On the Very Weak 0-1 Law for Random Graphs with Orders

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Abstract: Let us draw a graph R on $\{0, 1, \ldots, n-1\}$ by having an edge $\{i, j\}$ with probability $p_{|i-j|}$, where $\sum_i p_i < \infty$, and let $M_n = (n, <, R)$. For a first order sentence ψ let a^n_ψ be the probability of $M_n \models \psi$. We prove that $\lim_{n\to\infty}(a^n_\psi - a^{n+1}_\psi) = 0$.

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§0 Introduction

The kind of random models $M_n = (n, <, R)$ from the abstract are from Luczak Shelah [LuSh 435] where among other things, it is proved that the probability $\mathrm{Prob}(M_n \models \psi)$ of $M_n \models \psi$ does not necessarily converge (but if $\sum_i ip_i < \infty$ it converges, and the value of $\sum p_i$ is not enough, in general, to determine convergence). The theorem in the abstract says that it still behaves nicely. A sentence whose probability may not converge (by [LuSh 435]) is $(\exists x)\forall yz(y < x \leq z \rightarrow \neg yRz)$. But we can find a sequence $m_0 < m_1 < \dots$ such that the probability that $\vee_i(\exists yz)(x \leq m_i \wedge m_{i+1} \leq y \wedge xRy)$ is very small. So reformulation of making the random choice (in §1) + a model theoretic lemma discussed in §2 enables us to prove that theorem.

In §1 we also prove the very weak 0-1 law for a random partial order suggested by Luczak.

Notation

$\mathbb{N}$ is the set of natural numbers.

$\mathbb{Z}$ is the set of integers.

$\mathbb{R}$ the set of reals, $\mathbb{R}^+$ the set of reals which are positive (i.e. > 0).

$i, j, k, \ell, m, n, r, s, t$ are natural numbers.

ε, ζ are positive reals (or functions with values in $\mathbb{R}^+$).

f, g, h are functions.

τ denotes a vocabulary (for simplicity- set of predicates), $\mathcal{L}_\tau^{\mathrm{fo}}$ is the set of first order formulas in the vocabulary τ. (Generally, $\mathcal{L}$ is a logic $\mathcal{L}_\tau$ the set of sentences (or formulas) in $\mathcal{L}$ in the vocabulary τ). For a first order sentence (or formula) φ let $d_\varphi = d[\varphi]$ be its quantifier depth.

M, N denote models, but we do not distinguish strictly between a model and its universe= set of elements.

$\tau(M)$ is the vocabulary of M, for $R \in \tau(M)$, $n(R)$ is the number of places (=arity of R), R^M the interpretation of R in M.

A basic formula is one of the form $\pm R(x_{i_0}, \ldots, x_{i_{n(R)-1}})$

$\bar{a}$ denotes a sequence of elements of a model. $\ell g\,(\bar{a})$ is the length of $\bar{a}$. If $<$ belongs to the vocabulary τ, then in τ-models $<^M$ is a linear order, if not said otherwise.

If M_i are τ-models for $i < n$, $M = \sum_{i<n} M_i$ is (assuming for simplicity the universes are pairwise disjoint): universe $\bigcup_{i<n} M_i, R^M = \bigcup_{i<n} R^{M_i}$ for $i < n$ and (if $< \in \tau$) $x <^M y \Leftrightarrow \bigvee_{i<j<n}[x \in M_i \,\&\, y \in M_j] \vee \vee_i x <^{M_i} y$ (similarly with any linear order I instead of n). We write $M_0 + M_1$ instead of $\sum_{i<2} M_i$.

§1 The Very Weak Zero One Law.

1.1 Definition: 1) A 0-1 law context is a sequence $\bar{K} = \langle K_n, \mu_n : n < \omega \rangle$ such that

(a) for some vocabulary τ, for every n, K_n is a family of τ models, closed under isomorphism with the family of isomorphism types being a set.

(b) for each n, μ_n is a probability measure on the set of isomorphism types (of models from K_n).

2) $\bar{K}$ is finitary (countable) if for each n $\{M/\cong: M \in K_n\}$ is finite (countable).

3) For a sentence ψ (not necessarily f.o.) $\text{Prob}\mu_n(M_n \models \psi)$ or $\text{Prob}_{K_n}(M_n \models \psi)$ or $\text{Prob}_{\mu_n}(M_n \models \psi \Big| M_n \in K_m)$ means $\mu_n\{M/\cong: M \in K_n, M \models \psi\}$.

4) We may use K_n a set of models, μ_n a probability on K_n; particularly we introduce a random choice of M_n; the translation between the two contents should be clear.

1.2 Definition: 1) $\bar{K}$ satisfies the very weak zero-one law for $\mathcal{L}$ *if* for every sentence $\varphi \in \mathcal{L}_\tau$ we have: $a_n \overset{\text{def}}{=} \text{Prob}_{\mu_{n+1}}(M_{n+1} \models \varphi \Big| M_{n+1} \in K_{n+1}) - \text{Prob}_{\mu_{n+m}}(M_{n+m} \models \varphi \Big| M_{n+m} \in K_{n+1})$ converges to zero as $n \to \infty$.

2) $\bar{K}$ satisfies the h-very weak zero one law for $\mathcal{L}$ *if* for every sentence $\varphi \in \mathcal{L}_\tau$, $\max_{m \in [n,h(n)]} \Big[\text{Prob}_{\mu_m}(M_m \models \varphi \Big| M_m \in K_m) - \text{Prob}_{\mu_n}(M_n \models \varphi \Big| M_n \in K_n)\Big]$ converge to zero as $n \to \infty$. We shall concentrate on part 1.

3) $\bar{K}$ satisfies the convergence law for $\mathcal{L}$ if for every $\varphi \in \mathcal{L}_\tau$ we have: $\langle \text{Prob}_{\mu_n}(M_n \models \varphi \big| M_n \in K_n) : n < \omega\rangle$ converges (the same real $\in [0,1]_{\mathbb{R}}$). If it always converges to 0 or to 1 we say that $\bar{K}$ satisfies the 0-1 law for $\mathcal{L}$.

4) If $\mathcal{L}$ is the first order logic we may omit it.

The following $\bar{K}$ is from Luczak Shelah [LuSh 435].

1.3 Definition: Let $\bar{p} = \langle p_i : i \in \mathbb{N}\rangle$, p_i a real $0 \leq p_i \leq 1$, $p_0 = 0$. We define $\bar{K}^{\text{og}}_{\bar{p}}$ as follows:

the models from K_n are up to isomorphism $M = (n, <, R)$ when $n = \{0, \ldots, n-1\}$, $<$ the usual order, R a graph (i.e. a symmetric relation on n, $\neg xRx$) $\text{Prob}_{\mu_n}(M_n \cong M \big| M_n \in K_n)$ is $\prod\{p_{j-i} : i < j < n \text{ and } iRj\} \times \prod\{1 - p_{j-i} : i < j < n \text{ and } \neg iRj\}$ (i.e. for each $i < j < n$ we decide whether iRj by flipping a coin with probability p_{j-i} for yes, independently for the different pairs.

1.4 Theorem: $\bar{K}^{\text{og}}$ satisfies the very weak $0-1$ law if $\sum_i p_i < \infty$.

Proof: Let a real $\varepsilon > 0$ and a sentence $\theta \in \mathcal{L}^{\text{fo}}_\tau$ (τ -from 1.3) given. We shall define φ below (after $(*)_1$), and let $\psi = \psi_\varphi$, τ^*_1 (a vocabulary) and $m \in \mathbb{N}$ and $\varphi_i(x)$ for $i < m$ be defined as in 2.16 (for the φ here). Let d be the quantifier depth of ψ_φ (i.e. $d = d[\psi_\varphi]$). Let $k^* \in \mathbb{N}$ be large enough as in 1.5 (for the given

ε, m and τ). Let k be $(2k^*+2)(3^{d[\theta]}+1)$ (we could have waived the $3^{d[\theta]}+1$).

Now choose by induction on $r \leq k$, $m_r \in \mathbb{N}$ such that

$(*)$ (a) $0 = m_0$

(b) $m_r < m_{r+1} < \dots$

(c) for any n, $\varepsilon/3 > \mathrm{Prob}_{\mu_n}(M_n \models \vee_r\ (\exists x \leq m_r+1)\ (\exists y \geq m_{r+1} - 1)[xRy]\Big| M_n \in K_n)$.

Let n be given; let $n^* = n + k^* + 1$. Let $J = \{m_{(3^n+1)i} : 0 < i < 2k^*\}$. We first draw $\mathfrak{A}_n$, "a drunkard model $\mathfrak{A}_n$ for n". Drawing $\mathfrak{A}_n$ means:

laziness case=first case if $i < j < n$, $\vee_{i<k}[i \leq m_i + 1 \& j \geq m_{i+1} - 1]$ then $\{i, j\}$ is non edge (no drawing).

normal case= second case: if i, j, n^* is not in the first case but $\neg(\exists m \in J)[i \leq m \leq j]$ then we flip a coin and get $e_{i,j} \in \{\text{yes, no}\}$ with probability p_{j-i} (for yes).

drunkard case=third case: $i < j < n^*$ and no previous case apply; we make two draws. In one we get $e^1_{i,j} \in \{$ yes, no$\}$ with probability p_{j-i} (for yes) in the second we get $e^2_{i,j} \in \{$ yes, no$\}$ with probability p_{j-i-1} for yes (we may stipulate $p_0 = 0$).

