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A Geometric Approach for Denoting and Intersecting TR Subgroups of the Euclidean Group

by

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ABSTRACT

A group theoretical characterization of surface contacts between solids induces general yet concise expressions for computing relative positions between solids. Whether the intersections of subgroups of Euclidean group can be carried out effectively and efficiently on computers determines whether the group theoretical approach is computationally tractable and thus attractive in practice. Although there are many ways to denote a group and compute group intersections, it is not obvious that there exists a uniform computational denotation for the symmetry groups of surfaces, given the diversity of these groups — the finite, infinite, discrete and continuous subgroups of the Euclidean group. Furthermore, the denotation we are seeking should also accommodate the computation of group intersections.

In this article a geometric denotation in terms of *characteristic invariants* is proposed for an important family of subgroups of the Euclidean group, namely the **TR** subgroups, each of which is a semidirect product of a translation subgroup **T** and a rotation subgroup **R** of the Euclidean group. Under this denotation we prove that

- There exists a one-to-one correspondence between **TR** groups and their characteristic invariants;
- The intersections of **TR** groups are closed and can be obtained by some simple geometric calculations on their characteristic invariants;
- The algorithm for calculating **TR** group intersection using characteristic invariants is efficient, i.e. it has a worst case asymptotic time complexity $\mathcal{O}(n^2) + \mathcal{O}(logM)$ where n is the smaller number of isolated distinct rotation axes, if there is any, of the two **TR** groups involved in the intersection, and $\mathcal{O}(logM)$ is the order of magnitude for computing the greatest common divisor for two numbers a, b with $M = Max(a, b)$.

These results justify the group theoretic approach to formalizing contacts between solids as being general yet computationally tractable. Therefore, the group theoretical formalization of surface contacts is feasible in applications involving transformations between solids in Euclidean space, such as robotics, computer graphics, computer vision, assembly planning and mechanical design.

1 Introduction

All translations and rotations in Euclidean space $\mathbf{R}^3$ form a group called the proper[1] Euclidean group $\mathcal{E}^+$. Likewise, the set of all the symmetries of a subset S in $\mathbf{R}^3$ has a group structure and is thus called the *symmetry group* of S. Symmetries play an important role in studying the possible motions of rigid solids which are in contact. Here we consider a *solid* as a three-dimensional, connected and closed subset of Euclidean space. The coupling of solids constitutes a *kinematic chain* — rigid bodies related by means of mechanical constraints. This coupling takes place pairwise, and hence, it is given the name *kinematic pair* [1]. A *kinematic pair* thus produces a constrained relative motion of one body with respect to the other such that the specified contact remains. Under a group theoretical formalization [12, 14] these constrained motions of a solid can be expressed in terms of the symmetry groups belonging to the contacting surfaces of the solid, in general, in the form of

$$a_1 G_1 a_2 G_2 ... G_n a_{n+1} \tag{1}$$

where $a_i \in \mathcal{E}^+$ and G_i is the symmetry group of surface i that is in contact with other solids.

When the coupling between the two solids takes place along a surface, i.e. the two solids share an area, the possible relative motions between the two solids are determined by the computation of the group intersection in the form

$$aG'_1 \cap ... \cap G'_n b \tag{2}$$

where $a, b \in \mathcal{E}^+$ and G'_is are subgroups of $\mathcal{E}^+$ [12, 13, 14]. This article deals with the computation aspect of subgroup intersection in the form of formular 2. One feature of this work is to treat discrete and continuous groups uniformally under a geometric framework.

To find the relative positions of solids that are in contact is a fundamental problem in many fields, including robotics, graphics, CAD/CAM and computer vision. A group theoretical characterization of surface contacts between solids induces general yet concise expressions for computing relative positions between solids. Whether subgroup intersections can be carried out effectively and efficiently on computers, however, determines whether the group theoretical approach is computationally tractable and thus attractive in practice. We are led to the following questions:

- How can these symmetry groups be represented[2] on computers?
- If we can find such a representation for all the symmetry groups, can it be used effectively in the computation of symmetry group intersections?

[1] In our discussion we shall exclude reflections and deal with proper symmetries only, i.e. distance and handedness preserving isometries.

[2] When we use the term *representation* as applied to groups, we mean computational structures which serve to *denote* groups, and *not* a representation in terms of a homomorphism of the group into $GL(V)$[20], as is understood by group theorists.

- What will be the complexity of such a group intersection algorithm ?

This article is devoted to answer these questions. In section 2 we review related work from computational group theory to the analysis of mechanisms using continuous groups. Section 3 shows the main result of this article where we define **TR** subgroups of $\mathcal{E}^+$ and their characteristic invariants, prove that there is a one-to-one mapping between the **TR** subgroups and their invariants, and prove that the intersection of **TR** groups can be obtained from their characteristic invariants and that **TR** groups are closed under intersection. In section 4 an efficient **TR** subgroup intersection algorithm is given, and several examples are described. Section 5 concludes the article with discussions on open problems.

2 Related Work

The following is a short account of some related works with respect to the representation and computation (intersection) of groups.

2.1 Denotation of Groups on Computers

There are many ways to denote a group, such as, exhaustively listing all the members (e.g. $\{1\}$) or using the multiplication table for finite groups, exponent representation for Lie groups, or generators-and-relations for countable (finitely generated) groups. For the affine subgroups, for example, one can use homogeneous matrices; for rotations only, quaternions. However, to have a uniform denotation for the symmetry groups of all the solids and surfaces in Euclidean space is not a trivial problem. One cannot list all the members of an infinite group (e.g. the symmetry group of a sphere), nor can generators-and-relations be used to represent a Lie group. The exponent representation of Lie groups cannot be used to represent finitely generated groups such as crystallographic groups. Furthermore, the denotation we are seeking for should also accommodate the computation of group intersections.

2.2 Computation of Group intersections

Computational group theory is a branch of research between mathematics and computer science where the questions posed at the end of section 1 are studied. However, from the literature [4, 5] we are unable to find any existing algorithm for solving our problem directly. The existing algorithms for group intersections are for the *finitely generated* or *finitely presented* groups[3]. The symmetry groups we are concerned with, the subgroups of the Euclidean group, are not just finitely generated ones. Instead, they are a mixture of Lie groups ($SO(3), SO(2)$) and discrete groups (finite or infinite) which, topologically speaking, differ completely. Table 1 shows a sample of some symmetry groups where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are orthogonal unit vectors along axes X, Y, Z. $\mathbf{trans}(x, y, z)$ is a translation and $\mathbf{rot}(a, b)$ is a rotation

[3]Group G is *finitely generated* if there is a finite subset X of G such that $G =< X >$. ([27] p66)

