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Independence of Solution Sets and Minimal Asymptotic Bases

by

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ABSTRACT

Let $k \geq 2$. The set A of nonnegative integers is a minimal asymptotic basis of order k if every sufficiently large integer can be written as the sum of k elements of A, but no proper subset of A has this property. In this paper combinatorial methods are used to find criteria for an asymptotic basis of order k to contain a minimal asymptotic basis of order k, and probabilistic methods are used to prove the existence of asymptotic bases that satisfy these criteria.

1 Introduction

Let A be a set of positive integers, and let $k \geq 2$ be a fixed integer. Let $r_A(n)$ denote the number of representations of n in the form

$$n = a_1 + a_2 + \cdots + a_k, \tag{1}$$

where

$$0 < a_1 \leq a_2 \leq \cdots \leq a_k \tag{2}$$

and $a_i \in A$ for $i = 1, \ldots, k$. Let $r'_A(n)$ denote the number of "strict" representations of n in the form

$$n = a_1 + a_2 + \cdots + a_k, \tag{3}$$

where

$$0 < a_1 < a_2 < \cdots < a_k \tag{4}$$

and $a_i \in A$ for $i = 1, \ldots, k$. The set A is called an *asymptotic basis* of order k if there exists a natural number n_1 such that $r_A(n) > 0$ for all $n \geq n_1$. The set A is called a *strict asymptotic basis* of order k if there exists a natural number n_1 such that $r'_A(n) > 0$ for all $n \geq n_1$. All bases considered in this paper will be either asymptotic or strict asymptotic bases of order k. Erdős and Tetali [7] gave a probabilistic construction of a strict asymptotic basis $\mathcal{S}$ of order k whose representation function satisfies $\log n << r'_S(n) << \log n$.

An asymptotic basis (resp. strict asymptotic basis) A is called *minimal* if the removal of any element from the basis destroys all representations of some infinite sequence of numbers, that is, $A \setminus \{a\}$ is not an asymptotic basis (resp. strict asymptotic basis) for any $a \in A$. An asymptotic basis (resp. strict asymptotic basis) A is defined to be $\aleph_0$*–minimal* if $A \setminus F$ is an asymptotic basis (resp. strict asymptotic basis) for every finite subset F of A, but $A \setminus I$ fails to be a basis for every infinite subset I of A. Erdős and Nathanson [3, 4] survey results concerning minimal asymptotic bases. In [2], they derived conditions under which an asymptotic basis of order 2 contains a minimal asymptotic basis, and they also constructed in [1] a family of $\aleph_0$–minimal asymptotic bases of order 2.

This paper has two aims. First, we give a simple set of criteria under which an asymptotic basis (resp. strict asymptotic basis) contains a minimal asymptotic basis (resp. strict asymptotic basis). These criteria also enable us to construct $\aleph_0$–minimal bases. Second, we show that the strict asymptotic basis $\mathcal{S}$ constructed in [7] satisfies this set of criteria and so contains a minimal as well as an $\aleph_0$–minimal asymptotic basis. These results answer two important questions posed in [4].

Notation. Let kA denote the set of all sums of k elements of A, and let $k\hat{A}$ denote the set of all sums of k distinct elements of A. Let $r_A(n;a)$ (resp. $r'_A(n;a)$ denote the number of representations of n in the form (1)–(2) (resp. (3)–(4)) such that $a_i = a$ for some $i = 1, \ldots, k$. The solution set of n, denoted $S_A(n)$ (resp. $S'_A(n)$), is the set of integers in A that appear in some representation of n; that is,

$$S_A(n) = \{a \in A | r_A(n;a) > 0\}$$

and

$$S'_A(n) = \{a \in A | r'_A(n;a) > 0\}.$$

2 Minimal and $\aleph_0$–minimal asymptotic bases

Erdős and Nathanson [2] discovered a set of simple criteria for an asymptotic basis of order 2 to contain a minimal asymptotic basis of order 2. We shall generalize this result to asymptotic bases of order $k \geq 3$. The following theorem is a natural extension of Theorem 3 of [2]. Condition (ii) is trivially satisfied in the case $k = 2$, but is a nontrivial restriction for asymptotic bases of orders $k \geq 3$.

Theorem 1 *Let A be a strictly increasing sequence of positive integers, and let $k \geq 2$. If*

$$\begin{aligned} &(i) \quad \lim_{n\to\infty} r_A(n) = \infty, \\ &(ii) \quad r_A(n;a) \text{ is bounded for all } n \geq 1, a \in A, \\ &(iii) \quad |S_A(m) \cap S_A(n)| \text{ is bounded for all } m \neq n, \end{aligned}$$

then A contains

$$\begin{aligned} &(a) \quad \text{a minimal asymptotic basis of order } k, \text{and} \\ &(b) \quad \text{an } \aleph_0\text{–minimal asymptotic basis of order } k. \end{aligned}$$

Proof. Let $r_A(n;a) \leq c$ for all $n \geq 1$ and $a \in A$. It follows that if F is any finite subset of A and if $|F \cap S_A(n)| \leq w$, then

$$r_{A\setminus F}(n) \geq r_A(n) - cw$$

for all n, since the removal of any one element of $S_A(n)$ destroys at most c representations of n. Let $|S_A(m) \cap S_A(n)| \leq d$ for all $m \neq n$. If $F \subseteq S_A(m)$, then

$$|F \cap S_A(n)| \leq |S_A(m) \cap S_A(n)| \leq d,$$

and so

$$r_{A\setminus F}(n) \geq r_A(n) - cd$$

for all $n \neq m$.

