TOURNAMENT CERTIFICATES

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February, 1994

Abstract

A tournament certificate is a subdigraph of a labeled tournament T that in conjunction with unlabeled information I about T allows errorless reconstruction of T. The paper examines isomorphism certificates, where I is an unlabeled copy of T, and score certificates, where I is T's score sequence. We focus on aspects of edge-minimum certificates.

The work was motivated by Ariel Rubinstein's conjecture for isomorphism certificates that, with a few noted exceptions, every such certificate for an n-vertex tournament has at least $n-1$ edges. We resolve this only for $n \leq 7$, but prove elsewhere that the similar conjecture for score certificates is true. The paper also investigates maximum edge-minimum certificates for all n-vertex tournaments.

1. Introduction

A *certificate* for a labeled tournament T is a labeled subdigraph D of T with the same vertex set as T such that D and unlabeled information I about T allow errorless reconstruction of T. A certificate D is *minimum* if no other certificate for the same T and I has fewer edges than D. Our purpose here and in a companion paper [1] is to study characteristics of minimum certificates.

EXAMPLE 1. Teams $1, 2, 3, 4$ and 5 have completed a round-robin tournament in which each team won exactly two of its four matches. Hence the tournament's score sequence is $s = (2, 2, 2, 2, 2)$. Given s, we are interested in the minimum number of specific match outcomes needed to determine the results of all 10 matches. It is easily seen that the minimum is at least three. Moreover, three suffice. Suppose it is known that $1 \to 2$ (1 beat 2), $1 \to 3$ and $5 \to 2$. The information in s then implies $\{4, 5\} \to \{1\}$, $\{2\} \to \{3, 4\}$, $\{3, 4\} \to \{5\}$ and $3 \to 4$.

We consider two varieties of certificate that are defined after we fix some notation. Let $V(D)$ and $E(D) \subseteq V(D) \times V(D)$ denote the vertex set and edge set of digraph D, respectively. *We assume throughout that $E(D)$ is asymmetric*, so $ab \in E(D) \Rightarrow ba \notin E(D)$. We often write $ab \in E(D)$ as $a \to b$ and say that a beat b, or a beats b. For nonempty disjoint $A, B \subseteq V(D)$, $A \to B$ means that $A \times B \subseteq E(D)$. Digraph D' is a *subdigraph* of D, denoted by $D' \subseteq D$, if $V(D') \subseteq V(D)$ and $E(D') \subseteq E(D)$. Digraphs D and D' are *isomorphic*, denoted by $D \cong D'$, if there is a bijection $\varnothing : V(D) \to V(D')$ for which $ab \in E(D) \Leftrightarrow \varnothing(a)\varnothing(b) \in E(D')$. A *tournament* is a complete digraph, so $|E(T)| = \binom{n}{2}$ when $|V(T)| = n$. The *score sequence* $s(T) = (s_1, s_2, \ldots, s_n)$ of an n-vertex tournament T is the list in nondecreasing order of all n out-degrees of its vertices. Clearly $\sum s_j = \binom{n}{2}$. T is *regular* if $s_1 = s_n$, and T is *transitive* if $a \to b$ & $b \to c \Rightarrow a \to c$ for all $a, b, c \in V(T)$. We denote a transitive tournament (linear order) on n vertices by TT_n and note that $T \cong TT_n \Leftrightarrow s(T) = (0, 1, 2, \ldots, n-1)$. Numerous results for tournaments are presented in Moon [2] and Reid and Beineke [3].

Let D be a subdigraph of tournament T with $V(D) = V(T)$. We say that D is an *isomorphism certificate* for T if

$$T' \cong T \quad \text{and} \quad D \subseteq T' \Rightarrow T' = T \ ,$$

and that D is a *score certificate* for T if for every tournament T',

$$s(T') = s(T) \quad \text{and} \quad D \subseteq T' \Rightarrow T' = T \ .$$

These definitions capture our notion of errorless reconstruction by $T' = T$. Because $T' \cong T \Rightarrow s(T') = s(T)$, every score certificate is an isomorphism certificate.

Let T^0 denote an unlabeled copy of T. Then $I = T^0$ for an isomorphism certificate D because D and T^0 allow us to label the vertices of T^0 in complete fidelity to T. And $I = s(T)$ for a score certificate D because D and $s(T)$ allow accurate reconstruction of T.

EXAMPLE 2. Figure 1 pictures minimum isomorphism and score certificates for one of the three isomorphically distinct 5-vertex tournaments with score sequence $(1,2,2,2,3)$. Given D_1 and T^0, observe first that $5 \rightarrow 1$ because D_1 identifies 1 as the vertex (top of T^0) with out-degree three. Then $\{2,3,4\} \rightarrow \{5\}$ because T^0 reveals that the vertex that beats the out-degree-3 vertex is beaten by the other three. At this point, the partially labeled T^0 shows that $\{2,3,4\}$ forms a cyclic triple. Because D_1 has $3 \rightarrow 4$, it follows that $4 \rightarrow 2$ and $2 \rightarrow 3$. Score certificate D_2 shows that we must have $5 \rightarrow 1$, $2 \rightarrow 3$ and $4 \rightarrow 2$ to obtain $s(T)$.

Figure 1 about here

Let $\ell(T)$ and $L(T)$ denote the minimum number of edges in an isomorphism certificate for T and a score certificate for T, respectively. Clearly, $\ell(T) \leq L(T)$. For Figure 1, $\ell(T) = 4$ and $L(T) = 7$. Many tournaments, including all with $n \leq 4$ and all that have no nonisomorphic alternative with the same score sequence, have $\ell(T) = L(T)$.

We focus on properties of ℓ and L, including their extreme values on the set of all n-vertex tournaments. Our study is motivated by an intriguing conjecture in Rubinstein [4] concerning a lower bound on ℓ. It is easily seen that $\ell(TT_n) = L(TT_n) = n - 1$, so the minima of ℓ and L at n cannot exceed $n-1$. There are, however, tournaments for which $\ell(T) = L(T) < n - 1$. In particular, for each odd $n \geq 3$, let R_n denote the regular tournament on $\{1, 2, \ldots, n\}$ in which vertex j beats the first $(n-1)/2$ vertices to its right in $1, 2, \ldots, n, 1, 2, \ldots$. Then $\ell(R_3) = L(R_3) = 1$ and $\ell(R_5) = L(R_5) = 3$, the latter by Example 1. No other tournaments with $\ell < n - 1$ are known.

CONJECTURE 1. For all $n \geq 2$, $\ell(T) \geq n - 1$ for every n-vertex tournament except R_3 and R_5, where $\ell(T) = n - 2$.

We have not resolved the conjecture but know that it is true for $n \leq 7$. Its general truth is supported by a companion theorem for L that is proved in [1].

THEOREM 1. *For all* $n \geq 2$, $L(T) \geq n-1$ *for every* n*-vertex tournament except* R_3 *and* R_5.

It is also known, thanks to Noga Alon, that $\min \ell(n)$ is approximately n: see [4].

THEOREM 2. *For every* $\epsilon > 0$ *there is an* n_ϵ *such that* $\ell(T) \geq (1-\epsilon)n$ *for all tournaments on* $n \geq n_\epsilon$ *vertices.*

The next section proves several lemmas about characteristics of certificates that help to determine values of ℓ and L. We then concentrate on special tournaments and on extreme values defined as follows, where each min and max is taken over the set of all n-vertex tournaments:

$$\begin{aligned} f(n) &= \min \ell(T) \\ g(n) &= \max \ell(T) \\ F(n) &= \min L(T) \\ G(n) &= \max L(T) \ . \end{aligned}$$

Table 1 shows what we know about these extremes for $n \leq 7$.