Table 1: Typical Members of Some Canonical Subgroups of $\mathcal{E}^+$

Canonical Groups	Definition
$\mathcal{G}_{id}$	$\{1\}$
$SO(3)$	$\mathbf{gp}\{\mathbf{rot}(\mathbf{i},\theta)\mathbf{rot}(\mathbf{j},\sigma)\mathbf{rot}(\mathbf{k},\phi)\|\theta,\sigma,\phi\in R\}$
$O(2)$	$\mathbf{gp}\{\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi)\|\theta\in R, n\in\mathcal{N}\}$
$SO(2)$	$\mathbf{gp}\{\mathbf{rot}(\mathbf{k},\theta)\|\theta\in R\}$
D_{2n}	$\mathbf{gp}\{\mathbf{rot}(\mathbf{k},2\pi/n)\mathbf{rot}(\mathbf{i},m\pi)\|m,n\in\mathcal{N}\}$
C_n	$\mathbf{gp}\{\mathbf{rot}(\mathbf{k},2\pi/n)\|n\in\mathcal{N}\}$
$\mathcal{T}^1$	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\|z\in R\}$
$\mathcal{T}^1_{dis}(t_0)$	$\mathbf{gp}\{\mathbf{trans}(0,0,t_0)\|t_0\in R\}$
$\mathcal{T}^2$	$\mathbf{gp}\{\mathbf{trans}(x,y,0)\|x,y\in R\}$
$\mathcal{T}^3$	$\mathbf{gp}\{\mathbf{trans}(x,y,z)\|x,y,z\in R\}$
$\mathcal{G}_{cyl}$	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi)\|n\in\mathcal{N},\theta,z\in R\}$
$\mathcal{G}_{dir_cyl}$	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},\theta)\|z,\theta\in R\}$
$\mathcal{G}_{plane}$	$\mathbf{gp}\{\mathbf{trans}(x,y,0)\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi)\|x,y,\theta\in R, n\in\mathcal{N}\}$
$\mathcal{G}_{dir_plane}$	$\mathbf{gp}\{\mathbf{trans}(x,y,0)\mathbf{rot}(\mathbf{k},\theta)\|x,y,\theta\in R\}$
$\mathcal{G}_{screw}(p)$	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},2z\pi/p)\|z\in R\}$
$\mathcal{G}_{T_1C_2}$	$\mathbf{gp}\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{i},n\pi)\|n\in\mathcal{N},z\in R\}$

about axis a for angle b. **gp** indicates a group instead of a set. The doubled lines divide this set of subgroups into: the identity group, rotational subgroups, translational subgroups and subgroups containing both translations and rotations. These *canonical* subgroups are chosen with respect to a specified coordinate system in a systematic way: if they have a single axis of rotation, it is chosen to be the Z-axis, if they leave a single point of 3-space fixed, that point is chosen to be the origin. If they leave a plane setwise invariant, the plane is chosen as the X-Y plane. Many useful and important groups, for example $SO(3)$, the symmetry group of a sphere centered at the origin, cannot be finitely generated.

2.3 Analysis of mechanisms using group theory

In [10] Hervé introduced a rational classification of mechanisms by resorting to the theory of continuous groups. A *lower kinematic pair* exists when the coupling between the two bodies takes place along a surface (Figure 2.3). The *degree of freedom* (D.O.F.) of a kinematic chain is defined as the necessary and sufficient number of variables that define uniquely the position and orientation of all the links of the chain [1]. The problem of determining the number of D.O.F. of motion for mechanism which are constituted only of rigid bodies is represented as the whole set of rigid body positions, i.e. a nonempty subset of the group of displacements. Two kinds of operations between mechanical connections, the *intersection*

Figure 1: The six lower pairs

and the *composition*, allow characterization of *any connection between any pair of rigid bodies of any given mechanism which can be directly associated with the kinematic pairs.*

Three tables are given in [10]. Table 1 of [10] (See Table 2) shows a list of representatives for the conjugation classes of all the continuous subgroups of the proper Euclidean group. There are total 12 of them (including the identity group and the proper Euclidean group $\mathcal{E}^+$ itself), each of which is associated with its dimension(d.o.f.), a geometric element (which appears to be a crude form of the characteristic invariants to be defined in this article), and the corresponding lower pair (if any). Table 2 gives the subgroup relationship (a row-group is a subgroup of a column-group) among the 10 proper subgroups of $\mathcal{E}^+$ except the identity group. Table 3 lists the intersection and the composition (in regular form representation[4]) of those pairs of continuous subgroups of $\mathcal{E}^+$ whose intersection is neither the identity group nor one of the two groups to be composed. The final results of intersection and composition depend on the conditions of the geometric elements. It is illustrated clearly in his paper that the intersection of groups is the most crucial computation under such a group theoretic formalization, in that even the composition of liaisons among three or more rigid bodies depend on the intersection of corresponding groups. However, his article only addresses continuous subgroups of $\mathcal{E}^+$. The complete symmetry group of a cylinder (finite or infinite), for example, cannot be handled using this method.

2.4 Intersecting Groups in a Geometric Manner

Due to the fact that symbolic algebra is typically a costy computation on computers, an alternate though limited approach was proposed in [22] to deal with group intersections. An *invariant* S of a group G is defined to be a subset of $\mathbf{R}^3$ such that for all $g \in G, g(S) = S$. An invariant so defined is not unique for a group G. For example, all the infinite cylinders with the same central axis but different radius are invariants of the same symmetry group. As stated in [22], *we can use a unique set of characteristic invariants as a way of characterizing some important subgroups of the Euclidean Group.* This makes it possible to start with two symmetry groups, compute their characteristic invariants, combine these invariants, and then recover the intersection of the original groups from the combined invariants. Only a set of seven non-trivial groups treated in [22] and they are continuous groups only. Because of the arbitrariness in the choice of invariants, it led to difficulties in extending that work beyond a limited class of groups[5]

3 TR subgroups and their Characteristic Invariants

It is possible to define a similarity metric between two isometries. Under this metric $\mathcal{E}^+$ is a connected topological space. Subgroups of $\mathcal{E}^+$ that consist entirely of isolated points in

[4] *Regular form* means a group expressed as products of subgroups whose intersection is the identity group.

[5] Nevertheless, in this article, we inherited the term *characteristic invariants* from [22] though they are defined differently.

Table 2: Tale 1 in Hervé's 77 article: Conjugation classes of the subgroups of $\mathcal{E}^+$

Dimension D.O.F.	Notation	Element description	Geometric element definition	Associated kinematic pair
0	E	the identity transformation		rigid pair
1	T_D	straight translations	D is the direction of translation	(P) Prismatic
1	R_U	rotations about an axes	U determines the axes of rotation	(R) Revolute
1	$H_{u,p}$	helica l movements	u determines the rotation axes, and p is the tread pitch	(H) Screw
2	T_p	Translations on a plane	a plane defined by a normal p	
2	C_u	Bolt movements	an axes determined by vecter u	(C) Cylindrical
3	T	Spatial translations		
3	G_p	Sliding planar movements	a plane defined be a normal p	(E) Planar
3	S_O	Spherical rotations	a point O	(S) Spherical
3	$Y_{v,p}$	Y movements	a direction given by a free vecter v and a thread pitch p	
4	X_v	X movements	a direction of rotation given by a free vecter v	
6	D	Displacement		

Table 3: Hervé's notations versus our notations

Hervé's Notation	E	T_D	R_U	$H_{u,p}$	T_p	C_u
Our Notation	$\mathcal{G}_{id}$	T^1	$SO(2)$	$\mathcal{G}_{screw}(p)$	T^2	$\mathcal{G}_{dir_cyl}$
Hervé's Notation	T	G_p	S_O	$Y_{V,p}$	X_v	D
Our Notation	T^3	$\mathcal{G}_{plane}$	$SO(3)$	$\mathcal{G}_{plane}\mathcal{G}_{screw}(p)$	$SO(2)T^3$	$\mathcal{E}^+$

the topology are said to be *discrete*. The discrete symmetry groups may be either finite or countably infinite, and consist of:

- the *cyclic groups*, C_n, which are the symmetries of pyramids whose base is a regular polygon (except the regular tetrahedron);
- the *dihedral groups*, D_{2n}, which are the symmetries of regular prisms;
- the groups of the Platonic solids, and
- the crystallographic groups, which are the symmetries of spatial lattices of regularly repeated geometric patterns.

All non-discrete subgroups of $\mathcal{E}^+$ contain points which have other points of the subgroup arbitrarily close to them. They include, for example, the symmetry group of a sphere, $SO(3)$, and the symmetry group of the plane, $\mathcal{G}_{plane}$. These groups may contain isolated points. For example, the symmetry group of a line of equally placed, equal sized spheres or that of a set of equally placed parallel lines.

The fact that both discrete and continuous topological structures exist within one group presents challenging problems for computation. Pure algebraic manipulation using computers is usually very expensive, thus does not provide the computational tractability one desires. This motivates us to search for alternatives.