We shall use induction to construct a decreasing sequence of sets

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

such that

$$A^* = \cap_{j=0}^{\infty} A_j$$

is a minimal asymptotic basis of order k. We shall also construct a second decreasing sequence of subsets of A whose intersection is an $\aleph_0$–minimal asymptotic basis of order k.

Let $A_0 = A$. Since $\lim_{n\to\infty} r_A(n) = \infty$, we can choose an integer n_1 so that $r_{A_0}(n) = r_A(n) > c(1+d)$ for all $n \geq n_1$.
Choose $a_1, b_1 \in A_0$ such that $a_1 \leq n_1$ and $b_1 > kn_1$. Let

$$m_1 = a_1 + (k-1)b_1.$$

Then

$$kn_1 < b_1 \leq (k-1)b_1 < m_1 < kb_1.$$

We shall construct a set $A_1 \subseteq A_0$ such that $r_{A_1}(n) > 0$ for all $n \geq n_1$, but $m_1 \notin k(A_1 \setminus \{a_1\})$. Thus, every representation of m_1 as a sum of k elements of A_1 must include the integer a_1 as a summand.

We first determine a subset F_1 of A_0 that "destroys" every representation of m_1 that does not include a_1 as a summand. Every such representation is of the form

$$m_1 = a_1' + a_2' + \cdots + a_t' + (k-t)b_1,$$

where $a_i' \in A_0$ and $a_i' \neq a_1, b_1$ for $i = 1, 2, \ldots, t$. Note that $m_1 < kb_1$ implies that $t \neq 0$. If $t = 1$, then

$$a_1 + (k-1)b_1 = m_1 = a_1' + (k-1)b_1$$

implies that $a_1' = a_1$, which is false. Therefore, $2 \leq t \leq k$. Let

$$a_1' \leq a_2' \leq \cdots \leq a_t'.$$

Then

$$(k-1)b_1 < m_1 \leq ta_t' + (k-t)b_1 \leq ka_t' + (k-2)b_1$$

implies that

$$a_t' > b_1/k > n_1.$$

Let F_1 be the set of all such integers a_t', and let $A_1 = A_0 \setminus F_1$. Then $F_1 \subseteq [n_1+1, m_1]$. Since $a_1, b_1 \notin F_1$, it follows that $m_1 = a_1 + (k-1)b_1$ is a representation of m_1 as a sum of k elements of A_1, and so $r_{A_1}(m_1) > 0$. On the other hand, we have destroyed every representation of m_1 as the sum of k elements of A_0 all different from a_1, and so $m_1 \notin k(A_0 \setminus \{a_1\})$.

Let $n \geq n_1, n \neq m_1$. Since $F_1 \subseteq S_A(m_1)$, it follows that

$$r_{A_1}(n) = r_{A\setminus F_1}(n) \geq r_A(n) - cd > c(1+d) - cd = c > 0.$$

This completes the first step of the induction.

Let $j \geq 2$. Suppose we have constructed sets $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{j-1}$ and integers n_i, a_i, m_i for $i = 1, \ldots, j-1$ with the following properties:

(i) $kn_1 < m_1 < n_2 < kn_2 < m_2 < n_3 < \cdots < kn_{j-1} < m_{j-1}$

(ii) $F_i = A_{i-1} \setminus A_i \subseteq [n_i + 1, m_i]$ for $i = 1, \ldots, j-1$

(iii) $a_1, \ldots, a_{j-1} \in A_{j-1}$;

(iv) $r_{A_{j-1}}(n) > 0$ for $n \geq n_1$

(v) $m_i \notin k(A_i \setminus \{a_i\})$ for $i = 1, \ldots, j-1$.

We now construct the set A_j and integers n_j, a_j, and m_j.

Let $G_j = A \setminus A_{j-1} \subseteq [1, m_{j-1}]$. Choose $n_j > m_{j-1}$ such that

$$r_A(n) > c(j + d + |G_j|)$$

for all $n \geq n_j$. Choose $a_j, b_j \in A_{j-1}$ such that $a_j < n_j$ and $b_j > kn_j$. Let

$$m_j = a_j + (k-1)b_j.$$

Then

$$kn_j < b_j \leq (k-1)b_j < m_j < kb_j.$$

Exactly as in the first step of the induction, we shall determine a subset F_j of A_{j-1} that "destroys" every representation of m_j as a sum of k elements of A_{j-1} that does not include a_j as a summand. Every such representation is of the form

$$m_j = a'_1 + a'_2 + \cdots + a'_t + (k-t)b_j,$$

where $2 \leq t \leq k$, and $a'_i \in A_{j-1}, a'_i \neq a_j, b_j$ for $i = 1, 2, \ldots, t$. Let

$$a'_1 \leq a'_2 \leq \cdots \leq a'_t.$$

Then

$$(k-1)b_j < m_j \leq ta'_t + (k-t)b_j \leq ka'_t + (k-2)b_j$$

implies that

$$a'_t > b_j/k > n_j.$$

Let F_j be the set of all such integers a'_t, and let $A_j = A_{j-1} \setminus F_j$. Then

$$F_j \subseteq [n_j + 1, m_j] \cap S_{Aj-1}(m_j) \subseteq [n_j + 1, m_j] \cap S_A(m_j).$$

Since $a_j, b_j \notin F_j$, it follows that $m_j = a_j + (k-1)b_j$ is a representation of m_j as a sum of k elements of A_j, and so $r_{A_j}(m_j) > 0$. However, $m_j \notin k(A_j \setminus \{a_j\})$, since the set A_j was constructed so that every representation of m_j as the sum of k elements of A_j has at least one summand equal to a_j.

