppose $w = u$, since $T(v,u)v = u$, the left hand side equals

$$\begin{aligned}
T(u,v)\,(T(v,u)u) &= T(u,v)(u - \frac{<u, u+v>}{1+<u,v>}(u+v) + 2<u,v>u) \\
&= T(u,v)(-v + 2 < u, v > u) \\
&= (-1)(v - \frac{< v, u+v >}{1+ < u, v >}(u+v) + 2 < v, u > v) \\
&\quad +2 < u, v > v \\
&= (-1)\,v + \frac{1+ < v, u >}{1+ < u, v >}(u+v) - 2 \; < v, u > v \\
&\quad +2 < u, v > v \\
&= u.
\end{aligned}$$

Therefore, P7 is proved. Q.E.D.

Lemma A.5 *a) Let* $e^*(\sigma) := \Sigma_m \cap \prod_{i=1}^m \overline{H^{n-\sigma(i)+1}}$*. Then* $e^*(\sigma)$ *is homeomorphic to a direct product* $\prod_{i=1}^m D_i$*, where* $D_i = D(b_i, \cdots, b_m) \subset \mathbf{C}^n$ *is a closed unit disk of dimension* $(n - \sigma(i) + 1) - (m - i + 1)$ *for* $1 \leq i \leq m$*. In addition,* $e^*(\sigma) = \overline{e'(\sigma)}$ *and* $int(e^*(\sigma)) = e'(\sigma)$*.*
b) Let $h : \Sigma_m \to G_m(\mathbf{C}^n)$ *be the map defined by* $h(x_1, \cdots, x_m) = <x_1, \cdots, x_m>$*. Then* h *is a homeomorphism from* $e'(\sigma)$ *onto* $e(\sigma)$*.*

Proof of Lemma A.5. For any subset $S \subset \mathbf{C}^n$, let ∂S denote the boundary of S in $\mathbf{C}^n$. Our proof consists of four steps:

1. $e^*(\sigma)$ is homeomorphic to a direct product of closed disks such that $e'(\sigma)$ corresponds to the interior. Since $e^*(\sigma)$ is closed, $e^*(\sigma) = \overline{e'(\sigma)}$.

2. $h(\partial e'(\sigma)) \cap e(\sigma) = \emptyset$.

3. h is continuous.

4. h maps each relatively closed subset of $e'(\sigma)$ onto a relatively closed subset of $e(\sigma)$.

Step 1. Recall from Formula (48) that $x \in H^k$ means

$$x = (0, \cdots, 0, z_k, \ldots, z_2, z_1)$$

where $z_i \in \mathbf{C}$, $z_k \in \mathbf{R}$, and $z_k > 0$. By definition,

$$e^*(\sigma) = \{(x_1, \cdots, x_m) \in \prod_{i=1}^{m} \Delta^{n-\sigma(i)+1} \mid x_i \perp x_j \text{ for } i \neq j\}.$$

We use induction on m to show that $e^*(\sigma)$ is homeomorphic to a product $\prod_{i=1}^{m} D_i$, where $D_i = D(b_i, \cdots, b_m)$ is a closed unit disk of dimension $(n - \sigma(i) + 1) - (m - i + 1)$. When $m = 1$, the orthogonality condition is vacuous. Thus, $e^*(\sigma) = e^*(\sigma(1)) = \Delta^{n-\sigma(1)+1}$ is a closed unit disk of complex dimension $n - \sigma(1) + 1 - 1$.

The main idea of the induction is contained in the case $m = 2$ which we do separately for clarity. For $x_2 \in \Delta^{n-\sigma(2)+1}$ we have $D(x_2) = \{x_1 \in \Delta^{n-\sigma(1)+1} \mid x_1 \perp x_2\}$. Then

$$e^*(\sigma) = \bigcup_{x_2 \in \Delta^{n-\sigma(2)+1}} D(x_2) \times \{x_2\}.$$