3.1 The Basic Ideas of the Characteristic Invariants Approach

In order to find a uniform denotation for the subgroups of the Euclidean group that have diverse topological properties, we come to the realization of following crucial points:

- Firstly, the recognition of the separation of translations and rotations. It is not in the simplest sense that any motion can be expressed as a translation and a rotation, but rather in terms of a symmetry group G (a set of motions) that can be divided into a translation subgroup T and a rotation subgroup R — a semidirect product $G = TR$. We call such a group a **TR** subgroup of $\mathcal{E}^+$. For example, the symmetry group of an oriented plane is a **TR** group (it is composed of $T = T^2$ and $R = SO(2)$), but the symmetry group of a screw is not. For the latter, there is no infinitesimal symmetries of the screw which is a pure rotation or a pure translation.
- Secondly, the switch of computational domains. The separation of translations and rotations in a group does not only provide us a means for divide-and-conquer, but also the feasibility to do group intersection in a geometric domain instead of the algebraic domain where the problem is started. The separation of translations and rotations makes it possible to associate with each rotation or translation subgroup a distinct geometric entity, namely, its *characteristic invariant* (Figure 3.1).

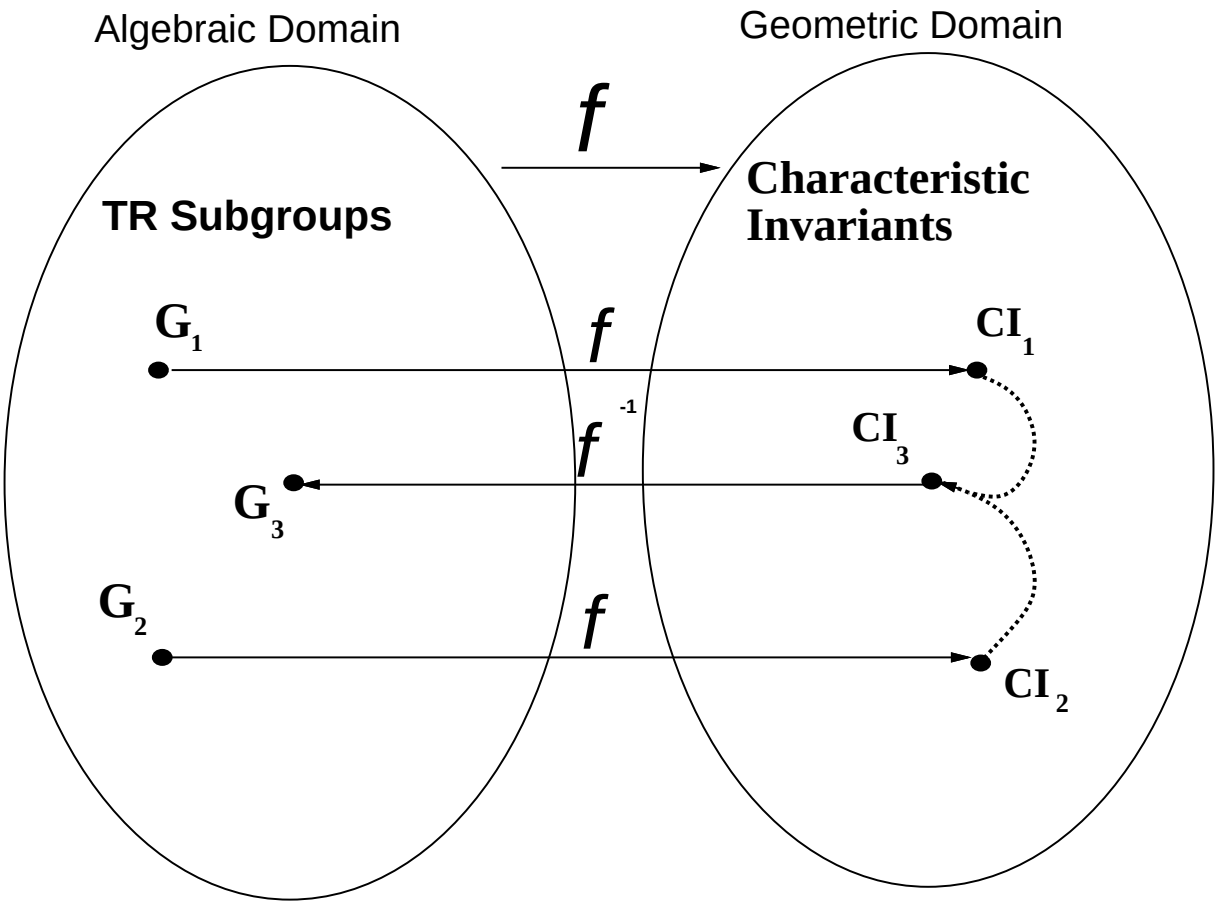

Figure 2: Mapping each group G_i to its invariant CI_i, carrying out some simple set intersections upon the two invariants to obtain CI_3, and finally mapping the resulted invariants CI_3 back to a group G_3, where $G_3 = G_1 \cap G_2$.

To justify that such a denotation is valid we first prove (Theorem 3.1) that there is a bijection between **TR** groups (algebraic) and their invariants (geometric). Then we prove (Theorem 3.2) that the intersection of any two **TR** groups is a **TR** group and can be obtained by

- mapping each group G_i to its characteristic invariants CI_i,
- carrying out some simple set intersections upon the two invariants to obtain a new invariant CI_3, and finally
- mapping the resulted invariants CI_3 back to a group G_3 which happens to be the group intersection so desired.

The approach reported in this article maintains the spirit of algebraic problem solving, while using a geometric methodology that provides a set of relatively intuitive proofs for the

justification of characteristic invariants, the closure property of **TR** groups and an efficient algorithm for **TR** group intersections.

In the rest of this section we first give a review of relevant concepts and properties of the subgroups of the Euclidean group in Section 3.2, then study translation and rotation subgroups separately in Sections 3.3 and 3.4. In section 3.5 we put translation and rotation subgroups together, where we define the characteristic invariants for **TR** groups, explore their properties and prove two theorems.

Before we start to show lemmas and proofs, there is one point worth of noting: since we only deal with subgroups of $\mathcal{E}^+$, the set equivalence of two subgroups implies a group equivalence. The following notational conventions are used in the rest of this article:

- We use the right associative function composition law $fg(x) = f(g(x))$.
- $\cong$ means *isomorphic*.
- $A \subset B$ means set A is a proper subset of B.
- $A \subseteq B$ means either $A \subset B$ or $A = B$.
- $\Leftrightarrow$ means *if and only if*, also *iff*.
- $\Rightarrow$ denotes *implication*.
- $\cup$ means set union, $\cap$ means set intersection.
- s_0 denotes the origin of the world coordinate system.
- S_0 represents the unit sphere centered at the origin
- If r is a rotation then $\bar{r}$ denotes the axis of the rotation and θ_r denotes the angle of the rotation.
- $\bigwedge$ denotes the greatest lower bound of a set.
- ∞ denotes the cardinal infinity corresponding to $|\mathbf{R}|$ i.e. the order of the set reals.
- $\mathcal{N}$ denotes the set of natural numbers.

3.2 Preliminaries

Those definitions and facts that can be found in basic group theory texts [6, 7, 8, 19, 20] will be assumed as given. However, in this section we shall state the group theoretic definitions and propositions that we need, and prove those which are not easily found in text books. Those readers who are familiar with notations listed here can skip to Section 3.3 (Page 22).

3.2.1 Group Actions

Definition 3.1 *A group G* **acts on** *a set X if there exists a mapping $\phi : G \times X \to X$, denoted by $(g, x) \mapsto g(x)$, which satisfies: $g(h(x)) = (gh)(x)$ and $e(x) = x$ where $g, h, e \in G, x \in X$.*

By this definition, one necessary condition for G to act on X is $G(X) = \{g(x) | x \in X, g \in G\} \subseteq X$.