Let $n_1 \leq n \leq n_j$. Since $A_{j-1} \setminus A_j = F_j \subseteq [n_j+1, m_j]$, it follows that $r_{A_j}(n) = r_{A_{j-1}}(n) > 0$. Let $n > n_j, n \neq m_j$. Since

$$A \setminus A_j = F_j \cup G_j$$

and

$$(F_j \cup G_j) \cap S_A(n) \subseteq (F_j \cap S_A(n)) \cup G_j \subseteq (S_A(m_j) \cap S_A(n)) \cup G_j,$$

it follows that

$$|\,(F_j \cup G_j) \cap S_A(n)| \leq d + |G_j|$$

and so

$$r_{A_j}(n) \geq r_A(n) - c(d + |G_j|) > c(j + d + |G_j|) - c(d + |G_j|) = cj > 0.$$

This completes the induction.

Let $A^* = \cap_{j=1}^{\infty} A_j$. Let $n \geq n_1$. Choose $j \geq 1$ so that $n_j \leq n < n_{j+1}$. Since $A_j \setminus A^* \subseteq [n_{j+1}+1, \infty)$, it follows that

$$r_{A^*}(n) = r_{A_j}(n) > cj > 0,$$

and so A^* is an asymptotic basis of order k. Moreover, since

$$m_j \notin k(A_j \setminus \{a_j\})$$

for every $j \geq 1$, it follows that

$$m_j \notin k(A^* \setminus \{a_j\}).$$

Recall that at each step j of the induction, we chose an integer a_j. We had complete freedom to select this integer, subject only to the conditions that $a_j \in A_{j-1}$ and $a_j \leq n_j$. Let us choose these integers in such a way that every element $a \in A^*$ is chosen infinitely often, that is, if $a \in A^*$, then $a = a_j$ for infinitely many j. Then the set A^* will be a minimal asymptotic basis of order k, since the deletion of any element $a \in A^*$ will destroy all representations of infinitely many integers m_j.

To construct an $\aleph_0$-minimal asymptotic basis, we choose the numbers a_j such that, if $a \in A^*$, then $a = a_j$ for exactly one integer j. If an infinite subset I is deleted from A^*, then there are infinitely many integers of the form m_j that cannot be written as the sum of k terms of $A \setminus I$, and so $A \setminus I$ is not an asymptotic basis of order k.

Let F be a finite subset of A^*, and let $|F| = f$. We shall show that $A^* \setminus F$ is an asymptotic basis of order k.

Since $a_j \in F$ for exactly f indices j, and since $b_j \in F$ for at most f indices j, it follows that $m_j \in k(A^* \setminus F)$ for all but at most $2f$ numbers m_j.

Let $n \geq n_f, n \neq m_j$ for all j. Choose j such that $n_j \leq n < n_{j+1}$. Then $j \geq f$. Since

$$r_{A^*}(n) = r_{A_j}(n) > cj > 0,$$

and since each element of F destroys at most c representations of n, it follows that

$$r_{A^* \setminus F}(n) > cj - cf \geq 0.$$

Thus, $A^* \setminus F$ is an asymptotic basis of order k, and so A^* is an $\aleph_0$–minimal asymptotic basis of order k. This completes the proof.

Theorem 2 *Let A be a strictly increasing sequence of positive integers, and let $k \geq 2$. If*

(i) $\lim_{n\to\infty} r'_A(n) = \infty$,

(ii) $r'_A(n;a)$ *is bounded for all* $n \geq 1, a \in A$,

(iii) $|S'_A(m) \cap S'_A(n)|$ *is bounded for all* $m \neq n$,

then A contains

(a) a minimal strict asymptotic basis of order k, and

(b) an $\aleph_0$–minimal strict asymptotic basis of order k.

Proof. Let $r'_A(n;a) \leq c$ for all $n \geq 1$ and $a \in A$. It follows that if F is any finite subset of A and if $|F \cap S'_A(n)| \leq w$, then

$$r'_{A\setminus F}(n) \geq r'_A(n) - cw$$

for all n, since the removal of any one element of $S'_A(n)$ destroys at most c representations of n. Let $|S'_A(m) \cap S'_A(n)| \leq d$ for all $m \neq n$. If $F \subseteq S'_A(m)$, then

$$|F \cap S'_A(n)| \leq |S'_A(m) \cap S'_A(n)| \leq d,$$

and so

$$r'_{A\setminus F}(n) \geq r'_A(n) - cd$$

for all $n \neq m$.

We shall use induction to construct a decreasing sequence of sets

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

such that

$$\hat{A} = \cap_{j=0}^{\infty} A_j$$

is a strict minimal asymptotic basis of order k. We shall also construct a second decreasing sequence of subsets of A whose intersection is an $\aleph_0$–minimal strict asymptotic basis of order k.

Let $A_0 = A$. Since $\lim_{n\to\infty} r'_A(n) = \infty$, we can choose an integer n_1 so that $r'_{A_0}(n) = r'_A(n) > c(1+d)$ for all $n \geq n_1$. Choose k integers $a_1, b_{1,1}, b_{1,2}, \ldots, b_{1,k-1} \in A_0$ such that

$$a_1 \leq n_1 < kn_1 < b_{1,1} < b_{1,2} < \cdots < b_{1,k-1}.$$

Let

$$m_1 = a_1 + b_{1,1} + b_{1,2} + \cdots + b_{1,k-1}.$$

We shall construct a set $A_1 \subseteq A_0$ such that $r'_{A_1}(n) > 0$ for all $n \geq n_1$, and with the additional property that every representation of m_1 as a sum of k distinct elements of A_1 must include the integer a_1 as a summand.