Since $D(x_2)$ is the intersection of $\Delta^{n-\sigma(1)+1}$ with the hyperplane orthogonal to x_2, it is a closed unit disk of complex dimension $n - \sigma(1)$. Let $b_2 = (0, \cdots, 0, 1, \underbrace{0, \cdots, 0}_{n-\sigma(2)})$. We will prove that $e^*(\sigma)$ is homeomorphic to $D(b_2) \times \Delta^{n-\sigma(2)+1}$. Since $< x_2, b_2 > = z_{2,n-\sigma(2)+1} \neq -1$, the map $T(x_2, b_2)$ is well defined. Also, $< x_2, b_2 > \in \mathbf{R}$, so we may apply P7 of Lemma A.4. Let $f : e^*(\sigma) \to D(b_2) \times \Delta^{n-\sigma(2)+1}$ be the map defined by

$$f(x_1, x_2) = (T(x_2, b_2)x_1, x_2).$$

Note that for $x_1 \in D(x_2)$, by P5 and P6,

$$< T(x_2, b_2)x_1, b_2 > = < T(x_2, b_2)x_1, T(x_2, b_2)x_2 > = < x_1, x_2 >= 0.$$

So $T(x_2, b_2)(D(x_2)) \subseteq D(b_2)$. Thus, $f(e^*(\sigma)) \subset D_1(b_2) \times D_2$. Similarly, we define $g : D(b_2) \times \Delta^{n-\sigma(2)+1} \to e^*(\sigma)$ by $g(u, v) = (T(b_2, v)u, v)$. Then, we need to check that

i) $T(b_2, v)u \perp v$, and

ii) $||T(b_2, v)u|| = 1$.

Item ii) is clear since we know that $||u|| = 1$ and $T(b_2, v)$ is unitary. As for item i), we have

$$< T(b_2, v)u, v > = < T(b_2, v)u, T(b_2, v)b_2 > = < u, b_2 > = 0.$$

Next, we show that $f \circ g =$ identity. By P7,

$$(f \circ g)(u, v) = f(T(b_2, v)u, v) = (T(v, b_2)T(b_2, v)u, v) = (u, v). \tag{57}$$

Similarly, $g \circ f =$ identity. Thus, f is bijective and $g = f^{-1}$. Since $(x_1, x_2) \mapsto T(x_2, b_2)x_1$ is continuous, f is continuous. Similarly, f^{-1} is continuous and hence f is a homeomorphism from $e^*(\sigma)$ onto $D(b_2) \times \Delta^{n-\sigma(2)+1}$ where $D(b_2)$ is a closed disk of dimension $n-\sigma(1)-1$ and $\Delta^{n-\sigma(2)+1}$ is a closed disk of dimension $n-\sigma(2)$. Thus, the first statement of part a) of the lemma is proved in case $m=2$. By P6, if $f(x_1,x_2)=(u,v)$, then $||x_1||=||u||$ and $||x_2||=||v||$. If $(x_1,x_2) \in int\, e^*(\sigma)$ with

$$x_i = (0,\cdots,0,z_{i,n-\sigma(i)+1},z_{i,n-\sigma(i)},\cdots,z_{i,1}), \text{ for } i=1,2.$$

then $z_{1,n-\sigma(1)+1} > 0$ and $z_{2,n-\sigma(2)+1} > 0$. Note that $x_2, b_2 \in \mathbf{C}^{n-\sigma(2)+1}$. By P2, $u = T(x_2,b_2)x_1 \equiv x_1 \bmod \mathbf{C}^{n-\sigma(2)+1}$. Thus, the $\sigma(1)$-th component of u (from the left) is positive. Since $v=x_2$, the $\sigma(2)$-th component of u (from the left) is $z_{2,n-\sigma(2)+1}>0$. Therefore,

$$f(int\, e^*(\sigma)) = int\, D(b_2) \times D_2 = f(e'(\sigma)).$$

Since f is bijective, $e'(\sigma)$ is the interior of $e^*(\sigma)$. Since $e^*(\sigma)$ is closed, it is the closure of $e'(\sigma)$.