Definition 3.2 *For each $x \in X$, $G(x) = \{g(x) | g \in G\}$ is called the* **orbit** *of x* **under the group** *G.*

If G acts on X then G may or may not act on a proper subset of X. For example, consider X as the set of all the vertices on a cube and Y be the set of vertices on one of the faces of the cube. The symmetry group G of the cube acts on X. Actually, $G(X) = X$. However, G cannot act on any proper subset Y of X, simply because some $g \in G$ brings some $y \in Y$ out of Y. Nevertheless a subgroup of G, C_4 a cyclic group of order 4, for example, can act on both X and Y (Figure 3). Note that when C_4 acts on Y there is only one orbit, while C_4 acting on X produces two disjoint orbits.

Definition 3.3 *If G acts on X and if for each pair $x, y \in X$, there exists at least one $g \in G$ such that $y = g(x)$, then* **group G acts transitively on set X** *i.e. the orbit of every element of X under group G is X itself.*

In Figure 3 the symmetry group G of the cube acts on both X and Y transitively although C_4 acts on Y transitively, it does not act transitively on X.

Definition 3.4 *For all $x \in X$ the set $G^x = \{g \in G | gx = x\}$ is a subgroup of G called the* **stabilizer subgroup (isotropy group)** *of G at x.*

G^x contains those elements of G which leave x invariant.

Definition 3.5 *If group G acts on X then $\mathcal{F}_G = \{x \in X | \forall g \in G, g(x) = x\}$ is said to be the* **fixed-point set** *of G.*

Every point in the fixed-point set of G has G as its stabilizer group.

In the following two propositions let G_1, G_2 be subgroups of G, and let G_1, G_2 be conjugate with each other, i.e. $G_1 = gG_2g^{-1}$ for some $g \in G$.

Proposition 3.6 $\mathcal{F}_{G_1} = g(\mathcal{F}_{G_2})$.

Proof :

Since $G_1 = gG_2g^{-1}$, for all $g_1 \in G_1$ there exists a $g_2 \in G_2$ such that $g_1 = gg_2g^{-1}$. For all $x \in \mathcal{F}_{G_2}, g_1(g(x)) = gg_2g^{-1}(g(x)) = gg_2(x) = g(x)$. Therefore $g(x) \in \mathcal{F}_{G_1}$ and so $g(\mathcal{F}_{G_2}) \subseteq \mathcal{F}_{G_1}$.

Similarly, since for all $g_2 \in G_2$ there exists a $g_1 \in G_1$ such that $g_2 = g^{-1}g_1g$, we have $\mathcal{F}_{G_1} \subseteq g(\mathcal{F}_{G_2})$.

Thus we conclude $\mathcal{F}_{G_1} = g(\mathcal{F}_{G_2})$. □

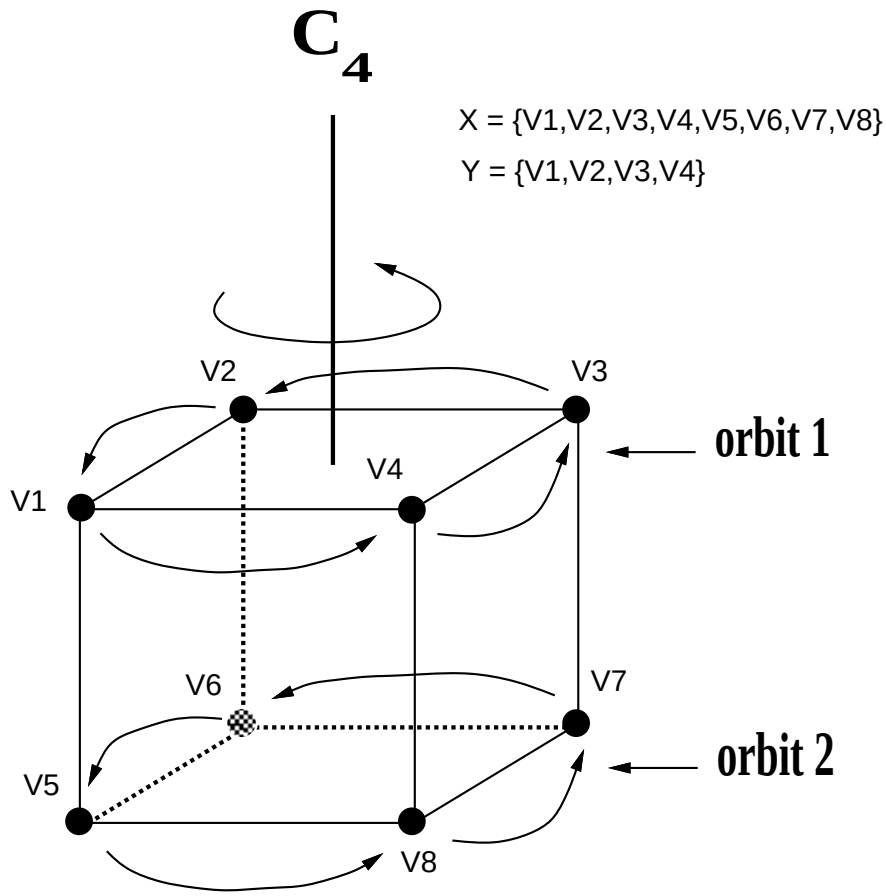

Figure 3: C_4 does not act on X transitively — there are two disjoint orbits; C_4 acts on Y transitively — there is only a single orbit.

Proposition 3.7 $G_1^{g(p)} = gG_2^p g^{-1}$.

Proof :

For all $g_1 \in G_1$ there exists $g_2 \in G_2$ such that $g_1 = gg_2g^{-1}$. In particular, if $g_1 \in G_1^{g(p)}$ then $g_1(g(p)) = gg_2g^{-1}(g(p)) = gg_2(p) = g(p)$. Thus $g_2(p) = p$, so $g_2 \in G_2^p$ and $g_1 \in gG_2^p g^{-1}$. Therefore $G_1^{g(p)} \subseteq gG_2^p g^{-1}$.

Similarly, since for all $g_2 \in G_2^p$ there exists a $g \in G_1$ such that $g_2 = g^{-1}g_1 g$, $gG_2^p g^{-1} \subseteq G_1^{g(p)}$.

So we conclude $G_1^{g(p)} = gG_2g^{-1}$. □

Proposition 3.8 *If* $G = G_1 \cap G_2$ *then* $G^p = G_1^p \cap G_2^p$.

Proof :

If $g \in G_1^p \cap G_2^p$ then $g(p) = p \Rightarrow g \in G^p$. Thus $G_1^p \cap G_2^p \subseteq G^p$.

If $g \in G^p$ then $g(p) = p$. Thus $G = G_1 \cap G_2$, $g \in G_1, g \in G_2$. Precisely, $g \in G_1^p, g \in G_2^p \Rightarrow g \in G_1^p \cap G_2^p$. Then $G^p \subseteq G_1^p \cap G_2^p$.

Hence $G^p = G_1^p \cap G_2^p$. □

3.2.2 The Euclidean Group and its Subgroups

The set of all the isometries of $\mathbf{R}^3$ forms a group with the composition of isometries as group product; this group is called the **Euclidean group**. If we restrict the set of isometries to those isometries that preserve handedness, thus excluding reflections, then the remaining set still has a group structure under isometry composition called the **Proper Euclidean group**, denoted by $\mathcal{E}^+$.

In this section we shall explore the internal structure of the proper Euclidean group $\mathcal{E}^+$ using its subgroups as building blocks. Let us start by defining the distance of two sets in $\mathbf{R}^3$ using the definition for the distance of two points in $\mathbf{R}^3$ **dist**.

Let $\mathbf{R}^3$ be three-dimensional real Euclidean space, $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbf{R}^3$.

