Combinatorial Bases in Systems of Simplices and Chambers

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May 12, 1994

Abstract

We consider a finite set E of points in the n-dimensional affine space and two sets of objects that are generated by the set E : the system Σ of n-dimensional simplices and the system Γ of chambers. The relation $(A; \Sigma, \Gamma)$ introduced by the incidence matrix $M = \| a_{\sigma,\gamma} \|$ defines the notion of linear independence and the rank of the system of simplices and of the system of chambers. We introduce the notion of a combinatorial basis. Combinatorial bases of chambers can be described in terms of a game. We describe the algorithm of decomposition of a convex polytope into shells. In the case of the affine plane, using the game and the algorithm we construct combinatorial basis B of chambers. Using the algorithm, we also construct a basis B' of simplices that together with the basis B of chambers form a "triangular pair".

Introduction

Let B and Γ be two finite sets. A relation [1] between B and Γ is a subset A of pairs $(\beta, \gamma) \in (B \times \Gamma)$. We will denote such a relation by $(A; B, \Gamma)$.

A relation can be presented by its incidence matrix M,

$$M = \|a_{\beta,\gamma}\|, \quad where \quad a_{\beta,\gamma} = 1, \ if \ (\beta,\gamma) \in A, \ a_{\beta,\gamma} = 0, \ if \ (\beta,\gamma) \notin A$$

Thus, the rows of the incidence matrix M correspond to the elements $\beta \in B$ and the columns of M correspond to the elements $\gamma \in \Gamma$.

Let us denote the row of the matrix M corresponding to an element β by f_β, $f_\beta \in \mathbf{R}^q$, where $q = \mid B \mid$ (i.e. q is equal to the number of elements in B). Let V_B be the linear subspace spanned by f_β, $\beta \in B$. Similarly, let us denote by g_γ, $\gamma \in \Gamma$, the columns of matrix M and by V_Γ the linear subspace spanned by $g_\gamma \in \mathbf{R}^p$, where $p = \mid \Gamma \mid$.

Since the rows of the matrix M are in one-to-one correspondence with the elements $\beta \in B$, instead of linear combinations $\sum c_\beta f_\beta$ of the rows of M we can consider linear combinations of elements of B, i.e. $\sum c_\beta \beta \in V_B$, where $\beta \in B$. We define that $\sum c_\beta \beta = 0$ if and only if $\sum c_\beta f_\beta = 0$.

Similarly, we will write linear combinations of the elements $\gamma \in \Gamma$, i.e. $\sum c_\gamma \gamma \in V_\Gamma$.

The linear spaces V_B and V_Γ are dual to each other. Indeed, we can construct the bilinear form

$$(\tilde{\beta}, \tilde{\gamma}) = \sum_{\beta,\gamma} a_{\beta,\gamma} c_\beta d_\gamma,$$

where $\tilde{\beta} = \sum_\beta c_\beta \beta \in V_B$, $\tilde{\gamma} = \sum_\gamma d_\gamma \gamma \in V_\Gamma$. It is easy to prove that the bilinear form $(\tilde{\beta}, \tilde{\gamma})$ is well defined and nondegenerate.

The important notion is the rank of a relation.

Definition 0.1 *Rank of the relation* $(A; B, \Gamma)$ *is the dimension of the subspace* V_B *or subspace* V_Γ.

Note that $dim V_B = dim V_\Gamma = rank M$.
The calculation of this rank is an important problem for different relations.

[1] The notion of relation was first introduced by MacLane.

In this paper we study relations connected with the following geometrical problem.

Consider a finite set E of N points in the n-dimensional affine space, $N > n$. Denote by P the convex hull of the set E. Any $n+1$ points in general position span a simplex σ. Denote by $\Sigma = \{\sigma\}$ the set of all these simplices. The set of simplices $\sigma \in \Sigma$ divide the polytope P into a finite number of chambers γ. Denote by Γ the set of all these chambers γ.

Thus we have two sets : the set Σ of all simplices σ and the set Γ of all chambers γ. [2] Let us denote by A the set of pairs (σ, γ), $\sigma \in S$, $\gamma \in \Gamma$, such that $\gamma \subset \sigma$. If $(\sigma, \gamma) \in A$ we will say that σ and γ are incident. Thus, for any finite set of points E in the n-dimensional affine space we have constructed a relation $(A; \Sigma, \Gamma)$.

Therefore, for the system of simplices and system of chambers the notion of rank is defined. These ranks are equal. The rank of the system of simplices has the following geometrical interpretation.

Let us denote by $\phi_\sigma(x)$ the characteristic function of a simplex σ, i.e.

$$\phi_\sigma(x) = 1, \ x \in \sigma \ \ and \ \ \phi_\sigma(x) = 0, \ \ if \ \ x \notin \sigma$$

It is easy to see that the following proposition holds.

Proposition 0.1 *A rank of the system of simplices Σ is the maximal number of linearly independent characteristic functions $\phi_\sigma(x)$.*

Systems of simplices and chambers appear in different problems (representation theory, Kostant partition functions , hypergeometric functions , etc.) and two important questions arise: 1) how to construct a basis of chambers, 2) how to construct a basis of chambers. It is important for combinatorial problems to give an explicit construction of bases consisting of chambers and bases of simplices and not of their linear combinations.

The combinatorial construction of a basis of simplices is contained in [AGZ]. In [AGZ] a theorem about the system of linear relations among simplices and a theorem about the system of linear relations among chambers are formulated. The approach in [AGZ] is based on the notion of marking (see also [B]) and is different from the geometrical approach used in the present

[2] In the general definition of a relations we denoted two sets as B and Γ.

paper. Note that, unlike [AGZ], in this paper we use only two systems of objects: the system of simplices and the system of chambers, and we do not consider the system of hyperplanes.

In section 1 we prove that among chambers in the n-dimensional affine space there are linear relations that have simple geometrical meaning (Theorem 1.1 , which was also formulated in [AGZ]).

On the affine plane these "basic" linear relations among chambers have the following form:

$$\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4, \tag{1}$$

where γ_i are chambers.

It turns out that all linear relations among chambers are linear combinations of linear relations (??) (see Theorem 5.3).

Let V_Σ be the linear space generated by simplices $\sigma \in \Sigma$ (or by rows of the incidence matrix M) and V_Γ be the linear space generated by chambers $\gamma \in \Gamma$ (or by columns of the incidence matrix M).

We construct combinatorial bases in V_Γ. A set B of chambers is called a combinatorial basis if any chamber (column of M) can be obtained from the set B by consequently applying the relation (??). We prove that such bases exist. Of course, any combinatorial basis is also a basis in the sense of linear algebra. [3]

In section 2 we introduce combinatorial bases in V_Γ. A combinatorial basis consists of chambers (and not of their linear combinations). Combinatorial bases of chambers naturally arise from some game. The game described in section 2 will be also used in the proof of Theorem 4.1, which is an important step in the proof of the main Theorem 5.2.

Section 3 contains an algorithm for the construction of some set of chambers B in the n-dimensional affine space. This set B will be studied in sections 4,5,6 in the case of the affine plane. However, it is also important that the described algorithm gives a decomposition of a convex polytope P into "shells" S_k (in case of the affine plane this decomposition into shells defines a triangulation). We describe geometrical properties of these shells S_k. In sections 4, 5, 6 we study in details the case of the affine plane.

[3] In the space V_Σ we can also define a combinatorial basis using the relations among simplices introduced in [AGZ]. The linear space V_Σ will be considered in another paper.

In section 4 we prove that on the affine plane any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma \in B$. In order to do this, we introduce a partial ordering of "new points" inside the shell S_k. We prove that any chamber in S_k has the minimal vertex. Then in terms of game described in section 2 we prove that the set B is a successfull initial painting (Theorem 4.1). From Theorem 4.1 it follows that any chamber is a linear combination of chambers from B.

In section 5 we prove that the chambers $\gamma \in B$ are linearly independent (Theorem 5.1) and that the set B is a combinatorial basis of chambers.

In section 6 we construct a basis of simplices B' using the algorithm from section 3. We show that there is a "triangular relation" between the basis of chambers B and the basis of simplices B'.

A pair of bases $e_1, \ldots, e_n$ in the space V and $f_1, \ldots, f_n$ in the dual space V' is called *triangular*, if $(e_i, f_k) = 0$, for $i > k$, and $(e_i, f_i) = 1$.

There are some remarks about the algorithm in the Appendix.

1 Basic linear relations among chambers

As we have mentioned in the Introduction we will consider a finite set of points $E = (e_1, e_2, \ldots, e_N)$ in the n-dimensional affine space V, $N > n$, and let $P = conv(E)$ be the convex hull of E. Let us assume that there are at least $n+1$ points in general position. If the points $e_{i_1}, \ldots, e_{i_{n+1}}$ are the points in general position we denote by $\sigma_{i_1,\ldots,i_{k+1}}$ (or sometimes simply by σ) the n-dimensional simplex spanned by these points. Denote by $\Sigma = \{\sigma\}$ the set of all these simplices. All simplices σ (usually overlapping) cover the polytope P. Simplices σ divide the polytope P into a finite number of chambers γ. (See examples of a system of simplices and a system of chambers in fig. 1).

Let us give the definition of a chamber. Denote by $\tilde{\sigma}$ the boundary of simplex σ and by $\tilde{\Sigma}$ the union of the boundaries of all simplices σ, i.e. $\tilde{\Sigma} = \bigcup_{\sigma \in \Sigma} \tilde{\sigma}$ and let us denote $\breve{P} = P \setminus \tilde{\Sigma}$. Let $\breve{\gamma}$ be a connected component of $\breve{P}$ and γ be closure of $\breve{\gamma}$.

Definition 1.1 *We will call γ a chamber and $\breve{\gamma}$ an open chamber.*

Denote by Γ the set of all chambers in P. Open chambers do not overlap and all the chambers cover the polytope P. Note that every chamber is a

polytope. We will say that a point is a *vertex of a chamber* if it is a vertex of the corresponding polytope.

Thus, for a finite set of points E we have constructed two sets of objects: a system of overlapping simplices Σ and a system of chambers Γ and we can consider a relation $(A; \Sigma, \Gamma)$ defined by the incidence matrix M.

For the system of simplices and chambers we repeat below the notations that we used in the Introduction. These notations will be used in all following sections.

M is the incidence matrix, i.e. $M = \| a_{\sigma,\gamma} \|$, $\sigma \in \Sigma$, $\gamma \in \Gamma$, , where

$$a_{\sigma,\gamma} = 1, \quad if \quad \gamma \subset \sigma, \quad a_{\sigma,\gamma} = 0, \quad if \quad \gamma \not\subset \sigma \tag{2}$$

f_σ is the row of M that corresponds to the simplex σ
g_γ is the column of M that corresponds to the chamber γ
V_Σ is the linear space spanned by simplices $\sigma \in \Sigma$
V_Γ is the linear space spanned by chambers $\gamma \in \Gamma$
r rank of the relation $(A; \Sigma, \Gamma)$.
$p = \mid \Sigma \mid$, $q = \mid \Gamma \mid$

As we have mentioned in the Introduction we can consider linear combinations of chambers instead of linear combinations of the corresponding columns of matrix M.

Definition 1.2 *Chambers $\gamma_1, \ldots, \gamma_k$ are linearly dependent if among these chambers there is a relation*

$$\sum_{i=1}^{k} c_i g_{\gamma_i} = 0$$

.

Below we will replace this relation by the following relation:

$$\sum_{i=1}^{k} c_i \gamma_i = 0 \tag{3}$$

The geometrical meaning of linear relations among chambers will be clear from this section. All linear relations among chambers are linear combinations of some simple basic " geometrical " relations (??).

In order to formulate these basic relations we will define a *new point*, and in order to determine the signs in the relation we will introduce an orientation around a new point.

Consider the set of all vertices of all chambers $\gamma \in \Gamma$. Some of these vertices are points from the set $E = \{e_i\}$ and some are not, see examples in fig.1.

Fig.1.

Definition 1.3 *A vertex w of a chamber $\gamma \in \Gamma$ is a new point if $w \notin E$.*

Let us denote by $W = \{w\}$ the set of all new points of all chambers $\gamma \in \Gamma$. The set of all vertices of all chambers $\gamma \in \Gamma$ is $E \cup W$.

Consider the case when through a new point $w \in W$ pass exactly n facets of simplices $\sigma \in \Sigma$, i.e. $(n-1)$-dimensional faces of simplices $\sigma \in \Sigma$. This means that all the facets of simplices $\sigma \in \Sigma$ are in general position.

We introduce the orientation around a new point. Let $w \in W$ be a new point. The new point w lies in the intersection of exactly n facets of some simplices $\sigma \in \Sigma$. Therefore we can introduce a local coordinate system with the origin at the point w if we choose these n facets as the coordinate hyperplanes and choose an arbitrary orientation.

Let us denote by $\gamma(w)$ the set of all chambers with the vertex w, i.e. $\gamma(w) = \{\gamma : w \in \gamma\}$. [4] Let $\gamma \in \gamma(w)$ and let $\xi \in \gamma$ be an arbitrary point with local coordinates $\xi_1, \ldots, \xi_n$. Then the chamber $\gamma \in \gamma(w)$ can be characterized

[4]Chambers $\gamma \in \gamma(w)$ are called chambers adjacent to the vertex w.

by the sequence of + and −, i.e. by $(sign(\xi_1), \ldots, sign(\xi_n))$. It is clear that this sequence does not depend on the point $\xi \in \gamma$.

If the point w lies inside the polytope P there are 2^n chambers in $\gamma(w)$ and they are coded by all the possible sequences of the length n consisting of + and −.

For a new point w we introduce the following function $\varepsilon_w(\gamma)$

$$\varepsilon_w(\gamma) = sign(\xi_1) \cdot sign(\xi_2) \cdot \ldots \cdot sign(\xi_n)$$

Remark 1. In case when not all the facets of simplices $\sigma \in \Sigma$ are in general position, a function $\varepsilon_w(\gamma)$ similar to the one defined here can be introduced. The formulation of Theorem 1.1 with this function $\varepsilon_w(\gamma)$ see in [AGZ]. We have this assumption in Theorem 1.1 to simplify the presentation of results that are used later. In sections 4,5,6 that contain the main results we do not make this assumption.

Note also that this restriction about general position of facets of simplices is applied only to a new point. Of course, there are no assumptions on an "old point" (a point from E).

Theorem 1.1 *For any new point $w \in W$ there is the following linear relation between chambers*

$$\sum_{\gamma \in \gamma(w)} \varepsilon_w(\gamma)\gamma = 0 \tag{4}$$

Note that the choice of another local coordinate system around w can only change simultaneously the signs of functions $\varepsilon_w(\gamma)$ for all $\gamma \in \gamma(w)$ and therefore will not change the relation (??).

For example, around the new point w_1 in fig.1 the relation (??) is $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0$.

Proof of Theorem 1.1. Let $w \in W$ be a new point. We have to prove that among the columns g_γ, $\gamma \in \gamma(w)$ of the incidence matrix M (see formula (??)) there is a linear relation

$$\sum_{\gamma \in \gamma(w)} \varepsilon_w(\gamma) g_\gamma = 0 \tag{5}$$

The relation (??) means that

$$\sum_{\gamma \in \gamma(w)} \varepsilon_w(\gamma) a_{\sigma,\gamma}(w) = 0, \quad for\ any\ \sigma \in \Sigma \tag{6}$$

Since $a_{\sigma,\gamma} = 1$, if $\gamma \subset \sigma$ and $a_{\sigma,\gamma} = 0$, if $\gamma \not\subset \sigma$, the relation (??) is equivalent to the following:

$$\sum_{\gamma \in \gamma(w), \gamma \subset \sigma} \varepsilon_w(\gamma) = 0, \ for\ any\ \sigma \in \Sigma \tag{7}$$

Let us choose a simplex $\sigma \in \Sigma$. The following cases are possible:

1) $w \notin \sigma$;

2) w belongs to the interior of the simplex σ;

3) w lies on the boundary of the simplex σ and lies inside the polytope P;

4) w lies on the boundary of the simplex σ and lies on the boundary of the polytope P.

Let us consider these cases. In the case 1) we have $\gamma \not\subset \sigma$ for any chamber $\gamma \in \gamma(w)$ and therefore the relation (??) holds.

Let us consider the case 2). Then w lies inside the simplex σ together with a small neigborhood. Since any chamber is connected, this implies that all chambers $\gamma \in \gamma(w)$ belong to the simplex σ, i.e. $\gamma \subset \sigma$, $for\ any\ \gamma \in \gamma(w)$.

As we have mentioned above, any chamber $\gamma \in \gamma(w)$ can be coded by the sequence $(sign(\xi_1), \ldots, sign(\xi_n))$, where $\xi \in \gamma$. Since $\gamma \subset \sigma$ for any $\gamma \in \gamma(w)$, all possible sequences $(sign(\xi_1), \ldots, sign(\xi_n))$ will be present in the sum (??). Therefore, due to the symmetry among all the sequences of $+$ and $-$ of the length n, half of the functions $\varepsilon_w(\gamma)$ will be equal to $+1$ and the other half equal to -1 and the relation (??) holds.

Consider the case 3). The point w lies on the boundary of the simplex σ and inside the polytope $\bar{E}$. Then $w \in q$, where $q \subset \sigma$ is an $(n-k)$-dimensional open face of σ. This means (due to the assumption about general position of faces of simplices) that w lies in k coordinate hyperplanes around the point w. Let the local coordinates around the point w be $\xi_1, \ldots, \xi_n$. Then the face q is described by the equations $\xi_{i_1} = 0, \ldots, \xi_{i_k} = 0$. The simplex σ lies to one

of the sides of each of these hyperplanes (since q is the face of σ). This means that the signs of the coordinates $\xi_{i_1}, \ldots, \xi_{i_k}$ are fixed for the simplex σ and therefore are fixed for all the chambers $\gamma \in \gamma(w), \ \gamma \subset \sigma$. The remaining $(n-k)$ coordinates can have arbitrary signs in the simplex σ. Therefore the chambers $\gamma \in \gamma(w), \ \gamma \subset \sigma$ are coded by all the possible sequences of $+$ and $-$ of the length $(n-k)$ and it is clear that the relation (??) holds.

Consider the case 4). Let the point w lie in the $(n-k)$-dimensional open face of the simplex σ and in the $(n-m)$-dimensional open face of the polytope P. In this case it is clear that for any chamber $\gamma \in \gamma(w), \ \gamma \subset \sigma \ \ k$, coordinates of a point $\xi \in \gamma$ will be fixed "due to " the face of the simplex and m coordinates will be fixed "due to" the face of the polytope P. The remaining coordinates can be arbitrary and therefore the chambers $\gamma \in \gamma(w), \ \gamma \subset \sigma$ are coded by all the possible sequences that has $+$ and $-$ in these coordinates. Therefore the relation (??) holds. $\square$

2 Combinatorial bases of chambers. Game.

Definition 2.1 *A basis of chambers is called combinatorial if any other chamber can be expressed in terms of basis chambers by consequently applying the relation (??) (see Theorem 1.1).*

Combinatorial bases arise naturally from the game that is formulated below. This game will be also used to prove Theorem 4.1. Again let us assume [5] that through a new point pass exactly n facets of simplicies $\sigma \in \Sigma$, i.e. all the facets of simplices $\sigma \in \Sigma$ are in general position.

Game. Let $E = \{e_1, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space. Let Σ be the set of all n-dimensional simplices spanned by points $e_i \in E$ and Γ be the set of all chambers (see Definition 1.1). Let $W = \{w\}$ be the set of all new points (see Definition 1.3). One has to paint initially some chambers with, for example, blue paint and pay for each blue chamber. After the initial painting has been completed, it is allowed to paint with green paint some other chambers according to the following rule:

Rule. *If all chambers adjacent to a new point $w \in W$ except one are already painted (either blue or green), then the last chamber adjacent to the new point w can be painted with green paint.*

[5] In the section 4 we reformulate this game on the affine plane without the assumption.

Green chambers are "for free", i.e. one does not pay for a green chamber [6].

Definition 2.2 *An initial painting B is called sufficient if all the chambers $\gamma \in \Gamma,\ \gamma \notin B$ can be painted with green paint according to the rule above.*

The purpose of the game is to construct a sufficient initial painting that has the lowest price.

Examples. In fig.2 there are two configurations of 6 points on the affine plane, $E = (e_1, \ldots, e_6)$. There are examples of sufficient initial paintings. Blue chambers are shaded and green chambers are marked by "+". The sufficient initial painting in fig.2a does not have the lowest price, which is equal to 10. The sufficient initial painting in fig.2b has the lowest price.

Fig.2.

Note that in the game the process of painting chambers green is actually the way of consequently applying the relations (??). Therefore, a combinatorial basis can be constructed from a sufficient initial painting that has the lowest price.

[6] It is clear that in this game there are iterations of painting chambers green, because after we have painted some green chambers according to the rule, more chambers that can be painted green can appear.