Now for every $Q \subseteq J$ we define $M_Q[\mathfrak{A}_n]$: it is a model $(I^Q, <, R^Q)$ where

I^Q is $\{0, \dots, n^* - 1\} \setminus Q$

$<$ is the usual order on I^Q

R^Q is $\{(i,j) : \{i,j\} = \{i',j'\}, i' < j' < n^*$, and:

(a) (i',j') fall into the second case above and $e_{i'j'} =$ yes

or (b) (i,j) fall into the third case say $i \leq m \leq j$ and $m \in J$

(m is unique by " not first case") $m \in Q$, and $e^1_{i',j'} =$ yes

or (c) (i,j) fall into the third case say $i \leq m \leq j$ and $m \in J$

(m is unique by " not first case") $m \notin Q$ and $e^2_{i',j'} =$ yes$\}$.

We also define a model $N[\mathfrak{A}_n]$:

the universe: $\{0, \ldots, n^* - 1\}$

relations: $<$ the usual order

$$R = \{\langle i,j\rangle : e_{i,j} = \text{yes},\ i < j\}$$

$$R^1 = \{\langle i,j\rangle : e^1_{i,j} = \text{yes},\ i < j\}$$

$$R^2 = \{\langle i,j\rangle : e^2_{i,j} = \text{yes},\ i < j\}$$

$$P = \{m_r : r \leq k\}$$

Observe

$(*)_1$ in $(N[\mathfrak{A}_n], Q)$ we can define $M_Q[\mathfrak{A}_n]$.

So for some first order φ depending on θ, τ but not on m (promised above in the beginning of the proof):

$(*)_2$ $M_Q[\mathfrak{A}_n] \models \theta$ iff $(N[\mathfrak{A}_n], Q) \models \varphi$.

By 2.16 (where $I[(N[\mathfrak{A}_n], Q)]$ is defined in 2.16 with M there standing for $(N[\mathfrak{A}_n], Q)$ here) and the choice of ψ_φ we have:

$(*)_3$ $(N[\mathfrak{A}_n], Q) \models \varphi$ iff $I[N[\mathfrak{A}_n], Q] \models \psi_\varphi$.

Looking at the definition of $I[N[\mathfrak{A}_n], Q]$ in 2.16 without loss of generality

$(*)_4$ $I[N[\mathfrak{A}_n], Q)] \models \psi_\varphi$ iff $(I([N[\mathfrak{A}_n]], Q) \models \psi_\varphi$.

1.4A Notation: 1) For a linear order I and $J \subseteq I$ let $\mathrm{npr}(I, J) = \{(Q^{\mathrm{no}}, Q^{\mathrm{yes}}) : Q^{\mathrm{no}} \subseteq Q^{\mathrm{yes}} \subseteq r$ and $Q^{\mathrm{no}} \setminus J = Q^{\mathrm{yes}} \setminus J$ and $|Q^{\mathrm{yes}} \setminus Q^{\mathrm{no}}| = 1\}$.

2) If I is a set of natural numbers we use the usual order.

3) If $J = I$ we write $\mathrm{npr}(I)$.

4) Any (I, J) is isomorphic to some (n, J') so we can use such pairs above.

Let $J = J^d \cup J^u$, J^d an initial segment, J^u an end segment, $|J^u| = k$ $|J^d| = k - 1$. Now we define further drawing; let μ_{J^u} be the distribution from 1.5(1) below on $\mathrm{npr}(J^u)$, and choose $(Q_0^u, Q_1^u) \in \mathrm{npr}(J^u)$ randomly by μ_{J^u} then choose $Q_1^d \subseteq J^d$ such that $|Q_1^d| = k + 1 - |Q_1^u|$ with equal probabilities, and let $Q^1 = Q_1^d \cup Q_1^u$, and let $Q_0^d = Q_1^d$, $Q^0 = Q_0^d \cup Q_0^u$.

Now for $\ell \in \{0,1\}$ we make a further drawing: if $i < j$ is a pair from the first possibility (in the drawing of $\mathfrak{A}$), we flip a coin for ${}^*e^\ell_{i,j} \in \{\text{yes}, \text{no}\}$ with probability $p_{|(i,j)\setminus Q^\ell|}$ for yes.

Let $M^\ell_{Q^\ell}$ be $(I^{Q^\ell}, <, R^{Q^\ell} \cup \{(i,j),(j,i) : i < j \text{ and: } {}^*e^\ell_{i,j} = \text{yes}\})$ (it depends on the choice of $\mathfrak{A}_n$ and on the further drawing).

Now reflecting we shall see

$(*)_5$ for $\ell = 0, 1$, the distribution of $M^\ell_{Q^\ell}$ is the same as that of $(K_{n+\ell}, \mu_{n+\ell})$.

$(*)_6$ for $\ell = 0, 1$, $\text{Prob}(M^\ell_{Q^\ell} \models \theta) = \text{Prob}(M_{n+\ell} \models \theta \big| M_{n+\ell} \in K_{n+\ell})$.

By the choice of m'_r's

$(*)_7$ $\text{Prob}(M^\ell_{Q^\ell} = M_{Q^\ell}[\mathfrak{A}_n])$ is $\geq 1 - \varepsilon/3$.

By 1.5 below (used above: the drawing of (Q^u_0, Q^u_1) was randomly by μ_{J^u}).

$(*)_8$ the absolute value of the differences between the following is $\leq \varepsilon/3$:

$\text{Prob}([N[\mathfrak{A}_{n^*}]], Q^0) \models \psi_\varphi)$

$\text{Prob}([N[\mathfrak{A}_{n^*}]], Q^1) \models \psi_\varphi)$.

So for $i = 0$,

(a) $\text{Prob}(M_{n+\ell} \models \theta \big| M_{n+\ell} \in K_{n+\ell}) = \text{Prob}(M^\ell_{Q^\ell} \models \theta)$, by $(*)_6$.

(b) $\text{Prob}(M^\ell_{Q^\ell} = M_{Q^\ell}[\mathfrak{A}_n]) \geq 1 - \varepsilon/3$, by $(*)_7$.

(c) $M_{Q^\ell}[\mathfrak{A}_n] \models \theta$ iff $(I[N[\mathfrak{A}_n)], Q^\ell) \models \psi_\varphi$, by $(*)_1 + (*)_2 + (*)_3 + (*)_4$.

By (a)+(b)+(c) it suffices to prove that the probability of "$(I[N[\mathfrak{A}_n]], Q^\ell)$ for $\ell = 0,1$ has difference $< \varepsilon/3$ but this holds by $(*)_8$. $\square_{1.4}$

1.5 Claim: Let $m, d \in \mathbb{N}$ be given, $\tau = \{<\} \cup \{P_i : i < m\}$, P_i a unary predicate, $K = \{M : M \text{ a } \tau\text{-model}, <^M \text{ a linear order}\}$,

1)For every $\varepsilon \in \mathbb{R}^+$ for every k large enough for some distribution μ_k on $\mathrm{npr}(k) = \{(Q^{\mathrm{no}}, Q^{\mathrm{yes}}) : Q^{\mathrm{no}} \subseteq Q^{\mathrm{yes}} \subseteq k, |Q^{\mathrm{yes}} \setminus Q^{\mathrm{no}}| = 1\}$ we have: *if* $M_i^{\boldsymbol{t}} \in K$, for $i < k$, $\boldsymbol{t} \in \{\text{yes no}\}$, and we choose μ_k-randomly $(Q^{\mathrm{no}}, Q^{\mathrm{yes}}) \in \mathrm{npr}(k)$, *then*

$(*)$ the probability of $\mathrm{th}^d(\sum_{i<k} M_i^{\text{if } (i \in Q^{\mathrm{yes}})}) = \mathrm{th}^d(\sum_{i<k} M_i^{\text{if } (i \in Q^{\mathrm{no}})})$ is at least

$1 - \varepsilon$ (th^d is defined as in 2.6(2)).

2) The same holds also if we first choose $(Q_u^{\mathrm{no}}, Q_{\mathrm{no}}^{\mathrm{yes}}) \in \mathrm{npr}\Big([[\frac{k+1}{2}], k)\Big)$ as above and then make a decision on a drawing of $Q_d^{\mathrm{no}} = Q_d^{\mathrm{yes}} \subseteq [0, [\frac{k+1}{2}])$ and let $Q^{\mathrm{yes}} = Q_d^{\mathrm{yes}} \cup Q_u^{\mathrm{yes}}$, $Q^{\mathrm{no}} = Q_d^{\mathrm{no}} \cup Q_u^{\mathrm{no}}$ then $(*)$ above still holds.

3) If we choose $Q^{\mathrm{yes}} \subseteq k$ such that $|Q^{\mathrm{yes}}| = [\frac{k+1}{2}]$ and then $Q^{\mathrm{no}} \subseteq Q^{\mathrm{yes}}$, $|Q^{\mathrm{no}}| + 1 = |Q^{\mathrm{yes}}|$ (all possibilities with probabilities) then with probability tending to 1 with k we get that $(*)$ holds.

Proof: Let V_k be the set of pairs (μ_k, ζ) such that: μ is a distribution on $\mathrm{npr}(k)$, $\zeta \in \mathbb{R}^+$ such that (we define $\mu_{[r,J]}$ when $|J| = k$, $J \subseteq r$, on $\mathrm{npr}(r, J)$ naturally)

$(*)_{(\mu_k,\zeta)}$ if $M_i^{\mathbf{t}} \in K$ for $\mathbf{t} \in \{\text{no}, \text{yes}\}, i < r (\in \mathbb{N})$ and $J = \{i : \text{th}^d(M_i^{\text{no}}) \neq \text{th}^d(M_i^{\text{yes}})\}$

has k members, then $(1 - \zeta_k) \geq$

$\text{Prob}_{\mu_k}\Big(\text{th}^d(\sum_{i<r} \mu_i^{\text{if } (i \in Q^{\text{no}})}) \neq \text{th}^d(\sum_{i<r} M_i^{\text{if } (i \in Q^{\text{yes}})})\Big|(Q^{\text{no}}, Q^{\text{yes}}) \in \text{npr}(r, J)\Big)$.