Definition 3.9 *The bilinear form*

$$< x, y >= \sum_{i=1}^{3} x_i y_i, \quad x, y \in \mathbf{R}^3$$

defines an **inner product** *on this space.*

Definition 3.10 *The inner product defines a* **norm** *for any point of* $\mathbf{R}^3$

$$||x|| =< x, x >^{1/2}$$

often called the **Euclidean length** *of* $x \in \mathbf{R}^3$*, and a metric,* **dist** *for any pair of points of* $\mathbf{R}^3$

$$\mathbf{dist}(x, y) = \|x - y\| =< x, y >^{1/2} .$$

Definition 3.11 *A map* $g : \mathbf{R}^3 \to \mathbf{R}^3$ *is an* **isometry** *if and only if* $\forall x, y \in \mathbf{R}^3$

$$||g(x) - g(y)|| = ||x - y||.$$

Definition 3.12 *If* S_1 *and* S_2 *are non-empty subsets of* $\mathbf{R}^3$*, then the distance of sets* S_1, S_2*,* $\mathbf{dist}(S_1, S_2)$*, is defined to be*

$$\mathbf{dist}(S_1, S_2) = \bigwedge_{s_1 \in S_1 s_2 \in S_2} \mathbf{dist}(s_1, s_2).$$

Note, it may happen that $\mathbf{dist}(S_1, S_2) \neq \mathbf{dist}(s_1, s_2)$ for any $s_1 \in S_1, s_2 \in S_2$.

Lemma 3.13 *If* $\mathbf{dist}(S_1, S_2) = \mathbf{dist}(s_1, s_2)$ *then for all subsets* H_1, H_2 *such that* $s_1 \in H_1 \subseteq S_1, s_2 \in H_2 \subseteq S_2$, $\mathbf{dist}(H_1, H_2) = \mathbf{dist}(S_1, S_2)$.

Isometries are **distance preserving** mappings in $\mathbf{R}^3$. Let us now consider all isometries that are **linear**. It has been proved [23] that all the linear isometries O are **length preserving**, i.e. for all $x \in \mathbf{R}^3, \|O(x)\| = \|x\|$. Furthermore, all linear, length preserving mappings are orthogonal and form a group. This is how the following group gets its name.

Definition 3.14 $O(3)$ *is defined to be the* **real orthogonal group** *in three-space that contains all the linear isometries in* $\mathbf{R}^3$.

We identify linear isometries in $\mathbf{R}^3$ with their matrix representation [2, 7, 23] $O(3) = \{3\times3$ matrices $O | OO^t = I_3\}$. Here I_3 is the 3×3 identity matrix, and for all $O \in O(3), \det(O) = \pm 1$.

$O(3)$ is the symmetry group of S_0. It contains rotations as well as reflections. To exclude reflections, we must constrain the determinant:

Definition 3.15 The special orthogonal group $SO(3)$ *is defined to be* $SO(3) = \{O \in O(3) | \det(O) = +1\}$.

Definition 3.16 *Any subgroup of* $SO(3)$ *is called a* **canonical rotation subgroup** *of* $\mathcal{E}^+$ *and will be denoted by* R_c.

Definition 3.17 R *is a* **rotation subgroup** *of* $\mathcal{E}^+$ *iff* R *belongs to the conjugation class of a canonical rotation group, i.e. there exist a canonical rotation subgroup* R_c *and some* $g \in \mathcal{E}^+$ *such that* $R = gR_cg^{-1}$.

Definition 3.18 *Suppose* R *is a rotation group. If* $\mathcal{F}_R$ *is a single point then* R *is called a* **point group***; if* $\mathcal{F}_R$ *is a line then* R *is called a* **line group**.

Note, this notation differs from that used in Crystallography and other fields where R is a point group if $\mathcal{F}_R$ is not empty. Here, each member of a rotation group R keeps all the points of its fixed point set $\mathcal{F}_R$ *pointwise* fixed.

Proposition 3.19 *The elements of $SO(3)$ are rotations about s_0, i.e. for all $r \in SO(3), s_0 \in \bar{r}$. ([20] Theorem 2.1)*

Corollary 3.20 *A nontrivial canonical rotation group R_c is either a line group or a point group.*

Corollary 3.21 *A nontrivial rotation group R is either a line group or a point group.*

Proof :

By Definition 3.17, $R = gR_cg^{-1}$. By Proposition 3.6, $\mathcal{F}_R = g(\mathcal{F}_{R_c})$. By Corollary 3.20, $\mathcal{F}_{R_c}$ is either a point or a line passing through the origin. Thus $\mathcal{F}_R$ is either a point or a line transformed by g from the origin. □

Note: Each rotation subgroup R treated in this article belongs to one of the following two categories:

- If R has infinite cardinality, it is conjugate to $SO(3)$, $O(2)$ or $SO(2)$[6].
- If R has a finite order, it is conjugate to a cyclic group C_n, a dihedral group D_{2n} or one of the platonic groups – the symmetry groups for regular polyhedrons.

Definition 3.22 *Let $\mathbf{T}^3$ be all the translations in the Euclidean group $\mathcal{E}$. $\mathbf{T}^3$ has a group structure and is called the* **translation group** *of $\mathcal{E}$.*

Informally, we identify each translation t in $\mathbf{T}^3$ with a point x in $\mathbf{R}^3$. A translation t acts on a point p of $\mathbf{R}^3$ by moving the point along the vector (Figure 4). That is, $\mathbf{T}^3$ is isomorphic to $\mathbf{R}^3$ under addition and therefore has a group structure which is connected and abelian. So there exists a bijection $\eta : \mathbf{T}^3 \to \mathbf{R}^3$ such that for all translations $t_x, t_p \in \mathbf{T}^3$, we have $x = \eta(t_x), p = \eta(t_p) \in \mathbf{R}^3$, and

$$\eta(t_x t_p) = \eta(t_x) + \eta(t_p) = x + p \qquad \textit{homomorphism property.}$$

In particular, a translation acting on a point of $\mathbf{R}^3$ can be written as $t_x(p) = p + \eta(t_x) = p + x = \eta(t_p) + x = t_p(x)$ (Figure 4).

Each non-identity translation $t \in \mathbf{T}^3$ is a non-linear isometry, since for any pair of points $p_1, p_2 \in \mathbf{R}^3, t_x(p_1 + p_2) = p_1 + p_2 + x$, while $t_x(p_1) + t_x(p_2) = p_1 + x + p_2 + x = p_1 + p_2 + 2x$. This non-linearity is the reason why translations can not be given a representation as 3×3 matrices.

An element of $\mathcal{E}^+$ can always be expressed as the product of a rotation and a translation, or can be seen as a screw-motion tr, where $t \in \mathbf{T}^3, r \in SO(3)$ and $t \| \bar{r}$, which includes the rotations and translations as 'extreme' cases [7]. **Note**, when $g = tr, t \perp \bar{r}$, g is a rotation [20].

Proposition 3.23 *If $g \in \mathcal{E}^+$ then $g = tr$ for a unique $t \in \mathbf{T}^3$ and a unique $r \in SO(3)$.*

[6] We exclude those infinite groups that are dense in $SO(3)$, e.g. isomorphic to the rationals. However, the *topological closure* of any infinite subgroup of SO(3) is one of these 3 groups.