Table 1 about here

Theorem 1 and $L(TT_n) = n-1$ extend column F to $F(n) = n-1$ for all $n > 7$. If Conjecture 1 is true then $f = F$.

Section 3 proves that $\ell(ATT_n) = n$ for $n \geq 5$, where ATT_n is the almost transitive tournament on $\{1, 2, \ldots, n\}$ with $i \to j$ whenever $i < j$, except that $n \to 1$. We then show that $L(ATT_5) = 7$ and $L(ATT_n) \leq 3n-9$ for $n \geq 6$.

Section 4 is based on the special regular tournaments R_n. We prove $\ell(R_n) \leq (3n-9)/2$ for odd $n \geq 7$, and suspect that $\ell(R_n) = (3n-9)/2$. We then show that $L(R_n) = (n^2-1)/8$ and use this to conclude that

$$G(n) > n^2/4 \quad \text{for all large } n \ .$$

Thus, there is no shortage of tournaments whose score certificates must contain more than half their edges. We do *not* have a quadratic lower bound on g.

Section 5 considers all small tournaments and outlines a proof that $\ell(T) = 7$ for a particular 6-vertex T. The paper's appendix lists all nonisomorphic tournaments for $n \in \{4, 5, 6\}$ along with isomorphism and score certificates for each. The certificates are not guaranteed to be minimum, but we believe that most are.

Section 6 outlines a proof that the only n-vertex tournaments for which $n \leq 7$ and $\ell(T) < n-1$ are R_3 and R_5.

Section 7 concludes with a brief discussion that includes challenging open questions not mentioned elsewhere in the paper.

The following notational conventions are adhered to throughout the rest of the paper.

CONVENTIONS. T is a tournament and T_n is an n-vertex tournament, arbitrary up to specified assumptions. C always denotes an isomorphism certificate for T or T_n understood in context, and D always denotes a score certificate for T or T_n understood in context.

2. Lemmas

Several aspects of certificates follow immediately from their definitions. Examples are: if $C \subseteq C' \subseteq T$ then C' is an isomorphism certificate for T; the converse (reverse all edge directions) of C is an isomorphism certificate for the converse of T; a bijection $\varnothing : V(T) \to S$ that maps T into $\varnothing(T) \cong T$ has $\varnothing(C)$ as an isomorphism certificate for $\varnothing(T)$. Similar properties hold for score certificates.

We now prove other elementary results that facilitate analysis of ℓ and L. Additional lemmas for score certificates are included in [1].

LEMMA 1. *C has edges incident to at least $n-1$ vertices of T_n.*

Proof. Suppose distinct $x, y \in V(T_n)$ are isolated vertices in an alleged isomorphism certificate C' for T_n. Let T'_n be obtained from T_n by interchanging vertex labels x and y. Then $T'_n \cong T_n$ and $C' \subseteq T'_n$, but $T'_n \neq T_n$. ■

Implications of Lemma 1 include $\ell(R_5) \geq 3$ and $\ell(T_n) \geq \lfloor n/2 \rfloor$. For the next lemma, we recall that a *circuit* in T is a directed cycle $(x_1 \to x_2 \to \cdots \to x_k \to x_1)$ of distinct vertices. The score or out-degree of vertex x is $s(x)$.

LEMMA 2. (a) *If $x \to y$ in T and either $\{x, y\} \to \{v\}$ or $\{v\} \to \{x, y\}$ for every $v \in V(T) \setminus \{x, y\}$, then $x \to y$ in C;*

(b) *If $x \to y$ and $s(x) = s(y) + 1$ in T, then $x \to y$ in D;*

(c) *Every circuit in T has an edge in D.*

Proof. (a) Given the hypotheses, let $T' = T$ except that $y \to x$ in T'. Then $T' \cong T$, $T' \neq T$, and T and T' are fully consistent with every subdigraph of T that excludes $x \to y$.

(b) Given the hypotheses and T' as in (a), $s(T') = s(T)$, so $x \to y$ must be in D to decide between $x \to y$ and $y \to x$.

(c) $s(T)$ is unchanged when all edges in a given circuit are reversed, so at least one edge of the circuit must be in D to orient the circuit in accordance with T. ■

Lemma 2(a) applied to TT_n shows that the unique minimum isomorphism certificate for a transitive tournament contains precisely those $x \to y$ for which no $z \in V(TT_n)$ satisfies $x \to z \to y$. Lemma 2(b) applied to T in Figure 1 says that $\{1\} \to \{2, 3, 4\} \to \{5\}$ in D, and Lemma 2(c) for circuit $2 \to 3 \to 4 \to 2$ requires another edge in D, so D_2 is indeed a minimum score certificate.

Our final lemma in this section gives an upper bound on $\ell(T)$ when $s(T)$ has certain properties.

LEMMA 3. *If no vertex of T_n has score $n - 1$ and exactly one vertex has score $n - 2$, then $\ell(T_n) \leq n - 2 + g(n - 2)$.*

Proof. Given the hypotheses, the $n - 2$ edges out of the vertex, say x, with score $n - 2$ uniquely identifies it and the one vertex y that beats it within an unlabeled copy of T_n. The other $n - 2$ vertices in that copy can then be identified from an isomorphism certificate for T_n with x and y deleted, so $\ell(T_n) \leq n - 2 + g(n - 2)$. ■

Reversal of all edges shows that $\ell(T_n) \leq n - 2 + g(n - 2)$ if no vertex of T_n has score 0 and exactly one vertex has score 1. We have found Lemma 3 and its converse companion helpful in proving that $\ell(T_6) \leq 7$ for all 6-vertex tournaments.

3. Almost Transitive Tournaments

Transitive tournaments and regular tournaments form extreme families within the class of all tournaments in regard to their abilities to rank members of $V(T)$ globally on a best-to-worst basis. Almost transitive tournaments, with score sequences $(1, 1, 2, 3 \ldots, n - 3, n - 2, n - 2)$, illustrate a family with nearly unambiguous rankings in which an apparently low-ranking team beats a team that beats all other teams. We consider their minimum certificates under the convention that $V(ATT_n) = \{1, 2, \ldots, n\}$ and $E(ATT_n) = \{ij : i < j, ij \neq (1, n)\} \cup \{n1\}$.

THEOREM 3. *$\ell(ATT_n) = n$ for all $n \geq 5$.*

Remark. $\ell(ATT_3) = \ell(R_3) = 1$ and $\ell(ATT_4) = 3$.

Proof. By Lemma 2(a), the edge set of an isomorphism certificate for ATT_n with $n \geq 5$ must include the $n-3$ edges $i \to i+1$ for $i = 2, \ldots, n-2$. Let $E(C)$ denote these edges in union with $\{1 \to 3, 3 \to n, n \to 1\}$, so $|E(C)| = n$. We claim that C is indeed an isomorphism certificate for ATT_n. Moreover, it is a minimum isomorphism certificate because, as is easily checked, two additions to the $n-3$ edges $i \to i+1$ are insufficient to reconstruct ATT_n unambiguously.