3 Algorithm of construction of set B.

In this section we will algorithmically construct the set of chambers B. We will prove (sections 4,5) that on the affine plane this set of chambers is a basis of chambers. In the n-dimensional affine space the set B is not neccessarily a basis of chambers. Nevertheless, we describe here this algorithm in the n-dimensional affine space since it gives the decomposition of the polytope P into shells which will be used for the construction of basis of chambers in the n-dimensional affine space.

Lemma 3.1 *Let $E = \{e_1, e_2, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space. A sequence of points $e_{i_1}, e_{i_2}, \ldots, e_{i_N}$, $e_{i_k} \in E$, can be constructed such that*

$$F_k \cap P_k = \emptyset, \ for\ k = 1, \ldots, N, \tag{8}$$

where

$$P_k = conv(E \setminus (e_{i_1}, \ldots, e_{i_k}) \tag{9}$$

$$F_k = conv(e_{i_1}, \ldots, e_{i_k}) \tag{10}$$

We assume also $F_0 = \emptyset, P_N = \emptyset, P_0 = conv(E) = P, F_N = conv(E) = P$.

Proof. In Lemma 3.1 two sequences of polytopes are defined:

$$P_0 \supset P_1 \supset \ldots \supset P_N$$

$$F_0 \subset F_1 \subset \ldots \subset F_N$$

Let us prove the stronger statement: there exists a sequence of points $\{e_{i_k}\}$ such that not only the condition (??) is satisfied but also for each k there exists a hyperplane that separates the polytopes P_k and F_k.

We will prove this by induction. On the zero step consider the polytope P and any hyperplane that does not intersect P. The statement is satisfied. Let us construct the first point of the sequence. To do this we move the hyperplane untill it meets a point of the polytope P. There are two possibilities. If the hyperplane meets a vertex of P we take this vertex as the point e_{i_1}. If the hyperplane meets more than one point from P we turn the hyperplane slightly so that it will meet only one point from the polytope P. It is

clear that $F_1 \cap P_1 = \emptyset$. In order to obtain the hyperplane that separates P_1 and F_1 let us move a little the hyperplane mentioned above parallel to itself after it has met the vertex e_{i_1}. This hyperplane will separate the polytopes P_1 and F_1.

Suppose that we have constructed the sequence of points $e_{i_1}, \ldots, e_{i_{k-1}}$ such that $F_{k-1} \cap P_{k-1} = \emptyset$ and there is a hyperplane that separates the polytopes P_{k-1} and F_{k-1}. Let us move this hyperplane parallel to itself towards P_{k-1} untill it meets a point from P_{k-1}. As in the first step there are two possibilities: 1) the hyperplane meets a vertex of a polygon P_{k-1} ; 2) the hyperplane meets more than one point of the polygon P_{k-1}.
It is clear that the same arguments as for the point e_{i_1} allow us to choose the point e_{i_k} such that $P_k \cap F_k = \emptyset$ and choose the position of the hyperplane that separates these polytopes. $\Box$

Algorithm. Let us construct the set of chambers B. Let $E = \{e_1, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space and let Σ be the set of simplices and Γ be the set of chambers (see section 1). Construction of the set of chambers B will be made by steps. Let us denote by B_k the set of chambers that will be constructed on the k-th step. We define then

$$B = \{\gamma : \gamma \in \bigcup_k B_k\} \quad .$$

First step. We choose a vertex of the polytope P and denote it by e_{i_1}. We define the set B_1 as a set of all chambers in P adjacent to the vertex e_{i_1}, i.e.

$$B_1 = \{\gamma : \quad \gamma \subset P, \ e_{i_1} \in \gamma\} \quad .$$

Second step. We choose a point $e_{i_2} \in E \setminus e_{i_1}$ such that $F_2 \cap P_2 = \emptyset$ (see notations (??) and (??) .) By Lemma 3.1, such a point exists. We define the set of chambers B_2 as the set of all chambers in the polytope P_1 adjacent to the point e_{i_2}, i.e.

$$B_2 = \{\gamma : \ \gamma \subset P_1, \quad e_{i_2} \in \gamma\} \quad .$$

Step k. We choose a point $e_{i_k} \in E \setminus (e_{i_1}, \ldots, e_{i_{k-1}})$ such that $F_k \cap P_k = \emptyset$ and define a set B_k as the set of all chambers in the polytope P_{k-1} that are adjacent to the point e_{i_k}, i.e.

$$B_k = \{\gamma : \ \gamma \in P_{k-1}, \ e_{i_k} \in \gamma\} \ .$$

Since in the set of polytopes $P_0 \supset P_1 \supset P_2 \supset \ldots$ every polytope has one vertex less than the preceding one, the algorithm ends after a finite number of steps. As the result, we obtain the set B of chambers

$$B = \{\gamma : \gamma \in \bigcup_k B_k\}. \tag{11}$$

Proposition 3.1 *The number of steps in the algorithm is equal to* $N - n$.

Proof. We continue the algorithm as long as there is at least one chamber in the polytope P_k. If the polygon P_k has only n vertices it contains no chambers. Therefore the algorithm ends after $N - n$ steps. □

On the complexity of the algorithm. From Proposition 3.1 it follows that the number of operations in the algorithm is $O(N)$.

3.1 Visibility and shells.

The algorithm that constructs a set of chambers B described in section 3 gives a decomposition of the polytope P into shells . We will describe here their geometrical properties.

Let us introduce a notion of visibility.

Definition 3.1 *Let* V *be an* n*-dimensional affine space ,* P *a convex polytope,* e *a point such that* $e \notin P$. *We say that the point* $p \in V$ *is* P*-visible from the point* e *if* $(e, p) \cap P = \emptyset$, *where* (e, p) *is an open segment.*

We have constructed in section 3 a sequence of points $\{e_{i_k}\}$ and two sets of polytopes P_k and F_k, $k = 1, \ldots, N - n$. Let us consider the polytope

$$S_k = \overline{P_{k-1} \setminus P_k}, \tag{12}$$

where $\bar{A}$ means the closure of the set A. From the definition of polytopes S_k it follows

Proposition 3.2 *Polytope P is the disjoint union of the polytopes S_k,*

$$P = \bigcup_{i=1}^{N-n} S_i \quad . \tag{13}$$

The polytopes S_k will be called *shells*. The following propositions describe the geometrical properties of the shells S_k.

Proposition 3.3 *The point $e_{i_k} \in P_{k-1}$ is $P_{k-1}-$visible point from the point $e_{i_{k-1}}$.*

Proof. By construction, the point e_{i_k} is one of the vertices of the polytope P_{k-1} . Suppose that the point e_{i_k} is not P_{k-1}-visible from the point $e_{i_{k-1}}$. Then $(e_{i_k-1}, e_{i_k}) \cap P_{k-1} \neq \emptyset$, where (e_{i_k-1}, e_{i_k}) is an open segment. Then this segment crosses one of the faces of P_{k-1}, for example, the face q. Notice that $q \subset P_k$ since P_k contains all the vertices of P_{k-1} except the vertex e_{i_k} (we recall that $P_k = conv(E \setminus (e_{i_1}, \ldots, e_{i_k}))$). Then we obtain that $P_k \cap (e_{i_k}, e_{i_{k-1}}) \neq \emptyset$. Since $e_{i_k}, e_{i_{k-1}} \in F_k$ we have $P_k \cap F_k \neq \emptyset$. This contradicts the condition (??). $\square$

Remark 2. From Proposition 3.3 it follows that:

1. The polytope S_k has the following shape: S_k is a part of the convex cone with the vertex e_{i_k} bounded by the "cover" $\mathcal{L}_k$, where $\mathcal{L}_k$ is the set of all points of the polytope P_k that are P_k-visible from the point e_{i_k}.

2. The "cover" $\mathcal{L}_k$ satisfies the following condition: if $x_1,\ x_2 \in \mathcal{L}_k$, then any point $x \subset (x_1, x_2)$ either belongs to $\mathcal{L}_k$ or does not lie in S_k.

3. Inside S_k there are no points of E.

In particular, if V is the affine plane, the shell S_k is a part of an angle with the vertex e_{i_k}, two sides q_1, q_2 and the "cover" $\mathcal{L}_k$ which is a convex polygonal line, see fig. 3.

Let $Q = \{q\}$ be the set of all faces of all simplices $\sigma \in \Sigma$ and let $star(e) = \{q \in Q : e \text{ is the vertex of } q\}$. It is easy to see that Proposition 3.3 implies the following result.

Proposition 3.4 *Let S_k and S_{k+1} be two neighbor shells, e_{ik} be the vertex of S_k and $e_{i_{k+1}}$ be the vertex of S_{k+1}. Then*

$$S_k \cap S_{k+1} = star(e_{i_{k+1}}) \cap \mathcal{L}_k,$$

where $\mathcal{L}_k$ is the set of all points P_k-visible from the point e_{i_k}.

Note that on the affine plane the decomposition of the polygon P into shells S_k defines also a triangulation of the polygon P, since the shell S_k already consists of triangles (see fig.3).

Fig.3.

4 Sufficient initial paintings of chambers

In section 3 we have constructed a set of chambers B in the n-dimensional affine space V. In this section we will prove that in case of the affine plane any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma \in B$.

Consider the affine plane V. Let E be a finite set of points in V, Σ be the set of simplices (triangles), Γ be the set of chambers, W be the set of all new points (see Definition 1.3), and $Q = \{q\}$ be the set of all edges of all triangles $\sigma \in \Sigma$.

Definition 4.1 *We say that a chamber $\gamma \in \Gamma$ is adjacent to some edge $q \in Q$, if there is an edge q' of the chamber γ, such that $q' \subset q$.*

Consider the following game.

Game on the affine plane. One must paint some chambers blue and pay for them (initial painting). After that some other chambers can be painted green according to the following rule.

Rule 1. *If there are three painted chambers (either blue or green) adjacent to some edge $q \in Q$ that pass through a new vertex $w \in W$ then the fourth chamber adjacent to q can be painted green.*
Green chambers are "for free", i.e. one does not have to pay for a green chamber.

Definition 4.2 *An initial painting B is said to be sufficient if any chamber $\gamma \in \Gamma$, $\gamma \notin B$, can be painted green color according to the rule above.*

The purpose of the game is to construct a sufficient initial coloring that has the lowest price.

Remark 3. In this game we do not impose any restrictions on the number of edges that pass through a new point. It is easy to see that in case when all the edges $q \in Q$ are in general position this game will coincide with the game defined in section 2.

Therefore, in this case (when all the edges $q \in Q$ are in general position) we can use the following Rule 2 instead of Rule 1.

Rule 2. *If there are three painted chambers adjacent to a new point $w \in W$, then the fourth chamber adjacent to w can be painted green (i.e. "for free").*

Note also that in case when more than two edges $q \in Q$ pass through a new point we can move slightly the points so that these edges will be in general position. As a result, some new chambers and new points will appear and through any new point only two edges will pass. Then we can use the Rule 2 instead of the Rule 1 . One can check that in either case the minimal number of chambers that we have to paint blue in order to be able to paint the rest of the chambers green, is the same.

We can now formulate the main theorem of this section. In section 3 we have constructed the set of chambers B. Consider the set B as an initial painting, i.e. let us paint all the chambers $\gamma \in B$ blue.

Theorem 4.1 *Let E be a finite set of points on the affine plane. The set of chambers $\{\gamma : \gamma \in B\}$ is a sufficient initial painting.*

Corollary 4.1 *Any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma_i \in B$, i.e.*

$$\gamma = \sum c_i \gamma_i, \quad \gamma_i \in B$$

Let us prove that Corollary 4.1 follows from Theorem 4.1. First, let us make the following remark. Rule 1 and Rule 2 mean that any green chamber γ is a linear combination of chambers $\gamma_1, \gamma_2, \gamma_3$, i.e. $\gamma = \gamma_1 - \gamma_2 + \gamma_3$, where chambers $\gamma_1, \gamma_2, \gamma_3$ are already painted (either blue or green). In Rule 1 chambers $\gamma, \gamma_1, \gamma_2, \gamma_3$ are the chambers adjacent to some edge $q \in Q$ and to some new point w, $w \in q$. In Rule 2 chambers $\gamma, \gamma_1, \gamma_2, \gamma_3$ are the chambers adjacent to some new point w. In Theorem 1.1 it is proved that the chambers adjacent to a new point satisfy the relation $\gamma = \gamma_1 - \gamma_2 + \gamma_3$, i.e. we get Rule 2. The proof for the case of Rule 1 is similar and we do not include it here.

Let chambers $\gamma \in B$ are painted blue. Consider $\gamma \in \Gamma$, $\gamma \notin B$. By Theorem 4.1, any chamber $\gamma \notin B$ can be painted green according to the rule of the game. Therefore, there exists a new point $w \in \gamma$ and an edge $q \in Q$, $w \in q$, $q \subset \gamma$ such that other three chambers $\gamma_1, \gamma_2, \gamma_3$ adjacent to q are already painted, i.e. we have the relation $\gamma = \gamma_1 - \gamma_2 + \gamma_3$. The chamber γ is a linear combination of chambers that have been already painted blue or green. Since initially we have painted only blue chambers and then green chambers were consequently painted using the rule, γ is a linear combination of blue chambers , i.e. of chambers $\gamma \in B$.

Before we can prove Theorem 4.1 we will prove some propositions describing shells S_k on the affine plane.

Consider the shell S_k. It has the shape of an angle with the vertex e_{i_k} , two sides $q_1, q_2 \in Q$ and the "cover" $\mathcal{L}_k$ (see Remark 2 and fig.3).

Proposition 4.1 *Any edge $q \in Q$ that intersects the polygon S_k can intersect only one of the edges q_1, q_2.*

Proof. Note that no edge $q \in Q$ ends inside S_k, (see Remark 2). Since q_1, q_2 are the sides of the cone S_k with the vertex e_{i_k} , any edge $q \in Q$ that intersects S_k either pass through the point e_{i_k} or intersects at least one of the sides q_1, q_2. Let us prove that an edge $q \in Q$ can intersect only one of the sides q_1, q_2.

Suppose that some $q \in Q$ intersects both edges q_1 and q_2. The edge q has two end points that belong to E, for example, $e', e'' \in E$. Consider the convex polygons $F_{k-1} = conv(e_{i_1}, \ldots, e_{i_{k-1}})$ and $P_{k-1} = conv(e_{i_k}, \ldots, e_{i_N})$ (see formulae (??) and (??). Any point $e \in E$ belongs either to F_{k-1} or to P_{k-1}. We will show that the points e', e'' cannot belong to either of these polygons.

We claim that $e', e'' \notin P_{k-1}$. Indeed, the convex polygon $P_{k-1} = S_k \cup P_k$ has edges q_1 and q_2. Since by assumption the segment (e', e'') intersects q_1 and q_2, we have $e', e'' \notin P_{k-1}$.

On the other hand, $e', e'' \notin F_{k-1}$. Indeed, otherwise $P_{k-1} \cap F_{k-1} \neq \emptyset$ which is forbidden by the condition (??).

We obtained that $e', e'' \in E$ and $e', e'' \notin P_{k-1} \cup F_{k-1}$. This contradiction proves the proposition. $\square$

Corollary 4.2 *Any edge $q \in Q$ that intersects S_k intersects the polygonal line $\mathcal{L}_k$ at exactly one point.*

This follows from Proposition 4.1 and the fact that the polygonal line $\mathcal{L}_k$ is convex (see Remark 2).

We introduce a partial ordering of the new points $w \in W$ in the polygon S_k as follows. As above, let q_1, q_2 be the sides of the shell S_k.

Definition 4.3 *Let w_1, w_2 be new points that lie on some edge $q \in Q$. There are two possibilities.*

1) The edge q passes through the point e_{i_k} , i.e. $e_{i_k} \in q$. Then we say that $w' < w''$, if $\mid e_{i_k}, w' \mid < \mid e_{i_k}, w'' \mid$, where $\mid e, w \mid$ is the length of the segment (e, w).

2) The edge q intersects one of the sides q_1, q_2 of the shell S_k, for example, q_1 and let w_0 be a new point in their intersection, i.e. $q \cap q_1 = w_0$, $w_0 \in W$. Then for any $w', w'' \in q$ we say that $w' < w''$, if $\mid w_0, w' \mid < \mid w_0, w'' \mid$.

We also define $e_{i_k} < w$, for any $w \in W \cap S_k$.

By Proposition 4.1, any $q \in Q$ can intersect only one of the sides q_1, q_2 of the shell S_k therefore this partial ordering is well defined.

Remark 4. Note that this partial ordering of new points in S_k induces a direction along each edge $q \in Q$ that intersects S_k. All these edges q are directed from inside the shell S_k towards the polygonal line $\mathcal{L}_k$, see fig.4 .

Definition 4.4 *Let $\gamma \in S_k$ be an arbitrary chamber $\gamma \in \Gamma$. A vertex $w \in \gamma$ is called a minimal new vertex of the chamber γ, if $w < w'$ for any $w' \in \gamma$ such that (w, w') is an edge of the chamber γ.*

Consider the shell S_k with the vertex e_{i_k} and the "cover" $\mathcal{L}_k$. As a special case, the polygonal line $\mathcal{L}_k$ can be a segment and the shell S_k will then be a triangle. We need a proposition concerning this special case.

Proposition 4.2 *Let S_k be the shell with the "cover" $\mathcal{L}_k$, where $\mathcal{L}_k$ is a segment with the end points e', e''. Any chamber $\gamma \in S_k$ has the minimal vertex.*

Proof.

Let $\gamma \in S_k$. Consider the vertex $w \in \gamma$ that has the largest distance [7] from (e', e''). Note that such point $w \in \gamma$ is unique since any edge $q \in Q$ that intersects the triangle $S_k = (e_{i_k}, e', e'')$ intersects also the side (e', e'') (by Corollary 4.2).

Both edges q', q'' of the chamber γ such that $w \in q',\ w \in q''$ are directed away from w, since all edges q that intersect S_k are directed towards (e', e'') (see Remark 4 and fig.4). Therefore, w is the minimal vertex of γ. $\square$

Proposition 4.3 *In the polygon S_k any chamber γ has the minimal vertex.*

Proof. The polygon S_k is the part of an angle with the vertex e_{i_k} bounded by the convex polygonal line $\mathcal{L}_k$. Let e', e'' be the end points of this polygonal line. Consider the triangle (e_{i_k}, e', e''), (see fig.4). We have $S_k \subset (e_{i_k}, e', e'')$. Let us extend all the edges $q \in Q$ that intersect S_k so that they will intersect the triangle (e_{i_k}, e', e''). Any such edge will intersect the side (e', e'') and is directed towards it (see Corollary 4.2 and Remark 4).

Consider a chamber $\gamma \in S_k$. There are two possibilities:

[7] More precisely this means that if we introduce the barycentric coordinates s_1, s_2, s_3 in the triangle (e', e'', e_{i_k}), and the equation of the side (e', e'') is $s_1 = 0$, then we consider the point $w \in \gamma$ with the largest coordinate s_1.

1) $\gamma \cap \mathcal{L}_k = \emptyset$ Then the chamber γ has the minimal vertex by the Proposition 4.2.

2) $\gamma \cap \mathcal{L}_k \neq \emptyset$. Let us remove the line $\mathcal{L}_k$. We obtain chambers in the triangle (e_{i_k}, e', e'') and by Proposition 4.2 any such chamber has the minimal vertex. It is clear that by removing the line $\mathcal{L}_k$ we obtained a chamber γ' such that $\gamma' \supset \gamma$. Let w be the minimal point of γ'. Obviously, (see fig.4) w lies "below" the polygonal line $\mathcal{L}_k$, i.e. inside the shell S_k and is the vertex of γ, moreover, w is the minimal vertex of γ. $\square$

Fig.4

Let S_k be a shell with the vertex e_{i_k} , the "cover" $\mathcal{L}_k$, and the sides q_1, q_2, (see fig 3). Consider $S_k \cap S_{k-1}$. From Proposition 3.4 it follows that two cases are possible:

1) $e_{i_k} \in \mathcal{L}_{k-1}$ is not an end point of the polygonal line $\mathcal{L}_{k-1}$. Then $S_k \cap S_{k-1} = q_1 \cup q_2$.

2) $e_{i_k} \in \mathcal{L}_{k-1}$ is an end point of $\mathcal{L}_{k-1}$. Then $S_k \cap S_{k-1}$ contains only one of the edges q_1, q_2.

The following proposition is related only to the case 2). [8]

Proposition 4.4 *Let edges q_1, q_2 be the sides of the shell S_k and let $q_1 \notin S_k \cap S_{k-1}$. If there exists an edge $q \in Q$ that intersects q_1 then for any point $x \in q_1$ we have $x \in S_1 \cup \ldots \cup S_{k-1}$.*

[8] In case 1) Proposition 4.4 is trivial.