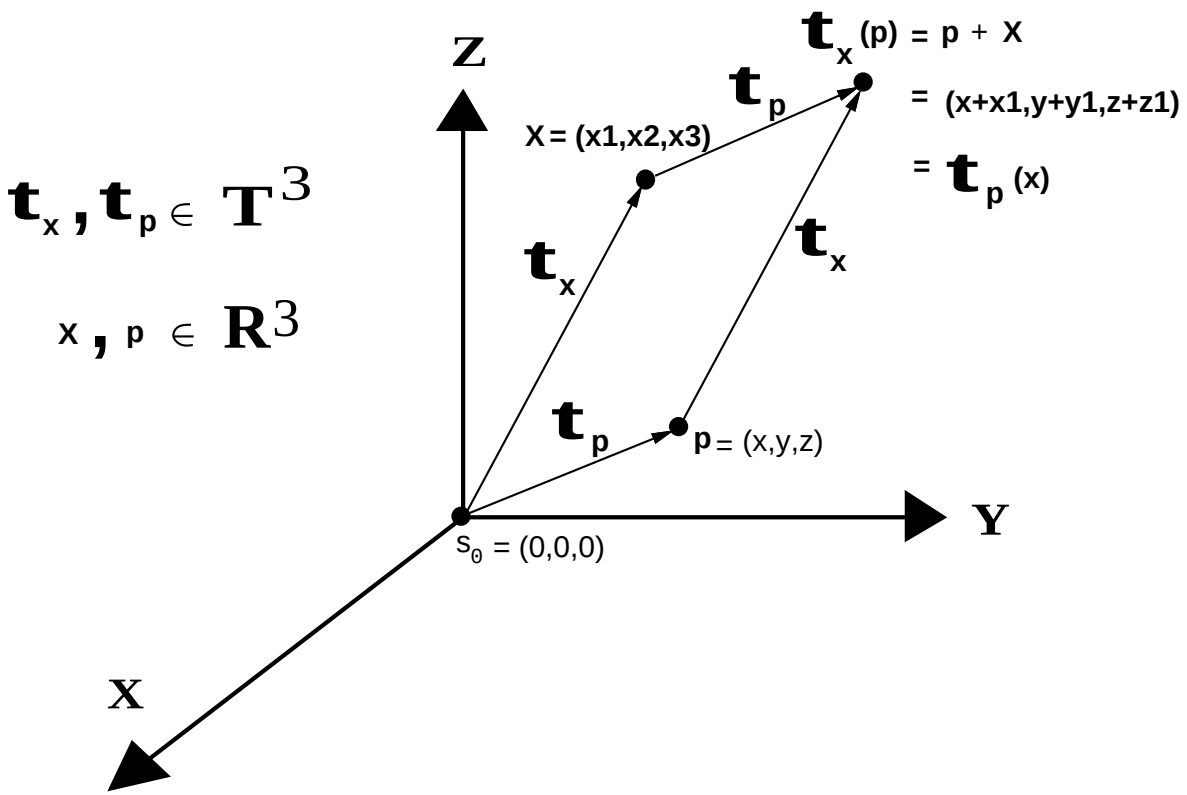

Figure 4: Identifying $\mathbf{T}^3$ with $\mathbf{R}^3$

Proof :

If $g = t_1 r_1 = t_2 r_2$ for some $t_1, t_2 \in \mathbf{T}^3, r_1, r_2 \in SO(3)$, then $t_2^{-1} t_1 = r_2 r_1^{-1}$. For $t_2^{-1} t_1$ to be a translation and to fix the origin s_0, it must be true that $t_2^{-1} t_1 = 1 \Rightarrow t_1 = t_2$. Thus $r_1 = r_2$. □

It has been proved that a member of the proper Euclidean group $\mathcal{E}^+$ can be represented by a 4×4 matrix [2] as:

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & a \\ r_{21} & r_{22} & r_{23} & b \\ r_{31} & r_{32} & r_{33} & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where the upper left 3×3 sub-matrix specifies the rotational part of the transformation (an element of $SO(3)$), and the right hand column specifies a translation by (a, b, c). Group multiplication is then simply matrix multiplication. This representation has been widely accepted in robotics [21] and is usually referred to as the 4×4 *homogeneous transformation* matrix.

Since any rotation r can be viewed as a rotation about some Z axis after a proper change of basis, w.l.o.g, let r represent a rotation about the Z axis and $t = (a, b, c)$ be an arbitrary translation under the orthogonal coordinate system whose Z axis coincides with $\bar{r}$.

Proposition 3.24 *Let* $t' = rtr^{-1}$. *Then* t' *is a translation and* $t' = (a\cos(\theta_r) - b\sin(\theta_r), a\sin(\theta_r) + b\cos(\theta_r), c)$.

Proof :

Using the 4×4 homogeneous transformation matrix representation for each element of $\mathcal{E}^+$ we have:

$$t = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, r = \begin{pmatrix} \cos\theta_r & -sin\theta_r & 0 & 0 \\ \sin\theta_r & \cos\theta_r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$rtr^{-1} = \begin{pmatrix} 1 & 0 & 0 & a\cos\theta_r - b\sin\theta_r \\ 0 & 1 & 0 & a\sin\theta_r + b\cos\theta_r \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Recall that the result of rotating a vector $t = (a, b)$ on the x-y plane by angle θ is:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{pmatrix}$$

Thus matrix (3) shows that rtr^{-1} is a translation resulting from rotating t about $\bar{r}$ by θ_r. □

Proposition 3.25 $\mathbf{T}^3$ *is a normal subgroup of* $\mathcal{E}^+$.

Proof :

For all $g = t_g r_g \in \mathcal{E}^+$ (Proposition 3.23) and all $t \in \mathbf{T}^3, gtg^{-1} = t_g r_g t r_g^{-1} t_g^{-1} = r_g t r_g^{-1} \in \mathbf{T}^3$ (Proposition 3.24). Therefore $\mathbf{T}^3 \lhd \mathcal{E}^+$ ([29] p. 23). □

Now, we are ready to describe the proper Euclidean group in terms of its subgroups $\mathbf{T}^3$ and $SO(3)$.

Proposition 3.26 *The* **Proper Euclidean group** $\mathcal{E}^+ = \mathbf{T}^3 \times_\theta SO(3)$ *is a semi-direct product of* $\mathbf{T}^3$ *and* $SO(3)$ *relative to* $\theta(r) : \mathbf{T}^3 \to \mathbf{T}^3$, *where for all* $r \in SO(3)$, *all* $t \in \mathbf{T}^3, \theta(r) : t \mapsto rtr^{-1}$. *The product rule of* $\mathcal{E}^+$ *follows:*

$$(t_1, r_1)(t_2, r_2) = (t_1(r_1 t_2 r_1^{-1}), r_1 r_2).$$

Proof :

This follows directly from Propositions 3.23 and 3.25, and [19] on semidirect products. □

Definition 3.27 *A subgroup of* $\mathcal{E}^+$ *is a* **translation subgroup** *iff it is a subgroup of* $\mathbf{T}^3$.

Proposition 3.28 *For any* $t_1, t_2 \in \mathbf{T}^3$ *and any* $x \in \mathbf{R}^3$, *if* $t_1(x) = t_2(x)$ *then* $t_1 = t_2$.

Proof :

Since $\eta : \mathbf{T}^3 \to \mathbf{R}^3$ is a bijection, $t_1(x) = x + \eta(t_1) = t_2(x) = x + \eta(t_2) \Rightarrow \eta(t_1) = \eta(t_2) \Rightarrow t_1 = t_2$. □

Proposition 3.29 *Let* T *be a translation subgroup. If* $t \in T^3, p \in \mathbf{R}^3$ *such that* $t(T(p)) = T(p)$ *then* $t \in T$.

Proof :

Since $p \in T(p), T(p) \neq \emptyset$. For all $x \in \mathbf{T}(p), t(x) \in \mathbf{T}(p)$. Since T acts on $T(p)$ transitively (Definition 3.3), for the two elements $x, t(x)$ of $T(p)$ there exists $t' \in T$ such that $t'(x) = t(x)$. By Proposition 3.28 $t = t' \in T$. □

Proposition 3.30 trt^{-1} *is a rotation about axis* $t(\bar{r})$ *with angle of rotation* θ_r.