To verify the claim for C, let i^* denote vertex i in the original ATT_n under our labeling convention. The only 3-circuits in ATT_n are $1^* \to j^* \to n^* \to 1^*$ for $2 \leq j \leq n-1$. Therefore $13n$ from C must be a permutation of some $1^*j^*n^*$. Because vertex 3 in C has two incoming and two outgoing edges in $E(C)$, it can not be 1^* or n^*. The orientation $1 \to 3 \to n \to 1$ in C then shows that $1 = 1^*$ and $n = n^*$. Removal of these two vertices leaves a TT_{n-2}, and the other $n-3$ edges in $E(C)$ then imply that $j = j^*$ for $j = 2, \ldots, n-1$. ■

LEMMA 4. *$L(ATT_5) = 7$.*

Proof outline. The score sequence for ATT_5 is $(1,1,2,3,3)$. Lemma 2(b) requires $\{1,2\} \to \{3\} \to \{4,5\}$ in D. This fixes 3 as the score-2 vertex. We complete D by adding $\{2\} \to \{4,5\}$ and $5 \to 1$, so $|E(D)| = 7$. This puts the score of vertex 2 up to 3, so $1 \to 2$. Then $s(1) \geq 2$, so in fact $s(1) = 3$ and we infer $1 \to 4$. This leaves 4 and 5 as the score-1 vertices, and since $5 \to 1$ we must have $4 \to 5$. Hence the original ATT_5 is completely determined from D and $s(ATT_5)$.

To prove that $L(ATT_5) \geq 7$, we consider each of the 15 ways that two of the remaining six edges can be added to the four required by Lemma 2(b) to form a potential score certificate with six edges. It is tedious but straightforward to show that none of these allows errorless reconstruction of ATT_5 in conjunction with the score sequence, so $L(ATT_5) = 7$. ■

We complete our analysis of almost transitive tournaments by establishing an upper bound on L for $n \geq 6$. We suspect that the bound is the correct value for L, but have not been able to prove this.

THEOREM 4. *$L(ATT_n) \leq 3n - 9$ for all $n \geq 6$.*

Proof. The digraph of Figure 2 has $(n-3)+(n-2)+(n-5)+1 = 3n-9$ edges. We prove that it is a score certificate for ATT_n. Given $s(ATT_n) = (1,1,2,3,\ldots,n-4,n-3,n-2,n-2)$, the figure identifies 3 as the vertex with score $n-3$. Because each of 4 through n has two or three incoming edges, vertices 1 and 2 must be the score-$(n-2)$ vertices. Then edge $n \to 1$ implies that 1 beats all others, including 2, which must beat all but 1. Because each of 4 through $n-2$ has two outgoing edges in the diagram, $n-1$ and n are the score-1 edges. Then $n \to 1$ implies that all others beat n, including $n-1$, which must be beaten by all but n. This leaves $\{4,5,\ldots,n-2\}$ not fully specified. However, all have three or four outgoing edges at this stage, except for vertex $n-2$, which is therefore the degree-2 vertex. Hence $n-2$ is beaten by all $i < n-2$. Then $n-3$ is the only remaining vertex with fewer than four outgoing edges, so it is the degree-3 vertex. The process continues back to vertex 4, which is the degree-$(n-4)$ vertex. The final result is ATT_n. ■

Figure 2 about here

4. Regular Tournaments and G

LEMMA 5. *$\ell(R_n) \le (3n-9)/2$ for odd $n \ge 7$.*

Proof. Let $m = (n-1)/2$, fix $2 \le j \le m$ and define C_j by $V(C_j) = \{1,2,\ldots,n\}$ and, with $n = 2m+1$,

$$\begin{aligned} E(C_j) &= \{i \to i+1 : \text{ all } i = 1,2,\ldots,2m \text{ except } j-1, m \text{ and } m+j\} \\ &\quad \cup \{m+1 \to i : \ i = m+3, m+4, \ldots, 2m+1\} \\ &\quad \cup \{j \to m+j\}\ , \end{aligned}$$

so $|E(C_j)| = (2m-3)+(m-1)+1 = 3m-3 = (3n-9)/2$. We are to prove that C_j is an isomorphism certificate for R_n. To see this, note first that the m edges out of vertex $m+1$ completely determine its status because it must have incoming edges from 1 through m. Hence

$$\{1,2,\ldots,m\} \to \{m+1\} \to \{m+2,\ldots,2m+1\}\ .$$

By the structure of R_n, the m vertices that beat $m+1$ form a TT_m, and the m beaten by $m+1$ form another TT_m. Hence $1 \to 2 \to \cdots \to m$ and $m+2 \to m+3 \to \cdots \to 2m+1$ would allow complete reconstruction of R_n. We can reach the same conclusion by using one less edge, as in C_j, by omitting $j-1 \to j$ and $m+j \to m+j+1$, and adding $j \to m+j$, where $m+j$ is the 'last' vertex beaten by j. By the aforementioned structure, j through $m+1$ form a linear order, as do $m+1$ through $m+j$. Because $j \to m+j$, j also beats the vertices between $m+1$ and $m+j$, and it follows that $j \to j+1 \to \cdots \to m+j$ gives a TT_{m+1}. Then $j-1 \to j$, else j would beat more than m vertices, and $m+j \to m+j+1$, else $m+j$ would be beaten by more than m vertices. ∎

We suspect that $\ell(R_n) = (3n-9)/2$ for odd $n \geq 7$. If true, then R_3, R_5, $R_7, \ldots$, which begins with the only tournaments known to have $\ell(T_n) = n-2$, has ℓ values near to $3n/2$.

Lemma 6. *$L(R_n) = (n^2-1)/8$ for odd $n \geq 3$.*

Proof. The lemma is known to be true for $n < 7$, so assume $n \geq 7$. Let D be a score certificate for R_n. Suppose every vertex has at least one incoming and one outgoing edge not in D. Then we have a circuit disjoint from D, and this contradicts Lemma 2(c). Therefore, some vertex has all $m = (n-1)/2$ outgoing edges in D, or all m incoming edges in D. In either case, regularity determines all edges for that vertex. If each of the remaining $n-1$ vertices has fewer than $m-1$ incoming edges in D and fewer than $m-1$ outgoing edges in D, we again have a circuit disjoint from D and repeat the contradiction. Continuation gives

$$|E(D)| \geq m + (m-1) + (m-2) + \cdots + 1 = \binom{m+1}{2} = \frac{n^2-1}{8} \, .$$

One score certificate with exactly this many edges has $\{1, 2, \ldots, m\} \to \{m+1\}$, $\{2, \ldots, m\} \to \{m+2\}$, $\{3, \ldots, m\} \to \{m+3\}, \ldots, m \to 2m$. ∎

The preceding proof shows that we need at least $(n^2-1)/8$ edges in a subset S of $E(R_n)$ so that every circuit of R_n has an edge in S. We use this fact in the following proof for large score certificates.

Theorem 5. *$G(n) > n^2/4$ for all $n \geq 69$.*

Remark. It is almost certainly true that $G(n) > n^2/4$ for all $n \geq 5$. Figure 1 implies $G(5) \geq 7$, and Figure 3 gives $G(6) \geq 11$ and $G(7) \geq 14$. Some other small n are covered by the following proof, but others need additional considerations that we do not pursue.