Proof. Let an edge $q \in Q$ intersect the edge q_1. Any edge $q \in Q$ has two end points that belong to E. Let e', e'' be the end points of q and $e_{i_k}, \tilde{e}$ be the end points of q_1. Due to Corollary 4.2 the edge q intersects the "cover" $\mathcal{L}_k$ of the shell S_k. Therefore, one of the points e', e'' lies in P_k, for example $e' \in P_k$ (see fig.5).

Fig.5

Consider the triangle (e_{i_k}, e', e''). Let x be a point such that $x \in int(e_{i_k}, e', e'')$, (where $int(A)$ is the set of all interior points of a set A), i.e. $x \in P = conv(E)$. Consider the polygon P,

$$P = S_1 \cup \ldots \cup S_{k-1} \cup S_k \cup \ldots \cup S_{N-2},$$

(see formula (??)). Since $S_k = \overline{P_{k-1} \setminus P_k}$, (see formula (??)) and $P_N = \emptyset$, we have

$$P = S_1 \cup \ldots \cup S_{k-1} \cup P_{k-1}.$$

It is clear that $x \notin P_k \cup S_k = P_{k-1}$, (see fig.5). Therefore, $x \in S_1 \cup \ldots \cup S_{k-1}$.

Since a shell is a closed set, any x that belongs to the boundary of (e_{i_k}, e', e''), in particular, any $x \in (e_{i_k}, \tilde{e}) = q_1$, belongs to the set $S_1 \cup \ldots \cup S_{k-1}$. Note also that $x \notin S_{k-1}$. $\square$

In further proofs we will use the game (formulated at the beginning of section 4) Let B be the set of chambers constructed in section 3. Consider the chambers $\gamma \in B$ as an initial painting for the game on the affine plane.

In other words, let us paint all chambers $\gamma \in B$ blue. According to the game we can now paint some chambers green if they satisfy Rule 1.

Let e_{i_k} be the vertex and q_1, q_2 be the sides of the shell S_k. We will explain how to paint chambers in S_k adjacent to the sides q_1, q_2 of the shell S_k.

Proposition 4.5 *Suppose that any $\gamma \in S_1 \cup \ldots \cup S_{k-1}$, $\gamma \notin B$ is painted green. Let $\gamma' \in S_k$ be a chamber adjacent to a new point $w \in q_j$. The chamber γ' can be painted green according to Rule 1 of the game.*

Proof. Let us first consider the case when all the edges $q \in Q$ are in general position, i.e. only two edges pass through a new point. In this case we can use Rule 2 in the game.

Consider the side q_j of the shell S_k. Let $e_{i_k} < w_1 < w_2 < \ldots < w_l$ be the ordering of e_{i_k} and all new points on the side q_j (see Definition 4.3 and fig.6).

Consider the point w_1 and four chambers adjacent to w_1. Two of these chambers lie in the shell S_k and two chambers lie in the domain that is adjacent to S_k by the edge q_1. We claim that two chambers that do not lie in S_k belong to $S_1 \cup \ldots \cup S_{k-1}$.

In order to prove this we will consider two possible cases (considered also in arguments preceding Proposition 4.4) :

1) $S_k \cap S_{k-1} = q_1 \cup q_2$. In this case edges q_1, q_2 of S_k are also the edges of S_{k-1}. Therefore, two chambers adjacent to $w_1 \in q_j$, which do not lie in S_k, lie in S_{k-1}.

2) $S_k \cap S_{k-1}$ consists of one of the edges q_1, q_2, for example, $S_k \cap S_{k-1} = q_2$. In this case, the edge q_2 of S_k is also the edge of S_{k-1}. Therefore, if $w_1 \in q_2$, then two chambers adjacent to w_1, which do not lie in S_k, lie in S_{k-1}. By Proposition 4.4 the edge q_1 of S_k belongs to $S_1 \cup \ldots \cup S_{k-1}$. Therefore, if $w_1 \in q_1$, then the two chambers adjacent to w_1, which do not lie in S_k, lie in $S_1 \cup \ldots \cup S_{k-1}$.

Therefore, in either case, there are two chambers adjacent to w_1 that do not lie in S_k, and that are already painted. There is a chamber adjacent to w_1 that lies in S_k and has the vertex e_{i_k}, since $e_{i_k} < w_1$, $e_{i_k} \in q_j$, $j = 1, 2$. This chamber belongs to B_k and is painted blue. Therefore the fourth chamber adjacent to the point $w_1 \in q_j$ can be painted green according to Rule 2 of the game. This chamber lies in S_k.

Consider the point w_2, see fig.6. The same reasoning as the one applied to the point w_1, shows us that there are three chambers adjacent to the point w_2 that are already painted. Therefore we can paint the fourth chamber adjacent to w_2 according to Rule 2.

Fig.6

Similarly, we can paint consequently all the chambers in S_k that are adjacent to any new point $w \in q_j$.

If there are more than two edges $q \in Q$ that pass through a new point $w \in q_j$, it is easy to see that in case of the game with the Rule 1 we can use similar arguments to show that the chambers adjacent to the points $w_1 < \ldots < w_l$ can be consequently painted. $\square$

Let us again consider the set of chambers B constructed in section 3 as an initial coloring, i.e. let any chamber $\gamma \in B$ be painted blue.

Lemma 4.1 *Suppose that any chamber $\gamma \in S_1 \cup \ldots \cup S_{k-1}$, $\gamma \notin B$ is painted green. Then any chamber $\gamma \in S_k, \gamma \notin B$ can be painted green according to Rule 1 of the game.*

Proof. Let us again first consider the case when all the edges $q \in Q$ are in general position. In this case we use the Rule 2 in the game.

Suppose that there is a chamber $\gamma \notin B$, $\gamma \in S_k$ that cannot be painted. Let w_0 be its minimal vertex. Then there is a chamber γ' adjacent to the

point w_0 that also cannot be painted. From Remark 4 it is easy to see that the chamber γ' has a vertex that is smaller than w_0. Consider the minimal vertex of γ'. By applying the same arguments as above to the minimal vertex of γ' and repeating this process we will finally conclude that there is a chamber that cannot be painted and that is adjacent to one of the sides $q_j,\ j \in (1,2)$ of the shell S_k. Note that any chamber in S_k with the vertex e_{i_k} belongs to B_k and is painted by blue color. By Proposition 4.5, any chamber in S_k that is adjacent to a new vertex $w \in q_j$ can be painted. We have obtained the contradiction.

Consider the case when there can be several edges $q \in Q$ that pass through a new point. Suppose again that there is a chamber $\gamma \in S_k$ that cannot be painted and let w_0 be its minimal vertex. ¿From Rule 1 it follows that there is another chamber γ' that is adjacent to w_0 and to some edge q_0 that pass through w_0. However, this chamber γ' can have the same minimal vertex w_0, see fig.7. Then there is another edge q' of the chamber γ' such that $w_0 \in q'$. Applying Rule 1 to the edge q' we will conclude that there exists a chamber γ'' adjacent to the point w_0 and to some edge $q'',\ w_0 \in q''$ that cannot be painted. However, among the chambers that are adjacent to the vertex w_0 only the chambers inside the angle (e', w_0, e''), where e', e'' are the end points of the polygonal line $\mathcal{L}_k$, have the minimal vertex equal to w_0, (see fig.7). This follows from the direction introduced by the ordering of new points (see Remark 4). Therefore, choosing consequently nonpainted chambers adjacent to w_0 and some edge $q,\ w_0 \in q$ we finally choose a chamber that is adjacent to w_0 and lies outside the angle (e', w_0, e''). It is clear that this chamber has the minimal vertex smaller than w_0.

Now we can apply similar arguments as in the case of general position of edges $q \in Q$. We will finally see that there exists a chamber that is adjacent to a new point $w \in q_j$, which contradicts Proposition 4.5. $\square$

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Let us consider as an initial painting the set B of chambers constructed in section 3. We must prove that any chamber $\gamma \in P = conv(E),\ \gamma \notin B$ can be painted green according to Rule 1 of the game.

Let us prove this by induction on the polygons S_k. On the first step of induction any chamber $\gamma \in S_1$ that is adjacent to the point e_1, is painted blue, since $\gamma \in B_1$ by the construction. There are no other chambers in S_1.

Suppose that all the chambers $\gamma \in S_1 \cup S_2 \cup \ldots \cup S_{k-1}$, $\gamma \notin B$ are already painted green. We must prove that all the chambers in S_k can be painted green according to Rule 1 of the game. This follows from Lemma 4.2. $\square$

Fig.7

5 Linear independence of chambers

In section 4 we have proved that any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma \in B$. In order to prove that the set of chambers B constructed in section 3 is a basis we will prove the following theorem.

Let E be a finite set of points on the affine plane and B be the set of chambers constructed in section 3.

Theorem 5.1 *All chambers $\gamma \in B$ are linearly independent.*

Proof. We must prove that the columns g_γ, $\gamma \in B$ of the incidence matrix M are linearly independent.

Let us construct a new incidence matrix M', which will be a submatrix of M. First let us choose the columns g_γ of M that correspond to the chambers $\gamma \in B_1$. We will enumerate these chambers as $\gamma_1^1, \ldots, \gamma_{b_1}^1$. Then we choose the columns g_γ that correspond to the chambers $\gamma \in B_2$, i.e. to $\gamma_1^2, \ldots, \gamma_{b_2}^2$. We continue to consequently choose the columns of M' in the same way, i.e. the matrix M' will contain all the columns g_γ, $\gamma \in B$ of the matrix M.

Now let us show how to choose the rows that will belong to M'. We associate a simplex $\sigma \in \Sigma$ to each chamber $\gamma \in B$ in the following way.

Consider a chamber $\gamma_j^1, \quad j \in (1, \ldots, b_1)$. This chamber (by construction of B_1) has the vertex e_{i_1} and lies between two neighbor edges that have the end point e_{i_1}. Since any edge $q \in Q$ has two end points that belong to E, then at the other end of each of these two edges there is a point from E. Thus, the chamber γ_j^1 defines uniquely a simplex σ_j^1. Note that the simplex σ_j^1 contains only one of the chambers $\gamma_i^1 \in B_1$ (it can contain some chambers from $B_2, \ldots, B_{N-2}$) . Therefore,

$$a_{\sigma_j^1,\gamma_j^1} = 1, \quad a_{\sigma_j^1,\gamma_i^1} = 0, \quad if \quad i \neq j$$

Consider a chamber $\gamma_j^2 \in B_2$. Similarly to the case above , this chamber defines a simplex $\sigma_j^2 \in \Sigma$. It is clear that the simplex σ_j^2 can contain only one chamber $\gamma_j^2 \in B_2$. Since $B_2 \subset P_1$ and $P_1 \cap F_1 = \emptyset$ (see (??) and (??)), the simplex σ_j^2 cannot contain a chamber $\gamma \in B_1$. Therefore,

$$a_{\sigma_j^2,\gamma_j^2} = 1, \quad a_{\sigma_j^2,\gamma_i^2} = 0, \; if \quad i \neq j, \quad a_{\sigma_j^2,\gamma_i^1} = 0, \quad for \; any \; i \; .$$

Similarly, we associate a simplex $\sigma \in \Sigma$ to any chamber $\gamma \in B$. As the rows of the matrix M' we choose the rows of M that correspond to these simplices. The submatrix M' of the matrix M is presented in fig.8.

Fig.8

Since the matrix M' is triangular, the columns of M' are linearly independent. By construction of M' these columns correspond to the chambers $\gamma \in B$, therefore the columns $g_\gamma,\ \gamma \in B$ of matrix M are linearly independent. □

¿From Corollary 4.1 and Theorem 5.1. follows

Theorem 5.2 *Let E be a finite set of points on the affine plane and B be the set of chambers constructed in section 3. The set B is a combinatorial basis in the linear space V_Γ.*

¿From Theorem 1.1 and Theorem 5.2 it follows also

Theorem 5.3 *On the affine plane all the linear relations among chambers are linear combinations of relations (??) (in Theorem 1.1).*

This theorem is formulated for the n-dimensional space in [AGZ].

Using the algorithm of construction of the set B we can calculate the rank r of the relation $(A; \Sigma, \Gamma)$.

Proposition 5.1

$$r = \binom{N-1}{2} - \sum_{q \in Q} \binom{m(q)-1}{2},$$

where $m(q)$ is the number of points $e \in E$ on the edge $q \in Q$.

6 Construction of basis of simplices

The algorithm for construction of the set of chambers B described in section 3 is also used to construct a basis of simplices, i.e. a basis in V_Σ. In the proof of Theorem 5.1 we associate a simplex $\sigma \in \Sigma$ to each chamber $\gamma \in B$. Let B' be the set of simplices associated to $\gamma \in B$.

Definition 6.1 *A pair of a basis $e_1, \ldots, e_n$ in the space V and a basis $f_1, \ldots, f_n$ in the dual space V' is called a triangular pair if $(e_i, f_k) = 0$ for $i > k$ and $(e_i, f_i) = 1$.*

Theorem 6.1 *1. The set of simplices $B' = \{\sigma\}$ is the basis in V_Σ.*

2. The basis of chambers B and the basis of simplices B' form a triangular pair.

The proof follows from the fact that the incidence between the chambers $\gamma \in B$ and simplices $\sigma \in B'$ is given by the submatrix $M' \subset M$, which is triangular (see the proof of Theorem 5.1).

7 Appendix

7.1 Remark on visibility

In the section 3 we used the algorithm to construct the set B of chambers, we have proved (Theorem 5.2) that on the affine plane the set B is a basis of chambers. In the algorithm of construction of the set B we considered a sequence of points $e_{i_k} \in E$ that satisfied the condition (??). In Proposition 3.3 we proved that the point e_{i_k}, satisfying the condition (??) is P_{k-1}-visible from the preceding point $e_{i_{k-1}}$. However, the visibility of the points e_{i_k} in the algorithm is not sufficient to obtain the basis of chambers.

Fig.9

Fig.9 shows an example of the points on the affine plane. The set E consists of 6 points. The condition of visibility, even from the two preceding points in the sequence e_{i_k}, is not sufficient to construct a basis. The sequence of points e_1, e_2, e_3, e_4 in fig.9 is such that each point is visible from the points of the two preceding steps. For example, the point e_3 is P_2-visible from the point e_2 and is also P_2-visible from the point e_1, the point e_4 is P_3-visible from the point e_3 and also P_3-visible from the point e_2. However, the set of chambers constructed by the algorithm for this sequence $\{e_{i_k}\}$ (i.e. the set of chambers in P adjacent to the point e_1, chambers in P_1 adjacent to the point e_2, etc.) is not a basis of chambers. Of course, for this sequence of points e_1, e_2, e_3, e_4 the condition (??) is not satisfied.

7.2 Iterations in painting chambers

In section 4 we have defined the game on the affine plane. Let B be a successfull initial painting, i.e. B is the set of chambers painted blue in the game. Then any chamber $\gamma \notin B$ can be painted by green color according to the rule of the game. We can introduce the "height" for each chamber $\gamma \notin B$.

Definition 7.1 *Any chamber $\gamma \in B$ has the height 0.*
We say that a chamber $\gamma \notin B$ has the height k, if γ can be painted by green color when applying the rule to chambers with the height $< k$.

The process of painting chambers with green paint is similar to diffusion and the height k means the time that is required for the process to reach a certain chamber. Such notions are useful if we consider the continuous analogue of the game formulated in this paper.

Acknowledgements

I am very grateful to I.Gelfand, S.Gelfand, G.Rybnikov and T.Trimble for the helpful discussions.

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Combinatorial Bases in Systems of Simplices and Chambers

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May 12, 1994

Abstract

We consider a finite set E of points in the n-dimensional affine space and two sets of objects that are generated by the set E : the system Σ of n-dimensional simplices and the system Γ of chambers. The relation $(A; \Sigma, \Gamma)$ introduced by the incidence matrix $M = \| a_{\sigma,\gamma} \|$ defines the notion of linear independence and the rank of the system of simplices and of the system of chambers. We introduce the notion of a combinatorial basis. Combinatorial bases of chambers can be described in terms of a game. We describe the algorithm of decomposition of a convex polytope into shells. In the case of the affine plane, using the game and the algorithm we construct combinatorial basis B of chambers. Using the algorithm, we also construct a basis B' of simplices that together with the basis B of chambers form a "triangular pair".

Introduction

Let B and Γ be two finite sets. A relation [1] between B and Γ is a subset A of pairs $(\beta, \gamma) \in (B \times \Gamma)$. We will denote such a relation by $(A; B, \Gamma)$.

A relation can be presented by its incidence matrix M,

$$M = \|a_{\beta,\gamma}\|, \quad where \quad a_{\beta,\gamma} = 1, \ if \ (\beta,\gamma) \in A, \ a_{\beta,\gamma} = 0, \ if \ (\beta,\gamma) \notin A$$

Thus, the rows of the incidence matrix M correspond to the elements $\beta \in B$ and the columns of M correspond to the elements $\gamma \in \Gamma$.

Let us denote the row of the matrix M corresponding to an element β by f_β, $f_\beta \in \mathbf{R}^q$, where $q = \mid B \mid$ (i.e. q is equal to the number of elements in B). Let V_B be the linear subspace spanned by f_β, $\beta \in B$. Similarly, let us denote by g_γ, $\gamma \in \Gamma$, the columns of matrix M and by V_Γ the linear subspace spanned by $g_\gamma \in \mathbf{R}^p$, where $p = \mid \Gamma \mid$.

Since the rows of the matrix M are in one-to-one correspondence with the elements $\beta \in B$, instead of linear combinations $\sum c_\beta f_\beta$ of the rows of M we can consider linear combinations of elements of B, i.e. $\sum c_\beta \beta \in V_B$, where $\beta \in B$. We define that $\sum c_\beta \beta = 0$ if and only if $\sum c_\beta f_\beta = 0$.

Similarly, we will write linear combinations of the elements $\gamma \in \Gamma$, i.e. $\sum c_\gamma \gamma \in V_\Gamma$.

The linear spaces V_B and V_Γ are dual to each other. Indeed, we can construct the bilinear form

$$(\tilde{\beta}, \tilde{\gamma}) = \sum_{\beta,\gamma} a_{\beta,\gamma} c_\beta d_\gamma,$$

where $\tilde{\beta} = \sum_\beta c_\beta \beta \in V_B$, $\tilde{\gamma} = \sum_\gamma d_\gamma \gamma \in V_\Gamma$. It is easy to prove that the bilinear form $(\tilde{\beta}, \tilde{\gamma})$ is well defined and nondegenerate.

The important notion is the rank of a relation.

Definition 0.1 *Rank of the relation* $(A; B, \Gamma)$ *is the dimension of the subspace* V_B *or subspace* V_Γ.

Note that $dim V_B = dim V_\Gamma = rank M$.
The calculation of this rank is an important problem for different relations.

[1] The notion of relation was first introduced by MacLane.

In this paper we study relations connected with the following geometrical problem.

Consider a finite set E of N points in the n-dimensional affine space, $N > n$. Denote by P the convex hull of the set E. Any $n+1$ points in general position span a simplex σ. Denote by $\Sigma = \{\sigma\}$ the set of all these simplices. The set of simplices $\sigma \in \Sigma$ divide the polytope P into a finite number of chambers γ. Denote by Γ the set of all these chambers γ.

Thus we have two sets : the set Σ of all simplices σ and the set Γ of all chambers γ. [2] Let us denote by A the set of pairs (σ, γ), $\sigma \in S$, $\gamma \in \Gamma$, such that $\gamma \subset \sigma$. If $(\sigma, \gamma) \in A$ we will say that σ and γ are incident. Thus, for any finite set of points E in the n-dimensional affine space we have constructed a relation $(A; \Sigma, \Gamma)$.

Therefore, for the system of simplices and system of chambers the notion of rank is defined. These ranks are equal. The rank of the system of simplices has the following geometrical interpretation.

Let us denote by $\phi_\sigma(x)$ the characteristic function of a simplex σ, i.e.

$$\phi_\sigma(x) = 1, \ x \in \sigma \ \ and \ \ \phi_\sigma(x) = 0, \ \ if \ \ x \notin \sigma$$

It is easy to see that the following proposition holds.

Proposition 0.1 *A rank of the system of simplices Σ is the maximal number of linearly independent characteristic functions $\phi_\sigma(x)$.*

Systems of simplices and chambers appear in different problems (representation theory, Kostant partition functions , hypergeometric functions , etc.) and two important questions arise: 1) how to construct a basis of chambers, 2) how to construct a basis of chambers. It is important for combinatorial problems to give an explicit construction of bases consisting of chambers and bases of simplices and not of their linear combinations.

The combinatorial construction of a basis of simplices is contained in [AGZ]. In [AGZ] a theorem about the system of linear relations among simplices and a theorem about the system of linear relations among chambers are formulated. The approach in [AGZ] is based on the notion of marking (see also [B]) and is different from the geometrical approach used in the present

[2] In the general definition of a relations we denoted two sets as B and Γ.

paper. Note that, unlike [AGZ], in this paper we use only two systems of objects: the system of simplices and the system of chambers, and we do not consider the system of hyperplanes.