We first determine a subset F_1 of A_0 that "destroys" every strict representation of m_1 that does not include a_1 as a summand. Every such representation is of the form

$$m_1 = a'_1 + a'_2 + \cdots + a'_t + b_{1,u_1} + b_{1,u_2} + \cdots + b_{1,u_{k-t}}.$$

where $2 \leq t \leq k$, and $a'_i \in A_0, a'_i \neq a_1, b_{1,u}$ for $i = 1, 2, \ldots, t$ and $u = 1, \ldots, k-1$. Let

$$a'_1 < a'_2 < \cdots < a'_t.$$

Since $(k-1) - (k-t) = t - 1 \geq 1$ it follows that

$$(b_{1,1} + b_{1,2} + \cdots + b_{1,k-1}) - (b_{1,u_1} + b_{1,u_2} + \cdots + b_{1,u_{k-t}}) = b_{1,v_1} + b_{1,v_2} + \cdots + b_{1,v_{t-1}} \geq b_{1,1}.$$

Then

$$\begin{aligned} m_1 &= a_1 + b_{1,1} + b_{1,2} + \cdots + b_{1,k-1} \\ &= a'_1 + a'_2 + \cdots + a'_t + b_{1,u_1} + b_{1,u_2} + \cdots + b_{1,u_{k-t}}. \end{aligned}$$

implies that

$$\begin{aligned} kn_1 &< b_{1,1} \\ &< a_1 + b_{1,1} \\ &\le a_1 + b_{1,v_1} + b_{1,v_2} + \cdots + b_{1,v_{t-1}} \\ &= m_i - (b_{1,u_1} + b_{1,u_2} + \cdots + b_{1,u_{k-t}}) \\ &= a'_1 + a'_2 + \cdots + a'_t \\ &< ta'_t \\ &\le ka'_t \end{aligned}$$

and so

$$a'_t > n_1.$$

Let F_1 be the set of all such integers a'_t, and let $A_1 = A_0 \setminus F_1$. Then $F_1 \subseteq [n_1 + 1, m_1]$. Since $a_1, b_{1,1}, b_{1,2}, \ldots, b_{1,k-1} \notin F_1$, it follows that $m_1 = a_1 + b_{1,1} + b_{1,2} + \cdots + b_{1,k-1}$ is a representation of m_1 as a sum of k distinct elements of A_1, and so $r'_{A_1}(m_1) > 0$. On the other hand, we have destroyed every representation of m_1 as the sum of k distinct elements of A_0 all different from a_1, and so $m_1 \notin k(A_0 \setminus \{a_1\})$.

Let $n \ge n_1, n \ne m_1$. Since $F_1 \subseteq S'_A(m_1)$, it follows that

$$r'_{A_1}(n) = r'_{A \setminus F_1}(n) \ge r'_A(n) - cd > c(1+d) - cd = c > 0.$$

This completes the first step of the induction.

Let $j \ge 2$. Suppose we have constructed sets $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_{j-1}$ and integers n_i, a_i, m_i for $i = 1, \ldots, j-1$ with the following properties:

(i) $kn_1 < m_1 < n_2 < kn_2 < m_2 < n_3 < \cdots < kn_{j-1} < m_{j-1}$

(ii) $F_i = A_{i-1} \setminus A_i \subseteq [n_i + 1, m_i]$ for $i = 1, \ldots, j-1$

(iii) $a_1, \ldots, a_{j-1} \in A_{j-1}$

(iv) $r'_{A_{j-1}}(n) > 0$ for $n \ge n_1$

(v) $m_i \notin k(A_i \setminus \{a_i\})$ for $i = 1, \ldots, j-1$.

We now construct the set A_j and integers n_j, a_j, and m_j.

Let $G_j = A \setminus A_{j-1} \subseteq [1, m_{j-1}]$. Choose $n_j > m_{j-1}$ such that

$$r'_A(n) > c(j + d + |G_j|)$$

for all $n \ge n_j$. Choose $a_j, b_{j,1}, b_{j,2}, \ldots, b_{j,k-1} \in A_{j-1}$ such that

$$a_j \le n_j < kn_j < b_{j,1} < b_{j,2} < \cdots < b_{j,k-1}.$$

Let

$$m_j = a_j + b_{j,1} + b_{j,2} + \cdots + b_{j,k-1}.$$

Exactly as in the first step of the induction, we shall determine a subset F_j of A_0 that "destroys" every representation of m_j as a sum of k distinct elements of A_{j-1} that does not include a_j as a summand.

Every such representation is of the form

$$m_j = a'_1 + a'_2 + \cdots + a'_t + b_{j,u_1} + b_{j,u_2} + \cdots + b_{j,u_{k-t}},$$

where $2 \leq t \leq k$, and $a'_i \in A_0, a'_i \neq a_j, b_{j,u}$ for $i = 1, 2, \ldots, t$ and $u = 1, 2, \ldots, k-1$. Let

$$a'_1 < a'_2 < \cdots < a'_t.$$

Since $(k-1) - (k-t) = t - 1 \geq 1$, it follows that

$$(b_{j,1} + b_{j,2} + \cdots + b_{j,k-1}) - (b_{j,u_1} + b_{j,u_2} + \cdots + b_{j,u_{k-t}}) = b_{j,v_1} + b_{j,v_2} + \cdots + b_{j,v_{t-1}} \geq b_{j,1}.$$