Figure 3 about here

Proof of Theorem 5. For odd $m \geq 1$ let $n_i = 3m + 2i$ for $i = 0, 1, 2$. Given m, we form T_{n_i}

from three vertex-disjoint copies of regular tournaments isomorphic to R_m or R_{m+2}. We denote the three by $R^{(1)}$, $R^{(2)}$ and $R^{(3)}$ with $V_j = V(R^{(j)})$, and in T_{n_i} always take $V_1 \to V_2 \to V_3$.

For T_{n_0}, $R^{(j)} \cong R_m$ for each j. Let α be a bijection from V_1 onto V_3. The edges between V_1 and V_3 in T_{n_0} are $x \to \alpha(x)$ and $[V_3 \setminus \{\alpha(x)\}] \to \{x\}$ for every $x \in V_1$. The scores for each vertex in V_1, V_2, and V_3 are $(3m+1)/2$, $(3m-1)/2$, and $(3m-3)/2$, respectively. Lemma 2(b) requires $2m^2$ edges in D for the $V_1 \to V_2$ and $V_2 \to V_3$ connections. We also require D to have $(m^2-1)/8$ edges within each $R^{(j)}$ by Lemma 6 and the paragraph that follows its proof. Therefore

$$|E(D)| \geq 2m^2 + 3(m^2-1)/8 = (19m^2-3)/8 \ .$$

The latter quantity is greater than $n_0^2/4 = 9m^2/4$ for all odd $m \geq 3$, so $G(n) > n^2/4$ for $n \in \{9, 15, 21, \ldots\}$.

Suppose $n = n_0 + 1$. Form T_n by adding a vertex to T_{n_0} that beats all other vertices. We then have $L(T_n) > n^2/4$ if $(19m^2-3)/8 > (3m+1)^2/4$, i.e., if $m^2 > 12m + 5$, which is true if $m \geq 13$. Hence $G(n) > n^2/4$ for $n \in \{40, 46, \ldots\}$.

For T_{n_1}, $R^{(1)} \cong R_m$, $R^{(2)} \cong R_{m+2}$ and $R^{(3)} \cong R_m$. In this case we take $V_3 \to V_1$, so the scores for each vertex in V_1, V_2, and V_3 are $(3m+3)/2$, $(3m+1)/2$, and $(3m-1)/2$, respectively. Lemma 2(b) gives $2m(m+2)$ D edges for $V_1 \to V_2$ and $V_2 \to V_3$, and there must be at least $2(m^2-1)/8 + [(m+2)^2-1]/8$ edges within the $R^{(j)}$ for D. Hence

$$|E(D)| \geq (19m^2 + 36m + 1)/8 \ .$$

This exceeds $n_1^2/4 = (3m+2)^2/4$ if $m^2 + 12m > 7$, which holds for all $m \geq 1$, so $G(n) > n^2/4$ for $n \in \{5, 11, 17, \ldots\}$.

Suppose $n = n_1 + 1$. Add a vertex to T_{n_1} that beats all other vertices. Then $G(n) > n^2/4$ if $(19m^2 + 36m + 1)/8 > (3m+3)^2/4$, which is true for $m \geq 5$, i.e., for $n \in \{18, 24, \ldots\}$.

For T_{n_2}, $R^{(1)} \cong R_{m+2}$, $R^{(2)} \cong R_m$ and $R^{(3)} \cong R_{m+2}$. Here we take two edges from each $x \in V_1$ to V_3, symmetrically, with all others from V_3 to V_1. Then the scores for each vertex in V_1, V_2, and V_3 are $(3m+5)/2$, $(3m+3)/2$, and $(3m+1)/2$, respectively. This gives

$$|E(D)| \geq 2m(m+2) + 2[(m+2)^2-1]/8 + (m^2-1)/8 = (19m^2 + 40m + 5)/8 \ .$$

This exceeds $n_2^2/4 = (3m+4)^2/4$ if $m^2 \geq 8m + 27$, which is true if $m \geq 11$, so $G(n) > n^2/4$ when $n \in \{37, 43, 49, \ldots\}$. For $n = n_2 + 1$, we have $(19m^2 + 40m + 5)/8 > (3m+5)^2/4$, or $m^2 > 20m + 45$, when $m \geq 23$, so $G(n) > n^2/4$ when $n \in \{74, 80, \ldots\}$.

Because the largest integer not covered for $G(n) > n^2/4$ by the preceding constructions is $n = 68$ ($m = 21$ in the preceding sentence), we have $G(n) > n^2/4$ for all $n \geq 69$. ∎

5. Small Tournaments

The Appendix pictures minimum or near-minimum isomorphism and score certificates for all tournaments on four, five and six vertices. Certificate vertex positions match their positions in T. The T list is from Moon [2]. Lemmas 2 and 3 guided the construction of most of the certificates.

We focus on extreme values for small n in this section and Section 6. The next section covers $f(n)$ for $n \leq 7$. The $G(n)$ for Table 1 are based on Figures 1 and 3 and the Appendix. Clearly, $g(4) = 3$, and $g(5) = 5$ by Theorem 3 and the Appendix. The Appendix shows that $g(6) \leq 7$. We know by Lemma 3 and $g(4) = 3$ that $\ell(T_6) \leq 7$ whenever $s(T_6)$ has no 0 and exactly one 1, or no 5 and exactly one 4. This covers 32 of the 56 nonisomorphic T_6's, and it appears that $\ell(T_6) = 7$ for most of these 32. At most four of the other 24 T_6's have $\ell(T_6) = 7$, and $g(6) \leq 7$ overall.

To verify $g(6) = 7$, we need to show that some T_6 has $\ell(T_6) = 7$.

LEMMA 7. *$\ell(T) = 7$ for T shown in Figure 4.*

Figure 4 about here

Proof outline. Given T in Figure 4, Lemma 2(a) says that C must contain $1 \to 3$, $3 \to 4$ and $5 \to 6$. It is easily checked that the addition of $\{1,3,4\} \to 6$ and $5 \to 2$ yields an isomorphism certificate for T. Hence $\ell(T) \leq 7$.

The proof is completed by showing that no three-edge addition to $\{1 \to 3 \to 4, 5 \to 6\}$ gives an isomorphism certificate. We observe first that, because $\{1,3\} \to \{4,5\} \to \{6\}$,

(i) C must contain one of $4 \to 5$, $5 \to 2$ and $2 \to 4$.

Similarly, because $\{1,2\} \to \{3,4\}$,

(ii) C must contain one of $2 \to 1$, $1 \to 5$, $5 \to 2$, $1 \to 6$ and $6 \to 2$.

The only edge common to (i) and (ii) is $5 \to 2$. We consider three-edge additions as follows:

step 1: $4 \to 5$ from (i), an edge from (ii), a third edge;

step 2: $2 \to 4$ from (i), an edge from (ii), a third edge (not $4 \to 5$);

step 3: $5 \to 2$ from (i) and (ii), two other edges (not $4 \to 5$ or $2 \to 4$).

The starting point for step 3 is shown on the right of Figure 4. Because 2 and 6 are equivalent there, at least one of the fifth and sixth edges must involve 2 or 6, so we consider the following in turn as the fifth edge: $6 \to 2$, $2 \to 1$, $2 \to 3$, $1 \to 6$, $3 \to 6$ and $4 \to 6$. A detailed analysis for steps 1–3 shows that there is no isomorphism certificate with six edges. ■

Our best guess for $g(7)$ is 10 or 11. We conclude the section with a proof for the upper bound on Table 1.