Now, we consider the general case $m \geq 2$. Recall that σ is an injective mapping from $[m]$ into $[n]$. Write the restriction of σ to $[2,m]$ as $\sigma|_{[2,m]}$. Let

$$e^*(\sigma|_{[2,m]}) := \Sigma_{m-1} \cap \prod_{i=2}^{m} \overline{H^{n-\sigma(i)+1}}.$$

Assume that $e^*(\sigma|_{[2,m]})$ is homeomorphic to a product $\prod_{i=2}^m D_i$ where D_i is a closed unit disk of dimension $(n-\sigma(i)+1)-(m-i+1)$. Write the homeomorphism as ζ. For $(x_2,\cdots,x_m) \in \zeta(e^*(\sigma|_{[2,m]})) = \prod_{i=2}^m D_i$, let

$$D(x_2,\cdots,x_m) = \{x_1 \in \Delta^{n-\sigma(1)+1} \mid x_1 \perp x_i, 2 \leq i \leq m\}.$$

Then

$$e^*(\sigma) = \bigcup_{(x_2,\cdots,x_m)\in e^*(\sigma|_{[2,m]})} D(x_2,\cdots,x_m) \times \{(x_2,\cdots,x_m)\}.$$

Note that when $m=2$,

$$e^*(\sigma|_{[2,2]}) = e^*(\sigma(2)) = \Delta^{n-\sigma(2)+1}$$

as in the case $m=2$. Define

$$T_m := T(x_2,b_2)\cdots T(x_m,b_m). \tag{58}$$

Since $b_i \perp b_j$ and $x_i \perp x_j$, for $1 \leq i \neq j \leq m$, T_m maps x_i to b_i for $2 \leq i \leq m$ and fixes all vectors orthogonal to $x_2,\cdots,x_m,b_2,\cdots,b_m$ by P4. Define $f_m : e^*(\sigma) \to D(b_2,\cdots,b_m) \times \zeta(e^*(\sigma|_{[2,m]}))$ by

$$f_m(x_1,\cdots,x_m) = (T_m(x_2,\cdots,x_m)x_1, x_2,\cdots,x_m).$$

where $D := D(b_2,\cdots,b_m) = \{u \in \Delta^{n-(\sigma(1)+1}|u \perp b_i, 2 \leq i \leq m\}$. Here, we need to verify that for $x_1 \in D_1(x_2,\cdots,x_m)$, $T_m x_1 \in D$. In fact,

$$< T_m x_1, b_i > = < T_m x_1, T_m x_i > = < x_1, x_i > = 0,$$

for $i \geq 2$. Exactly as in the case $m = 2$, we can verify that f_m is onto. Since $b_2, \cdots, b_m, x_2, \cdots, x_m \in \mathbf{C}^{n-\sigma(2)+1}$, we know, by P 2, that the $\sigma(1)$-th coordinate (from the left) of $T_m x_1$ is the same as that of x_1. For the same reason, all coordinates to the left of the $\sigma(1)$-th coordinate (from the left) of $T_m x_1$ are zero, i.e., we have

$$(0, \cdots, 0, z_{n-\sigma(1)+1}, z_{n-\sigma(1)}, \cdots, z_1).$$

The formula (58) for T_m and P4 imply that f_m has a continuous inverse. Thus f_m is a homeomorphism. Note that the boundary of $\Delta^{n-\sigma(i)+1}$ corresponds to $z_{i,n-\sigma(i)+1} = 0$ and the definition of D implies that $\partial D \subseteq \partial \Delta^{n-\sigma(1)+1}$. By the definition of $e'(\sigma)$ (see formula (52) and the description of $e^*(\sigma)$ in Lemma A.5,

$$e^*(\sigma) = e'(\sigma) \cup \partial e^*(\sigma).$$

Since $f(e^*(\sigma)) = D_1 \cdots D_m$,

$$D_1 \cdots D_m = f_m(e'(\sigma)) \cup f_m(\partial e^*(\sigma)).$$

Since f_m is a homeomorphism, $int(D_1 \cdots D_m) = f(int(e^*(\sigma))$. Note that f_m and f_m^{-1} are both one to one. Thus $f_m(\partial e^*(\sigma))$ is the boundary of the product $D_1 \cdots D_m$. Since T_k is unitary, for each k, by P 2 of Lemma A.4, $f_m(\partial e^*(\sigma)) = \partial(D_1 \cdots D_m)$. Moreover, since f_m is bijective, again by P 2 of Lemma A.4, the same homeomorphism f_m maps $e'(\sigma)$ onto the interior of the product of closed unit disks. Since $e^*(\sigma)$ is closed, $e^*(\sigma)$ is the closure of $e'(\sigma)$.
Step 2. Let $(x_1, \cdots, x_m) \in \partial e'(\sigma)$. Then, there is some i such that the leftmost nonzero entry in x_i is z_j with $j < \sigma(i)$. Let