Let $\zeta_k = \sup\{\zeta \in \mathbb{R}^+$: for some μ_k, $(\mu_k, \zeta) \in V_k\}$. Clearly without loss of generality $r \leq 2k+1$ in $(*)_{(\mu_k,\zeta)}$ (by additivity of th^d) hence the suprimum is obtained.

Let $c \in \mathbb{N}$ be the number of members in $\text{TH}^d(0, \tau)$ (the set of formally possible $th^d(\langle\rangle, M)$, $M \in K$, see 2.6). Let $k_0 \in \mathbb{N}$ be such that $k_0 \to (3^{d+8})^2_{c^2}$ (Ramsey theorem). We first note that:

$(*)_1 \zeta_{k_0} \geq \frac{1}{(k_0 2^{k_0})}$.

This is exemplified by the equi-probability distribution [Why? draw $Q \cap (k \setminus J)$].

It is enough to prove that now the probability of the equality is $\geq 1/(k_0 2^{k_0})$. For $i < j$ from J let $\langle \mathbf{t}_{i,j}^{\text{no}}, \mathbf{t}_{i,j}^{\text{yes}} \rangle$ be $\langle \text{th}^d(M_i^{\text{no}} + M_{i+1}^{\text{no}} + \ldots + M_{j-1}^{\text{no}}), \text{th}^d(M_i^{\text{yes}} + M_{i+1}^{\text{no}} + M_{i+2}^{\text{no}} + \ldots + M_{j-1}^{\text{no}}) \rangle$. So we can find $J' \subseteq J, |J| = 3^{d+8}$ and $\langle \mathbf{t}_0, \mathbf{t}_1 \rangle$ such that for $i < j$ in J', we have $\langle \mathbf{t}_{i,j}^{\text{no}}, \mathbf{t}_{i,j}^{\text{yes}} \rangle = \langle \mathbf{t}_0, \mathbf{t}_1 \rangle$. Let $J' = \{i_\ell : \ell < 3^d + 8\}$. For each $i \leq 3^{d+8}$ let $Q_\ell = \{i_r : r < i\}$. By addition theory for (first order theory) linear order, (see 2.7(2), (6)) for $\ell = [3^{d+8}/2]$, $\text{th}^d(\sum_i M_i^{(\text{if } i \in Q_\ell)}) = \text{th}^d(\sum_i M_i^{(\text{if } (i \in Q_{\ell+1})})$. So the probability for equality is at least the probability of $Q^{\mathrm{no}} = Q_\ell$, $Q^{\mathrm{yes}} = Q_{i+1}$ which is $\geq 1/(k_0 2^{k_0})$.

Also,

$(*)_2$ $(1-\zeta_{k\ell}) \leq (1-\zeta_k)(1-\zeta_\ell)$

[Why? Let μ_k, μ_ℓ exemplify the definition of ζ_k, ζ_ℓ. Now given $M_i^{\boldsymbol{t}}$ (for $i < r$) and J as in $(*)$ (so $|J| = k\ell$) first draw $Q^{\mathrm{yes}} \setminus J$. It is enough to prove that now the probability of inequality is $\leq (1-\zeta_k)(1-\zeta_\ell)$. Let $J = \bigcup_{j<\ell} J_j$, each J_j an interval of J of length k. Let $\langle I_j : j < \ell \rangle$ be a division of r to intervals with $J \cap I_j = J_j$. Now for each J draw $Q_j^{\mathrm{no}} \subseteq Q_j^{\mathrm{yes}} \subseteq J_j$ by $\mu_k[I_j, J_j]$. Next draw $Q_*^{\mathrm{no}} \subseteq Q_*^{\mathrm{yes}} \subseteq \ell$ by μ_ℓ and let $Q^{\boldsymbol{t}} = \cup\{Q_j^{\boldsymbol{t}} : j < \ell$ and: $(\boldsymbol{t} = \mathrm{yes} \wedge j \in Q_*^{\boldsymbol{t}}) \vee (\boldsymbol{t} = \mathrm{no} \wedge j \notin Q_*^{\boldsymbol{t}})\}$. Easily, $Q^{\mathrm{no}} \subseteq Q^{\mathrm{yes}}$ are drawn as required. Now we let for $j < \ell$, $\boldsymbol{t} \in \{\mathrm{yes}, \mathrm{no}\}$, $N_j^{\boldsymbol{t}} = \sum_{i \in I_j} M_i^{\mathrm{if}\,(i \in Q^{\boldsymbol{t}})}$. We ask "does the $j \in Q_*^{\mathrm{yes}} \setminus Q_*^{\mathrm{no}}$ satisfies $\mathrm{th}^d(N_j^{\mathrm{no}}) = \mathrm{th}^d(N_j^{\mathrm{yes}})$?" this has probability $\geq (1-\zeta_k)$. So it is enough to assume this fails. By additivity of expected values it is enough to show that the probability of equality is still $\geq \zeta_\ell$. This should be clear.]

Now by $(*)_1 + (*)_2$ we can finish easily.

2)Follow by (1) (and the addition theory see 2.7.)

3) Similar proof and not used (or see appendix). $\square_{1.5}$

Luczak [Lu] suggested another context.

1.6 Definition: $\bar{K}^{\rm opo}_{p_n}$ is the following 0-1 law context (p_n is a function of n, $0 \le p_n \le 1$; so more precisely we are defining $\langle K^{\rm opo}_{p_n} : n < \omega \rangle$. The models are up to isomorphism, $(n, <, <^*)$, $n = \{0, 1, \dots, n-1\}$, $<$-the usual order, $<^*$-the following partial order: we draw a random graph on n (edge relation $\mathbb{R}$) with edge probability p_n, $x <^* y$ iff there are $k \in N$ and $x_0 < x_1 < \dots < x_k < n$ such that $x_\ell R x_{\ell+1}$. The probability is derived from the probability distribution for R (which does not appear in the model.)

1.7 Theory Assume $p_n = \frac{1}{n^a}$, where $0 < a < 1$ (like [ShSp 304]). Then $\bar{K}^{opo}_{p_n}$ satisfies the very weak zero one law.

Proof: Similar to the previous theorem as easily

$(*)$ for $\varepsilon > 0$ Prob$(M_n \models (\exists x < y)[\neg x <^* y \& x + n^{1-\varepsilon} \le y)$ is very small.

§2 Model Theory

The main lemma (2.14) generalizes the addition theory and models with distances (both from model theory). Concerning the first, see Feferman Vaught [FV], and see representation of Gurevich [Gu]. Concerning models with distance see Gaifman [Gf], a forerunner in Marcus [M1]. We may look at it as dealing with sums with only local "disturbances", "semi sums". The connections are explained in AP2, AP3.

In 2.16 we draw the conclusion for linear order which we need §1.

Note: $\mathfrak{B}$ is a (two sorted) model.

2.1 Definition: 1) We call σ a vocabulary of systems if $\sigma = \langle \tau_1, \tau_2 \rangle$, τ_1, τ_2 are sets of predicates.

2) We call $\mathfrak{B}$ a σ-system if:

(A) $\mathfrak{B} = (M, I, h, d))$ $(= (M^{\mathfrak{B}}, I^{\mathfrak{B}}, h^{\mathfrak{B}}, d^{\mathfrak{B}}))$.

(B) M is a τ_1-model and I is a τ_2-model, (usually with finite vocabularies but not needed), but we use M, I also for their universes.

(C) h is a function from M onto I,

(D) d is a distance function on I, i.e.

(α) d is a symmetric two place function from I to $\mathbb{N} \cup \{\infty\} = \{0, 1, 2, 3, \ldots, \infty\}$,

(β) $d(x, x) = 0$

(γ) $d(x, z) \leq d(x, y) + d(y, z)$.

(E) M, I are disjoint.

3) Let $\sigma(\mathfrak{B}) = \sigma$ for $\mathfrak{B}$ a σ-system.

2.1A Discussion: The demands "M, I are disjoint and h is onto I" are not essential this is just for convenience. If h is not onto I we should allow relations on $M \cup I$, but then if M, I are not disjoint then for each predicate, for each of its places we should assign-does there appear a member of M or a member of I; also we should have two kinds of variables not allowing equality between variables of the different kinds. So in application we may use those alternative presentation.

2.2 Convention: (1) We may allow function symbols (and individual constants) but then treat them as relations.

(2) We stipulate $h(x) = x$ for $x \in I$.

(3) $\bar{a} \subseteq \mathcal{B}$ means Rang $\bar{a} \subseteq M^{\mathfrak{B}} \cup I^{\mathfrak{B}}$.

(4) A model M can be identified with the system $\mathfrak{B} = \mathfrak{B}^{\text{sim}}[M] : M^{\mathfrak{B}} = M$, $I^{\mathfrak{B}} = M$, $h^{\mathfrak{B}} = \text{id}_M$,

$$d(x,y) = \begin{cases} 0 & x = y \\ \infty & x \neq y \end{cases}$$

(you may of course take two disjoint copies of M as $M^{\mathfrak{B}}$ and $I^{\mathfrak{B}}$).