LEMMA 8. $g(7) \leq 11$.

Proof. Suppose $\min s(x) = 0$ for T_7. Remove the score-0 vertex to get a T_6. The Appendix shows that at most three edges need to be added to an isomorphism certificate for $n = 6$ with seven or fewer edges (added edges can go to the removed vertex) to supply an outgoing edge for each T_6 vertex. This identifies the score-0 vertex of T_7, so $\ell(T_7) \leq 3 + g(6) = 10$. Similarly, $\ell(T_7) \leq 10$ if $\max s(x) = 6$, so assume henceforth that $1 \leq s(x) \leq 5$ for all $x \in V(T_7)$.

Suppose exactly one vertex has score 1, or exactly one has score 5. Then $\ell(T_7) \leq 5 + g(5) = 10$ by Lemma 3. Assume henceforth that $|\{x : s(x) = 1\}| \neq 1 \neq |\{x : s(x) = 5\}|$.

Suppose exactly two vertices of T_7 have score 1, say x and y with $x \to y$. Consider the T_5 on $V(T_7) \setminus \{x, y\}$ and its isomorphism certificate in the Appendix. The only threats there to $\ell(T_7) \leq 11$ are the first three T_5's in the third row for $n = 5$. Each of these has $s(T_5) = (1, 2, 2, 2, 3)$. To prevent an already-excluded case (one with score 5), y must beat the degree-3 vertex in T_5, say z, with $y \to z$. For each of the three noted T_5's, add $z \to x$ to the edges in the isomorphism certificate for T_5. This identifies z as a score-4 vertex in T_7, and we can add $y \to z$ and the other edge into z at no cost (they are not in C). The minimum number of additional edges for C which give each T_5 vertex at least two outgoing edges brings the total for C to 11. This clearly identifies x and y as the score-1 vertices, and since $y \to z$ we must have $x \to y$. The original C edges for T_5 complete the unambiguous reconstruction of T_7. We assume henceforth that T_7 does not have exactly two score-1 vertices. By a symmetric proof, we can assume also that T_7 does not have exactly two score-5 vertices.

Suppose T_7 has three score-1 vertices. Then the three form a 3-circuit and are beaten by the other four. Based on $g(4) = 3$, it follows easily that $\ell(T_7) \leq 9$. Similarly, $\ell(T_7) \leq 9$ if T_7 has three score-5 vertices.

We conclude that $\ell(T_7) \leq 11$ except perhaps when all scores lie in $\{2, 3, 4\}$. Four score sequences cover these cases:

$$\begin{aligned} s^0 &= (3,3,3,3,3,3,3) \\ s^1 &= (2,3,3,3,3,3,4) \\ s^2 &= (2,2,3,3,3,4,4) \\ s^3 &= (2,2,2,3,4,4,4)\,. \end{aligned}$$

For s^0, Lemma 5 gives $\ell(R_7) \le 6$, and we have checked that $\ell(T_7) \le 7$ for the other two regular tournaments on seven vertices. When s^1 obtains, the score-4 vertex is completely identified by its four outgoing edges in C, so $\ell(T_7) \le 4 + g(6) = 11$.

We divide s^2 into six cases based on the score-4 vertices x and y with $x \to y$. C in each case includes the four edges out from y and the other three edges out from x. The first four edges imply $x \to y$ and the other edge into y, and the seven together tell how x and y relate to all other vertices. Let a and b denote the vertices for the edges into x, and let c be the non-x vertex with an edge into y.

Case 1: $a = c$, a score-3 vertex;

Case 2: $a = c$, a score-2 vertex;

Case 3: a, b and c are the three score-3 vertices;

Case 4: a is a score-3 vertex, b and c are the score-2 vertices;

Case 5: a and c are score-3 vertices, b is a score-2 vertex;

Case 6: a and b are score-3 vertices, c is a score-2 vertex.

For case 1 we add the third edge out from a to C to obtain all edges for a, then add up to three more edges to C for the four vertices not in $\{x, y, c\}$. This gives $|E(C)| \le 11$. Similarly, $|E(C)| \le 10$ for case 2. The other four cases also have $|E(C)| \le 10$. We illustrate only case 4 because analyses for the others are similar.

The situation for case 4 is shown on the upper left of Figure 5. The solid edges are the seven in C from x and y, the dashed edges are implied by those, and the double line connects vertices that are indistinguishable at this point. The $\{b, c\}$ edge could go either way, but it is known because b is the score-2 vertex that goes into the score-4 vertex beaten by a score-3 vertex, and c is the other score-2 vertex. The directed $\{b, c\}$ edge and an eighth edge in C for the double line completely determine the status of one of b and c, and of one at the end of the double line. Then we need at most one more edge in C, so $|E(C)| \le 9$.

Figure 5 about here

Our analysis for $s^3 = (2,2,2,3,4,4,4)$ depends on whether the same-score three-sets are 3-circuits or TT_3's. When both the score-2 set and the score-4 set are 3-circuits, it suffices to completely specify the score-3 vertex by putting all its edges into C. Up to five more edges for C may be needed for complete specification. The only cases that need $|E(C)| = 11$ under this procedure have exactly one score-2 vertex beating the score-3 vertex. It is illustrated on the upper right of Figure 5. The five edges added to the six for the score-3 vertex 4 identify 2 as a score-4 vertex and 6 as a score-2 vertex. Vertices 1 and 3 have score 4 because the others besides 6 that beat the score-3 vertex have score 4. Then 5 and 7 are the other score-2 vertices. We can complete the edges for $\{1,2,3\}$ and $\{5,6,7\}$ because they are 3-circuits with an edge in C. We then have $1 \rightarrow 6$ and $2 \rightarrow 7$. The other four edges (1 vs. 5, 1 vs. 7, 3 vs. 5, 3 vs. 7) will depend on which of two cases holds for T ($5 \rightarrow 1$ & $7 \rightarrow 3$ or $5 \rightarrow 3$ & $7 \rightarrow 1$), which can be ascertained from an unlabeled copy of T.

The remaining cases for s^3 have at least one transitive same-score set. We assume this for the three score-4 vertices. When the score-2 set is a 3-circuit, the procedure of the preceding paragraph gives $|E(C)| \leq 11$ in all cases, but this can be lowered to $|E(C)| \leq 10$ when the approach used for s^2 is applied. We also get $|E(C)| \leq 10$ by the s^2 approach when the score-2 set is transitive. An illustration is shown on the lower part of Figure 5. The nine solid edges for C out of vertices 1, 2 and 3 identify 3, then 2 (we get $2 \rightarrow 3$ freely), then 1 as the score-4 vertices, and all edges between $\{1,2,3\}$ and the others are known. An unlabeled copy of T shows that the score-4 vertex that is beaten by two smaller-score vertices has score 2 for the others. Therefore 6 and 7 have score 2. Moreover, the vertex beaten by all score-4 vertices is a score-2 vertex, so on C, 5 also is score 2 and 4 must be score 3. We know also that 5 beats 6 and 7 because the score-2 vertex beaten by all score-4 vertices beats the other score-2 vertices (and is therefore beaten by the score-3 vertex). The only thing left open is $\{4,6,7\}$, which forms a 3-circuit and is oriented correctly by the 10^{th} and final edge $6 \rightarrow 7$ in C.