Then

$$\begin{aligned} m_j &= a_j + b_{j,1} + b_{j,2} + \cdots + b_{j,k-1} \\ &= a'_1 + a'_2 + \cdots + a'_t + b_{j,u_1} + b_{j,u_2} + \cdots + b_{j,u_{k-t}} \end{aligned}$$

implies that

$$\begin{aligned} kn_j &< b_{j,1} \\ &< a_j + b_{j,1} \\ &\leq a_j + b_{j,v_1} + b_{j,v_2} + \cdots + b_{j,v_{t-1}} \\ &= m_j - (b_{j,u_1} + b_{j,u_2} + \cdots + b_{j,u_{k-t}}) \\ &= a'_1 + a'_2 + \cdots + a'_t \\ &< ta'_t \\ &\leq ka'_t \end{aligned}$$

and so

$$a'_t > n_j.$$

Let F_j be the set of all such integers a'_t, and let $A_j = A_{j-1} \setminus F_j$. Then $F_j \subseteq [n_j + 1, m_j]$. Since $a_j, b_{j,1}, b_{j,2}, \ldots, b_{j,k-1} \notin F_j$, it follows that $m_j = a_j + b_{j,1} + b_{j,2} + \cdots + b_{j,k-1}$ is a representation of m_j as a sum of k elements of A_j, and so $r'_{A_j}(m_j) > 0$. On the other hand, we have destroyed every representation of m_j as the sum of k distinct elements of A_{j-1} all different from a_j, and so $m_j \notin k(A_{j-1} \setminus \{a_j\})$. Let $n_1 \leq n \leq n_j$. Since $A_{j-1} \setminus A_j = F_j \subseteq [n_j + 1, m_j] \cap S'_A(m_j)$, it follows that $r'_{A_j}(n) = r'_{A_{j-1}}(n) > 0$. Let $n > n_j, n \neq m_j$. Since

$$A \setminus A_j = F_j \cup G_j$$

and

$$(F_j \cup G_j) \cap S'_A(n) \subseteq (F_j \cap S'_A(n)) \cup G_j \subseteq (S'_A(m_j) \cap S'_A(n)) \cup G_j,$$

it follows that

$$|F_j \cup G_j) \cap S'_A(n)| \leq d + |G_j|$$

and so

$$r'_{A_j}(n) \geq r'_A(n) - c(d + |G_j|) > cj > c(j + d + |G_j|) - c(d + |G_j|) = cj > 0.$$

This completes the induction.

Let $\hat{A} = \cap_{j=1}^{\infty} A_j$. Let $n \geq n_1$. Choose $j \geq 1$ so that $n_j \leq n < n_{j+1}$. Since $A_j \setminus \hat{A} \subseteq [n_{j+1} + 1, \infty)$, it follows that

$$r'_{\hat{A}}(n) = r'_{A_j}(n) > cj > 0,$$

and so $\hat{A}$ is a strict asymptotic basis of order k. Moreover, for every $j \geq 1$, since

$$m_j \notin k(A_j \setminus \{a_j\}),$$

it follows that

$$m_j \notin k(\hat{A} \setminus \{a_j\}).$$

Recall that at each step j of the induction, we chose an integer a_j. We had complete freedom to select this integer, subject only to the conditions that $a_j \in A_{j-1}$ and $a_j \leq n_j$. Let us choose these integers in such a way that every element $a \in \hat{A}$ is chosen infinitely often, that is, if $a \in \hat{A}$, then $a = a_j$ for infinitely many j. Then the set $\hat{A}$ will be a minimal asymptotic basis of order k, since the deletion of any element a will destroy all representations of infinitely many integers m_j.

To construct an $\aleph_0$-minimal strict asymptotic basis, we choose the numbers a_j such that, if $a \in \hat{A}$, then $a = a_j$ for exactly one integer j. If an infinite subset I is deleted from $\hat{A}$, then there is an infinite increasing sequence of integers of the form m_j that cannot be written as the sum of k terms of $\hat{A} \setminus I$, and so $\hat{A} \setminus I$ is not a strict asymptotic basis of order k.

Let F be a finite subset of $\hat{A}$, and let $|F| = f$. We shall show that $\hat{A} \setminus F$ is a strict asymptotic basis of order k.

Since $a_j \in F$ for exactly f indices j, and since $b_{j,u} \in F$ for at most f double indices (j, u), it follows that $m_j \in k(\hat{A} \setminus F)$ for all but at most $2f$ numbers m_j.

Let $n \geq n_f, n \neq m_j$ for all j. Choose j such that $n_j \leq n < n_{j+1}$. Then $j \geq f$. Since

$$r'_{\hat{A}}(n) = r'_{A_j}(n) > cj > 0,$$

and since each element of F destroys at most c representations of n, it follows that

$$r'_{\hat{A}\setminus F}(n) > cj - cf = c(j - f) \geq 0.$$

Thus, $\hat{A}\setminus F$ is a strict asymptotic basis of order k, and so $\hat{A}$ is an $\aleph_0$–minimal strict asymptotic basis of order k. This completes the proof.

3 Independence of solution sets

Let $\mathcal{S}$ be the asymptotic basis constructed in [7]. In this section we want to prove that $\mathcal{S}$ satisfies the conditions of Theorems 1 and 2. That is, we prove the following theorem.

Theorem 3 *The asymptotic basis $\mathcal{S}$ contains the following:*

(a) *a minimal asymptotic basis of order k,*
(b) *a minimal strict asymptotic basis of order k,*
(c) *an $\aleph_0$–minimal asymptotic basis of order k, and*
(d) *an $\aleph_0$–minimal strict asymptotic basis of order k.*

Proof. In view of the previous section, it suffices to verify that $\mathcal{S}$ satisfies the hypotheses of Theorems 1 and 2. We first prove, in Lemma 1 below, that it suffices to verify that $\mathcal{S}$ satisfies the hypothesis of Theorem 2. The first criterion of the hypothesis of Theorem 2 is satisfied by $\mathcal{S}$, since $r'_{\mathcal{S}}(n) = \Theta(\log n)$, which is the main result of [7]. Lemmas 2 and 3 in the following show that the asymptotic basis $\mathcal{S}$ does in fact satisfy the rest of the hypothesis of Theorem 2. (In short, Lemmas 1,2, and 3 below constitute the proof of this theorem.) □

Suppose that $\mathcal{S}$ satisfies the hypothesis of Theorem 2. The following argument shows that $\mathcal{S}$ satisfies the hypothesis of Theorem 1 as well.