$$x := h(x_1, \cdots, x_m) = < x_1, \cdots, x_m > .$$

Then, $\dim(x \cap \mathbf{C}^{\sigma(i)-1}) \geq m - i$. So $x = h(x_1, \cdots, x_m)$ is not in $e(\sigma)$.
Step 3. By the uniqueness of the orthonormal frame $x_1, \cdots, x_m$ in each m-dimensional subspace of $\mathbf{C}^n$, h is a one to one correspondence. We show that h is continuous. Take any relatively open set $O \subset e(\sigma)$. Since the topology is defined by the quotient topology in $G\backslash M$, there is an open set $O' \subset E(\sigma) \subset M$ such that $O = G\backslash O'$ where $E(\sigma)$ is the pre-image of $e(\sigma)$ under the projection π. Note that for each $x \in O$, x is an G-orbit of some matrix $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ which we identify with $(x_1, \cdots, x_m) \in \Sigma_m$. By the definition of subspace topology, $h^{-1}(O) = O' \cap \Sigma \cap H^\sigma$ is open in $e'(\sigma)$. Hence, h is continuous.
Step 4. Suppose $A \subset e'(\sigma)$ is relatively closed. Then, $\overline{A} \cap e'(\sigma) = A$ where $\overline{A} \subset \overline{e'(\sigma)}$ is compact, hence, $h(\overline{A})$ is closed as h is continuous by Step 3. By Step 2, $h(\overline{A}) \cap e(\sigma) = h(A)$, it follows that $h(A) \subset e(\sigma)$ is relatively closed.

Combining the results in 3 and 4, we know that h is a homeomorphism. Thus Lemma A.5 is proved. Q.E.D.

Lemma A.6 *$\overline{e'(\sigma)}$ is homeomorphic to $\overline{e(\sigma)}$.*

Proof. Let $\pi : M \to G\backslash M$ be the canonical map defined by $a \mapsto [a] = Ga$ for each $a \in M$. Let $E(\sigma) := \pi^{-1}(e(\sigma))$. First, we want to show that the map $h : \Sigma_m \to G_m(\mathbf{C}^n)$ defined by

$$h(x_1, \cdots, x_m) = < x_1, \cdots, x_m >$$

is continuous. Let U be an open subset of $G_m(\mathbf{C}^n) = G\backslash M$. There is a open subset $U_1 \subset M$ such that $U = G\backslash U_1$. For each $x \in U$, x is the $GL_m(\mathbf{C})$-orbit of some matrix $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in U_1$ of rank m. Identify such a matrix with the tuple $(x_1, \cdots, x_m) \in \Sigma_m$. By the definition of subspace topology on $\Sigma_m \cap (\cup_{\sigma \in I(m,n)} H^\sigma)$, $h^{-1}(U) = U_1 \cap \Sigma_m \cap (\cup_{\sigma \in I(m,n)} H^\sigma)$ is open in $\Sigma_m \cup_{\sigma \in I(m,n)} H^\sigma$. Thus h is continuous.

Next, we show that $h(\overline{e'(\sigma)}) = \overline{e(\sigma)}$. Note that h is continuous and $\overline{e'(\sigma)}$ is compact. So $h(\overline{e'(\sigma)})$ is closed in $G_m(\mathbf{C}^n)$ containing $e(\sigma)$. Thus, $\overline{e(\sigma)} \subset h(\overline{e'(\sigma)})$. On the other hand, $h^{-1}(\overline{e(\sigma)})$ is closed containing $e'(\sigma)$. This implies that $h(\overline{e'(\sigma)}) \subset \overline{e(\sigma)}$. Hence, we have $h(\overline{e'(\sigma)}) = \overline{e(\sigma)}$.

Finally, we show that h maps each closed subset of $\overline{e'(\sigma)}$ to a closed subset of $\overline{e(\sigma)}$. Let A be a closed subset of $\overline{e'(\sigma)}$. Then A is compact. By the equality proved in the previous paragraph, $h(A)$ is a closed subset of $\overline{e(\sigma)}$. Q.E.D.