6. Conjecture 1 for Small n

Our final theorem validates Conjecture 1 for $n \leq 7$.

THEOREM 6. *$\ell(T) \geq n-1$ if T has $n \leq 7$ vertices and $T \notin \{R_3, R_5\}$.*

Let A_n denote an asymmetric digraph on $\{1,2,\ldots,n\}$ for $n \leq 7$, such that $|E(A_n)| \leq n-2$.

CLAIM. A_n is not an isomorphism certificate for any T_n except when either $n = 3$ and $|E(A_n)| = 1$ or $n = 5$ and $|E(A_n)| = 3$ with $E(A_n)$ isomorphic to $\{a \to b, a \to c, d \to b\}$.

Because the exceptions provide isomorphism certificates only for R_3 and R_5, Theorem 6 is true if the Claim is true. The proof of the Claim is case intensive and will not be drawn in full detail, but we will present a complete outline and comment on several specific cases. Because the Claim is obvious if $n \leq 3$, we assume henceforth that $4 \leq n \leq 7$.

The first thing to observe is that $|E(A_n)| \leq n - 2$ implies that A_n has at least two components. If it has exactly two components then each must be a (directed) tree, for if the components have n_1 and n_2 vertices with $n_1 + n_2 = n$ and if both are trees then $|E(A_n)| = (n_1 - 1) + (n_2 - 1) = n - 2$, whereas if one is not a tree then $|E(A_n)| > n - 2$. A tree can consist of a single vertex, as for the minimum certificates for R_3 and R_5. If A_n has three components, then either all three are trees and $|E(A_n)| = n - 3$, or two are trees and the third has a single cycle with $|E(A_n)| = n - 2$.

The task of examining A_n's for certificate status is eased by a lemma which says that some things can not be isomorphism certificates. For any asymmetric digraph A let $x \sim y$ in A mean that x and y are distinct vertices in A and neither $x \to y$ in A nor $y \to x$ in A. When $x \sim y$ in A, let $A^{x \to y}$ denote $(V(A), E(A) \cup \{x \to y\})$. We then define another binary relation $\approx$ on $V(A)$ by

$$x \approx y \text{ in } A \text{ if } x \sim y \text{ in } A \text{ and } A^{x \to y} \cong A^{y \to x} .$$

Intuitively, $x \approx y$ in A says that the unlabeled view of A from x looks precisely like the unlabeled view of A from y: see Figure 6.

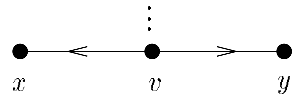
Figure 6 about here

LEMMA 9. *If an asymmetric digraph A contains x and y such that $x \approx y$ in A, then A is not an isomorphism certificate for any tournament.*

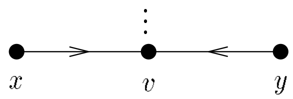
Proof. Contrary to the lemma, suppose A is an isomorphism certificate for T, and $x \approx y$ in A. Assume for definiteness that $x \to y$ in T. Then $A^{x \to y}$ is an isomorphism certificate for T. Let $\varnothing : V(A) \to V(A)$ be a bijection that establishes $A^{x \to y} \cong A^{y \to x}$, so $A^{y \to x} = \varnothing(A^{x \to y})$. Then $A \subseteq \varnothing(T)$ and $\varnothing(T) \cong T$, but $\varnothing(T) \neq T$ because $y \to x$ in $\varnothing(T)$ and $x \to y$ in T. This contradicts the supposition that A is an isomorphism certificate for T. ■

An immediate consequence of Lemma 9 is that an isomorphism certificate can have at most one singleton component and at most one two-vertex component. Therefore, if A_n is an isomorphism certificate and $|E(A_n)| \leq n-2$, then $n \leq 5$ implies that A_n has exactly two components, both trees, and $n \in \{6,7\}$ implies that A_n has two (both trees) or three components. If there are three components in the latter case, their orders must be $1,2,3$ for $n=6$ and either $1,2,4$ or $1,3,3$ for $n=7$. When $1,3,3$ applies, Lemma 9 forces one of the 3-vertex components to be like $\{x \to v, v \to y\}$ and the other to be R_3 or TT_3 with an edge count total of five.

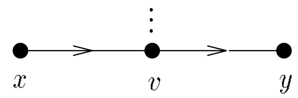
Another implication of Lemma 9, which we just used in the preceding sentence, is that if A_n is an isomorphism certificate with a tree component, then there cannot be terminal vertices x and y with configuration



or



for in either case $x \approx y$ in A_n. Moreover, if A_n has a singleton component $\{1\}$ and a terminal configuration in a tree like



then it cannot be an isomorphism certificate because $1 \approx x$ in $A_n^{1\to v}$ and $1 \approx y$ in $A_n^{v \to 1}$. This observation completes the proof of our Claim for $n=4$ since the only viable candidate for an A_4 isomorphism certificate subsequent to the preceding paragraph, with $|E(A_4)| = 2$, is $\{\{1,2,3,4\},\{2 \to 3, 3 \to 4\}\}$. It also shows that the $1,3,3$ case for three components at $n=7$ can not give an isomorphism certificate. The other three-component cases will be considered after we analyze the two-component cases.

Figure 7 identifies all (up to converses) two-tree A_n for $n \in \{5,6,7\}$ not previously eliminated by $\approx$ as possible isomorphism certificates. For example, we see no pair of three-vertex trees for $n=6$ because each would be like $\{x \to v, v \to y\}$ to prevent $\approx$, and a pair of isomorphic components can not be part of an isomorphism certificate.

Each A_n on Figure 7 is processed as follows. We suppose that A_n is an isomorphism certificate for T. Edges are added to A_n in the only ways possible to avoid $x \approx y$ or a contradiction to asymmetry during the process, for if $x \approx y$ arises then Lemma 9 and $[C \subseteq C' \subseteq T \Rightarrow C'$ is an isomorphism certificate$]$ contradict the certificate supposition. The process continues until we are forced to contradict asymmetry or are able to orient all edges without contradiction. The latter occurs only for the third A_5 on the figure, whose only noncontradictory completion is R_5. Contradictions arise for the others either directly or when we come to a point at which the edge-enhanced A_n includes an isomorph of the original A_n in which two vertices have exchanged positions. If x and y are those vertices, this would imply both $x \rightarrow y$ and $y \rightarrow x$ in T, contrary to asymmetry. We refer to this as *contradiction by interchange.* The process is illustrated for the A_n that are labeled on Figure 7.

Figure 7 about here

The first A_5. If $1 \rightarrow 3$ then $1 \approx 2$ with the addition of $1 \rightarrow 3$, and if $4 \rightarrow 1$ then $1 \approx 5$. We therefore require $3 \rightarrow 1$ and $1 \rightarrow 4$. These additions give the following isomorphs to A_5: $2 \rightarrow 3 \rightarrow 1 \rightarrow 4$ with isolated vertex 5, so $3 \rightarrow 5 \rightarrow 1$ by analogy to the preceding sentence; $3 \rightarrow 1 \rightarrow 4 \rightarrow 5$ with isolated vertex 2, so $1 \rightarrow 2 \rightarrow 4$ by analogy. Then we have $5 \rightarrow 1 \rightarrow 2 \rightarrow 4$ with isolated vertex 3, which gives $1 \rightarrow 3 \rightarrow 2$ by analogy, thus contradicting asymmetry. Alternatively, the final isomorph of A_5 gives a contradiction by interchange because 1 and 3 have exchanged positions from the original $2 \rightarrow 3 \rightarrow 4 \rightarrow 5$ with isolated vertex 1.