(5) A model M can be identified with the system $\mathfrak{B}^{\text{dis}}[M]$: $M^{\mathfrak{B}} = M$, $I^{\mathfrak{B}} = M$,

$h^{\mathfrak{B}} = \mathrm{id}_M$, and $d(x,y) \stackrel{\mathrm{def}}{=} \mathrm{Min}\{n :$ there are $z_0, \ldots, z_n \in M$, $x = z_0, y = z_n$ and for $\ell < n$ for some $R \in \tau[M]$, and sequence $\bar{a} \in R^M$, $\{z_\ell, z_{\ell+1}\} \subseteq \mathrm{Rang}\, \bar{a}\}$ (remember $\tau[M]$ is the vocabulary of M; this is the definition in [Gf]).

2.3 Definition: 1) For a system $\mathfrak{B} = (M, I, h, d)$:

for $x \in I$:

$N_r(x) \stackrel{\mathrm{def}}{=} \{y \in I : d(y,x) \leq r\}$

for $x \in M \cup I$,

$N_r^+(x) \stackrel{\mathrm{def}}{=} \{y \in M \cup I : d(h(y), h(x)) \leq r\}$.

2) $\mathcal{L}_\sigma$ is the set of first order formulas for σ; we have variables on M, variables on I, the predicates of τ_1, τ_2; $h(x) = y$; and "$h(y) \in N_r(h(x))$" for each r (see part (4) below).

3) $\mathfrak{B}^- = \langle M^{\mathfrak{B}}, I^{\mathfrak{B}}, h \rangle$, $\mathfrak{B}_r = \langle M^{\mathfrak{B}}, I^{\mathfrak{B}}, h^{\mathfrak{B}}, \text{"}d(x,y) \leq k\text{"} \rangle_{k \leq r}$ so $\mathfrak{B}_0 = \mathfrak{B}^-$ and we let $\sigma_r = \sigma(\mathfrak{B}_r)$.

4) So $\mathcal{L}_{\sigma_n}$ is defined as in part (2) but "$h(y) \in N_s(h(x))$" appear only for $s \leq n$.

We usually apply our theorems in the case

2.4 Definition: $\mathfrak{B}$ is *simple* if

(a) $d(x,y) \leq 1$ is equivalent to a quantified free formula in $\mathcal{L}_{\tau(I)}$.

(b) for $x, y \in I$ we have: $d(x,y) \le r$ *iff* there are $x_0, \dots, x_r \in I$ such that $x = x_0$, $y = x_r$, and $d(x_\ell, x_{\ell+1}) \le 1$ for $\ell < r$ (i.e., like 2.2(5)).

2.4A Remark: In this case, note that every formula in $\mathcal{L}_\sigma(\mathfrak{B})$ is equivalent to some formula in $\mathcal{L}_{\sigma_1}$.

2.5 Convention: 1) $f_n(r) = r + 3^n$ for $r, n \in \mathbb{N}$ or more generally it just satisfies $r < f_n(r) \in \mathbb{N}$ and $f_n^{(3)}(r) \le f_{n+1}(r)$ where $f_n^{(0)}(r) = r$, $f_n^{(\ell+1)}(r) = f_n(f_n^{(\ell)}(r))$.

2) We call f *nice* if $f_n^{(4)}(r) \le f_{n+1}(r)$.

3) Let $f(\langle r_\ell : \ell < m \rangle) = \langle f(r_\ell) : \ell < m \rangle$.

2.6 Definition: 1) For a system $\mathfrak{B}, m, n \in \mathbb{N}$, and $\bar{a} = \langle a_0, \dots, a_{m-1} \rangle \subseteq \mathfrak{B}$ we define $\mathrm{th}_r^n(\bar{a}, \mathfrak{B})$. We define it by induction† on n:

$$\mathrm{th}_r^0(\bar{a}, \mathfrak{B}) \text{ is } \{\varphi(x_0, \dots, x_{m-1}) : (M, I, h) \models \varphi[a_0, \dots, a_{m-1}],$$

$$\varphi \text{ a basic formula in } \mathcal{L}_{\sigma(\mathfrak{B}_r)}\}$$

$$\mathrm{th}_r^{n+1}(\bar{a}, \mathfrak{B}) = \cup\{\mathrm{th}_r^n(\bar{a}\frown\langle c \rangle, \mathfrak{B}) : c \in M \cup I\}.$$

2) If σ is a vocabulary of systems, $n, m \in \mathbb{N}$, $\mathrm{TH}_r^n(m, \sigma)$ is the set of formally possible $\mathrm{th}_r^n(\bar{a}, \mathfrak{B})$ for $\mathfrak{B}$ a σ-system; defined naturally. I.e., we define it by induction on n; <u>for $n = 0$</u> it is the family of sets $\mathbf{t}$ of basic formulas $\varphi(x_0, \dots, x_{m-1})$ of $\mathcal{L}_{\sigma_r}$ such that for each atomic $\varphi(x_0, \dots, x_{m-1})$ exactly one of

† In the following definition basic is atomic or a negation of atomic.

$\varphi(x_0, \dots, x_{m-1})$, $\neg\varphi(x_0, \dots, x_{m-1})$ belongs to $\mathbf{t}$.

$\mathrm{TH}^{n+1}_r(m, \sigma)$ is the family of subsets of $\mathrm{TH}^n_r(m + 1, \sigma)$.

3) If τ is a vocabulary of models (see 2.2(4)), $n, m \in \mathbb{N}$ then $\mathrm{TH}^n(m, \sigma)$ is the set of formally possible $\mathrm{th}^n(\bar{a}, \mathfrak{B})$, $\mathfrak{B}$ a τ-model, i.e. $\mathfrak{B} = \mathfrak{B}^{\mathrm{sim}} M$ for some τ-model M.

4) If $r = 0$ we may omit it, so for a model M, using $I = M$, h the identity we get the usual $th^n(\bar{a}, M)$ (but we do not assume its knowledge).

2.7 Claim: 1) For $\mathfrak{B}, n, m, \bar{a}$ as above, $\varphi = \varphi(x_0, \dots, x_{m-1})$ a (first order) formula in $\mathcal{L}_{\sigma(\mathfrak{B}_r)}$ of quantifier depth n, we have:

from $\mathrm{th}^n_r(\bar{a}, \mathfrak{B})$ we can compute the truth value of "$\mathfrak{B}_r \models \varphi[\bar{a}]$".

2) For any τ (vocabulary of models), $n, m \in \mathbb{N}$, if $\mathbf{t} \in \mathrm{TH}^n(m, \tau)$ *then* for some formula $\varphi(x_0, \dots, x_{m-1}) \in \mathcal{L}_\tau$ of quantifier depth n, for any $\bar{a} = \langle a_0, \dots, a_{m-1}\rangle \subseteq \mathfrak{B}$ we have: $\mathbf{t} = \mathrm{th}^n(\bar{a}, \mathfrak{B})$ iff $\mathfrak{B} \models \varphi[\bar{a}]$.

3) The functions $\langle \sigma, n, m\rangle \mapsto \mathrm{TH}^n_r(m, \sigma)$ and $\langle \tau, n, m\rangle \mapsto \mathrm{TH}^n(m, \tau)$ are computable.

4) From $\mathrm{th}^n_r(\bar{a}, \mathfrak{B})$ we can compute $\mathrm{th}^n_s(\bar{a}, \mathfrak{B})$ if $r \geq s$.

5) If $\mathfrak{B}$ is simple then from $\mathrm{th}^{n+r}_1(\bar{a}, \mathfrak{B})$ we can compute $\mathrm{th}^n_{2^r}(\bar{a}, \mathfrak{B})$.

6) If $n_1, n_2 \geq 2^d$ then $\mathrm{th}^d(\langle\rangle, (n_1, <)) = \mathrm{th}^d(\langle\rangle, (n_2, <))$. Also if $\mathrm{th}^d(\langle\rangle, M_0) = \mathbf{t}$

for $i < \max\{n_1, n_2\}$, $\tau = \{<, P_0, \ldots, P_{m+1}\}$ where each P_ℓ is a unary predicate, $M_i \in K_\tau$ then $\mathrm{th}^d(\langle\rangle, \sum_{i<n_1} M_i) = \mathrm{th}^d(\langle\rangle, \sum_{i<n_2} M_i)$.

7) For a given vocabulary τ, and $d \in \mathbb{N}$ there is an operation $\oplus$ on $\mathrm{TH}^d(0, \tau)$ such that $\mathrm{th}^d(\langle\rangle, \sum_{i<k} M_i) = \oplus\langle \mathrm{th}^d(\langle\rangle, M_i) : i < k\rangle$, this operation is associative (but in general not commutative).

8) Let $\mathfrak{B}$ be a system, for any $\langle a_i : i < m\rangle$ and $\langle r_i : i < m\rangle$ (where $a_i \in \mathfrak{B}, r_i \in \mathbb{N}$) for some $\langle n_i : i \in w\rangle$, $w \subseteq m$, $\sum_{i \in w} n_i \leq m - |w|$ and $\langle N^+_{f_{n_i}(\sum\{r_j : g(j)=i\})}(a_i) : i \in w\rangle$ are pairwise disjoint and $\bar{a}$ is included in their union.

Proof: Well known and/or easy (see e.g. [Gu]).

2.8 Definition: 1) For a system $\mathfrak{B}$, and $r, n, m \in \mathbb{N}$, and $\bar{a} = \langle a_0, \ldots, a_{m-1}\rangle \subseteq \mathfrak{B}$ we call $\bar{a}$ a $(\mathfrak{B}, r)$-component if $\bar{a} \subseteq N^+_r(a_0)$. In this case we define $\mathrm{bth}^n_r(\bar{a}, \mathfrak{B})$ (bth is for bounded theory). We do it by induction on n (for all r) (the function f from 2.5 is an implicit parameter.)