APPENDIX

B Combinatorial Description of Schubert Varieties in Grassmann Manifold

The results in this appendix are used in Section 6. These results are all known. But I do not find a suitable exposition in the literature for our application. So I write this appendix.

Lemma A.6 enables us to describe the closure of a Schubert cell combinatorially. Such a closure is called a Schubert variety. Let U be a subset of the Grassmann manifold $G_m(\mathbf{C}^n)$. We say that U has matrix representatives $S = \{x, y, \cdots\} \subset M$ if U is the set of row spaces of the elements in S. Note that a matrix determines its row space uniquely. On the other hand, for a fixed m dimensional subspace, there is a unique orthonomal m-frame $(v_1, \cdots, v_m)$ in $H^{n-\sigma(1)+1} \times \cdots \times H^{n-\sigma(m)+1}$ (see Lemma A.2). Use v_i as the i-th row vector for $1 \leq i \leq m$. We get a matrix a. This matrix is uniquely determined by the m dimensional subspace.

Lemma B.1 *For $m \leq n$, let $e(\sigma)$ be a Schubert cell in a Grassmann manifold $G_m(\mathbf{C}^n)$. Then, $e(\sigma)$ has matrix representatives*

$$\begin{pmatrix} 0 & \cdots & 0 & p_1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * \\ & & & & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ & & & & & & * & 0 & * & \cdots & * & 0 & * & \cdots & * \\ & & & & & & & p_{m-1} & * & \cdots & * & 0 & * & \cdots & * \\ & & & & & & & & & & & p_m & * & \cdots & * \end{pmatrix} \tag{59}$$

with real positive p_k in the $\sigma(k)$-th column, for $1 \leq k \leq m$ if and only if the corresponding Schubert variety $\overline{e(\sigma)}$ has matrix representatives

$$\begin{pmatrix} 0 & \cdots & 0 & * & * & \cdots & * & * & * & \cdots & * & * & * & \cdots & * \\ & & & & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ & & & & & & * & * & * & \cdots & * & * & * & \cdots & * \\ & & & & & & & * & * & \cdots & * & * & * & \cdots & * \\ & & & & & & & & & & & * & * & \cdots & * \end{pmatrix}. \tag{60}$$

with $n - \sigma(k) + 1$ $*$'s in the k-th row, for $1 \leq k \leq m$ where the $*$'s are complex numbers.

Remark. By Lemma A.5, the number of $*$'s in the matrix (59) equals the dimension of $e(\sigma)$.

Proof. By Lemma A.5 and A.6, we can identify $e(\sigma)$ with $e'(\sigma)$ and identify $\overline{e(\sigma)}$ with $\overline{e'(\sigma)}$. For each orthonormal frame $(x_1, \cdots, x_m) \in e'(\sigma)$, let x be the matrix which has x_i as the i-th row for $1 \leq i \leq m$. Recall that the componets of x_i are labelled backwards, i.e., the k-th component is $z_{i,n-k+1}$ for $1 \leq k \leq n$. By Lemma A.5, we know that the Schubert cell

$e'(\sigma)$ has matrix representatives as in the matrix (59) if and only if the Schubert cycle $\overline{e'(\sigma)}$ has matrix representatives as in the matrix (60) with $z_{i,n-\sigma(i)+1} \geq 0$, for $1 \leq i \leq m$. Since left multiplication by elements of G_m corresponds to the elementary operations on the rows of matrices in M, the $(i, \sigma(i))$ entry can be any complex number. Q.E.D.

In the following, we write a Ferrers board consisting of all the $*$'s in the matrix (60) as $F_{\lambda(\sigma)}$. For example, if $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, then $j_1 = 2$, and $j_2 = 3$, so $\overline{e(\sigma)}$ has a Ferrers board $\begin{pmatrix} * & * & * \\ & * & * \end{pmatrix}$ while $e(\sigma)$ has a matrix representative $\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \end{pmatrix}$.