Lemma 1

$$r_{\mathcal{S}}(n) - r'_{\mathcal{S}}(n) < \infty, \quad \textit{for all } n.$$

Proof. Consider the representations that contribute to $r_{\mathcal{S}}(n)$ but not to $r'_{\mathcal{S}}(n)$. The number of distinct elements in each such representation of n is at least one and at most $k-1$. Consider a representation of n with l distinct elements, where $1 \le l \le k-1$. i.e.

$$n = a_1 + \cdots + a_l + a_{l+1} + \cdots + a_k, \quad a_i \in \mathcal{S}, \quad a_1 < \cdots < a_l$$

We will be done by showing that there are only finitely many representations of this form for each n.
Consider $m = n - (a_{l+1} + \cdots + a_k) = a_1 + \cdots + a_l$. Equivalently, we want to show for each m, the number of representations (denoted $r'_l(m)$) as a sum of l distinct elements from $\mathcal{S}$ is bounded.

By Lemma 10 of [7], we know that the number of representations of n as a sum of l distinct elements is bounded for $l < k$. Hence the lemma. □

With this lemma, for the rest of this section it suffices to consider only the *distinct* representations, and verify that $\mathcal{S}$ satisfies the hypothesis of Theorem 2. The second criterion in Theorem 2 asserts that the number of representations of n that use a be bounded, for every $n \in N$, and $a \in \mathcal{S}$.

Lemma 2 *$r'_{\mathcal{S}}(n;a)$ is bounded for all $a \in \mathcal{S}$.*

Proof. Note that $r'_{\mathcal{S}}(n;a)$ = the number of representations (in $\mathcal{S}$) of $(n-a)$ as a sum of $k-1$ terms. Once again this follows from Lemma 10 of [7].

Finally, the following lemma proves that $\mathcal{S}$ meets the third criterion in Theorem 2.

Lemma 3 $|S'_{\mathcal{S}}(m) \cap S'_{\mathcal{S}}(n)|$ *is bounded for all* $m < n$.

Before we prove Lemma 3, we need a couple of technical lemmas. The idea is going to be similar to that of the proof of Lemma 10 of [7] – we first estimate the expected such number, and then bound the *disjoint* occurences.

Let $R_l(n,m)$ represent the number of representations of n and m that overlap in l numbers. (Note that $l \in [1, k-1]$.) Further, let $R^*_l(n,m)$ represent a maximal collection of "disjoint overlaps" – each overlapping pair of representations for n and m is disjoint from the other overlapping pairs. Also, let $R(n,m)$ and $R^*(n,m)$ denote the corresponding terms when no restriction is made on the size (l) of the overlap.

Lemma 4

$$E[R(n,m)] \leq n^{-\frac{l}{2k}+o(1)}$$

Proof. Without loss of generality, let $m < n$. Then, for fixed n and m, a typical overlapping pair of representations is of the following form.

$$z_1 + \cdots + z_l + x_1 + \cdots + x_{k-l} = n, \quad z_1 + \cdots + z_l + y_1 + \cdots + y_{k-l} = m,$$

$$\text{where } z_1 + \cdots + z_l = t, \quad 1 \leq t < m.$$

Thus the expected value of $R_l(n,m)$ equals

$$\begin{aligned}
&= \sum_{1 \leq t < m} \sum_{\substack{z_1+\cdots+z_l=t \\ x_1+\cdots+x_{k-l}=n-t \\ y_1+\cdots+y_{k-l}=m-t}} \Pr[z_1]\cdots\Pr[z_l] \\
&\quad \times (\Pr[x_1]\cdots\Pr[x_{k-l}])\,(\Pr[y_1]\cdots\Pr[y_{k-l}]) \\
&= \sum_t \left(\sum_{z_1+\cdots+z_l=t} \Pr[z_1]\cdots\Pr[z_l] \right) \\
&\quad \times \left(\sum_{x_1+\cdots+x_{k-l}=n-t} \Pr[x_1]\cdots\Pr[x_{k-l}] \right) \left(\sum_{y_1+\cdots+y_{k-l}=m-t} \Pr[y_1]\cdots\Pr[y_{k-l}] \right) \\
&= \sum_t \mu_l(t)\mu_{k-l}(n-t)\mu_{k-l}(m-t) \\
&= \Delta \text{ (say).}
\end{aligned}$$

We are going to show that $\Delta \leq n^{-\frac{l}{2k}+o(1)}$ by making use of the following estimates for $\mu_l(n)$ from [7] (Lemma 8, page 252):

$$\mu_l(n) \leq n^{-1+l/k+o(1)}, \qquad \text{for } 2 \leq l \leq (k-1).$$

For technical reasons, fix $\varepsilon = \frac{l}{4k}$. Now pick t_0 such that

$$\mu_l(t) \leq t^{-1+\frac{l}{k}+\varepsilon}, \qquad \text{for } t > t_0.$$

The proof that $\Delta \leq n^{-\frac{l}{2k}+o(1)}$ gets quite technical, and can be omitted on the first reading without loss of understanding of the rest of the paper.

Case 1. Let us assume that $m = O(n^\delta)$, for $\delta < 1$.