$(\alpha)\mathrm{bth}^0_r(\bar{a}, \mathfrak{B}) = \mathrm{th}^0(\bar{a}, \mathfrak{B})$

$(\beta)\mathrm{bth}_r^{n+1}(\bar a, \mathfrak{B})$ is $\langle \mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2\rangle$ where

$$\text{(i) } \mathbf{t}_0 = \mathrm{bth}_r^n(\bar a, \mathfrak{B})$$

$$\text{(ii) } \mathbf{t}_1 = \{\mathrm{bth}_r^n(c) : c \in N^+_{f_n^{(3)}(r)}(a_0) \setminus N^+_{f_n(r)}(a_0)\}$$

$$\text{(iii) } \mathbf{t}_2 = \{\mathrm{bth}^n_{f_n^{(2)}(r)}(\bar a {}^\frown \langle c\rangle, \mathfrak{B}) : c \in N^+_{f_n(r)}(a_0)\}$$

2) If σ is a vocabulary of systems and $r, n, m \in \mathbb{N}$ then $BTH_r^n(m, \sigma)$ is the set of formally possible $\mathrm{bth}_r^n(\bar a, \mathfrak{B})$, ($\mathfrak{B}$ a σ-system, $\bar a = \langle a_0, \ldots, a_{m-1}\rangle \subseteq \mathfrak{B}$ and $\bar a$ is a $(\mathfrak{B}, r)$-component).

2.9 Claim: 1) For any σ (vocabulary of systems), numbers $n, m, r \in \mathbb{N}$, and $\mathbf{t} \in \mathrm{BTH}_r^n(m, \sigma)$, *there is* $\varphi = \varphi(x_0, \ldots, x_{m-1}) \in \mathcal{L}_\sigma$ (even $\mathcal{L}_{\sigma(\mathfrak{B}_{f_n(r)})}$) of quantifier depth n such that for any σ-system $\mathfrak{B}$, and $\bar a = \langle a_0, \ldots, a_{m-1}\rangle \subseteq \mathfrak{B}$ an $(\mathfrak{B}, r)$-component we have: $\mathbf{t} = \mathrm{bth}_r^n(\bar a, \mathfrak{B})$ *iff* $\mathfrak{B}_{f_n(r)} \restriction N^+_{f_n(r)}(a_0) \models \varphi[\bar a]$.

2) if f is nice, $\mathrm{bth}_r^n(\bar a, \mathfrak{B})$ (for $\bar a$ *a* $(\mathfrak{B}, r)$-component) depends just on $\mathfrak{B} \restriction N^+_{f_{n(r)}}(a_0)$

2.9A Remark: 1) if $\mathfrak{B}$ is simple, also a suitable converse holds.

2) If f is not nice, in 2.9(2) we still can find k depending on r, n and f such that $\mathrm{bth}(\bar a, \mathfrak{B})$ depends just on $\mathfrak{B} \restriction N_k^+(\bar a)$.

Proof: Should be clear.

2.10 Definition: For a σ-system $\mathfrak{B}$, and $r, n, m \in \mathbb{N}$ we define $\mathfrak{B}^n_{r,m}$ as the expansion of $\mathfrak{B}$ by the relations $R^\ell_{\mathbf{t}} = \{\bar{a} : \bar{a}$ a $(\mathfrak{B}, r)$ component , $\mathbf{t} = \mathrm{bth}^n_r(\bar{a}, \mathfrak{B})$, $[\ell = 1 \Rightarrow \bar{a} \subseteq M]$, $[\ell = 2 \Rightarrow \bar{a} \subseteq I]\}$ for each $\mathbf{t} \in \mathrm{BTH}^n_r(m, \sigma), \ell \in \{1, 2\}$. We let $I^n_{r,m}[\mathfrak{B}] = \mathfrak{B}^n_{r,m} \restriction I$. Writing $\bar{r} = \langle r_\ell : \ell < k \rangle$ we mean $\max \bar{r}$. Writing $\leq r$ we mean for every $r' \leq r$. .

2.11 Claim: $\mathfrak{B}^{n+1}_{r,k}$ essentially expands $\mathfrak{B}^n_{r,k}$ (essentially expand means that every predictable in the later is equivalent to a quantifier free formula in the former).

Proof: Should be clear by (i) of (β) of 2.8(1).

2.12 Definition: For a system $\mathfrak{B}$ and $n, m \in \mathbb{N}$ and $\bar{r} = \langle r_\ell : \ell < m \rangle$ such that $r_\ell \in \mathbb{N}$ and $\bar{a} = \langle a_0, \ldots, a_{k-1} \rangle \subseteq I^{\mathfrak{B}}$

1) We call $\bar{a}$ $(n, \bar{r})$-sparse for $\mathfrak{B}$ if $\bar{r} = \langle r_\ell : \ell < m \rangle$ and $\ell < k < m \Rightarrow N^+_{f_n(r_\ell)}(a_\ell) \cap N^+_{f_n(r_k)}(a_k) = \emptyset$ (slightly stronger but causes no damage is requiring $d(a_\ell, a_k) > f_n(r_\ell) + f_n(r_k) + 1$).

2) We define $\mathrm{uth}^n_{\bar{r}}(\bar{a}, \mathfrak{B})$ by induction on n: $\mathrm{uht}^0_{\bar{r}}(\bar{a}, \mathfrak{B}) = \mathrm{th}^0_{\bar{r}}(\bar{a}, \mathfrak{B})$ and $\mathrm{uth}^{n+1}_{\bar{r}}(\bar{a}, \mathfrak{B}) = \langle \mathbf{t}_0, \mathbf{t}_1 \rangle$ where:

$\mathbf{t}_0 = \mathrm{uht}^n_{f^{(2)}_n(\bar{r})}(\bar{a}, \mathfrak{B})$ (see 2.5, $f^{(2)}_n(\bar{r}) = \langle f_n(f_n(r_\ell)) : \ell < \ell g\, \bar{a} \rangle$)

$\mathbf{t}_1 = \{\mathrm{uht}^n_{\bar{r}\frown\langle 0 \rangle}(\bar{a}\frown\langle c \rangle, \mathfrak{B}) :\ c \in I^{\mathfrak{B}}$ and $\bar{a}\frown\langle c \rangle$ is $(n, \bar{r}\frown\langle 0 \rangle)$-sparse i.e.

$N^+_{f_n(0)}(c)$ is disjoint to $N^+_{f_n(r_\ell)}(a_\ell)$ for $\ell < \ell g\,\bar a\}$.

3) $\mathrm{UTH}^n_{\bar r}(m,\sigma)$ is the set of formally possible $\mathrm{uth}^n_{\bar r}(\bar a, \mathfrak{B})$ ($\bar a$ is $(n,\bar r)$-sparse for $\mathfrak{B}$ of length m, $\mathfrak{B}$ a σ-system, etc.).

4) We may use $I^2_{f_n(r),m}[\mathfrak{B}]$ instead of $\mathfrak{B}$ in part (2) (see 2.13(2) below).

2.13 Claim: 1) For every $\mathbf{t} \in \mathrm{UTH}^n_r(m,\sigma)$ there is a formula $\varphi = \varphi(x_0,\dots,x_{k-1}) \in \mathcal{L}_{\sigma_{f_n(\bar r)}})$ where $f_n(\bar r) = \max_{\ell<\ell g\,\bar r} f_n(r_\ell)$, of quantifier depth n such that: For a σ-system $\mathfrak{B}$ and $(n,\bar r)$-sparse $\bar a = \langle a_0,\dots,a_{m-1}\rangle \subseteq I^{\mathfrak{B}}$, we have: $\mathfrak{B} \models \varphi[a_0,\dots,a_{m-1}]$ *iff* $\mathbf{t} = \mathrm{uth}^n_{\bar r}(\bar a,\mathfrak{B})$ (and being $(n,\bar r)$-sparse is equivalent to some quantifier free formula).

2) In Definition 2.12 only $\mathfrak{B}\restriction I$ matters and in part (1), the quantifications are on I only.

Proof: Should be clear.

2.14 Main Lemma: Let σ be a system-vocabulary; if $\otimes_0$ holds then $\otimes_n$ holds for every n where:

$\otimes_n$ there are functions $F_{n,\bar r,\bar m}$, for $\bar r = \langle r_\ell : \ell < k\rangle$, $\bar m = \langle m_\ell : \ell < k\rangle$, where $k, r_\ell, m_\ell \in \mathbb{N}$, such that:

$(*)$ *if* $\mathfrak{B}$ is a σ-system, $\bar a = \bar a^{0\frown}\dots{}^\frown\bar a^{k-1}$, $\bar a^\ell$ an $(\mathfrak{B}, r_\ell)$-component, $\langle a^\ell_0 : \ell < k\rangle$ is

$(n,\bar r)$-sparse and $\bar m = \langle \ell g\, \bar a^\ell : \ell < k\rangle$ *then* letting $\mathbf{t}_\ell = \mathrm{bth}^n_{r_\ell}(\bar a^\ell, \mathfrak{B})$ (for $\ell < k$) and $\mathbf{t} = \mathrm{uth}^n_{\bar r}(\langle h(a^\ell_0) : \ell < k\rangle, I^n_{\bar r, f_n(\bar r)}[\mathfrak{B}])$ we have $\mathrm{th}^n_0(\bar a, \mathfrak{B}) = F_{n,\bar r,\bar m}(\mathbf{t}, \mathbf{t}_0, \dots, \mathbf{t}_{k-1})$.

$(**)$ $F_{n,\bar r,\bar m}$ is recursive in its variables, $n, \bar r, \bar m$ and the function $F_{0,r',\bar m'}$ where $\bar m' = \langle m'_i : i < k'\rangle$, $k' \geq k$, $r' \leq f_n(\bar r)$, for $i < k$ we have $m'_i \geq m_i$ and $\sum_{i<k}(m'_i - m_i) + \sum_{i=k}^{k'-1} m'_i \leq n$.