Corollary B.2 *For $\sigma, \sigma' \in J(m,n)$, $e(\sigma) \subseteq \overline{e(\sigma')}$ if and only if $F_{\lambda(\sigma)} \subseteq F_{\lambda(\sigma')}$.*

Proof. By the definition of Schubert cells (Definition A.1), each Schubert cell has matrix representatives as in the matrix (59) for some $\sigma \in J(m,n)$. Thus, every Schubert cycle has matrix representatives as in the matrix (60) for the corresponding $\sigma \in J(m,n)$. Conversely, for each $\sigma \in J(m,n)$, there is a unique Schubert cell and its closure which have the matrix representatives of the shape defined by σ. Therefore, the Schubert cycles are in a one to one correspondence with the Ferrers boards consisting of the stars in the matrix (60). Since the $*$'s in the matrix (60) can be arbitrary complex numbers, the conclusion of the corollary follows from the shape of the Ferrers board formed by the $*$'s in matrix (59) Q.E.D.

Theorem B.3 *The Grassmann manifold $G\backslash M$ has a CW-complex structure.*

Proof. We need only to show that for each $[a] \in (\overline{e(\sigma)} - e(\sigma))$, there is some $\sigma' \in J(m,n)$ such that $[a] \in e(\sigma')$ and $\dim e(\sigma') < \dim e(\sigma)$. Since $G\backslash M = \cup_{\sigma \in J(m,n)} e(\sigma)$, there is some $\sigma' \in J(m,n)$ such that $[a] \in e(\sigma')$. For $[a] \in \overline{e(\sigma)}$, each nonzero entry of the matrix representative of $e(\sigma')$ is contained in the Ferrers board defined by the matrix representative of $\overline{e(\sigma)}$ by B.2. So $\sigma'(i) \geq \sigma(i)$, for $1 \leq i \leq m$. But, the fact that $[a] \notin e(\sigma)$ implies that there is some i such that the inequality is strict. Therefore, the number of $*$'s in the matrix representative of $e(\sigma')$ is strictly less than that in the matrix representative of $e(\sigma)$. Hence, $dim(e(\sigma')) < dim(e(\sigma))$. Q.E.D.

APPENDIX

C Flag Manifolds

A flag is a sequence of nested subspaces in $\mathbf{C}^m$

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = \mathbf{C}^m \tag{61}$$

where $dim\, V_i = i$, for $1 \leq i \leq m$. Let $\mathcal{F}(\mathbf{C}^m)$ be the set of all flags as in (61). Take a vector $v_i \in V_i - V_{i-1}$ for $1 \leq i \leq m$. Use v_i as the i-th row for $1 \leq i \leq m$ to get an m by m matrix $a \in G_m$. Recall that we write a_k for the submatrix of a consisting of the last k rows of a. Then V_i is the row space of a_i. Write $V_i =< a_i >$. Note that B is upper triangular. For two matrices $a, a' \in G_m$, $< a_i >=< a'_i >$ for $1 \leq i \leq m$ if and only if $Ba = Ba'$. This gives a one to one correspondence between $\mathcal{F}(\mathbf{C}^m)$ and the quotient space $B\backslash G$. Identify a flag with its corresponding B-orbit. Transport the topology of $B\backslash G$ to the set $\mathcal{F}(\mathbf{C}^m)$. Then $\mathcal{F}(\mathbf{C}^m)$ is a complex manifold, called the flag manifold.

On the other hand, by the definition of Grassmann manifolds, we can view a flag manifold as the subset of all $(V_1, \cdots, V_m)$ in the direct product of Grassmann manifolds $\prod_{i=1}^m G_i(\mathbf{C}^m)$ such that $V_i \in G_i(\mathbf{C}^m)$, for $1 \leq i \leq m$ and $V_i \subset V_{i+1}$, for $1 \leq i \leq m-1$.

Thus, we have two different ways to define a topology on the flag manifold: either transport the quotient topology from $B\backslash G$ to all the flags or let the flags inherit the topology from the direct product $\prod_{i=1}^m G_i(\mathbf{C}^m)$ where the flag in (61) corresponds to $(V_1, \cdots, V_m) \in \prod_{i=1}^m G_i(\mathbf{C}^m)$. In the second case, the topology on the flags as in (61) is the subspace topology given by the product topology on $\prod_{i=1}^m G_i(\mathbf{C}^m)$. The following lemma says that these two topologies are equivalent, i.e., there is a homeomorphism between them.