The third A_5: $3 \leftarrow 2 \rightarrow 4 \leftarrow 5$ with isolated vertex 1. We must have $1 \rightarrow 2$ and $4 \rightarrow 1$ to avoid $\approx$, e.g., $2 \rightarrow 1$ gives $1 \approx 3$. Moreover, $3 \rightarrow 5$, for otherwise the isomorph $4 \leftarrow 2 \rightarrow 3 \leftarrow 5$ of A_5 creates a contradiction by interchange for 3 and 4. Also, $1 \rightarrow 5$, else $1 \leftarrow 5 \rightarrow 4 \leftarrow 2$ gives a contradiction by interchange, and $3 \rightarrow 1$, else $4 \leftarrow 2 \rightarrow 3 \leftarrow 1$ gives a contradiction by interchange. If $2 \rightarrow 5$ then $3 \leftarrow 2 \rightarrow 5 \leftarrow 1$ with isolated vertex 4 is an isomorph of A_5 which requires $4 \rightarrow 2$ (else $3 \approx 4$), contrary to $2 \rightarrow 4$. Therefore $5 \rightarrow 2$. Similarly, $4 \rightarrow 3$. We have specified all edges in the only way possible to avoid a contradiction. Each vertex beats exactly two others, so T is R_5.

The last A_6: $1 \rightarrow 2$ and $3 \rightarrow 4 \rightarrow 5 \leftarrow 6$. Then $5 \rightarrow 2$ (else $1 \approx 3$ and $2 \approx 4$) and $2 \rightarrow 4$ (else $1 \approx 6$ and $2 \approx 5$). If $1 \rightarrow 5$, we have $3 \rightarrow 4 \rightarrow 5 \leftarrow 1$: if also $2 \rightarrow 6$, analogy to $5 \rightarrow 2$ of the preceding sentence gives $5 \rightarrow 6$, a contradiction; if $6 \rightarrow 2$, 1 and 6 show a contradiction by interchange. We conclude that $5 \rightarrow 1$. Thus $A_6 \Rightarrow \{5 \rightarrow 2, 2 \rightarrow 4, 5 \rightarrow 1\}$. The isomorphs

$$\{6 \rightarrow 5, 1 \rightarrow 2 \rightarrow 4 \leftarrow 3\} \quad \text{and} \quad \{3 \rightarrow 4, 6 \rightarrow 5 \rightarrow 2 \leftarrow 1\}$$

imply $4 \rightarrow 6$ and $2 \rightarrow 3$ respectively in analogy to $5 \rightarrow 1$ for the original A_6. Because $1 \rightarrow 4$ gives a contradiction by interchange (consider $6 \rightarrow 5$ and $2 \rightarrow 3 \rightarrow 4 \leftarrow 1$ for 1 and 6), we also have $4 \to 1$, hence also $5 \to 3$ and $2 \to 6$ by analogy from the preceding isomorphs. We then have $\{2 \to 3, 6 \to 5 \to 1 \leftarrow 4\}$ which implies $3 \to 5$ by analogy to $2 \to 4$ for A_6 at the outset of the proof. This brings us to the inescapable contradiction that $5 \to 3$ and $3 \to 5$.

A_7, first row: $2 \leftarrow 3 \leftarrow 4 \to 5 \to 6 \to 7$ with isolated 1. To avoid $\approx$, we need $1 \to 3$ and $1 \to 6$. Then $5 \to 1$, else a contradiction by interchange from $2 \leftarrow 3 \leftarrow 1 \to 5 \to 6 \to 7$, and $1 \to 4$, else a contradiction by interchange from $2 \leftarrow 3 \leftarrow 4 \to 1 \to 6 \to 7$. We have isomorph $2 \leftarrow 3 \leftarrow 1 \to 4 \to 5 \to 6$, hence by analogy, $\{7\} \to \{1,3,5\}$ and $4 \to 7$. Backward analogy with $5 \to 1$ gives $6 \to 4$ in the original A_7, thus producing isomorph $5 \leftarrow 7 \leftarrow 6 \to 4 \to 3 \to 2$ and the contradiction $1 \to 7$.

A_7, second row. To avoid $\approx$ in $2 \to 3 \leftarrow 4 \leftarrow 5$ with $\{6\} \to \{3,7\}$ and isolated vertex 1, we require $3 \to 1$, $4 \to 1$, $1 \to 6$, $2 \to 5$ (else $2 \approx 4$) and $7 \to 2$ (else $2 \approx 6$). Then $1 \to 2$ implies a contradiction by interchange for isomorph $\{4 \to 3 \leftarrow 2 \leftarrow 1, \{6\} \to \{3,7\}\}$ with isolated 5, and $2 \to 1$ implies a similar contradiction for $\{6 \to 3 \leftarrow 4 \leftarrow 5, \{2\} \to \{1,3\}\}$ via 2 and 6.

A_7, third row (a): $1 \to 2$, $3 \to 4 \to 5 \to 6$ and $5 \leftarrow 7$. We need $5 \to 2$ (else $1 \approx 3$) and $6 \to 1$ (else $2 \approx 6$), then conclude from isomorph $\{6 \to 1, 3 \to 4 \to 5 \to 2$ and $5 \leftarrow 7\}$ that $5 \to 1$ and $2 \to 6$. A new isomorph at this point is $\{3 \to 4, 7 \to 5 \to 6 \to 1$ and $2 \to 6\}$. This requires $6 \to 4$, whose positions in the original A_7 imply $1 \to 5$ in the new isomorph. Hence 1 and 5 contradict asymmetry.

A_7, third row (b): $1 \to 2 \to 3$ and $4 \leftarrow 5 \to 6 \leftarrow 7$. To avoid contradiction by interchange when 3 and 4 are switched, we require $4 \to 2$ or $3 \to 5$; similarly (switch 1 and 4), $2 \to 4$ or $1 \to 5$. Suppose $4 \to 2$. Then also $1 \to 5$, so we have an isomorph $\{a \to b \to c, 5 \leftarrow 1 \to 2 \leftarrow 4\}$ where abc is a permutation of 376 that need not be made specific. Because $5 \to 4$ in A_7, we see that $5 \to 4$ in the isomorph translates back to the original to give $4 \to 7$. This gives a second isomorph whose 4-vertex component is $7 \leftarrow 4 \to 2 \leftarrow 1$, so analogy with $4 \to 7$ in A_7 gives $7 \to 1$. Then a third isomorph has component $1 \leftarrow 7 \to 6 \leftarrow 5$, for contradiction by interchange with the original $4 \leftarrow 5 \to 6 \leftarrow 7$ (5 versus 7).