In section 1 we prove that among chambers in the n-dimensional affine space there are linear relations that have simple geometrical meaning (Theorem 1.1 , which was also formulated in [AGZ]).

On the affine plane these "basic" linear relations among chambers have the following form:

$$\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4, \tag{1}$$

where γ_i are chambers.

It turns out that all linear relations among chambers are linear combinations of linear relations (??) (see Theorem 5.3).

Let V_Σ be the linear space generated by simplices $\sigma \in \Sigma$ (or by rows of the incidence matrix M) and V_Γ be the linear space generated by chambers $\gamma \in \Gamma$ (or by columns of the incidence matrix M).

We construct combinatorial bases in V_Γ. A set B of chambers is called a combinatorial basis if any chamber (column of M) can be obtained from the set B by consequently applying the relation (??). We prove that such bases exist. Of course, any combinatorial basis is also a basis in the sense of linear algebra. [3]

In section 2 we introduce combinatorial bases in V_Γ. A combinatorial basis consists of chambers (and not of their linear combinations). Combinatorial bases of chambers naturally arise from some game. The game described in section 2 will be also used in the proof of Theorem 4.1, which is an important step in the proof of the main Theorem 5.2.

Section 3 contains an algorithm for the construction of some set of chambers B in the n-dimensional affine space. This set B will be studied in sections 4,5,6 in the case of the affine plane. However, it is also important that the described algorithm gives a decomposition of a convex polytope P into "shells" S_k (in case of the affine plane this decomposition into shells defines a triangulation). We describe geometrical properties of these shells S_k. In sections 4, 5, 6 we study in details the case of the affine plane.

[3] In the space V_Σ we can also define a combinatorial basis using the relations among simplices introduced in [AGZ]. The linear space V_Σ will be considered in another paper.

In section 4 we prove that on the affine plane any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma \in B$. In order to do this, we introduce a partial ordering of "new points" inside the shell S_k. We prove that any chamber in S_k has the minimal vertex. Then in terms of game described in section 2 we prove that the set B is a successfull initial painting (Theorem 4.1). From Theorem 4.1 it follows that any chamber is a linear combination of chambers from B.

In section 5 we prove that the chambers $\gamma \in B$ are linearly independent (Theorem 5.1) and that the set B is a combinatorial basis of chambers.

In section 6 we construct a basis of simplices B' using the algorithm from section 3. We show that there is a "triangular relation" between the basis of chambers B and the basis of simplices B'.

A pair of bases $e_1, \ldots, e_n$ in the space V and $f_1, \ldots, f_n$ in the dual space V' is called *triangular*, if $(e_i, f_k) = 0$, for $i > k$, and $(e_i, f_i) = 1$.

There are some remarks about the algorithm in the Appendix.

1 Basic linear relations among chambers

As we have mentioned in the Introduction we will consider a finite set of points $E = (e_1, e_2, \ldots, e_N)$ in the n-dimensional affine space V, $N > n$, and let $P = conv(E)$ be the convex hull of E. Let us assume that there are at least $n+1$ points in general position. If the points $e_{i_1}, \ldots, e_{i_{n+1}}$ are the points in general position we denote by $\sigma_{i_1,\ldots,i_{k+1}}$ (or sometimes simply by σ) the n-dimensional simplex spanned by these points. Denote by $\Sigma = \{\sigma\}$ the set of all these simplices. All simplices σ (usually overlapping) cover the polytope P. Simplices σ divide the polytope P into a finite number of chambers γ. (See examples of a system of simplices and a system of chambers in fig. 1).

Let us give the definition of a chamber. Denote by $\tilde{\sigma}$ the boundary of simplex σ and by $\tilde{\Sigma}$ the union of the boundaries of all simplices σ, i.e. $\tilde{\Sigma} = \bigcup_{\sigma \in \Sigma} \tilde{\sigma}$ and let us denote $\breve{P} = P \setminus \tilde{\Sigma}$. Let $\breve{\gamma}$ be a connected component of $\breve{P}$ and γ be closure of $\breve{\gamma}$.

Definition 1.1 *We will call γ a chamber and $\breve{\gamma}$ an open chamber.*

Denote by Γ the set of all chambers in P. Open chambers do not overlap and all the chambers cover the polytope P. Note that every chamber is a

polytope. We will say that a point is a *vertex of a chamber* if it is a vertex of the corresponding polytope.

Thus, for a finite set of points E we have constructed two sets of objects: a system of overlapping simplices Σ and a system of chambers Γ and we can consider a relation $(A; \Sigma, \Gamma)$ defined by the incidence matrix M.

For the system of simplices and chambers we repeat below the notations that we used in the Introduction. These notations will be used in all following sections.

M is the incidence matrix, i.e. $M = \| a_{\sigma,\gamma} \|$, $\sigma \in \Sigma$, $\gamma \in \Gamma$, , where

$$a_{\sigma,\gamma} = 1, \quad if \quad \gamma \subset \sigma, \quad a_{\sigma,\gamma} = 0, \quad if \quad \gamma \not\subset \sigma \tag{2}$$

f_σ is the row of M that corresponds to the simplex σ
g_γ is the column of M that corresponds to the chamber γ
V_Σ is the linear space spanned by simplices $\sigma \in \Sigma$
V_Γ is the linear space spanned by chambers $\gamma \in \Gamma$
r rank of the relation $(A; \Sigma, \Gamma)$.
$p = \mid \Sigma \mid$, $q = \mid \Gamma \mid$

As we have mentioned in the Introduction we can consider linear combinations of chambers instead of linear combinations of the corresponding columns of matrix M.

Definition 1.2 *Chambers $\gamma_1, \ldots, \gamma_k$ are linearly dependent if among these chambers there is a relation*

$$\sum_{i=1}^{k} c_i g_{\gamma_i} = 0$$

.

Below we will replace this relation by the following relation:

$$\sum_{i=1}^{k} c_i \gamma_i = 0 \tag{3}$$

The geometrical meaning of linear relations among chambers will be clear from this section. All linear relations among chambers are linear combinations of some simple basic " geometrical " relations (??).

In order to formulate these basic relations we will define a *new point*, and in order to determine the signs in the relation we will introduce an orientation around a new point.

Consider the set of all vertices of all chambers $\gamma \in \Gamma$. Some of these vertices are points from the set $E = \{e_i\}$ and some are not, see examples in fig.1.

Fig.1.

Definition 1.3 *A vertex w of a chamber $\gamma \in \Gamma$ is a new point if $w \notin E$.*

Let us denote by $W = \{w\}$ the set of all new points of all chambers $\gamma \in \Gamma$. The set of all vertices of all chambers $\gamma \in \Gamma$ is $E \cup W$.

Consider the case when through a new point $w \in W$ pass exactly n facets of simplices $\sigma \in \Sigma$, i.e. $(n-1)$-dimensional faces of simplices $\sigma \in \Sigma$. This means that all the facets of simplices $\sigma \in \Sigma$ are in general position.

We introduce the orientation around a new point. Let $w \in W$ be a new point. The new point w lies in the intersection of exactly n facets of some simplices $\sigma \in \Sigma$. Therefore we can introduce a local coordinate system with the origin at the point w if we choose these n facets as the coordinate hyperplanes and choose an arbitrary orientation.

Let us denote by $\gamma(w)$ the set of all chambers with the vertex w, i.e. $\gamma(w) = \{\gamma : w \in \gamma\}$. [4] Let $\gamma \in \gamma(w)$ and let $\xi \in \gamma$ be an arbitrary point with local coordinates $\xi_1, \ldots, \xi_n$. Then the chamber $\gamma \in \gamma(w)$ can be characterized

[4]Chambers $\gamma \in \gamma(w)$ are called chambers adjacent to the vertex w.

by the sequence of + and −, i.e. by $(sign(\xi_1), \ldots, sign(\xi_n))$. It is clear that this sequence does not depend on the point $\xi \in \gamma$.

If the point w lies inside the polytope P there are 2^n chambers in $\gamma(w)$ and they are coded by all the possible sequences of the length n consisting of + and −.

For a new point w we introduce the following function $\varepsilon_w(\gamma)$

$$\varepsilon_w(\gamma) = sign(\xi_1) \cdot sign(\xi_2) \cdot \ldots \cdot sign(\xi_n)$$

Remark 1. In case when not all the facets of simplices $\sigma \in \Sigma$ are in general position, a function $\varepsilon_w(\gamma)$ similar to the one defined here can be introduced. The formulation of Theorem 1.1 with this function $\varepsilon_w(\gamma)$ see in [AGZ]. We have this assumption in Theorem 1.1 to simplify the presentation of results that are used later. In sections 4,5,6 that contain the main results we do not make this assumption.

Note also that this restriction about general position of facets of simplices is applied only to a new point. Of course, there are no assumptions on an "old point" (a point from E).

Theorem 1.1 *For any new point $w \in W$ there is the following linear relation between chambers*

$$\sum_{\gamma \in \gamma(w)} \varepsilon_w(\gamma)\gamma = 0 \tag{4}$$

Note that the choice of another local coordinate system around w can only change simultaneously the signs of functions $\varepsilon_w(\gamma)$ for all $\gamma \in \gamma(w)$ and therefore will not change the relation (??).

For example, around the new point w_1 in fig.1 the relation (??) is $\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0$.

Proof of Theorem 1.1. Let $w \in W$ be a new point. We have to prove that among the columns g_γ, $\gamma \in \gamma(w)$ of the incidence matrix M (see formula (??)) there is a linear relation

$$\sum_{\gamma \in \gamma(w)} \varepsilon_w(\gamma) g_\gamma = 0 \tag{5}$$

The relation (??) means that

$$\sum_{\gamma \in \gamma(w)} \varepsilon_w(\gamma) a_{\sigma,\gamma}(w) = 0, \quad for\ any\ \sigma \in \Sigma \tag{6}$$

Since $a_{\sigma,\gamma} = 1$, if $\gamma \subset \sigma$ and $a_{\sigma,\gamma} = 0$, if $\gamma \not\subset \sigma$, the relation (??) is equivalent to the following:

$$\sum_{\gamma \in \gamma(w), \gamma \subset \sigma} \varepsilon_w(\gamma) = 0, \ for\ any\ \sigma \in \Sigma \tag{7}$$

Let us choose a simplex $\sigma \in \Sigma$. The following cases are possible:

1) $w \notin \sigma$;

2) w belongs to the interior of the simplex σ;

3) w lies on the boundary of the simplex σ and lies inside the polytope P;

4) w lies on the boundary of the simplex σ and lies on the boundary of the polytope P.

Let us consider these cases. In the case 1) we have $\gamma \not\subset \sigma$ for any chamber $\gamma \in \gamma(w)$ and therefore the relation (??) holds.

Let us consider the case 2). Then w lies inside the simplex σ together with a small neigborhood. Since any chamber is connected, this implies that all chambers $\gamma \in \gamma(w)$ belong to the simplex σ, i.e. $\gamma \subset \sigma$, $for\ any\ \gamma \in \gamma(w)$.

As we have mentioned above, any chamber $\gamma \in \gamma(w)$ can be coded by the sequence $(sign(\xi_1), \ldots, sign(\xi_n))$, where $\xi \in \gamma$. Since $\gamma \subset \sigma$ for any $\gamma \in \gamma(w)$, all possible sequences $(sign(\xi_1), \ldots, sign(\xi_n))$ will be present in the sum (??). Therefore, due to the symmetry among all the sequences of $+$ and $-$ of the length n, half of the functions $\varepsilon_w(\gamma)$ will be equal to $+1$ and the other half equal to -1 and the relation (??) holds.

Consider the case 3). The point w lies on the boundary of the simplex σ and inside the polytope $\bar{E}$. Then $w \in q$, where $q \subset \sigma$ is an $(n-k)$-dimensional open face of σ. This means (due to the assumption about general position of faces of simplices) that w lies in k coordinate hyperplanes around the point w. Let the local coordinates around the point w be $\xi_1, \ldots, \xi_n$. Then the face q is described by the equations $\xi_{i_1} = 0, \ldots, \xi_{i_k} = 0$. The simplex σ lies to one

of the sides of each of these hyperplanes (since q is the face of σ). This means that the signs of the coordinates $\xi_{i_1}, \ldots, \xi_{i_k}$ are fixed for the simplex σ and therefore are fixed for all the chambers $\gamma \in \gamma(w), \ \gamma \subset \sigma$. The remaining $(n-k)$ coordinates can have arbitrary signs in the simplex σ. Therefore the chambers $\gamma \in \gamma(w), \ \gamma \subset \sigma$ are coded by all the possible sequences of $+$ and $-$ of the length $(n-k)$ and it is clear that the relation (??) holds.

Consider the case 4). Let the point w lie in the $(n-k)$-dimensional open face of the simplex σ and in the $(n-m)$-dimensional open face of the polytope P. In this case it is clear that for any chamber $\gamma \in \gamma(w), \ \gamma \subset \sigma \ \ k$, coordinates of a point $\xi \in \gamma$ will be fixed "due to " the face of the simplex and m coordinates will be fixed "due to" the face of the polytope P. The remaining coordinates can be arbitrary and therefore the chambers $\gamma \in \gamma(w), \ \gamma \subset \sigma$ are coded by all the possible sequences that has $+$ and $-$ in these coordinates. Therefore the relation (??) holds. $\square$

2 Combinatorial bases of chambers. Game.

Definition 2.1 *A basis of chambers is called combinatorial if any other chamber can be expressed in terms of basis chambers by consequently applying the relation (??) (see Theorem 1.1).*

Combinatorial bases arise naturally from the game that is formulated below. This game will be also used to prove Theorem 4.1. Again let us assume [5] that through a new point pass exactly n facets of simplicies $\sigma \in \Sigma$, i.e. all the facets of simplices $\sigma \in \Sigma$ are in general position.

Game. Let $E = \{e_1, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space. Let Σ be the set of all n-dimensional simplices spanned by points $e_i \in E$ and Γ be the set of all chambers (see Definition 1.1). Let $W = \{w\}$ be the set of all new points (see Definition 1.3). One has to paint initially some chambers with, for example, blue paint and pay for each blue chamber. After the initial painting has been completed, it is allowed to paint with green paint some other chambers according to the following rule:

Rule. *If all chambers adjacent to a new point $w \in W$ except one are already painted (either blue or green), then the last chamber adjacent to the new point w can be painted with green paint.*

[5] In the section 4 we reformulate this game on the affine plane without the assumption.

Green chambers are "for free", i.e. one does not pay for a green chamber [6].

Definition 2.2 *An initial painting B is called sufficient if all the chambers $\gamma \in \Gamma,\ \gamma \notin B$ can be painted with green paint according to the rule above.*

The purpose of the game is to construct a sufficient initial painting that has the lowest price.

Examples. In fig.2 there are two configurations of 6 points on the affine plane, $E = (e_1, \ldots, e_6)$. There are examples of sufficient initial paintings. Blue chambers are shaded and green chambers are marked by "+". The sufficient initial painting in fig.2a does not have the lowest price, which is equal to 10. The sufficient initial painting in fig.2b has the lowest price.

Fig.2.

Note that in the game the process of painting chambers green is actually the way of consequently applying the relations (??). Therefore, a combinatorial basis can be constructed from a sufficient initial painting that has the lowest price.

[6] It is clear that in this game there are iterations of painting chambers green, because after we have painted some green chambers according to the rule, more chambers that can be painted green can appear.

3 Algorithm of construction of set B.

In this section we will algorithmically construct the set of chambers B. We will prove (sections 4,5) that on the affine plane this set of chambers is a basis of chambers. In the n-dimensional affine space the set B is not neccessarily a basis of chambers. Nevertheless, we describe here this algorithm in the n-dimensional affine space since it gives the decomposition of the polytope P into shells which will be used for the construction of basis of chambers in the n-dimensional affine space.

Lemma 3.1 *Let $E = \{e_1, e_2, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space. A sequence of points $e_{i_1}, e_{i_2}, \ldots, e_{i_N}$, $e_{i_k} \in E$, can be constructed such that*

$$F_k \cap P_k = \emptyset, \ for\ k = 1, \ldots, N, \tag{8}$$

where

$$P_k = conv(E \setminus (e_{i_1}, \ldots, e_{i_k}) \tag{9}$$

$$F_k = conv(e_{i_1}, \ldots, e_{i_k}) \tag{10}$$

We assume also $F_0 = \emptyset, P_N = \emptyset, P_0 = conv(E) = P, F_N = conv(E) = P$.

Proof. In Lemma 3.1 two sequences of polytopes are defined:

$$P_0 \supset P_1 \supset \ldots \supset P_N$$

$$F_0 \subset F_1 \subset \ldots \subset F_N$$

Let us prove the stronger statement: there exists a sequence of points $\{e_{i_k}\}$ such that not only the condition (??) is satisfied but also for each k there exists a hyperplane that separates the polytopes P_k and F_k.

We will prove this by induction. On the zero step consider the polytope P and any hyperplane that does not intersect P. The statement is satisfied. Let us construct the first point of the sequence. To do this we move the hyperplane untill it meets a point of the polytope P. There are two possibilities. If the hyperplane meets a vertex of P we take this vertex as the point e_{i_1}. If the hyperplane meets more than one point from P we turn the hyperplane slightly so that it will meet only one point from the polytope P. It is

clear that $F_1 \cap P_1 = \emptyset$. In order to obtain the hyperplane that separates P_1 and F_1 let us move a little the hyperplane mentioned above parallel to itself after it has met the vertex e_{i_1}. This hyperplane will separate the polytopes P_1 and F_1.

Suppose that we have constructed the sequence of points $e_{i_1}, \ldots, e_{i_{k-1}}$ such that $F_{k-1} \cap P_{k-1} = \emptyset$ and there is a hyperplane that separates the polytopes P_{k-1} and F_{k-1}. Let us move this hyperplane parallel to itself towards P_{k-1} untill it meets a point from P_{k-1}. As in the first step there are two possibilities: 1) the hyperplane meets a vertex of a polygon P_{k-1} ; 2) the hyperplane meets more than one point of the polygon P_{k-1}.
It is clear that the same arguments as for the point e_{i_1} allow us to choose the point e_{i_k} such that $P_k \cap F_k = \emptyset$ and choose the position of the hyperplane that separates these polytopes. $\Box$

Algorithm. Let us construct the set of chambers B. Let $E = \{e_1, \ldots, e_N\}$ be a finite set of points in the n-dimensional affine space and let Σ be the set of simplices and Γ be the set of chambers (see section 1). Construction of the set of chambers B will be made by steps. Let us denote by B_k the set of chambers that will be constructed on the k-th step. We define then

$$B = \{\gamma : \gamma \in \bigcup_k B_k\} \quad .$$

First step. We choose a vertex of the polytope P and denote it by e_{i_1}. We define the set B_1 as a set of all chambers in P adjacent to the vertex e_{i_1}, i.e.

$$B_1 = \{\gamma : \quad \gamma \subset P, \ e_{i_1} \in \gamma\} \quad .$$

Second step. We choose a point $e_{i_2} \in E \setminus e_{i_1}$ such that $F_2 \cap P_2 = \emptyset$ (see notations (??) and (??) .) By Lemma 3.1, such a point exists. We define the set of chambers B_2 as the set of all chambers in the polytope P_1 adjacent to the point e_{i_2}, i.e.

$$B_2 = \{\gamma : \ \gamma \subset P_1, \quad e_{i_2} \in \gamma\} \quad .$$

Step k. We choose a point $e_{i_k} \in E \setminus (e_{i_1}, \ldots, e_{i_{k-1}})$ such that $F_k \cap P_k = \emptyset$ and define a set B_k as the set of all chambers in the polytope P_{k-1} that are adjacent to the point e_{i_k}, i.e.

$$B_k = \{\gamma : \ \gamma \in P_{k-1}, \ e_{i_k} \in \gamma\} \ .$$

Since in the set of polytopes $P_0 \supset P_1 \supset P_2 \supset \ldots$ every polytope has one vertex less than the preceding one, the algorithm ends after a finite number of steps. As the result, we obtain the set B of chambers

$$B = \{\gamma : \gamma \in \bigcup_k B_k\}. \tag{11}$$

Proposition 3.1 *The number of steps in the algorithm is equal to* $N - n$.

Proof. We continue the algorithm as long as there is at least one chamber in the polytope P_k. If the polygon P_k has only n vertices it contains no chambers. Therefore the algorithm ends after $N - n$ steps. □

On the complexity of the algorithm. From Proposition 3.1 it follows that the number of operations in the algorithm is $O(N)$.

3.1 Visibility and shells.

The algorithm that constructs a set of chambers B described in section 3 gives a decomposition of the polytope P into shells . We will describe here their geometrical properties.

Let us introduce a notion of visibility.