Lemma C.1 *For each matrix $a \in G$, let a_k be the submatrix consisting of the last k rows of a. Let $< a_i >$ be the row space of a_i. Let $\theta : B\backslash G \to \mathcal{F}(\mathbf{C}^m)$ be the map defined by $[a] \mapsto (V_1, \cdots, V_m)$ where $V_i =< a_i >$. Then, θ is a homeomorphism.*

Proof. This is a corollary of Lemma 6.1.

Let $a \in M = M_{m,n}^m$ and as before let a_k be the submatrix of a consisting of the last k rows in a. Let V_k be the subspace of $\mathbf{C}^n$ which has the k rows in a_k as a basis, $1 \leq k \leq m$. Then, the nested sequence $V_1 \subset V_2 \subset \cdots \subset V_m$ is a partial flag in $\mathbf{C}^n$ with $dim\, V_i = i$. Denote the set of partial flags by

$$\mathcal{F}_m(\mathbf{C}^n) = \{F := (V_1, \cdots, V_m) \,|\, \dim\, V_i = i \text{ and } V_i \subset V_{i+1}, \forall i\}. \tag{62}$$

If $b \in B$, the row space of $(b\,a)_k$ is equal to the row space of a_k. Thus $b\,F = F$, for $F \in \mathcal{F}_m(\mathbf{C}^n)$. There is a bijection between the sets $B\backslash M$ and $\mathcal{F}_m(\mathbf{C}^n)$ which is a homeomorphism when the sets are given the topologies analogous to the ones in Lemma C.1. This is proved in Lemma 6.1.

APPENDIX

D Embedding of Complex Manifolds

I learned the following proposition from Prof. Terao's class on algebraic geometry. Since I do not have a reference, I quote it here together with the proof.

Proposition D.1 *Let M and N be complex manifolds of dimensions m and n, respectively, where $m \geq n$. If N is compact and $f : N \to M$ is an embedding then $f(N)$ is a (smooth) analytic subvariety of M*

Proof. Let $z = (z_1, \cdots, z_n)$ be local coordinates of $z \in N$ and $f(z) = (w_1, \cdots, w_m)$ be local coordinates around $f(z)$ in M, Then, f has a local expression

$$f(z) = (f_1(z), \cdots, f_m(z)).$$

Since f is an embedding it is full ranked. Without loss of generality we can assume that $det(\frac{\partial f_i}{\partial z_j})_{1 \leq i,j \leq n} \neq 0$, on a neighborhood U' of z in N. So, $f_1, \cdots, f_n$ can be chosen as coordinates on U'. Then, we can assume

$$f(z) = (z_1, \cdots, z_n, f_{n+1}(z), \cdots, f_m(z)).$$

Thus, $f(U')$ is defined by

$$h_i(w) = w_i - f_i(w_1, \cdots, w_n) = 0, n < i \leq m.$$

Also, $rank(\frac{\partial h_i}{\partial w_j}) = m - n$. Since f is injective, $f(z)$ is not in $f(N - U')$. Since N is compact, $N - U'$ is compact. Again, the fact that f is an embedding implies that f is continuous. Hence, $f(N - U')$ is closed in M.

Thus, we know that there is some open neighborhood W of $f(z)$ in M such that $W \cap f(N - U') = \emptyset$. Therefore, $W \cap f(N) \subseteq f(U')$.

Let $U = f^{-1}(W \cap f(N)) \subset U'$. Then, $f(U) = W \cap f(N) \subset f(U')$ is defined by the equations $w_i - f_i(w_1, \cdots, w_n) = 0, n < i \leq m$. So $f(N)$ is an analytic subvariety of M. Q.E.D.

References

[1] G. E. Andrews, A theorem on reciprocal polynomials with applications to permutations and compositions, American Math. Monthly 82:8 (1975) 830-833.

[2] A. Björner, L. Lovász, S. T. Vrećica and R. T. Živaljević, Chessboard complexes and matching complexes, J. London Math. Soc. (2) 49 (1994), 25-39.

[3] R. A. Brualdi, Introductory Combinatorics, 2nd edition, North-Holland, 1992, p. 90.

[4] K. Ding, Invisible Permutations and Rook Placements on a Ferrers Board, Proceedings of the Fourth Conference on Formal Series and Algebraic Combinatorics, Publications du LACIM, Université du Québec à Montréal, Québec, 1992, p.137-154. To appear in Discrete Mathematics.