Case 1(a). $t \leq t_0$:

$$\begin{aligned}
\Delta_1 &= \sum_{t \leq t_0} (\mu_l(t)\mu_{k-l}(n-t)\mu_{k-l}(m-t)) \\
&< n^{-1+\frac{k-l}{k}+o(1)} m^{-1+\frac{k-l}{k}+o(1)} \sum_{t \leq t_0} H \\
&< n^{-1+\frac{k-l}{k}+o(1)} = n^{-l/k+o(1)}
\end{aligned}$$

Case 1(b). $m - t_0 < t \leq m$:

$$\begin{aligned}
\Delta_2 &= \sum_{m-t_0 < t \leq m} (\mu_l(t)\mu_{k-l}(n-t)\mu_{k-l}(m-t)) \\
&< \left(m^{-1+\frac{l}{k}+o(1)} n^{-1+\frac{k-l}{k}+o(1)}\right) \sum_{m-t_0 < t \leq m} \mu_{k-l}(m-t) \\
&< \left(n^{-1+\frac{k-l}{k}+o(1)}\right) \sum_{m-t_0 < t \leq m} H \\
&< \left(n^{-1+\frac{k-l}{k}+o(1)}\right) = n^{-l/k+o(1)}
\end{aligned}$$

Case 1(c). $t_0 < t \leq m - t_0$:

$$\begin{aligned}
\Delta_3 &= \sum_{t_0 < t \leq m-t_0} (\mu_l(t)\mu_{k-l}(n-t)\mu_{k-l}(m-t)) \\
&< \left(n^{-1+\frac{k-l}{k}+o(1)} m^{-1+\frac{k-l}{k}+o(1)}\right) \sum_{t_0 < t \leq m-t_0} t^{-1+\frac{l}{k}+\varepsilon}
\end{aligned}$$

We can now estimate the sum by an integral over the full range $0 \leq t \leq m$:

$$\begin{aligned}
\Delta_3 &< \left(n^{-1+\frac{k-l}{k}+o(1)} m^{-1+\frac{k-l}{k}+o(1)}\right) \left(\int_{t=0}^{m} t^{-1+\frac{l}{k}+\varepsilon} + O(1)\right) \\
&= \left(n^{-1+\frac{k-l}{k}+o(1)} m^{-1+\frac{k-l}{k}+o(1)}\right) \left(m^{l/k+\varepsilon} + O(1)\right) \\
&< n^{-1+\frac{k-l}{k}+o(1)} = n^{-l/k+o(1)}
\end{aligned}$$

Case 2. In this case, we let $m = \Theta(n)$.

Case 2(a). $t \leq t_0$:

$$\begin{aligned}
\Delta_1' &= \sum_{t \leq t_0} (\mu_l(t)\mu_{k-l}(n-t)\mu_{k-l}(m-t)) \\
&< n^{-1+\frac{k-l}{k}+o(1)} m^{-1+\frac{k-l}{k}+o(1)} \sum_{t \leq t_0} H \\
&< n^{-1+\frac{k-l}{k}+o(1)} m^{-1+\frac{k-l}{k}+o(1)} \\
&< n^{-2+2\frac{k-l}{k}+o(1)}, \text{ since } m = \Theta(n) \\
&= n^{-l/k+o(1)}
\end{aligned}$$

Case 2(b). $m - t_0 < t \leq m$:

$$\begin{aligned}
\Delta_2' &= \sum_{m-t_0 < t \leq m} (\mu_l(t)\mu_{k-l}(n-t)\mu_{k-l}(m-t)) \\
&< \left(m^{-1+\frac{l}{k}+o(1)}\right) \sum_{m-t_0 < t \leq m} (n-t)^{-1+\frac{k-l}{k}+o(1)} H \\
&= \left(m^{-1+\frac{l}{k}+o(1)}\right) \sum_{m-t_0 < t \leq m} (n-t)^{-1+\frac{k-l}{k}+o(1)} \\
&< \left(m^{-1+\frac{l}{k}+o(1)}\right) \left(t_0 \times (n-m)^{-1+\frac{k-l}{k}+o(1)}\right) \\
&= \left(m^{-1+\frac{l}{k}+o(1)}\right) \left(n^{-1+\frac{k-l}{k}+o(1)}\right) \\
&= \left(n^{-1+\frac{l}{k}+o(1)}\right) \left(n^{-l/k+o(1)}\right) \\
&< n^{-1+o(1)}
\end{aligned}$$

Case 2(c). $t_0 < t \leq m/2$:

$$\begin{aligned}
\Delta_3' &= \sum_{t_0 \leq t \leq m/2} (\mu_l(t)\mu_{k-l}(n-t)\mu_{k-l}(m-t)) \\
&< \left(n^{-1+\frac{k-l}{k}+o(1)} m^{-1+\frac{k-l}{k}+o(1)}\right) \sum_{t_0 \leq t \leq m/2} t^{-1+\frac{l}{k}+\varepsilon}
\end{aligned}$$

We can now estimate the sum by an integral over the full range $0 \leq t \leq m$:

$$\begin{aligned}
\Delta_3' &< \left(n^{-1+\frac{k-l}{k}+o(1)}\right) \left(m^{-1+\frac{k-l}{k}+o(1)}\right) \left(\int_{t=0}^{m} t^{-1+\frac{l}{k}+\varepsilon} + O(1)\right) \\
&= \left(n^{-1+\frac{k-l}{k}+o(1)}\right) \left(m^{-1+\frac{k-l}{k}+o(1)}\right) \left(m^{l/k+\varepsilon} + O(1)\right) \\
&= \left(n^{-\frac{l}{k}+o(1)} m^{\varepsilon+o(1)}\right) \\
&< n^{-\frac{l}{k}+\varepsilon+o(1)} = n^{-\frac{3l}{4k}+o(1)}
\end{aligned}$$