Definition 3.1 *Let* V *be an* n*-dimensional affine space ,* P *a convex polytope,* e *a point such that* $e \notin P$. *We say that the point* $p \in V$ *is* P*-visible from the point* e *if* $(e, p) \cap P = \emptyset$, *where* (e, p) *is an open segment.*

We have constructed in section 3 a sequence of points $\{e_{i_k}\}$ and two sets of polytopes P_k and F_k, $k = 1, \ldots, N - n$. Let us consider the polytope

$$S_k = \overline{P_{k-1} \setminus P_k}, \tag{12}$$

where $\bar{A}$ means the closure of the set A. From the definition of polytopes S_k it follows

Proposition 3.2 *Polytope P is the disjoint union of the polytopes S_k,*

$$P = \bigcup_{i=1}^{N-n} S_i \quad . \tag{13}$$

The polytopes S_k will be called *shells*. The following propositions describe the geometrical properties of the shells S_k.

Proposition 3.3 *The point $e_{i_k} \in P_{k-1}$ is $P_{k-1}-$visible point from the point $e_{i_{k-1}}$.*

Proof. By construction, the point e_{i_k} is one of the vertices of the polytope P_{k-1} . Suppose that the point e_{i_k} is not P_{k-1}-visible from the point $e_{i_{k-1}}$. Then $(e_{i_k-1}, e_{i_k}) \cap P_{k-1} \neq \emptyset$, where (e_{i_k-1}, e_{i_k}) is an open segment. Then this segment crosses one of the faces of P_{k-1}, for example, the face q. Notice that $q \subset P_k$ since P_k contains all the vertices of P_{k-1} except the vertex e_{i_k} (we recall that $P_k = conv(E \setminus (e_{i_1}, \ldots, e_{i_k}))$). Then we obtain that $P_k \cap (e_{i_k}, e_{i_{k-1}}) \neq \emptyset$. Since $e_{i_k}, e_{i_{k-1}} \in F_k$ we have $P_k \cap F_k \neq \emptyset$. This contradicts the condition (??). $\square$

Remark 2. From Proposition 3.3 it follows that:

1. The polytope S_k has the following shape: S_k is a part of the convex cone with the vertex e_{i_k} bounded by the "cover" $\mathcal{L}_k$, where $\mathcal{L}_k$ is the set of all points of the polytope P_k that are P_k-visible from the point e_{i_k}.

2. The "cover" $\mathcal{L}_k$ satisfies the following condition: if $x_1,\ x_2 \in \mathcal{L}_k$, then any point $x \subset (x_1, x_2)$ either belongs to $\mathcal{L}_k$ or does not lie in S_k.

3. Inside S_k there are no points of E.

In particular, if V is the affine plane, the shell S_k is a part of an angle with the vertex e_{i_k}, two sides q_1, q_2 and the "cover" $\mathcal{L}_k$ which is a convex polygonal line, see fig. 3.

Let $Q = \{q\}$ be the set of all faces of all simplices $\sigma \in \Sigma$ and let $star(e) = \{q \in Q : e \text{ is the vertex of } q\}$. It is easy to see that Proposition 3.3 implies the following result.

Proposition 3.4 *Let S_k and S_{k+1} be two neighbor shells, e_{ik} be the vertex of S_k and $e_{i_{k+1}}$ be the vertex of S_{k+1}. Then*

$$S_k \cap S_{k+1} = star(e_{i_{k+1}}) \cap \mathcal{L}_k,$$

where $\mathcal{L}_k$ is the set of all points P_k-visible from the point e_{i_k}.

Note that on the affine plane the decomposition of the polygon P into shells S_k defines also a triangulation of the polygon P, since the shell S_k already consists of triangles (see fig.3).

Fig.3.

4 Sufficient initial paintings of chambers

In section 3 we have constructed a set of chambers B in the n-dimensional affine space V. In this section we will prove that in case of the affine plane any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma \in B$.

Consider the affine plane V. Let E be a finite set of points in V, Σ be the set of simplices (triangles), Γ be the set of chambers, W be the set of all new points (see Definition 1.3), and $Q = \{q\}$ be the set of all edges of all triangles $\sigma \in \Sigma$.

Definition 4.1 *We say that a chamber $\gamma \in \Gamma$ is adjacent to some edge $q \in Q$, if there is an edge q' of the chamber γ, such that $q' \subset q$.*

Consider the following game.

Game on the affine plane. One must paint some chambers blue and pay for them (initial painting). After that some other chambers can be painted green according to the following rule.

Rule 1. *If there are three painted chambers (either blue or green) adjacent to some edge $q \in Q$ that pass through a new vertex $w \in W$ then the fourth chamber adjacent to q can be painted green.*
Green chambers are "for free", i.e. one does not have to pay for a green chamber.

Definition 4.2 *An initial painting B is said to be sufficient if any chamber $\gamma \in \Gamma$, $\gamma \notin B$, can be painted green color according to the rule above.*

The purpose of the game is to construct a sufficient initial coloring that has the lowest price.

Remark 3. In this game we do not impose any restrictions on the number of edges that pass through a new point. It is easy to see that in case when all the edges $q \in Q$ are in general position this game will coincide with the game defined in section 2.

Therefore, in this case (when all the edges $q \in Q$ are in general position) we can use the following Rule 2 instead of Rule 1.

Rule 2. *If there are three painted chambers adjacent to a new point $w \in W$, then the fourth chamber adjacent to w can be painted green (i.e. "for free").*

Note also that in case when more than two edges $q \in Q$ pass through a new point we can move slightly the points so that these edges will be in general position. As a result, some new chambers and new points will appear and through any new point only two edges will pass. Then we can use the Rule 2 instead of the Rule 1 . One can check that in either case the minimal number of chambers that we have to paint blue in order to be able to paint the rest of the chambers green, is the same.

We can now formulate the main theorem of this section. In section 3 we have constructed the set of chambers B. Consider the set B as an initial painting, i.e. let us paint all the chambers $\gamma \in B$ blue.

Theorem 4.1 *Let E be a finite set of points on the affine plane. The set of chambers $\{\gamma : \gamma \in B\}$ is a sufficient initial painting.*

Corollary 4.1 *Any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma_i \in B$, i.e.*

$$\gamma = \sum c_i \gamma_i, \quad \gamma_i \in B$$

Let us prove that Corollary 4.1 follows from Theorem 4.1. First, let us make the following remark. Rule 1 and Rule 2 mean that any green chamber γ is a linear combination of chambers $\gamma_1, \gamma_2, \gamma_3$, i.e. $\gamma = \gamma_1 - \gamma_2 + \gamma_3$, where chambers $\gamma_1, \gamma_2, \gamma_3$ are already painted (either blue or green). In Rule 1 chambers $\gamma, \gamma_1, \gamma_2, \gamma_3$ are the chambers adjacent to some edge $q \in Q$ and to some new point w, $w \in q$. In Rule 2 chambers $\gamma, \gamma_1, \gamma_2, \gamma_3$ are the chambers adjacent to some new point w. In Theorem 1.1 it is proved that the chambers adjacent to a new point satisfy the relation $\gamma = \gamma_1 - \gamma_2 + \gamma_3$, i.e. we get Rule 2. The proof for the case of Rule 1 is similar and we do not include it here.

Let chambers $\gamma \in B$ are painted blue. Consider $\gamma \in \Gamma$, $\gamma \notin B$. By Theorem 4.1, any chamber $\gamma \notin B$ can be painted green according to the rule of the game. Therefore, there exists a new point $w \in \gamma$ and an edge $q \in Q$, $w \in q$, $q \subset \gamma$ such that other three chambers $\gamma_1, \gamma_2, \gamma_3$ adjacent to q are already painted, i.e. we have the relation $\gamma = \gamma_1 - \gamma_2 + \gamma_3$. The chamber γ is a linear combination of chambers that have been already painted blue or green. Since initially we have painted only blue chambers and then green chambers were consequently painted using the rule, γ is a linear combination of blue chambers , i.e. of chambers $\gamma \in B$.

Before we can prove Theorem 4.1 we will prove some propositions describing shells S_k on the affine plane.

Consider the shell S_k. It has the shape of an angle with the vertex e_{i_k} , two sides $q_1, q_2 \in Q$ and the "cover" $\mathcal{L}_k$ (see Remark 2 and fig.3).

Proposition 4.1 *Any edge $q \in Q$ that intersects the polygon S_k can intersect only one of the edges q_1, q_2.*

Proof. Note that no edge $q \in Q$ ends inside S_k, (see Remark 2). Since q_1, q_2 are the sides of the cone S_k with the vertex e_{i_k} , any edge $q \in Q$ that intersects S_k either pass through the point e_{i_k} or intersects at least one of the sides q_1, q_2. Let us prove that an edge $q \in Q$ can intersect only one of the sides q_1, q_2.

Suppose that some $q \in Q$ intersects both edges q_1 and q_2. The edge q has two end points that belong to E, for example, $e', e'' \in E$. Consider the convex polygons $F_{k-1} = conv(e_{i_1}, \ldots, e_{i_{k-1}})$ and $P_{k-1} = conv(e_{i_k}, \ldots, e_{i_N})$ (see formulae (??) and (??). Any point $e \in E$ belongs either to F_{k-1} or to P_{k-1}. We will show that the points e', e'' cannot belong to either of these polygons.

We claim that $e', e'' \notin P_{k-1}$. Indeed, the convex polygon $P_{k-1} = S_k \cup P_k$ has edges q_1 and q_2. Since by assumption the segment (e', e'') intersects q_1 and q_2, we have $e', e'' \notin P_{k-1}$.

On the other hand, $e', e'' \notin F_{k-1}$. Indeed, otherwise $P_{k-1} \cap F_{k-1} \neq \emptyset$ which is forbidden by the condition (??).

We obtained that $e', e'' \in E$ and $e', e'' \notin P_{k-1} \cup F_{k-1}$. This contradiction proves the proposition. $\square$

Corollary 4.2 *Any edge $q \in Q$ that intersects S_k intersects the polygonal line $\mathcal{L}_k$ at exactly one point.*

This follows from Proposition 4.1 and the fact that the polygonal line $\mathcal{L}_k$ is convex (see Remark 2).

We introduce a partial ordering of the new points $w \in W$ in the polygon S_k as follows. As above, let q_1, q_2 be the sides of the shell S_k.

Definition 4.3 *Let w_1, w_2 be new points that lie on some edge $q \in Q$. There are two possibilities.*

1) The edge q passes through the point e_{i_k} , i.e. $e_{i_k} \in q$. Then we say that $w' < w''$, if $\mid e_{i_k}, w' \mid < \mid e_{i_k}, w'' \mid$, where $\mid e, w \mid$ is the length of the segment (e, w).

2) The edge q intersects one of the sides q_1, q_2 of the shell S_k, for example, q_1 and let w_0 be a new point in their intersection, i.e. $q \cap q_1 = w_0$, $w_0 \in W$. Then for any $w', w'' \in q$ we say that $w' < w''$, if $\mid w_0, w' \mid < \mid w_0, w'' \mid$.

We also define $e_{i_k} < w$, for any $w \in W \cap S_k$.

By Proposition 4.1, any $q \in Q$ can intersect only one of the sides q_1, q_2 of the shell S_k therefore this partial ordering is well defined.

Remark 4. Note that this partial ordering of new points in S_k induces a direction along each edge $q \in Q$ that intersects S_k. All these edges q are directed from inside the shell S_k towards the polygonal line $\mathcal{L}_k$, see fig.4 .

Definition 4.4 *Let $\gamma \in S_k$ be an arbitrary chamber $\gamma \in \Gamma$. A vertex $w \in \gamma$ is called a minimal new vertex of the chamber γ, if $w < w'$ for any $w' \in \gamma$ such that (w, w') is an edge of the chamber γ.*

Consider the shell S_k with the vertex e_{i_k} and the "cover" $\mathcal{L}_k$. As a special case, the polygonal line $\mathcal{L}_k$ can be a segment and the shell S_k will then be a triangle. We need a proposition concerning this special case.

Proposition 4.2 *Let S_k be the shell with the "cover" $\mathcal{L}_k$, where $\mathcal{L}_k$ is a segment with the end points e', e''. Any chamber $\gamma \in S_k$ has the minimal vertex.*

Proof.

Let $\gamma \in S_k$. Consider the vertex $w \in \gamma$ that has the largest distance [7] from (e', e''). Note that such point $w \in \gamma$ is unique since any edge $q \in Q$ that intersects the triangle $S_k = (e_{i_k}, e', e'')$ intersects also the side (e', e'') (by Corollary 4.2).

Both edges q', q'' of the chamber γ such that $w \in q',\ w \in q''$ are directed away from w, since all edges q that intersect S_k are directed towards (e', e'') (see Remark 4 and fig.4). Therefore, w is the minimal vertex of γ. $\square$

Proposition 4.3 *In the polygon S_k any chamber γ has the minimal vertex.*

Proof. The polygon S_k is the part of an angle with the vertex e_{i_k} bounded by the convex polygonal line $\mathcal{L}_k$. Let e', e'' be the end points of this polygonal line. Consider the triangle (e_{i_k}, e', e''), (see fig.4). We have $S_k \subset (e_{i_k}, e', e'')$. Let us extend all the edges $q \in Q$ that intersect S_k so that they will intersect the triangle (e_{i_k}, e', e''). Any such edge will intersect the side (e', e'') and is directed towards it (see Corollary 4.2 and Remark 4).

Consider a chamber $\gamma \in S_k$. There are two possibilities:

[7] More precisely this means that if we introduce the barycentric coordinates s_1, s_2, s_3 in the triangle (e', e'', e_{i_k}), and the equation of the side (e', e'') is $s_1 = 0$, then we consider the point $w \in \gamma$ with the largest coordinate s_1.

1) $\gamma \cap \mathcal{L}_k = \emptyset$ Then the chamber γ has the minimal vertex by the Proposition 4.2.

2) $\gamma \cap \mathcal{L}_k \neq \emptyset$. Let us remove the line $\mathcal{L}_k$. We obtain chambers in the triangle (e_{i_k}, e', e'') and by Proposition 4.2 any such chamber has the minimal vertex. It is clear that by removing the line $\mathcal{L}_k$ we obtained a chamber γ' such that $\gamma' \supset \gamma$. Let w be the minimal point of γ'. Obviously, (see fig.4) w lies "below" the polygonal line $\mathcal{L}_k$, i.e. inside the shell S_k and is the vertex of γ, moreover, w is the minimal vertex of γ. $\square$

Fig.4

Let S_k be a shell with the vertex e_{i_k} , the "cover" $\mathcal{L}_k$, and the sides q_1, q_2, (see fig 3). Consider $S_k \cap S_{k-1}$. From Proposition 3.4 it follows that two cases are possible:

1) $e_{i_k} \in \mathcal{L}_{k-1}$ is not an end point of the polygonal line $\mathcal{L}_{k-1}$. Then $S_k \cap S_{k-1} = q_1 \cup q_2$.

2) $e_{i_k} \in \mathcal{L}_{k-1}$ is an end point of $\mathcal{L}_{k-1}$. Then $S_k \cap S_{k-1}$ contains only one of the edges q_1, q_2.

The following proposition is related only to the case 2). [8]

Proposition 4.4 *Let edges q_1, q_2 be the sides of the shell S_k and let $q_1 \notin S_k \cap S_{k-1}$. If there exists an edge $q \in Q$ that intersects q_1 then for any point $x \in q_1$ we have $x \in S_1 \cup \ldots \cup S_{k-1}$.*

[8] In case 1) Proposition 4.4 is trivial.

Proof. Let an edge $q \in Q$ intersect the edge q_1. Any edge $q \in Q$ has two end points that belong to E. Let e', e'' be the end points of q and $e_{i_k}, \tilde{e}$ be the end points of q_1. Due to Corollary 4.2 the edge q intersects the "cover" $\mathcal{L}_k$ of the shell S_k. Therefore, one of the points e', e'' lies in P_k, for example $e' \in P_k$ (see fig.5).

Fig.5

Consider the triangle (e_{i_k}, e', e''). Let x be a point such that $x \in int(e_{i_k}, e', e'')$, (where $int(A)$ is the set of all interior points of a set A), i.e. $x \in P = conv(E)$. Consider the polygon P,

$$P = S_1 \cup \ldots \cup S_{k-1} \cup S_k \cup \ldots \cup S_{N-2},$$

(see formula (??)). Since $S_k = \overline{P_{k-1} \setminus P_k}$, (see formula (??)) and $P_N = \emptyset$, we have

$$P = S_1 \cup \ldots \cup S_{k-1} \cup P_{k-1}.$$

It is clear that $x \notin P_k \cup S_k = P_{k-1}$, (see fig.5). Therefore, $x \in S_1 \cup \ldots \cup S_{k-1}$.

Since a shell is a closed set, any x that belongs to the boundary of (e_{i_k}, e', e''), in particular, any $x \in (e_{i_k}, \tilde{e}) = q_1$, belongs to the set $S_1 \cup \ldots \cup S_{k-1}$. Note also that $x \notin S_{k-1}$. $\square$

In further proofs we will use the game (formulated at the beginning of section 4) Let B be the set of chambers constructed in section 3. Consider the chambers $\gamma \in B$ as an initial painting for the game on the affine plane.

In other words, let us paint all chambers $\gamma \in B$ blue. According to the game we can now paint some chambers green if they satisfy Rule 1.

Let e_{i_k} be the vertex and q_1, q_2 be the sides of the shell S_k. We will explain how to paint chambers in S_k adjacent to the sides q_1, q_2 of the shell S_k.

Proposition 4.5 *Suppose that any $\gamma \in S_1 \cup \ldots \cup S_{k-1}$, $\gamma \notin B$ is painted green. Let $\gamma' \in S_k$ be a chamber adjacent to a new point $w \in q_j$. The chamber γ' can be painted green according to Rule 1 of the game.*

Proof. Let us first consider the case when all the edges $q \in Q$ are in general position, i.e. only two edges pass through a new point. In this case we can use Rule 2 in the game.

Consider the side q_j of the shell S_k. Let $e_{i_k} < w_1 < w_2 < \ldots < w_l$ be the ordering of e_{i_k} and all new points on the side q_j (see Definition 4.3 and fig.6).

Consider the point w_1 and four chambers adjacent to w_1. Two of these chambers lie in the shell S_k and two chambers lie in the domain that is adjacent to S_k by the edge q_1. We claim that two chambers that do not lie in S_k belong to $S_1 \cup \ldots \cup S_{k-1}$.

In order to prove this we will consider two possible cases (considered also in arguments preceding Proposition 4.4) :

1) $S_k \cap S_{k-1} = q_1 \cup q_2$. In this case edges q_1, q_2 of S_k are also the edges of S_{k-1}. Therefore, two chambers adjacent to $w_1 \in q_j$, which do not lie in S_k, lie in S_{k-1}.

2) $S_k \cap S_{k-1}$ consists of one of the edges q_1, q_2, for example, $S_k \cap S_{k-1} = q_2$. In this case, the edge q_2 of S_k is also the edge of S_{k-1}. Therefore, if $w_1 \in q_2$, then two chambers adjacent to w_1, which do not lie in S_k, lie in S_{k-1}. By Proposition 4.4 the edge q_1 of S_k belongs to $S_1 \cup \ldots \cup S_{k-1}$. Therefore, if $w_1 \in q_1$, then the two chambers adjacent to w_1, which do not lie in S_k, lie in $S_1 \cup \ldots \cup S_{k-1}$.

Therefore, in either case, there are two chambers adjacent to w_1 that do not lie in S_k, and that are already painted. There is a chamber adjacent to w_1 that lies in S_k and has the vertex e_{i_k}, since $e_{i_k} < w_1$, $e_{i_k} \in q_j$, $j = 1, 2$. This chamber belongs to B_k and is painted blue. Therefore the fourth chamber adjacent to the point $w_1 \in q_j$ can be painted green according to Rule 2 of the game. This chamber lies in S_k.

Consider the point w_2, see fig.6. The same reasoning as the one applied to the point w_1, shows us that there are three chambers adjacent to the point w_2 that are already painted. Therefore we can paint the fourth chamber adjacent to w_2 according to Rule 2.

Fig.6

Similarly, we can paint consequently all the chambers in S_k that are adjacent to any new point $w \in q_j$.

If there are more than two edges $q \in Q$ that pass through a new point $w \in q_j$, it is easy to see that in case of the game with the Rule 1 we can use similar arguments to show that the chambers adjacent to the points $w_1 < \ldots < w_l$ can be consequently painted. $\square$

Let us again consider the set of chambers B constructed in section 3 as an initial coloring, i.e. let any chamber $\gamma \in B$ be painted blue.

Lemma 4.1 *Suppose that any chamber $\gamma \in S_1 \cup \ldots \cup S_{k-1}$, $\gamma \notin B$ is painted green. Then any chamber $\gamma \in S_k, \gamma \notin B$ can be painted green according to Rule 1 of the game.*

Proof. Let us again first consider the case when all the edges $q \in Q$ are in general position. In this case we use the Rule 2 in the game.