[5] D. Foata and M. Schützenberger, On the rook polynomials of Ferrers relations, Colloq. Math. Soc. János Bolyai, 4, Combinatorial Theory and its Applications, Vol. 2, (P. Erdös et al. eds.) North-Holland, Armsterdam, 1970, p. 413-436.

[6] A. M. Garsia and J. B. Remmel, q-counting rook configurations and a formula of Frobenius, J. Combin. Theory Ser. A 41(1986) 246-275.

[7] P. F. Garst, Cohen-Macaulay complexes and group actions, Ph D thesis, University of Wisconsin-Madison, 1979.

[8] I. M. Gessel, A q-analog of the exponential formula, Discrete Mathematics 40(1982) 69-80.

[9] J. R. Goldman, J. T. Joichi, and D. E. White, Rook theory I, Rook equivalence of Ferrers boards, Proc. Amer. Math. Soc. 52(1975), 485-492.

[10] J. R. Goldman, J. T. Joichi, D. L. Reiner, and D. E. White, Rook theory II, Boards of binomial type, SIAM J. Appl. Math. 31 (1976), 618-633.

[11] J. R. Goldman, J. T. Joichi, and D. E. White, Rook theory III, Rook polynomials and the chromatic structure of graphs, J. Combin. Theory, Ser. b 25 (1978) 135-142.

[12] J. R. Goldman, J. T. Joichi, and D. E. White, Rook theory IV, Orthogonal sequences of rook polynomials, Studies in Appl. Math., 56(1977) 267-272.

[13] J. R. Goldman, J. T. Joichi, and D. E. White, Rook theory V, Rook polynomials, Möbius inversion and the umbral calculus, J. Combin. Theory, Ser. A 21(1976), 230-239.

[14] H. Gould, The q-Stirling numbers of the first and second kinds, Duke Math. J. , 28(1961), 281-289.

[15] P. Griffiths and J. Harris, Principles of Algebraic Geometry, John Wiley & Sons, Inc., New York , 1978.

[16] M. Hall Jr, Proceedings Symposium in Pure Mathematics, AMS, Providence, 6(1963), 203.

[17] J. Harris, Algebraic Geometry, a first course, Springer-Verlag, 1992.

[18] H. Hiller, Geometry of Coxeter Groups, Pitman Books Limited, 1982.

[19] J. E. Humphreys, Linear Algebraic Groups, Springer-Verlag, New York, 1975.

[20] D. Husemoller, Fibre Bundles, Springer-Verlag, Second Edition, 1975.

[21] M. Ishikawa, a private letter, December 28, 1991.

[22] Macdonald, Notes on Schubert Polynomials, Publications du LACIM, Univ. du Québec à Montréal, 1991.

[23] J. Milnor and J. Stasheff, Characteristic Classes, Annals of Mathematics Studies, no. 76, Princeton University Press, 1974.

[24] J. R. Munkres, Topology a First Course, Prentice-Hall Inc., 1975.

[25] A. Nijenhuis, A. E. Solow and H. S. Wilf, Bijective methods in the theory of finite vector spaces, J. Combin. Theory Ser. A, 37(1984) 80-84.

[26] K. M. O'Hara, Unimodality of Gaussian coefficients: a constructive proof, J. Combin. Theory, Ser. A. 53:1(1990) 29-52.

[27] J. Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958.

[28] B. Sagan, A maj statistic for set partitions, Europ. J. Combin., 12(1991), 69-79.

[29] L. Solomon, The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field, Geometriae Dedicata 36(1990) 15-49.

[30] R. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, SIAM J. Alg. Disc. Meth. 1:2(1980) 168-184.

[31] M. Wachs and D. White, p, q-Stirling numbers and set partition statistics, J. Combin. Theory Ser. A 56:1(1991) 27-46.

[32] F. W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Springer-Verlag, 1983.

[33] G. M. Ziegler, Shellability of chessboard complexes, Isreal J. of Math. 87(1994), 97-110.

Address from fall of 1995:
Kequan Ding
Department of Mathematics
University of Illinois
Urbana, IL 61801
e-mail: k-ding@orion.math.uiuc.edu