Assume henceforth for A_7 (b) that $2 \to 4$, so also $3 \to 5$. We can also assume that $7 \to 2$ and $6 \to 1$ because switches for 3 and 7 and for 1 and 4 in the original A_7 give $\{7 \to 2$ or $6 \to 3\}$ and $\{6 \to 1$ or $2 \to 7\}$, and $\{2 \to 7, 6 \to 3\}$ yields a contradiction as in the preceding paragraph for $\{4 \to 2, 1 \to 5\}$. This gives isomorphs $\{3 \to 5 \to 4, 6 \leftarrow 7 \to 2 \leftarrow 1\}$ and $\{7 \to 6 \to 1, 3 \leftarrow 2 \to 4 \leftarrow 5\}$, which in conjunction with A_7 yield the additional edges $4 \to 7$, $1 \to 5$, $6 \to 3$, $3 \to 7$ and $4 \to 1$. A new isomorph for the four-vertex component is

$1 \leftarrow 4 \rightarrow 7 \leftarrow 3$, which in analogy to $4 \rightarrow 7$ for A_7 implies $1 \rightarrow 3$, then other edges, including $3 \rightarrow 4$. Finally, we have isomorph $\{6 \rightarrow 1 \rightarrow 5, 7 \leftarrow 3 \rightarrow 4 \leftarrow 2\}$, which switches the positions of 3 and 5 in A_7 and gives a contradiction by interchange.

This concludes our remarks for two-tree cases. The three-component cases that require further consideration are $n = 6$ with orders $1, 2, 3$ and $n = 7$ with orders $1, 2, 4$ for the components. For $n = 6$ let A_6 have singleton $\{1\}$, doubleton $\{2 \rightarrow 3\}$, and a two or three-edge component on $\{4, 5, 6\}$. The first two components require R_3 on $\{1, 2, 3\}$ to avoid $\approx$, and the third must be TT_3 to avoid $\approx$ internally, or with $\{1\}$, or in comparison to R_3. Let the third component be $\{4 \rightarrow 5 \rightarrow 6, 4 \rightarrow 6\}$. Because the first two components are isomorphic to $\{2\}$ and $\{3 \rightarrow 1\}$, and to $\{3\}$ and $\{1 \rightarrow 2\}$, for each $k \in \{4, 5, 6\}$ we require $\{k\} \rightarrow \{1, 2, 3\}$ or $\{1, 2, 3\} \rightarrow \{k\}$. To avoid contradiction by interchange under the supposition that A_6 is an isomorphism certificate, it follows that $5 \rightarrow 1$ or $6 \rightarrow 1$ (switch 1 and 4), $1 \rightarrow 4$ or $6 \rightarrow 1$ (switch 1 and 5), and $1 \rightarrow 5$ or $1 \rightarrow 4$ (switch 1 and 6). If $4 \rightarrow 1$ then $1 \rightarrow 5$ and $6 \rightarrow 1$, and the A_6 isomorph with $\{6 \rightarrow 3\}$ and $\{4 \rightarrow 1 \rightarrow 5, 4 \rightarrow 5\}$ implies $2 \rightarrow 1$ by analogy to $1 \rightarrow 5$ for A_6. This contradicts $1 \rightarrow 2$. Therefore $1 \rightarrow 4$; similarly, $6 \rightarrow 1$. Then, if $1 \rightarrow 5$ we get the contradiction $2 \rightarrow 1$ with isomorph $[\{2\}, \{6 \rightarrow 3\}, \{1 \rightarrow 4 \rightarrow 5, 1 \rightarrow 5\}]$, and if $5 \rightarrow 1$ we have the contradiction $1 \rightarrow 3$ with isomorph $[\{3\}, \{2 \rightarrow 4\}, \{5 \rightarrow 6 \rightarrow 1, 5 \rightarrow 1\}]$. Therefore A_6 is not an isomorphism certificate.

For $n = 7$, suppose that A_7 is an isomorphism certificate with components $\{1\}$, $\{2 \rightarrow 3\}$ and a tree or four-edge digraph on $\{4, 5, 6, 7\}$. As before, this implies that $\{1, 2, 3\}$ is R_3 and for each $k \in \{4, 5, 6, 7\}$ that $\{k\} \rightarrow \{1, 2, 3\}$ or $\{1, 2, 3\} \rightarrow \{k\}$. There are eight candidates for the four-vertex component that do not give a direct instance of $x \approx y$. Up to permutations of labels and converses, they are the paths $4 \rightarrow 5 \rightarrow 6 \rightarrow 7$, $4 \leftarrow 5 \rightarrow 6 \rightarrow 7$ and $4 \leftarrow 5 \rightarrow 6 \leftarrow 7$, the four-cycles $4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 4$ and $4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \leftarrow 4$, R_3 on $\{4, 5, 6\}$ with $7 \rightarrow 4$, and two cases of $TT_3 = \{4 \rightarrow 5 \rightarrow 6, 4 \rightarrow 6\}$ with either $7 \rightarrow 4$ or $4 \rightarrow 7$. Straightforward analysis of each candidate shows that its A_7 can not be an isomorphism certificate.

7. Discussion

We prove in [1] that the only tournaments on n vertices that have score certificates with less than $n - 1$ edges are the regular tournaments R_3 and R_5. The preceding section shows the same thing for isomorphism certificates when $n \leq 7$, but Conjecture 1 remains unresolved. Other open questions concern exact values of $g(7)$ and $G(7)$, the growth rate of g, and whether, for regular tournaments,

$$\ell(R_n) = (3n - 9)/2 \quad \text{for odd} \quad n \geq 7 \ ,$$

and, for almost transitive tournaments,

$$L(ATT_n) = 3n - 9 \quad \text{for} \quad n \geq 6 \ .$$

We conclude with a few substantial questions not raised earlier. Call T *pseudo-linear* if it can be written as a linear order $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_k$ with each A_i a singleton or a three-vertex set that forms a 3-circuit. It is easily seen that pseudo-linear tournaments have unique score sequences, so $L = \ell$ for the pseudo-linear family. Pseudo-linear tournaments in the Appendix are shown as vertical stacks of single vertices and R_3's. There are six of them for $n = 6$ and each of those six has $\ell(T_6) = 5$ and a minimum isomorphic certificate that can be written as a directed Hamiltonian path $(1 \rightarrow 2 \rightarrow \cdots \rightarrow 6)$. Moreover, there appear to be no other T_6's for which $\ell(T_6) = 5$. At $n = 7$, we observed that $\ell(R_7) = 6$, but we know of no nonregular T_7 for which $\ell(T_7) = 6$. These observations motivate a companion to Conjecture 1.

CONJECTURE 2. For each $n \geq 6$, the following statements are mutually equivalent for all nonregular T_n:

(i) T_n is pseudo-linear;

(ii) $\ell(T_n) = n - 1$;

(iii) T_n has a directed path isomorphism certificate.

If true, we also wonder if there are any regular tournaments for $n \geq 9$ for which $\ell(T_n) < n$.

Our final questions concern minimum certificate monotonicity.

1. Does $T \subseteq T' \Rightarrow \ell(T) \leq \ell(T')$ for all tournaments T and T'?

2. Does $T \subseteq T' \Rightarrow L(T) \leq L(T')$ for all tournaments T and T'?

It is hard to imagine any possibility under which ℓ or L would violate monotonicity, but we have resolved neither question.

Appendix

All tournaments on 4, 5 and 6 vertices are shown with their score sequences. Omitted edges go from higher to lower vertices. Isomorphism and score certificates appear to the right of the tournament, in that order. If the two certificates are identical, only one diagram appears with T. The numbers below the certificates are their edge cardinalities.

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TABLE 1

n	ℓ: $f(n)$	ℓ: $g(n)$	L: $F(n)$	L: $G(n)$
2	1	1	1	1
3	1	2	1	2
4	3	3	3	3
5	3	5	3	7
6	5	7	5	11
7	6	≤ 11	6	≥ 14