Proof :

Let us prove this using some basic geometric arguments. Figure 5 illustrates the effect of trt^{-1} acting on an arbitrary point $x \in \mathbf{R}^3$. Here point B is a point on $t(\bar{r})$ that realizes $\mathbf{dist}(x, t(\bar{r})) = \mathbf{dist}(x, B)$, i.e. B is the perpendicular projection of point x on line $t(\bar{r})$. Point A is point B translated by t^{-1}, obviously, $A = t^{-1}(B) \in \bar{r}$. To prove that trt^{-1} is a rotation about axis $t(\bar{r})$ with rotation angle θ_r is to prove that the two triangles $\{A, t^{-1}(x), rt^{-1}(x)\}$ and $\{B, x, trt^{-1}(x)\}$ are congruent to each other. Due to the fact that t and t^{-1} form parallel, equal length line-segments as shown in Figure 5, one can easily verify that all the corresponding edges of the two triangles are of equal length. Hence they are congruent. □

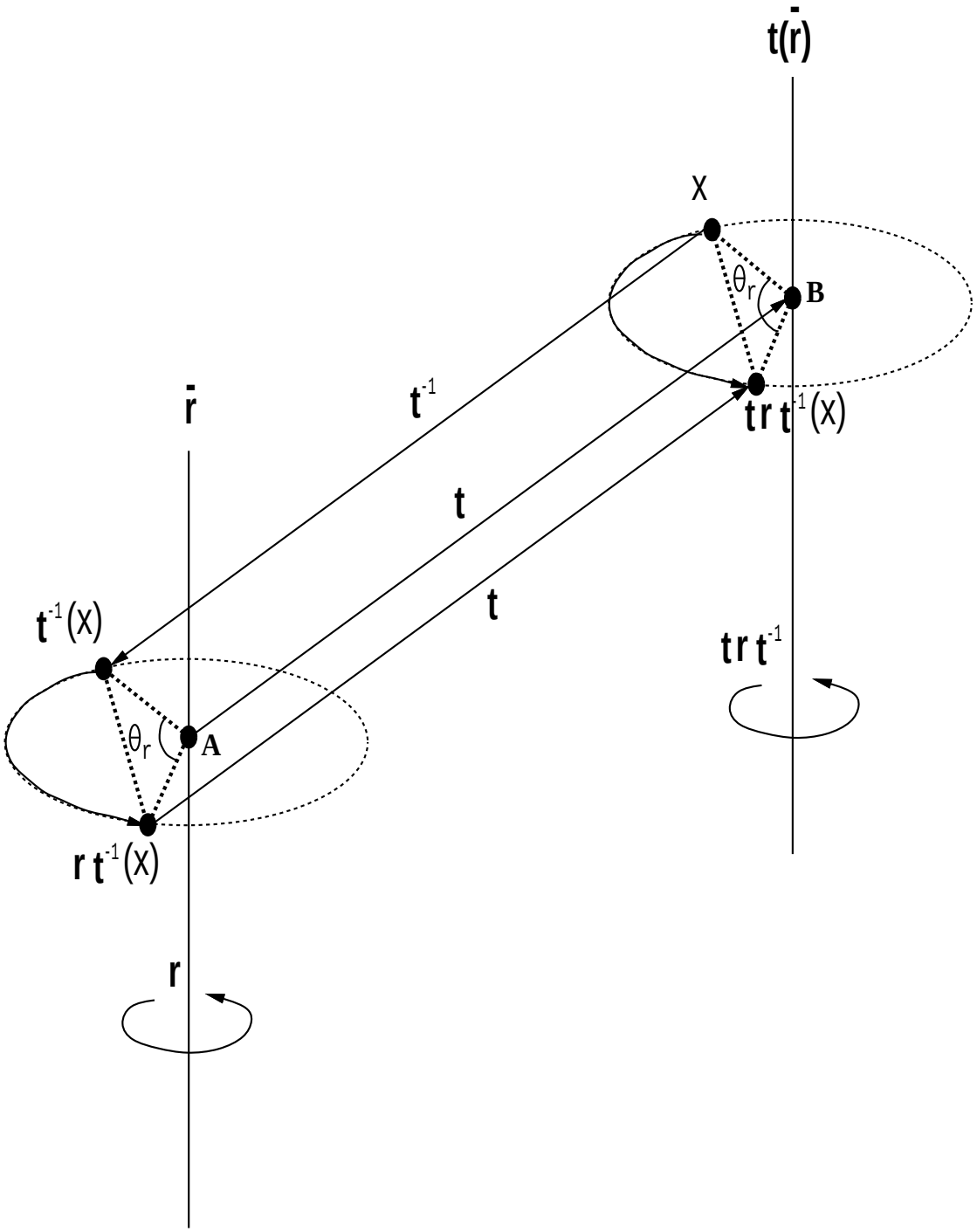

Figure 5: This shows the effect of trt^{-1} acting on an arbitrary point x in order to prove that trt^{-1} is a rotation about axis $t(\bar{r})$ with angle of rotation θ_r.

That trt^{-1} is a rotation about a new parallel rotation axis $t(\bar{r})$ with angle of rotation θ_r is not obvious from the matrix product of trt^{-1},

$$trt^{-1} = \begin{pmatrix} \cos\theta_r & -\sin\theta_r & 0 & a-(a\cos\theta_r - b\sin\theta_r) \\ \sin\theta_r & \cos\theta_r & 0 & b-(a\sin\theta_r + b\cos\theta_r) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is because trt^{-1} is not a linear rotation (a rotation about the origin) any more.

Corollary 3.31 *If $\bar{r}_1 \| \bar{r}_2$ and $\theta_{r_1} = \theta_{r_2}$, with respect to the same reference vector, then for all $t \in T^3$, $t(\bar{r}_1) = \bar{r}_2 \Rightarrow tr_1t^{-1} = r_2$.*

Proposition 3.32 *Let R_1, R_2 be rotation groups. If $R_1 = tR_2t^{-1}$ for some $t \in \mathbf{T}^3$ and $\mathcal{F}_{R_1} \cap \mathcal{F}_{R_2} \neq \emptyset$ then $R_1 = R_2$.*

Proof:

For all $r_1 \in R_1$ there exists $r_2 \in R_2$ such that $r_1 = tr_2t^{-1}$. By Proposition 3.30, $\bar{r}_1 \| \bar{r}_2$ and $\theta_{r_1} = \theta_{r_2}$. Since $\mathcal{F}_{R_1} \cap \mathcal{F}_{R_2} = \mathcal{F} \neq \emptyset$, there exists at least one point $x \in \mathcal{F}$ such that $x \in \bar{r}_1$ and $x \in \bar{r}_2$. Therefore $r_1 = r_2 \in R_2$ and so $R_1 \subseteq R_2$. Similarly, $R_2 \subseteq R_1$.

Thus we conclude $R_1 = R_2$.

Corollary 3.33 *If R_1, R_2 are canonical rotation groups and $R_1 = gR_2g^{-1}$ for some $g = tr$ with $t \in \mathbf{T}^3, r \in SO(3)$ then $R_1 = rR_2r^{-1}$.*

Proof: $R_1 = trR_2r^{-1}t^{-1}$. Since $rR_2r^{-1} \subseteq SO(3)$, $R_1 = rR_2r^{-1}$ (Proposition 3.32). □

Proposition 3.34 *If R is a rotation group then there exists a* **unique** *canonical rotation group R_c such that $R = tR_ct^{-1}$ for some $t \in \mathbf{T}^3$.*

Proof:

By Definition 3.17, R being a rotation subgroup means $R = gR'_cg^{-1}$ for some $g = tr \in \mathcal{E}^+$ where $t \in \mathbf{T}^3, r \in SO(3)$ and $R'_c \subseteq SO(3)$. Since $rR'_cr^{-1} = R_c \subseteq SO(3)$, we have $R = tR_ct^{-1}$.