Proof: We prove this by induction on n, (for all $\bar r, \bar m$), so for $n = 0$ clearly $\otimes_n$ holds. So assume $\otimes_n$ and we shall prove $\otimes_{n+1}$. We shall now describe a value $\mathbf{t}$ computed from $\mathrm{bth}^{n+1}_{r_\ell}(\bar a^\ell, \mathfrak{B})$ (for $\ell < k$) and $\mathrm{uth}^n(\langle h(a^\ell_0) : \ell < k\rangle, \mathfrak{B}^n_{r, f_{n+1}(r)})$.

Now $\mathbf{t}$ will consist of two parts (see Definition).

We use an informal description as it is clearer.

First Part: The set of $\mathrm{th}^n(\bar a {}^\frown \langle c\rangle, \mathfrak{B})$, where for some $\ell(*) < k, c \in N^+_{f^{(2)}_n(r_{\ell(*)})}(a^{\ell(*)}_0)$.

Why can we compute this (using the induction hypothesis of course)? for each such c we have: let $\bar b^\ell = \bar a^\ell$ if $\ell < k, \ell \neq \ell(*)$ and let $\bar b^{\ell(*)} = \bar a^{(\ell(*))}{}^\frown\langle c\rangle$ (i.e. $b^{\ell(*)}_{m_{\ell(*)}} = c$); next r'_ℓ is: r_ℓ for $\ell \neq \ell(*)$ and $f^{(2)}_n(r_{\ell(*)})$ if $\ell = \ell(*)$. Now:

(α) $N^+_{f_n(r'_\ell)}(\bar b^\ell_0)$ for $\ell < k$ are pairwise disjoint (as $f^{(3)}_n(r)) \leq f_{n+1}(r)$ clearly $f_n(r'_\ell) \leq f_{n+1}(r_\ell)$ and $b^\ell_0 = a^\ell_0$).

(β) $\bar b^\ell \subseteq N^+_{r'_\ell}(b^\ell_0)$[when $\ell = \ell(*)$ as $r_\ell \leq f^{(2)}_n(r_\ell) = r'_\ell$ and assumption on c; for $\ell \neq \ell(*)$ trivial.]

(γ) We can compute $\mathrm{bth}^n_{r'_\ell}(\bar{b}^\ell, \mathfrak{B})$ [for $\ell \neq \ell(*)$ by monotonicity properties of bth (see $\mathbf{t}_0$ in Definition 2.8(1)); for $\ell = \ell(*)$ this possibility is listed in bth^{n+1} (see $\mathbf{t}_1$ there).]

(δ) We can compute $\mathrm{uth}^n_{\bar{r}'}(\langle h(b^\ell_0) : \ell < h \rangle, \mathfrak{B}^n_{r, f_n(r)})$ [by monotonicity of $\mathrm{th}^n_{\bar{r}}$ and 2.13].

Second Part: The set of $\mathrm{th}^n(\bar{a}{}^\frown\langle c\rangle, \mathfrak{B})$ where for each $\ell < k$, $c \notin N^+_{f^{(2)}_n(r_\ell)}(a^\ell_0)$. So $N^+_{f_n(0)}(c) \subseteq N^+_{f_n(r_\ell)}$ is disjoint to such $N^+_{f_n(r_\ell)}(a^\ell_0)$. Why can we compute this? for such a c we can let $\bar{a}^k = \langle c \rangle$ (so $c = a^k_0$). Let $r'_\ell = r_\ell, r'_k = 0$.

(α) $N^+_{f_n(r_\ell)}(a^\ell_0)$ for $\ell \leq k$ are pairwise disjoint [as $f_n(r) \leq f_{n+1}(r)$ and the discussion above].

(β) $\bar{a}^\ell \subseteq N^+_{r'_\ell}(a^\ell_0)$ [for $\ell < k$ as $r^\dagger_\ell = r_\ell$, for $\ell = k$ as $\bar{a}^k = \langle a^k_0 \rangle (= \langle c \rangle)$].

(γ) we can compute $\mathrm{bth}^n_{hr}(\bar{a}^\ell, \mathfrak{B})$ [for $\ell < k$ by monotonicity properties of bth; for $\ell = k$: from $\mathrm{uth}^{n+1}(\langle h(a^\ell_0) : \ell < k\rangle, \mathfrak{B}^n_{r,f(r)})$. We can compute the possibilities for it.]

(δ) we can compute the possibilities of $\mathrm{uth}^n_{\bar{r}'}(\langle h(a^\ell_0) : \ell < k \rangle{}^\frown\langle h(c)\rangle, I^n_{\bar{r},k+1}[\mathfrak{B}])$.

$\square_{2.14}$

2.15 Conclusion: For any system vocabulary σ, and (first order) sentence of quantifier depth n, given $F_{0,\bar{r},\bar{m}}$'s satisfying $\otimes_0$ of 2.14 we can compute a sentence

ψ_φ of quantifier depth n, (and vocabulary explained below) and numbers r, m such that:

$(*)$ if $\mathfrak{B}$ is a σ-system which satisfied $\otimes_0$ as exemplified by $\langle F_{0,\bar r,\bar m} : \bar r, \bar m\rangle$ then $\mathfrak{B} \models \varphi \Leftrightarrow I^m_{r,m}[\mathfrak{B}] \models \psi_\varphi$.

2.16 Conclusion: Let τ be a vocabulary (finite for simplicity) including a binary relation $<$, and $\varphi \in \mathcal{L}_\tau$. *Then* we can compute a $m < \omega$, formulas $\varphi_i(x) \in \mathcal{L}_\tau$ for $i < m$ and sentence $\psi_\varphi \in \mathcal{L}_{\tau_i}$ where $\tau^*_1 = \{<\} \cup \{P_i : i < m\}$ ($m \in \mathbb{N}$ computable from φ, each P_i a unary predicate) satisfying the following:

$(*)$ Assume M is a finite τ-model, $<^M$ a linear order, $P \in \tau$ is unary, such that: if $R \in \tau \setminus \{<\}$ and $\bar a = \langle a_0, \ldots, a_{n(R)-1}\rangle \in R^M$, then $P^M \cap [\,\min \bar a,\ \max \bar a]_M$ has at most one member.

Define $I[M] = \langle P^M, \ldots, P_\ell \ldots\rangle$ where $P_\ell = \{a \in P^M : \varphi_\ell[a]$ is satisfied in $M{\restriction}\{x : x \le a, |[x,a]_M \cap P^M| \le 3^{d[\varphi]}$ or $a \le x,\ |[a,x]_M \cap P^M| \le 3^{d[\varphi]}\}$ ($d[\varphi]$ in the quantifier depth of φ). *Then* $M \models \varphi \Leftrightarrow I[M] \models \psi_\varphi$.

Proof: Should be clear.

Appendix

AP1 *Proof of 1.5(3):*

AP1.1 Notation: 1) For a finite set $J \subseteq I$ let $\mathrm{spr}(I,J)$ be the set of pairs $(Q^{\mathrm{no}}, Q^{\mathrm{yes}})$, where $Q^{\mathrm{no}} \subseteq I$, $Q^{\mathrm{yes}} \subseteq I$, $Q^{\mathrm{no}} \setminus J = Q^{\mathrm{yes}} \setminus J$, $|Q^{\mathrm{yes}} \triangle Q^{\mathrm{no}}| = 1$ ($A \triangle B$ is the symmetric difference).

2) For finite $J \subseteq I$ let $\mu^*(I,J)$ be the following distribution on $\mathrm{spr}(I,J)$; it is enough to describe a drawing:

first choose $Q^{\mathrm{no}} \subseteq I$ (all possibilities with probability $1/2^{[I]}$)

then choose $j \in J$ (all possibilities with probability $1/|J|$)

$$\text{finally, let } Q^{\mathrm{yes}} = \begin{cases} Q^{\mathrm{no}} \cup \{j\} & \textit{if } j \neq Q^{\mathrm{no}} \\ Q^{\mathrm{no}} \setminus \{j\} & \textit{if } j \in Q^{\mathrm{no}} \end{cases}$$

AP1.2 Remark: Note that the distribution is symmetric for $Q^{\mathrm{yes}}, Q^{\mathrm{no}}$.

AP1.3 Notation: For the τ, K, d from 1.5(3), let ζ_k be the real in $[0,1]$ such that: if $M_i^t \in K$ for $i < r$, $t \in \{\mathrm{yes}, \mathrm{no}\}$ (where $r \in \mathbb{N}$), $J = \{i < r : \mathrm{th}^d(M_i^{\mathrm{no}}) = \mathrm{th}^d(M_i^{\mathrm{yes}})\}$ have k elements, then $\frac{k}{r}(1-\zeta_k) \geq \mathrm{Prob}_{\mu^*}\Big(\mathrm{th}^d(\sum_{i<r} M_i^{\mathrm{if}\ (i \in Q^{\mathrm{no}})}) \neq \mathrm{th}^d(\sum_{i<r} M_i^{\mathrm{if}\ (i \in Q^{\mathrm{yes}})}) \Big| (Q^{\mathrm{no}}, Q^{\mathrm{yes}}) \in \mathrm{spr}(r,J)\Big)$ (clearly well defined; by addition theory without loss of generality $r \leq 2k+1$, and the number of possible $\langle \mathrm{th}(M_i^t) : i,t \rangle$ is finite). Note $0 \leq \zeta_k \leq 1$.

As in the proof of 1.5(1)

AP1.4 Observation: For large enough k, $\zeta_k > 0$.