Suppose that there is a chamber $\gamma \notin B$, $\gamma \in S_k$ that cannot be painted. Let w_0 be its minimal vertex. Then there is a chamber γ' adjacent to the

point w_0 that also cannot be painted. From Remark 4 it is easy to see that the chamber γ' has a vertex that is smaller than w_0. Consider the minimal vertex of γ'. By applying the same arguments as above to the minimal vertex of γ' and repeating this process we will finally conclude that there is a chamber that cannot be painted and that is adjacent to one of the sides $q_j,\ j \in (1,2)$ of the shell S_k. Note that any chamber in S_k with the vertex e_{i_k} belongs to B_k and is painted by blue color. By Proposition 4.5, any chamber in S_k that is adjacent to a new vertex $w \in q_j$ can be painted. We have obtained the contradiction.

Consider the case when there can be several edges $q \in Q$ that pass through a new point. Suppose again that there is a chamber $\gamma \in S_k$ that cannot be painted and let w_0 be its minimal vertex. ¿From Rule 1 it follows that there is another chamber γ' that is adjacent to w_0 and to some edge q_0 that pass through w_0. However, this chamber γ' can have the same minimal vertex w_0, see fig.7. Then there is another edge q' of the chamber γ' such that $w_0 \in q'$. Applying Rule 1 to the edge q' we will conclude that there exists a chamber γ'' adjacent to the point w_0 and to some edge $q'',\ w_0 \in q''$ that cannot be painted. However, among the chambers that are adjacent to the vertex w_0 only the chambers inside the angle (e', w_0, e''), where e', e'' are the end points of the polygonal line $\mathcal{L}_k$, have the minimal vertex equal to w_0, (see fig.7). This follows from the direction introduced by the ordering of new points (see Remark 4). Therefore, choosing consequently nonpainted chambers adjacent to w_0 and some edge $q,\ w_0 \in q$ we finally choose a chamber that is adjacent to w_0 and lies outside the angle (e', w_0, e''). It is clear that this chamber has the minimal vertex smaller than w_0.

Now we can apply similar arguments as in the case of general position of edges $q \in Q$. We will finally see that there exists a chamber that is adjacent to a new point $w \in q_j$, which contradicts Proposition 4.5. $\square$

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Let us consider as an initial painting the set B of chambers constructed in section 3. We must prove that any chamber $\gamma \in P = conv(E),\ \gamma \notin B$ can be painted green according to Rule 1 of the game.

Let us prove this by induction on the polygons S_k. On the first step of induction any chamber $\gamma \in S_1$ that is adjacent to the point e_1, is painted blue, since $\gamma \in B_1$ by the construction. There are no other chambers in S_1.

Suppose that all the chambers $\gamma \in S_1 \cup S_2 \cup \ldots \cup S_{k-1}$, $\gamma \notin B$ are already painted green. We must prove that all the chambers in S_k can be painted green according to Rule 1 of the game. This follows from Lemma 4.2. $\square$

Fig.7

5 Linear independence of chambers

In section 4 we have proved that any chamber $\gamma \in \Gamma$ is a linear combination of chambers $\gamma \in B$. In order to prove that the set of chambers B constructed in section 3 is a basis we will prove the following theorem.

Let E be a finite set of points on the affine plane and B be the set of chambers constructed in section 3.

Theorem 5.1 *All chambers $\gamma \in B$ are linearly independent.*

Proof. We must prove that the columns g_γ, $\gamma \in B$ of the incidence matrix M are linearly independent.

Let us construct a new incidence matrix M', which will be a submatrix of M. First let us choose the columns g_γ of M that correspond to the chambers $\gamma \in B_1$. We will enumerate these chambers as $\gamma_1^1, \ldots, \gamma_{b_1}^1$. Then we choose the columns g_γ that correspond to the chambers $\gamma \in B_2$, i.e. to $\gamma_1^2, \ldots, \gamma_{b_2}^2$. We continue to consequently choose the columns of M' in the same way, i.e. the matrix M' will contain all the columns g_γ, $\gamma \in B$ of the matrix M.

Now let us show how to choose the rows that will belong to M'. We associate a simplex $\sigma \in \Sigma$ to each chamber $\gamma \in B$ in the following way.

Consider a chamber $\gamma_j^1, \quad j \in (1, \ldots, b_1)$. This chamber (by construction of B_1) has the vertex e_{i_1} and lies between two neighbor edges that have the end point e_{i_1}. Since any edge $q \in Q$ has two end points that belong to E, then at the other end of each of these two edges there is a point from E. Thus, the chamber γ_j^1 defines uniquely a simplex σ_j^1. Note that the simplex σ_j^1 contains only one of the chambers $\gamma_i^1 \in B_1$ (it can contain some chambers from $B_2, \ldots, B_{N-2}$) . Therefore,

$$a_{\sigma_j^1,\gamma_j^1} = 1, \quad a_{\sigma_j^1,\gamma_i^1} = 0, \quad if \quad i \neq j$$

Consider a chamber $\gamma_j^2 \in B_2$. Similarly to the case above , this chamber defines a simplex $\sigma_j^2 \in \Sigma$. It is clear that the simplex σ_j^2 can contain only one chamber $\gamma_j^2 \in B_2$. Since $B_2 \subset P_1$ and $P_1 \cap F_1 = \emptyset$ (see (??) and (??)), the simplex σ_j^2 cannot contain a chamber $\gamma \in B_1$. Therefore,

$$a_{\sigma_j^2,\gamma_j^2} = 1, \quad a_{\sigma_j^2,\gamma_i^2} = 0, \; if \quad i \neq j, \quad a_{\sigma_j^2,\gamma_i^1} = 0, \quad for \; any \; i \; .$$

Similarly, we associate a simplex $\sigma \in \Sigma$ to any chamber $\gamma \in B$. As the rows of the matrix M' we choose the rows of M that correspond to these simplices. The submatrix M' of the matrix M is presented in fig.8.

Fig.8

Since the matrix M' is triangular, the columns of M' are linearly independent. By construction of M' these columns correspond to the chambers $\gamma \in B$, therefore the columns $g_\gamma,\ \gamma \in B$ of matrix M are linearly independent. □

¿From Corollary 4.1 and Theorem 5.1. follows

Theorem 5.2 *Let E be a finite set of points on the affine plane and B be the set of chambers constructed in section 3. The set B is a combinatorial basis in the linear space V_Γ.*

¿From Theorem 1.1 and Theorem 5.2 it follows also

Theorem 5.3 *On the affine plane all the linear relations among chambers are linear combinations of relations (??) (in Theorem 1.1).*

This theorem is formulated for the n-dimensional space in [AGZ].

Using the algorithm of construction of the set B we can calculate the rank r of the relation $(A; \Sigma, \Gamma)$.

Proposition 5.1

$$r = \binom{N-1}{2} - \sum_{q \in Q} \binom{m(q)-1}{2},$$

where $m(q)$ is the number of points $e \in E$ on the edge $q \in Q$.

6 Construction of basis of simplices

The algorithm for construction of the set of chambers B described in section 3 is also used to construct a basis of simplices, i.e. a basis in V_Σ. In the proof of Theorem 5.1 we associate a simplex $\sigma \in \Sigma$ to each chamber $\gamma \in B$. Let B' be the set of simplices associated to $\gamma \in B$.

Definition 6.1 *A pair of a basis $e_1, \ldots, e_n$ in the space V and a basis $f_1, \ldots, f_n$ in the dual space V' is called a triangular pair if $(e_i, f_k) = 0$ for $i > k$ and $(e_i, f_i) = 1$.*

Theorem 6.1 *1. The set of simplices $B' = \{\sigma\}$ is the basis in V_Σ.*

2. The basis of chambers B and the basis of simplices B' form a triangular pair.

The proof follows from the fact that the incidence between the chambers $\gamma \in B$ and simplices $\sigma \in B'$ is given by the submatrix $M' \subset M$, which is triangular (see the proof of Theorem 5.1).

7 Appendix

7.1 Remark on visibility

In the section 3 we used the algorithm to construct the set B of chambers, we have proved (Theorem 5.2) that on the affine plane the set B is a basis of chambers. In the algorithm of construction of the set B we considered a sequence of points $e_{i_k} \in E$ that satisfied the condition (??). In Proposition 3.3 we proved that the point e_{i_k}, satisfying the condition (??) is P_{k-1}-visible from the preceding point $e_{i_{k-1}}$. However, the visibility of the points e_{i_k} in the algorithm is not sufficient to obtain the basis of chambers.

Fig.9

Fig.9 shows an example of the points on the affine plane. The set E consists of 6 points. The condition of visibility, even from the two preceding points in the sequence e_{i_k}, is not sufficient to construct a basis. The sequence of points e_1, e_2, e_3, e_4 in fig.9 is such that each point is visible from the points of the two preceding steps. For example, the point e_3 is P_2-visible from the point e_2 and is also P_2-visible from the point e_1, the point e_4 is P_3-visible from the point e_3 and also P_3-visible from the point e_2. However, the set of chambers constructed by the algorithm for this sequence $\{e_{i_k}\}$ (i.e. the set of chambers in P adjacent to the point e_1, chambers in P_1 adjacent to the point e_2, etc.) is not a basis of chambers. Of course, for this sequence of points e_1, e_2, e_3, e_4 the condition (??) is not satisfied.

7.2 Iterations in painting chambers

In section 4 we have defined the game on the affine plane. Let B be a successfull initial painting, i.e. B is the set of chambers painted blue in the game. Then any chamber $\gamma \notin B$ can be painted by green color according to the rule of the game. We can introduce the "height" for each chamber $\gamma \notin B$.

Definition 7.1 *Any chamber $\gamma \in B$ has the height 0.*
We say that a chamber $\gamma \notin B$ has the height k, if γ can be painted by green color when applying the rule to chambers with the height $< k$.

The process of painting chambers with green paint is similar to diffusion and the height k means the time that is required for the process to reach a certain chamber. Such notions are useful if we consider the continuous analogue of the game formulated in this paper.

Acknowledgements

I am very grateful to I.Gelfand, S.Gelfand, G.Rybnikov and T.Trimble for the helpful discussions.

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On the fundamental group of the complement of a complex hyperplane arrangement

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May 12, 1994

Abstract

We construct two combinatorially equivalent line arrangements in the complex projective plane such that the fundamental groups of their complements are not isomorphic. In the proof we use a new invariant of the fundamental group of the complement of a line arrangement with prescribed combinatorial type with respect to isomorphisms inducing the canonical isomorphism of first homology groups.

1 Introduction

Many topological properties of a complex hyperplane arrangement can be expressed in terms of its combinatorial structure (matroid). For example, it is true for the cohomology ring of the complement of the arrangement [4].However, for the fundamental group of this complement all known algorithms of computation use non-matroidal data. (In the case of complexified real arrangement the answer can be expressed in terms of oriented matroid,

*Supported in part by DIMACS
†Supported in part by the AMS grant for mathematicians in the former Soviet Union

which requires more information than just dimensions of all intersections of hyperplanes as for ordinary matroid.)

Due to the classical theorem of Zariski, the fundamental group of the complement of a hypersurface in $\mathbb{C}P^n$ is isomorphic to the fundamental group of the complement in a generic 2-dimensional plane of it's intersection with the hypersurface. Hence, in our case it is sufficient to study fundamental groups of complements of line arrangements in $\mathbb{C}P^2$. In this paper we use a description of combinatorial structure of line arrangements in terms of projective configurations, which is essentially equivalent to using matroid language, but has an advantage of being more visual.

The aim of this paper is to construct two combinatorially equivalent line arrangements in $\mathbb{C}P^2$ whose complements have different fundamental groups. Our approach to this is the following.

We consider the canonical map $\pi_1(X) \to H_1(X, \mathbb{Z})$ where X is the complement of a line arrangement L. The first homology group $H = H_1(X, \mathbb{Z})$ is a free Abelian group which is canonically determined by the configuration C describing the combinatorial structure of L. We construct for any C an invariant of the group $\pi_1(X)$ with respect to isomorphisms $\pi_1(X) \to \pi_1(X')$ inducing trivial automorphism of H, where X and X' are complements of line arrangements combinatorially described by C. This invariant depends only on $\pi_1(X)/\gamma_4\pi_1(X)$, where $\gamma_k G$ denotes the k-th member of the lower central series of a group G.

Next we consider two conjugate realizations of MacLane configuration C_8 corresponding to MacLane matroid [2] (see also [7]) and show that our invariant distinguishes them. Then, at last, we use these computations to prove that configuration C_{13}, obtained by gluing together two MacLane configurations, has at least two realizations with different fundamental groups.

I am deeply grateful for helpful discussions to T. Alekseyevskaya, A. Dress, A. Goncharov, S. Lvovsky, N. Mnev, G. Noskov, S. Orevkov, D. Stone, G. Ziegler, and especially to I. M. Gelfand, without whom this paper would never have been written.

2 Construction of distinguishing invariant

Definition 1 *A projective configuration is a triple* $C = (\mathcal{L}, \mathcal{P}, \succ)$, *consisting*

of two sets: $\mathcal{L}$, called the set of lines, and $\mathcal{P}$, called the set of points, and the binary relation $\succ$, called incidence, between $\mathcal{L}$ and $\mathcal{P}$, provided the following axioms hold:

(1) For any two distinct lines l and l' there is a unique point p such that $l \succ p$ and $l' \succ p$;

(2) Any point is incident to at least two lines.

For lines l and l' and a point p as in axiom (1) we write $p = l \cap l'$ and call p the intersection point of the lines l and l'. We call a configuration $C = (\mathcal{L}, \mathcal{P}, \succ)$ non-degenerate if $|\mathcal{P}| > 1$, hence not all the lines pass through one point.[1]

Definition 2 *A complex realization of a projective configuration $C = (\mathcal{L}, \mathcal{P}, \succ)$ is an injective map ϕ from $\mathcal{L}$ to the set of lines in $\mathbb{C}P^2$ and from $\mathcal{P}$ to the set of points in $\mathbb{C}P^2$ such that $\phi p \in \phi l$ if and only if $p \prec l$.*

Let $C = (\mathcal{L}, \mathcal{P}, \succ)$ be a finite non-degenerate projective configuration. We order the elements of $\mathcal{L}$: $\mathcal{L} = \{l_0, l_1, \ldots, l_n\}$, and call l_0 line at infinity. Let $\mathcal{P}_0 = \{p \in \mathcal{P} \mid p \not\prec l_0\}$. Now we shall model in abstract terms the presentation of the fundamental group of the complement of a line arrangement in $\mathbb{C}^2$, as stated in [1] (see also [5]).

Let F be a free group with free generators $w_1, \ldots, w_n$, and let $\mathcal{A} = \{(i,p) \in \{1, \ldots, n\} \times \mathcal{P}_0 \mid p \prec l_i\}$. For any $(i,p) \in \mathcal{A}$ we choose $w_i(p) \in F$ which is conjugate in F to w_i: $w_i(p) = (g(i,p))^{-1} w_i g(i,p)$ for some $g(i,p) \in F$. Thus the elements $w_i(p)$ are determined by some mapping $g : \mathcal{A} \to F$.

Let $p \in \mathcal{P}_0$ and $\{i_1, \ldots, i_k\} = \{i \in \{1, \ldots, n\} \mid l_i \succ p\}$ with $i_1 < \ldots < i_k$. We set $c_\alpha(p) = w_{i_\alpha}(p) \cdot w_{i_{\alpha-1}}(p) \cdot \ldots \cdot w_{i_1}(p) \cdot w_{i_k}(p) \cdot \ldots \cdot w_{i_{\alpha+1}}(p)$ for $\alpha = 1, \ldots, k$, and $r(i_\alpha, p) = c_{\alpha-1}(p)^{-1} \cdot c_\alpha(p)$ for $\alpha = 1, \ldots, k$, where $c_0(p) = c_k(p)$. It is easy to see that $r(i_\alpha, p) = [w_{i_\alpha}, c_\alpha(p)]$ and $\prod_{\alpha=1}^{k} r(i_\alpha, p) = 1$, hence only $k-1$ of them are independent. We denote by $[w_{i_k}(p), w_{i_{k-1}}(p), \ldots, w_{i_1}(p)]$ the set $\{r(i_2, p), \ldots, r(i_k, p)\}$. Let $\mathcal{R} = \mathcal{R}(g) = \{r(i,p) \mid l_i \succ p \text{ and } i \neq \min\{i \mid l_i \succ p\}\}$ be the union of these sets. We set $G = G(\mathcal{R}) = F/\langle\mathcal{R}\rangle$, where $\langle\mathcal{R}\rangle$ is the minimal normal subgroup of F containing $\mathcal{R}$.

Let ϕ be a complex realization of the configuration C, and let $X = \mathbb{C}P^2 \setminus \bigcup_{l \in \mathcal{L}} \phi l$. Then $\pi_1(X) \cong G(\mathcal{R}(g))$ for some particular choice of g. The

[1] The notion of a non-degenerate configuration is essentially equivalent to the notion of a simple rank 3 matroid on the set of lines.

image of the generator w_i corresponds under this isomorphism to a small loop around ϕl_i passed in the positive direction (the lines in $\mathbb{C}^2$ have canonical coorientation).

Let us consider the lower central series $(\gamma_k F)$ of the group F ($\gamma_1 F = F$, $\gamma_{k+1} F = [F, \gamma_k F]$). It is well known that the groups $\mathrm{gr}_k F = \gamma_k F / \gamma_{k+1} F$ are free Abelian groups of finite rank. Moreover, $\mathrm{gr} F = \bigoplus_{k=1}^{\infty} \mathrm{gr}_k F$ is naturally isomorphic (as a graded Abelian group) to the free Lie algebra L over $\mathbb{Z}$ with n generators $x_1, \ldots, x_n$ ($L = \bigoplus_{k=1}^{\infty} L_k$ is graded by degree of monomials) [3]. Under this isomorphism the Lie algebra commutator in L corresponds to group commutator in F. Further on we will identify $\mathrm{gr}_k F$ with L_k.

The group $\langle \mathcal{R} \rangle$ is generated by $r(i,p)$ (which are commutators) and the elements of the form $[w_{i_1}^{\pm 1}, [w_{i_2}^{\pm 1}, \ldots, [w_{i_k}^{\pm 1}, r(i,P)] \ldots]]$. Therefore $\langle \mathcal{R} \rangle \subset \gamma_2 F$. It is easy to see that the images of the elements of $\mathcal{R}$ in L_2 are $\bar{r}(i,p) = [x_i, \sum_{j : p \prec l_j} x_j]$, hence they depend only on C.

We will need in the further considerations a broader class of groups G. Let now $\mathcal{R} = \{ r(i,p) \mid (i,p) \in \mathcal{A} \ \& \ i \neq \min\{ i \mid l_i \succ p \} \}$ be a set of arbitrary elements of $\gamma_2 F$ with the only condition that the images of $r(i,p)$ in L_2 must be the same $\bar{r}(i,p)$ as above. We call such set $\mathcal{R}$ admissible for C. The group $G = G(\mathcal{R})$ is defined as before.

As $\bar{r}(i,p)$ are linearly independent in L_2, it follows that not only $\mathrm{gr}_2 \langle \mathcal{R} \rangle = R_2$ depend only on C, but it is also true for $\mathrm{gr}_3 \langle \mathcal{R} \rangle = R_3 = [H, R_2]$, where $H = L_1 \cong \mathbb{Z}^n$. Hence $\mathrm{gr}_1 G \cong H$, $\mathrm{gr}_2 G \cong P_2 = L_2 / R_2$, and $\mathrm{gr}_3 G \cong P_3 = L_3 / R_3$ are canonically determined by C. This means that in each of these groups we have a canonical basis.

In the case when $G = \pi_1(X)$ for some complex realization of C, we have $H = H_1(X, \mathbb{Z})$ and the canonical basis in it is dual to the canonical basis of $H^1(X, \mathbb{Z})$ described in [4].

The group $G / \gamma_3 G$ is an extension of the group H (which is free Abelian) with a trivial H-module P_2. Such extensions are parameterized by the group $H^2(H, P_2) \cong \mathrm{Hom}(\Lambda^2 H, P_2)$. But as $\Lambda^2 H \cong L_2$, we have a natural projection $\chi_2 : \Lambda^2 H \to P_2$. It is easy to see that χ_2 is exactly the element of $H^2(H, P_2)$, corresponding to the group extension $1 \to P_2 \to G / \gamma_3 G \to H \to 1$. Hence $G / \gamma_3 G$ is canonically determined by the configuration up to an automorphism trivial on H and P_2. The group of such automorphisms is isomorphic to $\mathrm{Hom}(H, P_2)$.

The group $M = \gamma_2 G / \gamma_4 G$ is Abelian, but it has a non-trivial structure of H-module. The arguments similar to those used for $G / \gamma_3 G$ show that this

module is also canonically determined by C up to the action of $\mathrm{Hom}(P_2, P_3)$.