Case 2(d). $m/2 < t \leq m - t_0$:

$$\begin{aligned}
\Delta_4' &= \sum_{m/2<t\leq m-t_0} (\mu_l(t)\mu_{k-l}(n-t)\mu_{k-l}(m-t)) \\
&< \left(m^{-1+\frac{l}{k}+o(1)}\right) \sum_{m/2<t\leq m-t_0} (n-t)^{-1+\frac{k-l}{k}+\varepsilon}(m-t)^{-1+\frac{k-l}{k}+\varepsilon} \\
&< \left(m^{-1+\frac{l}{k}+o(1)}\right) \sum_{m/2<t\leq m-t_0} (m-t)^{-2l/k+2\varepsilon}
\end{aligned}$$

Once again, we estimate the sum by an integral over the full range $0 \leq t \leq m$:

$$\begin{aligned}
\Delta_4' &< m^{-1+\frac{l}{k}+o(1)} \left(\int_{t=0}^{m} (m-t)^{-2l/k+2\varepsilon} + O(1)\right) \\
&< m^{-1+\frac{l}{k}+o(1)} \left(-(m-t)^{1-2l/k+2\varepsilon}|_0^m + O(1)\right) \\
&= m^{-l/k+2\varepsilon+o(1)} \\
&= n^{-l/k+2\varepsilon+o(1)} = n^{-\frac{l}{2k}+o(1)}
\end{aligned}$$

¿From Cases 1 and 2, we can conclude that

$$E[R_l(m,n)] \leq n^{-\frac{l}{2k}+o(1)}$$

This implies

$$E[R(m,n)] = \sum_{l=1}^{k-1} E[R_l(m,n)] \leq n^{-\frac{l}{2k}+o(1)}.$$

Lemma 5 *(i)* $\Pr[R^*(m,n) > 8k] < n^{-4+o(1)}$
(ii) *a.a.* $\exists c^*$, *s.t.* $R^*(m,n) < c^*$ *for all* $m < n$.

Proof. (i) We use the *disjointness lemma* from [7] to prove the first part; thus

$$\begin{aligned}
\Pr[R^*(m,n) > 8k] &< \frac{(E[R(m,n)])^{8k}}{(8k)!} \\
&< \frac{1}{(8k)!}\left(n^{-\frac{l}{2k}+o(1)}\right)^{8k} \\
&= n^{-4l+o(1)} \\
&< n^{-4+o(1)}, \text{ since } l \geq 1.
\end{aligned}$$

(ii) Let A_{mn} denote the event that $R^*(m,n) > 8k$. Then the first part of this lemma implies that $\Pr[A_{mn}] < n^{-4+o(1)}$. There are at most n^2 pairs (m,n) such that $m < n$, and since $n^2 \Pr[A(m,n)] < \infty$, by the Borel-Cantelli lemma (see e.g. [7]), this implies that

$$\text{a.a} \quad \exists n^* \quad \text{s.t. } R^*(m,n) < 8k \text{ for all } m < n, \text{ whenever } n > n^*.$$

But for any finite n^*, $R^*(m,n^*)$ is certainly bounded for all $m < n^*$. Thus we conclude that

$$\text{a.a.} \quad \exists c^*, \quad \text{s.t. } R^*(m,n) < c^* \text{ for all } m < n.$$

□

Proof of Lemma 3. Let us define the following equivalence relation "**o**" on the numbers in $|S_{\mathcal{S}}(m) \cap S_{\mathcal{S}}(n)|$.
$x\mathbf{o}y$ iff

$$x + a_1 + \cdots + a_{k-1} = m \text{ and } y + a'_1 + \cdots + a'_{k-1} = n$$

and moreover,

$$\{a_1, \ldots, a_{k-1}\} \cap \{a'_1, \ldots, a'_{k-1}\} \neq \phi.$$

(Thus x and y are related iff x and y belong to some overlapping pair of representations for m and n.) The number of equivalence classes defined by **o** is bounded since $R^*(m,n)$ is bounded. Moreover, for each $x \in$class C_x, there are at most a bounded number of $y \in C_x$, since both $r'_{\mathcal{S}}(m;a)$ and $r'_{\mathcal{S}}(n;a)$ are bounded. Thus each equivalence class is also bounded, and hence $|S'_{\mathcal{S}}(m) \cap S'_{\mathcal{S}}(n)|$ is bounded. □

4 Conclusions

Theorems 1 and 2 along with Lemmas 1, 2, and 3 imply that the asymptotic basis constructed in [7] contains a minimal (strict) and an $\aleph_0$–minimal (strict) asymptotic basis.

Erdős and Nathanson [3] obtained the following very simple criterion for an asymptotic basis A of order 2 to contain a minimal asymptotic basis.

Theorem 4 (EN) *If there exists a constant $c > 1/\log(\frac{4}{3})$, such that $r'_A(n) > c\log n$ for all sufficiently large n, then A contains a minimal asymptotic basis of order 2.*

The combinatorial lemma at the heart of this theorem has since been generalized by Jia [8] and Nathanson [10]. However, the search for an analogue of Theorem [**EN**] remains open for bases of order $k > 2$. Clearly, this question requires some new ideas.

Another very interesting problem, which is open even for bases of order 2, is if the weaker condition that $r'_A(n) \to \infty$ is sufficient to imply that A must contain a minimal asymptotic basis. Perhaps this conjecture is too optimistic, but it is possible that $r'_A(n) > c\log n$, for every $c > 0$, is sufficient to imply that A must contain a minimal asymptotic basis.

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