AP1.5 Observation: $(1-\zeta_{k_0k_1}) \leq (1-\zeta_{k_0})(1-\zeta_{k_1})$ [Why? as in the proof of 1.5(1), but the situation is symmetric.]

AP1.6 Observation: $\zeta_{k_0k_1} \geq \zeta_{k_0}$

AP1.7 Observation: $(1-\zeta_{k_0+k_1}) \leq \frac{k_0}{k_0+k_1}(1-\zeta_{k_0}) + \frac{k_1}{k_0+k_1}(1-\zeta_{k_0})$

AP1.8 Conclusion: 1) For some $\varepsilon > 0$ (and given d, τ), for every k large enough $1-\zeta_k \leq n^{-\varepsilon}$

2) Hence for k large enough:

$(*)$ Assume $M_i^t \in K$ for $i < r$, $J \subseteq r$, $|J| = 2k+1$ large enough, $[i \in r = \{0,\dots,r-1\} \setminus J \Rightarrow \mathrm{th}^d(M_i^{\mathrm{yes}}) = \mathrm{th}^d(M_i^{\mathrm{no}})]$. Let $Q^{\mathrm{yes}} \setminus J$, $Q^{\mathrm{no}} \setminus J \subseteq r$ be whatever (e.g. $\emptyset$). If we choose randomly $Q^{\mathrm{yes}} \cap J \subseteq J$, $|Q^{\mathrm{yes}}| = k+1$ (i.e. all possibilities with the same probability) and then choose randomly $Q^{\mathrm{no}} \cap J \subseteq Q^{\mathrm{yes}} \cap J$, $|Q^{\mathrm{yes}} - Q^{\mathrm{no}}| = 1$ *then* the probability that $\mathrm{th}^d(\sum_{i<r} M_i^{\mathrm{if}\,(i \in Q^{\mathrm{no}})}) \neq \mathrm{th}^d(\sum_{i<r} M_i^{\mathrm{if}\,(i \in Q^{\mathrm{no}})})$ is $\leq 2\sqrt{2\pi n}/n^{\varepsilon}$.

[Why? as the probability by μ^* (from AP1.1(2)) that $(Q^{\mathrm{no}}, Q^{\mathrm{yes}}) \in \mathrm{npr}(2k+1, r)$ satisfies $|Q^{\mathrm{yes}}| = k+1$, $|Q^{\mathrm{no}}| = k$ is $\simeq \frac{1}{2}\frac{1}{\sqrt{2\pi n}}$ (we can get rid of the half)].

AP2 Generalized sums and Distortions:

We try here to explain the results on §2 as distorted generalized sums (and the connection with generalized sums). First we give the definition of the generalized sum.

AP2.1 Definition: Let τ_0, τ_1, τ_2 be vocabularies of models. For a τ_0-model I (serving as index models), τ_1-models pairwise disjoint for simplicity $M_t (t \in I)$, and function F (explained below) we say that a τ_2-model M is the F-sum of $\langle M_t : t \in I \rangle)$ in symbols $M = \oplus_F \{M_t : t \in I\}$ if:

(1) the universe $|M|$ of M is $\bigcup_{t \in I} |M_t|$ (if the M_t's are not pairwise disjoint: $\{(t,a) : t \in I, a \in M_t\}$) and we define $h_\alpha : M \to I$, $h(a) = t$ if $a \in M_t$ (if not disjoint $h(\langle t,a \rangle) = a, h_t(\langle t,a \rangle) = a$),

(2) if $t_1, \ldots, t_k \in I$ are pairwise distinct, $\bar a = \bar a_1 {}^\frown \ldots {}^\frown \bar a_k$ and $\bar a_\ell \in M_{t_\ell}$ (finite sequences) *then* $\mathrm{tp}_{\mathrm{qf}}(\bar a, \emptyset, M) = F(\mathrm{tp}_{\mathrm{qf}}(\langle t_1, \ldots, t_k \rangle, \emptyset, I), \mathrm{tp}_{\mathrm{qf}}(\bar a_0, \emptyset, M_t), \ldots, \mathrm{tp}_{\mathrm{qf}}(\bar a_k, \emptyset, M_{t_k}))$.

Another way to say it is:

(2)$'$ if $a_0, \ldots, a_{k-1} \in M$ then $\mathrm{tp}_{\mathrm{qf}}(\langle a_0, \ldots, a_{k-1} \rangle, \emptyset, M) = F(\mathrm{tp}_{\mathrm{qf}}(\langle h(a_i) : \ell < k \rangle, \ldots, \mathrm{tp}_{\mathrm{qf}}(\langle b_\ell : \ell < k, t = h(a_\ell) \rangle, \emptyset, M_\delta), \ \ldots)_{t \in \{h(b_\ell) : \ell < k\}}$.

AP2.3 Remark: 1) So the form of F is implicitely defined, it suffices to look at sequences $\bar{a}_0{}^\frown \ldots {}^\frown \bar{a}_k$ (in (b)) or $\langle a_0, \ldots, a_k \rangle$ of length the arity of τ_2 (if it is finite).

2) We can consider a generalization where the universe of M and equality are defined like any other relation.

AP2.4 The generalized Sum Theorem: In the notation above if Δ_n is the set of formulas of quantifier depth n (see below) then we can compute from F the following function: like F in (2) (or $(2)'$) replacing $\mathrm{tp}_{\mathrm{qf}}$ by tp_{Δ_n}.

AP2.5 Discussion: So looking at a sequence $\bar{a}$ from M, to find its [quantifier free] types we need to know two things:

(α) the [quantifier free] type of its restriction to each $h^{-1}(\{t\}) = \{b \in M : h(b) = t\}$ (in, essential or you may express it to each "part" $\{t\} \times M_t$, *in* essentially M_t.

(β) the [quantifier free] type of the sequence of parts, $\langle h(b_\ell) : \ell < k \rangle$ *in* I.

AP2.6 Definition: Now M is a d-distorted F-sum of $\{M_t : t \in I\}$ if

(0) d is a distance function on I.

(1) $|M|$ is the disjoint axiom of A_t $(t \in I)$ and for each t we have a model M_t with universe $\cup\{A_s : s \in I, d(s,t) \leq 1\}$ and

(2)$^+$ if $b_0, \ldots, b_{k-1} \in M$, then $\mathrm{tp}_{\mathrm{qf}}(\langle b_0, \ldots, b_{k-1}\rangle, \emptyset, M) = F(\mathrm{tp}_{\mathrm{qf}}(\langle h(b_0) \ldots, h(b_{k-1}), \emptyset, I),$

$\ldots, \mathrm{tp}_{\mathrm{qf}}(\langle b_\ell : (\langle b_\ell : n < k, d(h(b_\ell), t) \leq 1), \emptyset, M_t), \ldots)_{t \in \{h(b_m) : m < k\}}$.

AP2.7 Remark: Note: by $\langle b_\ell : \ell < k, \Pr(\ell)\rangle$ we mean the function g with domain $\{\ell < k : \Pr(\ell)\}$, satisfying $g(\ell) = b_\ell$).

AP2.8 Discussion: Our main Lemma 2.14, generalizes the generalized sum theorem, to distorted sum but naturally the distortion "expands" with the quantifier depth.

AP2.9 The Distorted Generalized Lemma: In the notation above, for m let M^m_t be the model with

universe $\cup\{A_s : d(s,t) \leq m\} \cup \{s \in I : d(s,t) \leq m\}$

relations: those of the M_s's i.e. for $R \in \tau_1$, a k-place predicate we let $Q^{t,m}_R = \{\langle s, a_1, \ldots, a_k\rangle : s \in I, d(s,t) \leq m, \langle a_1, \ldots, a_k\rangle \in R^{M_s}$ (so $\{a_1, \ldots, a_s\} \subseteq \cup\{A_s :$

$d(s,t) \leq 1\})$

$Q^{t,\ell,m}_d = \{(s_1, s_2) : d(s,t) \leq \ell, d(s_2,t) \leq \ell,\ d(s_1,s_2) \leq m\}$

$Q_h = \{(s,a) : a \in A_s, s \in I, d(s,t) \leq m\}$.

Now there are functions F_n and a number $m(n)$ computable from F (and n) such that:

$\otimes$ for $b_0, \ldots, b_{k-1} \in M$

$\mathrm{tp}_{\Delta_n}(\langle b_0, \ldots, b_k\rangle, \emptyset, M) = F_n(\mathrm{tp}_{\Delta_n}(\langle h(b_0), \ldots, h(b_{k-1})\rangle, \emptyset, I), \ldots, \mathrm{tp}_{\Delta_n}(\langle b_\ell : \ell < k, d(h(b_\ell), t) \leq \ell\rangle, \emptyset, M_t), \ldots)_{t \in \{h(b_m) : m < k\}}$.

AP2.10 Discussion: Now if d is trivial:

$$d(x,g) = \begin{cases} 0 & x = y \\ \infty & x \neq y \end{cases}$$

then $M_t^n = M_t$, and AP 2.9 (the distorted generalized sum Lemma) becomes degenerated to AP2.4 (the generalized sum Lemma).

AP3 Models with Distance

AP3.1 Discussion: We try here to explain the results of §2 as models with distance functions “weakly suitable” for the model and the connection models with a distance function for the whore vocabulary which is suitable for the model. This is another variant of the distorted sums.

AP3.2 Content: Let τ be a fixed vocabulary.

1) Let K be the class of $\mathfrak{A} = (M, d)$, M a τ-model, d a distance on M (i.e. a two place symmetric function from $|M|$ to $\omega \cup \{\infty\}$, $d(x, x) = 0$, satisfying the triangular inequality).