Let us now consider the group extension $1 \to M \to G/\gamma_4 G \to H \to 1$. It corresponds to some $\chi_3 \in H^2(H, M)$. ¿From the short exact sequence of H-modules $0 \to P_3 \to M \to P_2 \to 0$ we get a long exact sequence

$$\ldots \to H^1(H, P_2) \stackrel{\delta}{\to} H^2(H, P_3) \stackrel{\alpha}{\to} H^2(H, M) \stackrel{\beta}{\to} H^2(H, P_2) \to \ldots.$$

Let us rewrite it in the following form:

$$\ldots \to \mathrm{Hom}(H, P_2) \stackrel{\delta}{\to} \mathrm{Hom}(\Lambda^2 H, P_3) \stackrel{\alpha}{\to} H^2(H, M) \stackrel{\beta}{\to} \mathrm{Hom}(\Lambda^2 H, P_2) \to \ldots.$$

We have $\beta(\chi_3) = \chi_2$, thus χ_3 lies in $\beta^{-1}(\chi_2)$ which is a principal homogeneous space over an Abelian group $\mathrm{Hom}(\Lambda^2 H, P_3)/\delta\mathrm{Hom}(H, P_2)$.

Recall that M is determined by C only up to the action of an Abelian group $\mathrm{Hom}(P_2, P_3)$, hence only the orbit $\bar{\chi}_3$ of χ_3 with respect to this action has an invariant sense. The set of orbits $Y = \beta^{-1}(\chi_2)/\mathrm{Hom}(P_2, P_3)$ is a principle homogeneous space over the group $T = \mathrm{Hom}(R_2, P_3)/\bar{\delta}\mathrm{Hom}(H, P_2)$, where $\bar{\delta}$ is the composition of δ with the natural projection $\mathrm{Hom}(\Lambda^2 H, P_3) \to \mathrm{Hom}(R_2, P_3)$. We see that Y, T, and the action of T on Y are determined by C canonically, thus any choice of admissible $\mathcal{R}$ provides us with a well-defined element $\bar{\chi}_3(\mathcal{R}) \in Y$.

Now let us have two admissible sets $\mathcal{R}$ and $\mathcal{R}'$. We consider corresponding groups $G = G(\mathcal{R})$ and $G' = G(\mathcal{R}')$. Let $\kappa(\mathcal{R}, \mathcal{R}') = \bar{\chi}_3(\mathcal{R}) - \bar{\chi}_3(\mathcal{R}') \in T$. ¿From this definition it is clear that $\kappa(\mathcal{R}, \mathcal{R}'') = \kappa(\mathcal{R}, \mathcal{R}') + \kappa(\mathcal{R}', \mathcal{R}'')$ for any admissible $\mathcal{R}$, $\mathcal{R}'$, and $\mathcal{R}''$.

The groups $G/\gamma_2 G$ and $G'/\gamma_2 G'$ are both canonically isomorphic to H. Thus we have a canonical isomorphism $\lambda_1 : G/\gamma_2 G \to G'/\gamma_2 G'$. As we noted above, an isomorphism $\lambda_2 : G/\gamma_3 G \to G'/\gamma_3 G'$ extending λ_1 always exists, although it is not determined canonically.

Theorem 2.1 *An isomorphism $\lambda_3 : G/\gamma_4 G \to G'/\gamma_4 G'$ extending λ_1 exists if and only if $\kappa(\mathcal{R}, \mathcal{R}') = 0$ in T.*

Proof: As we have shown, the Lie algebras $\mathrm{gr}(G/\gamma_4 G)$ and $\mathrm{gr}(G'/\gamma_4 G')$ are canonically isomorphic and generated by $G/\gamma_2 G$ and $G'/\gamma_2 G'$ correspondingly. Therefore, if λ_3 exists, then $\mathrm{gr}\lambda_3$ is uniquely determined and coincides with this canonical isomorphism. Hence $\kappa(\mathcal{R}, \mathcal{R}') = 0$.

In the opposite direction, if $\kappa(\mathcal{R},\mathcal{R}')=0$, then the extensions $1\to M\to G/\gamma_4 G\to H\to 1$ and $1\to M\to G'/\gamma_4 G'\to H\to 1$ differ by an automorphism of M, thus after applying this automorphism they become equivalent. Hence there exists λ_3 extending λ_1. $\square$

In order to use the invariant provided by this theorem, we must describe P_2 and P_3. What can be said in the general case is the following.

Let us identify L_2 with $\Lambda^2 H$. We denote by (x_i^*) the basis of H^* dual to the basis (x_i) of H. Then $L_2^*=\Lambda^2 H^*$ and $R_2^\perp\subset\Lambda^2 H^*$ is generated by the following elements: $\omega_{ijk}=(x_i^*-x_j^*)\wedge(x_j^*-x_k^*)=x_i^*\wedge x_j^*+x_j^*\wedge x_k^*+x_k^*\wedge x_i^*$ for $l_i\succ l_j\cap l_k\in\mathcal{P}_0$ and $\omega_{ij}=x_i^*\wedge x_j^*$ for $l_0\succ l_i\cap l_j$. It is easy to show that $(R_2^\perp)^\perp=R_2$, hence P_2 is a free Abelian group and $P_2^*\cong R_2^\perp$. Therefore we will consider the forms ω_{ijk} and ω_{ij} defined above as the elements of P_2^*.

We have an exact sequence of Abelian groups:

$$0\to\Lambda^3 H\xrightarrow{d}H\otimes L_2\xrightarrow{c}L_3\to 0$$

where $d(x\wedge y\wedge z)=x\otimes[y,z]+y\otimes[z,x]+z\otimes[x,y]$ and $c(x\otimes f)=[x,f]$. As we know, $R_3=[H,R_2]$, thus $R_3=c(H\otimes R_2)$. From the dual exact sequence

$$0\leftarrow\Lambda^3 H^*\xleftarrow{d^*}H^*\otimes\Lambda^2 H^*\xleftarrow{c^*}L_3^*\leftarrow 0$$

we get that $R_3^\perp=\ker d^*|_{H^*\otimes R_2^\perp}$. Therefore $R_3^\perp$ contains the elements of the form $S_{ijk}=(x_i^*-x_j^*)\otimes\omega_{ijk}$ for $l_i\succ l_j\cap l_k\in\mathcal{P}_0$ and $S_{ij}=x_i^*\otimes\omega_{ij}$ for $l_0\succ l_i\cap l_j$. I don't claim that these elements generate $R_3^\perp$, as we shall see in the next section it is not so, but I don't know any general description of the rest of $R_3^\perp$.

Let us obtain now a more explicit method of calculating $\kappa(\mathcal{R},\mathcal{R}')$ in the case when $\mathcal{R}=\mathcal{R}(g)$ and $\mathcal{R}'=\mathcal{R}(g')$ for some mappings $g,g':\mathcal{A}\to F$. The following propositions follow easily from definitions.

Proposition 2.2 *If* $g(i,p)\equiv g'(i,p)\pmod{\gamma_2 F}$ *for any* $i=1,\dots,n$ *and* $p\prec l_i$, *then* $\kappa(\mathcal{R},\mathcal{R}')=0$. *In other words* $\bar\chi_3(\mathcal{R}(g))$ *depends only on* $\bar g:\mathcal{A}\to H$.

Let us consider an Abelian group $A=H^{\mathcal{A}}$. For $a\in A$ we define $\tilde\tau a\in\mathrm{Hom}(R_2,P_3)$ by setting

$$\tilde\tau a(\bar r(i,p))=[[x_i,a(i,p)],\sum_{j:p\prec l_j}x_j]+[x_i,\sum_{j:p\prec l_j}[x_j,a(j,p)]]+R_3.$$

Let $\tau a=\tilde\tau a+\bar\delta\mathrm{Hom}(H,P_2)$.

Proposition 2.3 *If $\mathcal{R} = \mathcal{R}(g)$ and $\mathcal{R}' = \mathcal{R}(g')$, then $\kappa(\mathcal{R}, \mathcal{R}') = \tau(\bar{g} - \bar{g}')$.*

It would be interesting to find $\ker \tau$. Here are some partial results in this direction.

Let us define the following elements of A: $a^{(0)}_{i,p}$ for $i \in \{1, \dots, n$, $p \in \mathcal{P}_0$, $p \prec l_i$; $a^{(1)}_{i,p}$ for $i \in \{1, \dots, n$, $p \in \mathcal{P}_0$; $a^{(2)}_{i,p_1,p_2}$ for $i \in \{1, \dots, n$, $p_1, p_2 \in \mathcal{P}_0$, $p_1, p_2 \prec l_i$ (p_1 and p_2 not necessary distinct).

$$a^{(0)}_{i,p}(j,q) = \delta_{i,j}\delta_{p,q}x_i;$$

$$a^{(1)}_{i,p}(j,q) = \delta_{p,q}x_i;$$

$$a^{(2)}_{i,p_1,p_2}(j,q) = \delta_{i,j}\delta_{p_1,q}\sum_{k:p_2 \succ l_k} x_k.$$

Let U be a subgroup of A generated by them.

Proposition 2.4 *$U \subseteq \ker \tilde{\tau}$.*

Let B be a subgroup of A consisting of functions which don't depend on p.

Proposition 2.5 *$B \subseteq \tilde{\tau}^{-1}(\bar{\delta}\mathrm{Hom}(H, P_2))$.*

Let $W = A/(U + B)$. The homomorphism τ is a composition of the projection $\pi : A \to W$ and a homomorphism $\bar{\tau} : W \to T$. I don't know whether $\bar{\tau}$ is always injective, but it is so for an example which we will consider in the next section.

3 Computations for MacLane configuration

The MacLane matroid ML_8 can be defined as an affine plane over $\mathbb{F}_3$ with one element deleted. The correspondent configuration C_8 has 8 lines $l_1, \dots, l_7$ and 12 points p_{012}, p_{034}, p_{056}, p_{07}, p_{135}, p_{147}, p_{16}, p_{23}, p_{246}, p_{257}, p_{367}, p_{45} with $p_{ijk} \prec l_i, l_j, l_k$ and $p_{ij} \prec l_i, l_j$. Hence, $\mathcal{P}_0$ consists of the last 8 of these points.

It is easy to show that any complex realization of C_8 is projectively equivalent to a realization of the following form:

$$\begin{aligned}
\phi l_0 &= \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_0 = 0\},\\
\phi l_1 &= \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_1 = 0\},\\
\phi l_2 &= \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_1 = z_0\},\\
\phi l_3 &= \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_2 = 0\},\\
\phi l_4 &= \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_2 = z_0\},\\
\phi l_5 &= \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_2 + \omega z_1 = 0\},\\
\phi l_6 &= \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid z_2 + \omega z_1 = (\omega + 1) z_0\},\\
\phi l_7 &= \{(z_0 : z_1 : z_2) \in \mathbb{C}P^2 \mid (\omega + 1) z_1 + z_2 = z_0\},
\end{aligned}$$

where ω is a root of a polynomial $x^2 + x + 1$. The points ϕp for $p \in \mathcal{P}$ are defined as intersections of corresponding lines.

Let us denote the realization of C_8, corresponding to $\omega = \exp(2\pi i/3)$, by ϕ^+, and the realization, corresponding to $\omega = \exp(-2\pi i/3)$, by ϕ^-. Let $G^+ = \pi_1(X^+)/\gamma_4\pi_1(X^+)$ and $G^- = \pi_1(X^-)/\gamma_4\pi_1(X^-)$

Theorem 3.1 *There exists no isomorphism $G^+ \to G^-$ such that the induced isomorphism $H_1(X^+, \mathbb{Z}) \to H_1(X^-, \mathbb{Z})$ maps the elements of the canonical basis of $H_1(X^+, \mathbb{Z})$ to the corresponding elements of the canonical basis of $H_1(X^-, \mathbb{Z})$.*

Proof: Let $X^\pm = \mathbb{C}P^2 \setminus \bigcup \phi^\pm l$. The implementation of Arvola's algorithm gives us the following sets of defining relators for $\pi_1(X^\pm)$ (both in a free group F with generators $w_1, \ldots, w_7$): for $\pi_1(X^+)$

$$[w_5, w_3, w_1],$$

$$[w_7, w_4, w_1],$$

$$[w_6, w_1],$$

$$[w_3, w_2],$$

$$[w_6, w_4, w_2],$$

$$[w_7, w_5, w_5^{-1} w_2 w_5],$$

$$[w_7, w_6, w_6^{-1} w_3 w_6],$$
$$[w_5, w_3 w_6 w_4 w_6^{-1} w_3^{-1}];$$

and for $\pi_1(X^-)$

$$[w_5, w_3, w_1],$$
$$[w_7, w_4, w_1],$$
$$[w_6, w_1],$$
$$[w_3, w_5^{-1} w_2 w_5],$$
$$[w_7^{-1} w_6 w_7, w_4, w_2],$$
$$[w_7, w_5, w_5^{-1} w_2 w_5],$$
$$[w_7, w_6, w_6^{-1} w_3 w_6],$$
$$[w_5, w_7 w_4 w_7^{-1}].$$

These sets of relators correspond to the mappings $g^{\pm} : \mathcal{A} \to F$, whose values are 1 except for the following:

$$\begin{aligned}
g^+(3, p_{367}) &= w_6, \\
g^+(4, p_{45}) &= w_6^{-1} w_3^{-1}, \\
g^+(2, p_{257}) &= w_5, \\
g^-(3, p_{367}) &= w_6, \\
g^-(4, p_{45}) &= w_7^{-1}, \\
g^-(2, p_{23}) &= w_5, \\
g^-(2, p_{257}) &= w_5, \\
g^-(6, p_{246}) &= w_7.
\end{aligned}$$

Let $a_0 = \bar{g^+} - \bar{g^-} : \mathcal{A} \to H$. This mapping has only the following non-zero values:

$$\begin{aligned}
a_0(4, p_{45}) &= x_7 - x_6 - x_3, \\
a_0(2, p_{23}) &= -x_5, \\
a_0(6, p_{246}) &= -x_7.
\end{aligned}$$

By Theorem 2.1 and Proposition 2.3 it is enough to show that $\tau a_0 \neq 0$ in T.

Lemma 3.2 *The image of a_0 in W is non-zero.*

Proof: First we compute $\tilde{W} = A/U$. As any of the generators $a_{i,p}^{(\alpha)}$ of U is zero on (j,q) with $q \neq p$, we may consider decompositions $A = \bigoplus_{p\in\mathcal{P}_0} A_p$ $U = \bigoplus_{p\in\mathcal{P}_0} U_p$ and $\tilde{W} = \bigoplus_{p\in\mathcal{P}_0} \tilde{W}_p$, where A_p, U_p, and $\tilde{W}_p$ are defined in the obvious way. For each p the computation is quite simple. All of $\tilde{W}_p$ appear to be free Abelian groups, thus $\tilde{W}$ is also free Abelian, and hence it's dual $\tilde{W}^*$ is naturally isomorphic to $U^\perp \subset A^*$.

Let $e_{ij}(p) \in A^*$ be defined by $(e_{ij}(p), a) = (x_j^*, a(i,p))$ for $a \in A$, where (x_i^*) is the basis of H^* dual to the basis (x_i) of H. The basis of $U^\perp$ consists of the following elements:

$$
\begin{aligned}
I(p_{135}) &= e_{13}(p_{135}) - e_{15}(p_{135}) + e_{35}(p_{135}) \\
&\quad - e_{31}(p_{135}) + e_{51}(p_{135}) - e_{53}(p_{135}), \\
I(p_{147}) &= e_{14}(p_{147}) - e_{17}(p_{147}) + e_{47}(p_{147}) \\
&\quad - e_{41}(p_{147}) + e_{71}(p_{147}) - e_{74}(p_{147}), \\
I(p_{367}) &= e_{36}(p_{367}) - e_{37}(p_{367}) + e_{67}(p_{367}) \\
&\quad - e_{63}(p_{367}) + e_{73}(p_{367}) - e_{76}(p_{367}), \\
I(p_{257}) &= e_{25}(p_{257}) - e_{27}(p_{257}) + e_{57}(p_{257}) \\
&\quad - e_{52}(p_{257}) + e_{72}(p_{257}) - e_{75}(p_{257}), \\
I(p_{246}) &= e_{24}(p_{246}) - e_{26}(p_{246}) + e_{46}(p_{246}) \\
&\quad - e_{42}(p_{246}) + e_{62}(p_{246}) - e_{64}(p_{246}), \\
J(p_{16}) &= e_{12}(p_{16}) - e_{62}(p_{16}) + e_{64}(p_{16}) \\
&\quad - e_{14}(p_{16}) + e_{17}(p_{16}) - e_{67}(p_{16}) \\
&\quad + e_{63}(p_{16}) - e_{13}(p_{16}) + e_{15}(p_{16}) - e_{65}(p_{16}), \\
J(p_{45}) &= e_{43}(p_{45}) - e_{53}(p_{45}) + e_{51}(p_{45}) \\
&\quad - e_{41}(p_{45}) + e_{47}(p_{45}) - e_{57}(p_{45}) \\
&\quad + e_{52}(p_{45}) - e_{42}(p_{45}) + e_{46}(p_{45}) - e_{56}(p_{45}), \\
J(p_{23}) &= e_{21}(p_{23}) - e_{31}(p_{23}) + e_{35}(p_{23}) \\
&\quad - e_{25}(p_{23}) + e_{27}(p_{23}) - e_{37}(p_{23}) \\
&\quad + e_{36}(p_{23}) - e_{26}(p_{23}) + e_{24}(p_{23}) - e_{34}(p_{23}), \\
K_1(p_{135}) &= e_{12}(p_{135}) + e_{36}(p_{135}) - e_{37}(p_{135})
\end{aligned}
$$

$$
\begin{aligned}
& + e_{57}(p_{135}) - e_{52}(p_{135}) - e_{56}(p_{135}), \\
K_2(p_{135}) &= e_{14}(p_{135}) - e_{17}(p_{135}) + e_{37}(p_{135}) \\
& - e_{34}(p_{135}) - e_{36}(p_{135}) + e_{56}(p_{135}), \\
K_1(p_{147}) &= e_{12}(p_{147}) + e_{13}(p_{147}) - e_{15}(p_{147}) \\
& - e_{43}(p_{147}) + e_{75}(p_{147}) - e_{72}(p_{147}), \\
K_2(p_{147}) &= e_{12}(p_{147}) + e_{46}(p_{147}) - e_{42}(p_{147}) \\
& - e_{43}(p_{147}) + e_{73}(p_{147}) - e_{76}(p_{147}), \\
K_1(p_{367}) &= e_{31}(p_{367}) - e_{34}(p_{367}) - e_{35}(p_{367}) \\
& + e_{65}(p_{367}) + e_{74}(p_{367}) - e_{71}(p_{367}), \\
K_2(p_{367}) &= e_{34}(p_{367}) + e_{62}(p_{367}) - e_{64}(p_{367}) \\
& - e_{65}(p_{367}) + e_{75}(p_{367}) - e_{72}(p_{367}), \\
K_1(p_{257}) &= e_{21}(p_{257}) + e_{53}(p_{257}) - e_{51}(p_{257}) \\
& - e_{56}(p_{257}) + e_{76}(p_{257}) - e_{73}(p_{257}), \\
K_2(p_{257}) &= e_{24}(p_{257}) - e_{21}(p_{257}) - e_{26}(p_{257}) \\
& + e_{56}(p_{257}) + e_{71}(p_{257}) - e_{74}(p_{257}), \\
K_1(p_{246}) &= e_{21}(p_{246}) + e_{47}(p_{246}) - e_{41}(p_{246}) \\
& - e_{43}(p_{246}) + e_{63}(p_{246}) - e_{67}(p_{246}), \\
K_2(p_{246}) &= e_{25}(p_{246}) - e_{27}(p_{246}) + e_{43}(p_{246}) \\
& + e_{67}(p_{246}) - e_{63}(p_{246}) - e_{65}(p_{246}).
\end{aligned}
$$

Let us consider the following linear functional $t : \tilde{W} \to \mathbb{Z}/3\mathbb{Z}$

$$t = 2I(p_{367}) + J(p_{45}) + K_1(p_{135}) - K_2(p_{147}) + K_1(p_{257}) - K_1(p_{246}) + 3\mathbb{Z}.$$

It is not hard to check that $t|_B = 0$ and $t(a_0) = 1$. Hence the image of a_0 in $W = \tilde{W}/B$ is non-zero. $\square$

Lemma 3.3 *$U = \ker \tilde{\tau}$ for the configuration C_8.*

Proof: In order to prove this, we must first study P_3. It turns out that the basis of $R_3^{\perp}$ consists of $S_{ij} = x_i^* \otimes x_i^* \wedge x_j^*$ for $l_0 \succ l_i \cap l_j$, $S_{ijk} = (x_i^* - x_j^*) \otimes (x_i^* - x_j^*) \wedge (x_j^* - x_k^*)$ for $l_i \succ l_j \cap l_k \in \mathcal{P}_0$ with $i < j$, $i < k$ and the following additional elements:

$$T_0 = x_7^* \otimes (x_1^* - x_3^*) \wedge (x_3^* - x_5^*) + (x_2^* - x_3^*) \otimes (x_1^* - x_4^*) \wedge (x_4^* - x_7^*)$$

$$
\begin{aligned}
&+ (x_4^* - x_5^*) \otimes (x_3^* - x_6^*) \wedge (x_6^* - x_7^*) \\
&+ (x_1^* - x_6^*) \otimes (x_2^* - x_5^*) \wedge (x_5^* - x_7^*) \\
&- x_7^* \otimes (x_2^* - x_4^*) \wedge (x_4^* - x_6^*) + (x_4^* - x_5^*) \otimes x_1^* \wedge x_2^* \\
&- (x_1^* - x_6^*) \otimes x_3^* \wedge x_4^* + (x_2^* - x_3^*) \otimes x_5^* \wedge x_6^*, \\
T_1 \;=\;& x_7^* \otimes (x_1^* - x_3^*) \wedge (x_3^* - x_5^*) - x_3^* \otimes (x_1^* - x_4^*) \wedge (x_4^* - x_7^*) \\
&- x_5^* \otimes (x_3^* - x_6^*) \wedge (x_6^* - x_7^*) + x_1^* \otimes (x_2^* - x_5^*) \wedge (x_5^* - x_7^*) \\
&- (x_5^* - x_7^*) \otimes x_1^* \wedge x_2^* - (x_1^* - x_7^*) \otimes x_3^* \wedge x_4^* \\
&- (x_3^* - x_7^*) \otimes x_5^* \wedge x_6^*, \\
T_2 \;=\;& x_2^* \otimes (x_1^* - x_3^*) \wedge (x_3^* - x_5^*) + (x_2^* - x_5^*) \otimes (x_3^* - x_6^*) \wedge (x_6^* - x_7^*) \\
&+ (x_3^* - x_6^*) \otimes (x_2^* - x_5^*) \wedge (x_5^* - x_7^*) - x_3^* \otimes (x_2^* - x_4^*) \wedge (x_4^* - x_6^*) \\
&+ (x_3^* - x_5^*) \otimes x_1^* \wedge x_2^* - (x_2^* - x_6^*) \otimes x_3^* \wedge x_4^* \\
&+ (x_2^* - x_3^*) \otimes x_5^* \wedge x_6^*, \\
T_3 \;=\;& x_6^* \otimes (x_1^* - x_3^*) \wedge (x_3^* - x_5^*) - (x_3^* - x_6^*) \otimes (x_1^* - x_4^*) \wedge (x_4^* - x_7^*) \\
&- (x_1^* - x_4^*) \otimes (x_3^* - x_6^*) \wedge (x_6^* - x_7^*) - x_1^* \otimes (x_2^* - x_4^*) \wedge (x_4^* - x_6^*) \\
&+ (x_4^* - x_6^*) \otimes x_1^* \wedge x_2^* - (x_1^* - x_6^*) \otimes x_3^* \wedge x_4^* \\
&+ (x_1^* - x_3^*) \otimes x_5^* \wedge x_6^*, \\
T_4 \;=\;& x_4^* \otimes (x_1^* - x_3^*) \wedge (x_3^* - x_5^*) + (x_2^* - x_5^*) \otimes (x_1^* - x_4^*) \wedge (x_4^* - x_7^*) \\
&+ (x_1^* - x_4^*) \otimes (x_2^* - x_5^*) \wedge (x_5^* - x_7^*) - x_5^* \otimes (x_2^* - x_4^*) \wedge (x_4^* - x_6^*) \\
&+ (x_4^* - x_5^*) \otimes x_1^* \wedge x_2^* - (x_1^* - x_5^*) \otimes x_3^* \wedge x_4^* \\
&+ (x_2^* - x_4^*) \otimes x_5^* \wedge x_6^*.
\end{aligned}
$$

What is more, $(R_3^\perp)^\perp = R_3$, hence P_3 is a free Abelian group and $P_3^* \cong R_3^\perp$. Therefore it is enough to show that $U^\perp = \operatorname{Im} \tilde{\tau}^*$.

We identify $(\operatorname{Hom}(R_2, P_3))^*$ with $R_2 \otimes P_3^*$. It is easy to see that

$$
\begin{aligned}
I(p_{135}) &= \tilde{\tau}^*(r(1, p_{135}) \otimes S_{135}), \\
I(p_{147}) &= \tilde{\tau}^*(r(1, p_{147}) \otimes S_{147}), \\
J(p_{16}) &= \tilde{\tau}^*(r(1, p_{16}) \otimes T_3), \\
K_1(p_{135}) &= \tilde{\tau}^*(r(1, p_{135}) \otimes (T_0 - T_3)), \\
K_2(p_{135}) &= \tilde{\tau}^*(r(1, p_{135}) \otimes T_3), \\
K_1(p_{147}) &= -\tilde{\tau}^*(r(1, p_{147}) \otimes T_0), \\
K_2(p_{147}) &= -\tilde{\tau}^*(r(1, p_{147}) \otimes T_3).
\end{aligned}
$$

Thus $(U^\perp)_p \subset \operatorname{Im}\tilde{\tau}^*$ for $p = p_{135}, p_{147}, p_{16}$. Now we can use the fact that the group, generated by permutations (16)(25)(34) and (135)(246), naturally acts on C_8 by automorphisms to prove that it is true for any $p \in \mathcal{P}_0$. Hence $U^\perp \subset \operatorname{Im}\tilde{\tau}^*$. Combining it with Proposition 2.4 we get the statement of the Lemma. $\square$

Lemma 3.4 *$B = \tilde{\tau}^{-1}(\bar{\delta}\mathrm{Hom}(H, P_2))$ for the configuration C_8.*

Proof: We will first show that if $\bar{\delta}f \in \tilde{\tau}A$ for some $f \in \mathrm{Hom}(H, P_2)$, then
1) $(\omega_{ij}, f(x_k)) = 0$ for any i, j, k such that $l_i \cap l_j \prec l_0$, $k \neq i, j$ and
2) $(\omega_{ijk}, f(x_m)) = 0$ for any i, j, k, m such that $l_i \succ l_j \cap l_k \in \mathcal{P}_0$, $m \neq i, j, k$.

Suppose that 1) doesn't hold, then, as no four pairwise distinct lines of C_8 meet in one point, we have $p = l_i \cap l_k \in \mathcal{P}_0$ and $(S_{ij}, \bar{\delta}f(r(k,p))) = (\omega_{ij}, f(x_k)) \neq 0$. But $(S_{ij}, \tilde{\tau}a(r(k,p))) = 0$ for any $a \in A$, hence 1) must hold.

Suppose that 2) doesn't hold, then we have either $p = l_i \cap l_m \in \mathcal{P}_0$ or $p' = l_j \cap l_m \in \mathcal{P}_0$. Due to the symmetry of (ijk) we may suppose that $p = l_i \cap l_k \in \mathcal{P}_0$. Then $(S_{ijk}, \bar{\delta}f(r(m,p))) = (\omega_{ijk}, f(x_m)) \neq 0$. But $(S_{ijk}, \tilde{\tau}a(r(m,p))) = 0$ for any $a \in A$, hence 2) must also hold.

It follows that we can choose $\tilde{f} \in \mathrm{Hom}(H, \Lambda^2 H)$ with $f(x) = \tilde{f}(x) + R_2$ in such a way that $\tilde{x_k} = x_k \wedge b(k)$ for some $b : \{1, \ldots, 7\} \to H$. Hence $\bar{\delta}f = \tilde{\tau}a$, where $a \in B$, $a(k,p) = b(k)$ for any k and p. $\square$

¿From the last two lemmas it follows that $\bar{\tau} : W \to T$ is an inclusion, hence $\tau a_0 \neq 0$. This concludes the proof of the Theorem. $\square$

Note that the groups $\pi_1(X^+)$ and $\pi_1(X^-)$ are, in fact, isomorphic, for there exists a homeomorphism $X^+ \to X^-$ induced by complex conjugation. But as this homeomorphism reverses coorientation of lines, the corresponding isomorphism of the first homology groups maps canonical basis of one of them to minus canonical basis of the other.

4 Combinatorially equivalent arrangements with different fundamental groups

We define the configuration C_{13} by gluing together two MacLane configurations in the following way. Let $C_8 = (\mathcal{L}, \mathcal{P}, \succ)$ and $C_8' = (\mathcal{L}', \mathcal{P}', \succ)$ be two copies of MacLane configuration ($\mathcal{L} = \{l_0, \ldots, l_7\}$, $\mathcal{P} = \{p_{012}, \ldots, p_{45}\}$, $\mathcal{L}' =$

$\{l'_0, \ldots, l'_7\}$, $\mathcal{P}' = \{p'_{012}, \ldots, p'_{45}\}$). Let us introduce an equivalence relation on $\mathcal{L} \cup \mathcal{L}'$ by saying that $l_0 \sim l'_0$, $l_1 \sim l'_1$, and $l_2 \sim l'_2$, and an equivalence relation on $\mathcal{P} \cup \mathcal{P}'$ by saying that $p_{012} \sim p'_{012}$. Let $\mathcal{P}'' = \{p''_{ij} \mid i, j \in \{3, \ldots, 7\}\}$. We set $\tilde{\mathcal{L}} = (\mathcal{L} \cup \mathcal{L}')/\sim$, $\tilde{\mathcal{P}} = ((\mathcal{P} \cup \mathcal{P}')/\sim) \cup \mathcal{P}''$ and the incidence relation between $\tilde{\mathcal{L}}$ and $(\mathcal{P} \cup \mathcal{P}')/\sim$ induced by that of C_8, C'_8, and between $\tilde{\mathcal{L}}$ and $\mathcal{P}''$ defined by $p''_{ij} \prec l_i, l'_j$. Let $C_{13} = (\tilde{\mathcal{L}}, \tilde{\mathcal{P}}, \succ)$. We denote by $\bar{l}_i$ the equivalence class of l_i $(i = 0, 1, 2)$, and by $\bar{p}_{012}$ the equivalence class of p_{012}.

Let us construct realizations ϕ^{++} and ϕ^{+-} of C_{13} by gluing together realizatons ϕ^+ and ϕ^- of C_8. Let ψ be a generic projective transformation of $\mathbb{C}P^2$ with the condition that $\psi\phi^+ l_i = \phi^+ l_i (= \phi^- l_i)$ for $i = 0, 1, 2$. We set $\phi^{+\pm}\bar{l}_i = \phi^+ l_i$ for $i = 0, 1, 2$, $\phi^{+\pm} l_i = \phi^+ l_i$ and $\phi^{+\pm} l'_i = \psi\phi^{\pm} l_i$ for $i = 3, \ldots, 7$. For a generic ψ the intersection points of the complex lines $\phi^{+\pm} l$ for all $l \in \tilde{\mathcal{L}}$ are in one-to-one correspondence with the elements of $\tilde{\mathcal{P}}$, hence we get realizations of C_{13}.

Further on we will write $C_{13} = (\mathcal{L}, \mathcal{P}, \succ)$ instead of $C_{13} = (\tilde{\mathcal{L}}, \tilde{\mathcal{P}}, \succ)$, l_i instead of $\bar{l}_i$, and l_{i+5} instead of l'_i, in order to make the notations compatible to that of section 2.

Let $X^{+\pm} = \mathbb{C}P^2 \setminus \bigcup \phi^{+\pm} l$.

Theorem 4.1 *The groups $\pi_1(X^{++})$ and $\pi_1(X^{+-})$ are not isomorphic.*

Proof: Let $G^{++} = \pi_1(X^{++})$ and $G^{+-} = \pi_1(X^{+-})$. It is sufficient to show that the groups $G^{++}/\gamma_4 G^{++}$ and $G^{+-}/\gamma_4 G^{+-}$ are not isomorphic. We will start with the study of the group Γ of automorphisms of C_{13}.

It is easy to see that any automorphism $\sigma \in \Gamma$ preserves the set $\{l_0, l_1, l_2\}$ and either preserves each of the sets $\{l_3, \ldots, l_7\}$ and $\{l_8, \ldots, l_{12}\}$, or interchanges them. Moreover, the action of σ on $\{l_0, l_1, l_2\}$ may be arbitrary, but after it is determined there are only two possibilities, with $\sigma l_3 \in \{l_3, \ldots, l_7\}$ for one of them and $\sigma l_3 \in \{l_8, \ldots, l_{12}\}$ for the other. Hence $\Gamma \cong \mathrm{S}_3 \times \mathbb{Z}_2$, where S_3 is a symmetric group on the set $\{0, 1, 2\}$.

Let $x_0 = -\sum_{i=1}^{12} x_i$. There is a natural action of the group Γ on H induced by permutations of the elements x_i, thus we may consider Γ as a subgroup of $GL(H)$. Let $\tilde{\Gamma} = \gamma \times \{\pm \mathrm{id}_H\}$.

Let $G_2 = G^{++}/\gamma_3 G^{++}$. It is isomorphic to $G^{+-}/\gamma_3 G^{+-}$, though not canonically. Any automorphism of G_2 preserves $[G_2, G_2] = P_2$, hence it acts naturally on $H = G_2/P_2$. Let us denote the corresponding homomorphism $Aut(G_2) \to GL(H)$ by π.

Lemma 4.2 *The image of π equals $\tilde{\Gamma}$.*

Proof: This is really a sketch of the proof, which involves simple, but long computations.

The map $\Lambda^2 H \to P_2$ induced by the group commutator in G_2 is equivariant with respect to the action of $Aut(G_2)$. Hence $R_2 = \ker(\Lambda^2 H \to P_2)$ and $R_2^\perp \subset \Lambda^2 H^*$ are preserved by the natural action of this group. It is not hard to check that for C_{13} any decomposable element of $R_2^\perp$ is proportional to some ω_{ijk} or ω_{ij}. Generating $\mathbb{Z}$-submodules by several decomposable forms and computing ranks of their generic elements we can reconstruct all $\mathbb{Z}\omega_{ij}$ and $\mathbb{Z}\omega_{ijk}$ up to an automorphism of C_{13}. Taking kernels of suitable linear combinations of decomposable forms we reconstruct all x_i up to a sign of each and an automorphism of C_{12}. At last we use the unique linear dependance between x_i to reconstruct them up to multiplication of all of them simultaneously by ± 1 and an automorphism of C_{13}, in other words up to applying some element of $\tilde{\Gamma}$. It follows that $\pi(u) \in \tilde{\Gamma}$ for any $u \in Aut(G_2)$. On the other hand, it is fairly easy to construct for each $\sigma \in \tilde{\Gamma}$ some $u \in Aut(G_2)$ such that $\sigma = \pi(u)$. Hence $\pi Aut(G_2) = \tilde{\Gamma}$. $\square$

Let G be either G^{++} or G^{+-}. Let us choose some arbitrary isomorphism $\zeta : G/\gamma_3 G \to G_2$. From ζ with the help of natural projections $G/\gamma_4 G \to G/\gamma_3 G$ and $G_2 \to H$ we construct an homomorphism $\xi : G/\gamma_4 G \to H$. Let $\bar{w}_1^{(\alpha)}, \ldots, \bar{w}_7^{(\alpha)}$ be arbitrary pre-images with respect to ξ of $x_1, \ldots, x_7$, and let $\bar{w}_1^{(\beta)}, \ldots, \bar{w}_7^{(\beta)}$ be arbitrary pre-images of $x_1, x_2, x_8, \ldots, x_{12}$. We set $\bar{w}_0^{(\alpha)} = (\bar{w}_1^{(\alpha)})^{-1} \cdot \ldots \cdot (\bar{w}_7^{(\alpha)})^{-1}$ and $\bar{w}_0^{(\beta)} = (\bar{w}_1^{(\beta)})^{-1} \cdot \ldots \cdot (\bar{w}_7^{(\beta)})^{-1}$. Let $\bar{G}^{(\alpha)}$ be a subgroup of $G/\gamma_4 G$, generated by $\bar{w}_1^{(\alpha)}, \ldots, \bar{w}_7^{(\alpha)}$ and let $\bar{G}^{(\beta)}$ be a subgroup of $G/\gamma_4 G$, generated by $\bar{w}_1^{(\beta)}, \ldots, \bar{w}_7^{(\beta)}$. We say that G is of class 0 if there is an isomorphism $\mu : \bar{G}^{(\alpha)} \to \bar{G}^{(\beta)}$ such that $\mu \bar{w}_i^{(\alpha)} \in \bar{w}_i^{(\beta)} \cdot [\bar{G}^{(\beta)}, \bar{G}^{(\beta)}]$ for any $i = 1, \ldots, 7$, and that G is of class 1 if there is no such isomorphism.

Lemma 4.3 *The class of G is correctly defined, G^{++} is of class 0 and G^{+-} is of class 1.*

Proof: By Lemma 4.2 any automorphism of G_2 acts on H by an element of $\tilde{\Gamma}$. It follows that for any $u \in Aut(G_2)$ we can change $\bar{w}_i^{(\alpha)}$ and $\bar{w}_i^{(\beta)}$ in such a way that they will correspond to $u\zeta$ instead of ζ, but the groups $\bar{G}^{(\alpha)}$ and $\bar{G}^{(\beta)}$ will remain the same (they can only change places). Moreover, the condition of compatibility of μ with the new chosen generators of these

groups will be equivalent to that for original generators, hence the class will remain the same. This follows from the fact that any $\sigma \in \Gamma$ preserves the partition $\mathcal{L} = \bigcup_{i=0}^{2}\{l_i\} \cup \bigcup_{i=3}^{7}\{l_i, l_{i+5}\}$.

Therefore we may assume that the homomorphism $\xi : G/\gamma_4 G \to H$ is canonical. It is easy to see that the groups $\bar{G}^{(\alpha)}$ and $\bar{G}^{(\beta)}$ may be described as $G(\mathcal{R}^{(\alpha)})/\gamma_4 G(\mathcal{R}^{(\alpha)})$ and $G(\mathcal{R}^{(\beta)})/\gamma_4 G(\mathcal{R}^{(\beta)})$ correspondingly for the configuration C_8, where the sets of relators $\mathcal{R}^{(\alpha)}$ and $\mathcal{R}^{(\beta)}$ satisfy the condition from section 2, i. g. modulo $\gamma_3 F$ they coinside with $\{\bar{r}(i,p)\}$.

Let $\tilde{w}_1, \ldots, \tilde{w}_{12}$ be Arvola's generators of G corresponding to small loops around lines, and let $\tilde{G}^{(\alpha)} = G/\langle \tilde{w}_8, \ldots, \tilde{w}_{12}\rangle$, $\tilde{G}^{(\beta)} = G/\langle \tilde{w}_3, \ldots, \tilde{w}_7\rangle$. Then $\tilde{G}^{(\alpha)} \cong \pi_1(X^+)$, while $\tilde{G}^{(\beta)} \cong \pi_1(X^+)$ for $G = G^{++}$ and $\tilde{G}^{(\beta)} \cong \pi_1(X^-)$ for $G = G^{+-}$. Let us consider the natural projections $\nu^{(\alpha)} : G \to \tilde{G}^{(\alpha)}$ and $\nu^{(\beta)} : G \to \tilde{G}^{(\beta)}$. They induce projections $\bar{\nu}^{(\alpha)} : G/\gamma_4 G \to \tilde{G}^{(\alpha)}/\gamma_4\tilde{G}^{(\alpha)}$ and $\bar{\nu}^{(\beta)} : G/\gamma_4 G \to \tilde{G}^{(\beta)}/\gamma_4\tilde{G}^{(\beta)}$. Restricting them to $\bar{G}^{(\alpha)}$ and $\bar{G}^{(\beta)}$ correspondingly we obtain isomorphisms $\bar{G}^{(\alpha)} \cong \tilde{G}^{(\alpha)}/\gamma_4\tilde{G}^{(\alpha)}$ and $\bar{G}^{(\beta)} \cong \tilde{G}^{(\beta)}/\gamma_4\tilde{G}^{(\beta)}$. All the constructed homomorphisms are canonical modulo commutants, hence from Theorem 3.1 it follows that an isomorphism $\mu : \bar{G}^{(\alpha)} \to \bar{G}^{(\beta)}$ mapping any $\bar{w}_i^{(\alpha)}$ to $\bar{w}_i^{(\beta)}$ modulo $\gamma_2\bar{G}^{(\beta)}$ exists for $G = G^{++}$ and doesn't exist for $G = G^{+-}$. $\square$

As in the computation of the class of G only $G/\gamma_4 G$ was used, and all the arbitrary choices do not affect the result which is different for G^{++} and G^{+-}, we conclude that $G^{++}/\gamma_4 G^{++}$ is not isomorphic to $G^{+-}/\gamma_4 G^{+-}$. $\square$

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