2) $K^{\rm sut} \subseteq K$ is the class of $(M, d) \in K$ such that a (which is a distance on M) suitable for the model, i.e.

$\otimes_1$ $\langle a_0, \ldots, a_{k-1}\rangle \in R^M, R \in \tau_2 \Rightarrow \wedge_{\ell<m<k} d(a_\ell, a_m) \leq 1$.

3) $K^{\rm sim} \subseteq K$ is the class of $(M, d) \in K$ which are simple, i.e

$\otimes_2$ $d(x, y) = {\rm Min}\{n :$ there are $z_0, \ldots, z_n$ such that $x = z_0, z_n = y$ and

$$\wedge_{\ell<n} \vee_{R\in T_2} (\exists \bar{a} \in R^M)[\{z_\ell, z_{\ell+1}\} \subseteq {\rm Rang}\, \bar{a}]\}.$$

4) $K_F^{\rm ws}$ is the class of $(M, d) \in K$ which are F-weakly suitable which means:

$\otimes_3$ for every m_i *if* $\bar{a}^i = \langle a_0^i, \ldots, a_{n_i-1}^i\rangle$ for $i < k$, $a_\ell^i \in$M$, d(a_0^i, a_\ell^i) \leq m_i$ and $i < j < k \Rightarrow d(a_0^i, a_0^j) > m_i + m_j + 1$ *then* the quantifier free type of $\bar{a}^0{}^\frown\bar{a}^1{}^\frown \ldots {}^\frown\bar{a}^{k-1}$ is computed by F for the quantifier free types of $\bar{a}^0, \bar{a}^1, \ldots, \bar{a}^{k-1}$ and of $\langle a_0^0, a_0^1, \ldots, a_0^{k-1}\rangle$

Note: we can strengthen the demands e.g.

$(*)$ $d(a_0^i, a_\ell^i) \leq r_i$, $d(a^i, a^j) > f_0(r_i) + f_0(r_j) + 1$ or at least $\neg(\exists x, y)[d(x, a_i) \leq f_0(r_i) \wedge d(y, a_j) \leq f_0(r_j) \wedge d(x, y) \leq 1]$.

5) $K^{\rm as}$ is the family of $\mathfrak{B} = (M, d) \in K$ which are almost simple: "$d(x, y) \leq 1$" is defined by quantifier free formula and $d(x, y) = \text{Min}\{n$: we can find $z_0, \ldots, z_n, x = z_0, z_n = y$, $d(z_\ell, z_{\ell+1}) \leq 1\}$.

AP3.3 Discussion: For $(M, d) \in K_F^{\rm ws}$ we want to "separate" the quantification to bounded ones and to distant ones. We can either note that it fits the context of §2 or repeat it.

AP3.4 Definition: 1) For $\mathfrak{B} = (M, d) \in K$, $x \in M$, $m < \omega$ let $N_m^+(x) = \{y \in M : d(y, x) \leq m\}$.

2) We define "$\bar{a}$ is a $(\mathfrak{B}, r)$-component" and $\text{bth}_r^n(\bar{a}, \mathfrak{B})$ as in Definition 2.8(1), and $\text{BTH}_r^n(m, \tau)$ as in 2.8(2).

3) We define $\mathfrak{B}_{\eta,m}^2$ as expanding $\mathfrak{B}$ by the relation $R_{\mathbf{t}} = \{\bar{a} : \bar{a}$ is a $(\mathfrak{B}, r)$-component, $\mathbf{t} = \text{bth}_r^n(\bar{a}, \mathfrak{B})\}$ for $\mathbf{t} \in \text{BTH}_r^\eta(m, \tau)$.

4) We define "$\bar{a}$ is $(\mathfrak{B}, \bar{r})$-sparse" and $\text{uth}_{\bar{r}}^n(\bar{a}, \mathfrak{B})$", $\text{UTH}_{\bar{r}}^n(m, \tau)$ as in 2.12.

AP3.5 Theorem: the parallel of 2.14.

AP 3.6 Gaifman's Theorem: We can now conclude Gaifman's Theorem.

AP4 Better numerical bound

AP4.1 Discussion: In our main case (see 1.16 we have a linear order and "local" relations) we can improve the numerical bound i.e. to 2^n rather than 3^n; we shall prove it in AP4.7 below.

In general instead of using $(\mathfrak{B}, r)$-components, given $\bar{a} = \langle a_\ell : \ell < k \rangle$, and r we can divide $\{0, \ldots, k-1\}$ to $\bar{w} = \langle w_\ell : \ell < \ell(*) \rangle$ as follows: define a graph on $\{0, \ldots, k-1\}$ by the w_ℓ are components in the following graph: $\ell \neq m$ are connected iff $N_r^+(a_\ell) \cap N_r^+(a_m) \neq \emptyset$ and $\langle \mathrm{Min}(w_\ell) : \ell < \ell(*) \rangle$ is strictly increasing. We call $\bar{w}$ the r-partition of $\bar{a}$ (in $\mathfrak{B}$). Now we can denote it by $\bar{w}^r[\bar{a}, \mathfrak{B}] = \langle w_\ell^r[\bar{a}, \mathfrak{B}] : \ell < \ell^r[\bar{c}, [\mathfrak{B}] \rangle$ and $u^r[\bar{a}, \mathfrak{B}] = \{\mathrm{Min}(w_\ell) : \ell < k\}$; we let $N_r^+(\bar{a}) = \bigcup_{\ell < \ell g\, \bar{a}} N_r^+(a_\ell)$ and $\bar{N}_r^+[\bar{a}] = \langle \bigcup_{i \in w_\ell} N_r^+(a_i) : \ell < \ell(*) \rangle$. Now we can phrase another variant of Lemma 2.14. For diversion we do it in the content of AP3.

AP4.2 Definition: For $\mathfrak{B} = (M, d) \in K$, $\bar{a}$ a finite sequence of members of M. We define $\mathrm{bth}^n(\bar{a}, \mathfrak{B})$, by induction on n:

<u>$n = 0$</u> this is $\bar{w}^r[\bar{a}, \mathfrak{B}]$ and the quantifier free types of each $\bar{a} \restriction w_\ell^r[\bar{a}, \mathfrak{B}]$ in M equivalently set of basic (atomic and negation of atomic) formulas it satisfies.

<u>$n + 1$:</u> $\mathrm{bth}^{n+1}(\bar{a}, \mathfrak{B}) = \{\langle \ell, \mathrm{bth}^n(\bar{a} {}^\frown \langle c \rangle, \mathfrak{B}) \rangle : c \in N_{2^n}^+(a_\ell),\, \ell < \ell g\, \bar{a}\}$.

AP4.3 Claim: 1) for $\mathfrak{B} = (M, d) \in K$ and a finite sequence $\bar a$ with $\bar w = \langle w_\ell : \ell < \ell(*)\rangle = \bar w^{2^n}[\bar a, \mathfrak{B}]$, we can compute $\mathrm{bth}^n(\bar a, \mathfrak{B})$ from $\bar w$ and $\langle \mathrm{bth}^n(\bar a \restriction w_\ell, \mathfrak{B}) : \ell < \ell(*)\rangle$.

2)letting $M^n_{\bar a} = (M \restriction N^+_{2^n+1-1}(\bar a), \dots, \{(x,y) : d(x,y) = \ell\}, \dots)_{\ell \le 2^{n+1}}$ we can compute $\mathrm{bth}^n(\bar a, \mathfrak{B})$ from $\mathrm{th}^n(\bar a, M^n_{\bar a})$.

AP4.4 Definition: For $\mathfrak{B} = (M, I) \in K$, $\bar a$ a finite sequence of members of M, we define $\mathrm{uth}^n(\bar a, \mathfrak{B})$ by induction on n.

$\underline{n = 0}$ $\mathrm{uth}^0(\bar a, \mathfrak{B})$ is the quantifier free type of $\bar a$.

$\underline{n = 1}$ $\mathrm{uth}^{n+1}(\bar a, \mathfrak{B}) = \langle \mathbf{t}_0, \mathbf{t}_1\rangle$, where $\mathbf{t}_0 = \mathrm{uth}^n(\bar a, \mathfrak{B})$, $\mathbf{t}_1 = \{\langle \mathrm{bth}^n(c, \mathfrak{B}), \mathrm{uth}^n(\bar a {}^\frown c, \mathfrak{B})\rangle : c \in M \setminus N^+_{2^n}(\bar a)\}$.

AP4.5 The Main Lemma: For every function F_0 (as described below) we can compute (from F_0) functions F_n, (for $n < \omega$) such that: if $\mathfrak{B} = (M, d) \in K$ satisfies $\otimes_0$ then for every n it satisfies $\otimes_n$:

$\otimes_n$ if $\bar a$ is a finite sequence of length m from M, then

$$\mathrm{tp}^n(\bar a, M) = F_n\Big(\bar w^r[\bar a, \mathfrak{B}], \langle \mathrm{bth}^n(\bar a \restriction w^r_\ell) : \ell < \ell(*)\rangle,\ \mathrm{uth}^n(\bar a \restriction u^r[\bar a, \mathfrak{B}], \mathfrak{B})\Big).$$

AP4.6 Claim: 1) Assume (a) $(M, d) \in K$, M a τ-model.

(b) $<^M$ linearly orders $|M|$.

(c) if $R \in \tau \setminus \{<\}$, $\bar{a} \in R^M$, then $\wedge_{i,m} d(a_i, a_m) \leq 1$.

(d) for $x \in M$, $\{y : d(y,x) \leq m\}$ is an interval of $(|M|, <^M)$

then $\otimes_0$ of AP4.5 holds.

2) Like part one but instead of (b):

(b)$^-$ $< \notin \tau$

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