Now assume there are two canonical rotation subgroups R_{c_1}, R_{c_2} and translations t_1 and t_2 such that $R = t_1R_{c_1}t^{-1} = t_2R_{c_2}t_2^{-1}$. Then $R_{c_1} = t_0R_{c_2}t_0^{-1}, t_0 = t_1^{-1}t_2 \in \mathbf{T}^3$. Since at least $s_0 \in \mathcal{F}_{R_{c_1}}, s_0 \in \mathcal{F}_{R_{c_2}}$, by Proposition 3.32 $R_{c_1} = R_{c_2} = R_c$. □

Definition 3.35 *Let R_1, R_2 be nontrivial rotation groups, $R_1 = tR_2t^{-1}$. Among all the $t_i \in \mathbf{T}^3$ such that $R_1 = t_iR_2t_i^{-1}$, let t_{min} be the* **minimum translation** *such that $||t_{min}|| = min\{||t_i||\}$.*

Proposition 3.36 *The minimum translation t_{min} is* **unique** *where $||t_{min}|| =$ **dist**$(\mathcal{F}_{R_1}, \mathcal{F}_{R_2})$.*

Proof:

By Proposition 3.6, $R_1 = tR_2t^{-1} \Rightarrow \mathcal{F}_{R_1} = t(\mathcal{F}_{R_2})$. Thus $\mathcal{F}_{R_1}, \mathcal{F}_{R_2}$ are both points of $\mathbf{R}^3$, or are two parallel lines (Figure 6).

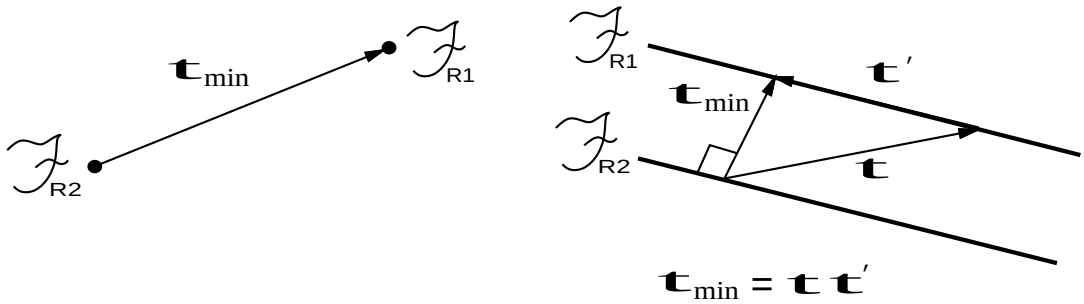

Figure 6: Minimum translation t_{min}

1. $\mathcal{F}_{R_1}, \mathcal{F}_{R_2}$ are two points in $\mathbf{R}^3$. The only translation which moves one point to the other is $||t|| = \mathbf{dist}(\mathcal{F}_{R_1}, \mathcal{F}_{R_2}) = ||t_{min}||$ and so $t_{min} = t$ is the minimum translation as desired.

2. $\mathcal{F}_{R_1}, \mathcal{F}_{R_2}$ are parallel lines in $\mathbf{R}^3$. Since $\mathcal{F}_{R_1}, \mathcal{F}_{R_2}$ are closed subspaces of $\mathbf{R}^3$, $\mathbf{dist}(\mathcal{F}_{R_1}, \mathcal{F}_{R_2})$ can be identified with the distance of a pair of points from $\mathcal{F}_{R_1}, \mathcal{F}_{R_2}$ respectively. This determines the minimum translation between $\mathcal{F}_{R_1}$ and $\mathcal{F}_{R_2}$ to be the one that is perpendicular to both lines as shown in Figure 6. So $||t_{min}|| = \mathbf{dist}(\mathcal{F}_{R_1}, \mathcal{F}_{R_2})$.

Now we need to prove that $R_1 = t_{min} R_2 t_{min}^{-1}$. Let t' be a translation along $\mathcal{F}_{R_1}$ such that $t_{min} = tt'$. From $R_1 = tR_2t^{-1}$, for all $r_1 \in R_1$ there exists $r_2 \in R_2$ such that $r_1 = tr_2t^{-1}$. Since $t' \| \bar{r}_2, r_2 = t'r_2t'^{-1}$ (Proposition 3.30). Thus $r_1 = tt'r_2t'^{-1}t^{-1} = t_{min}r_2t_{min}^{-1} \in t_{min}R_2t_{min}^{-1}$. Hence $R_1 \subseteq t_{min}R_2t_{min}^{-1}$. Similarly, we can prove $t_{min}R_2t_{min}^{-1} \subseteq R_1$. Therefore $R_1 = t_{min}R_2t_{min}^{-1}$. □

From now on, whenever we have $R_1 = tR_2t^{-1}$, t should be understood to be the unique minimum translation whose existence has just been proved.

Definition 3.37 *A* **pole** *of a rotation group $R = tR_ct^{-1}$ is a point p on S_0 which is left fixed by some rotation of the group R_c other than the identity. I.e. for some $p \in S_0, \exists r \in R_c, r \neq 1$ such that $r(p) = p$.*

Each rotation in $SO(3)$ fixes a line passing through the origin. This axis of rotation pierces the unit sphere to form two poles, so poles always appear in pairs (Figure 7). Different rotations about the same axis have the same pair of poles. Therefore poles are not sufficient to be used to distinguish different rotation groups. Nevertheless, we shall see in the next proposition that the number of poles of a rotation group can determine whether a rotation group is a line or point group.

Lemma 3.38 *$R_c = \cup R_c^p$ where R_c^p is the stabilizer subgroup of R_c at a pole p.*

Proof:
$R_c^p \subseteq R_c \Rightarrow \cup R_c^p \subseteq R_c$.

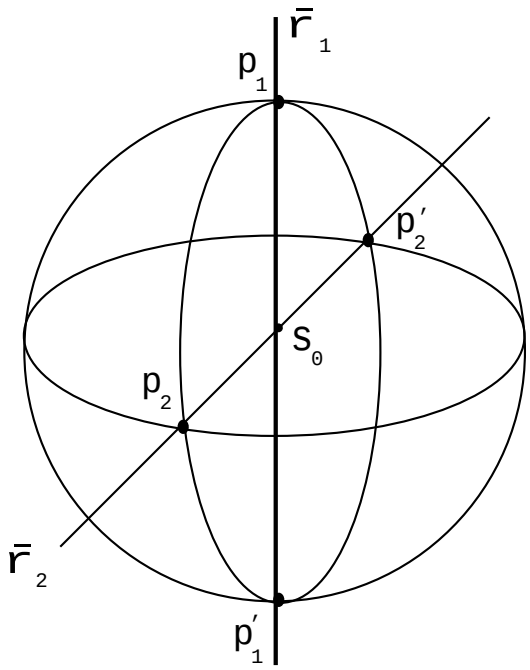

Figure 7: When a rotation group has only two poles p_1 and p_2, they have to belong to the same rotation axis; otherwise more than two poles would appear.

For each nontrivial rotation $r \in R_c$, there exists a $p \in S_0$ such that $r(p) = p$. By Definition 3.4 and Definition 3.37 there exists a stabilizer subgroup $R_c^p \subseteq R_c$ and $r \in R_c^p$. Therefore $r \in \cup R_c^p$. So $R_c \subseteq \cup R_c^p$ completes the proof. □

Corollary 3.39 *If $R = tR_ct^{-1}$, then $R = \cup R^{t(p)}$ for all the poles p of R.*

Proof: From Lemma 3.38 $R_c = \cup R_c^p \Rightarrow R = tR_ct^{-1} = t(\cup R_c^p)t^{-1} = \cup tR_c^pt^{-1} = \cup R^{t(p)}$. Here $R^{t(p)} = tR_c^pt^{-1}$ is a special case of Proposition 3.7. □

Proposition 3.40 *Let P be the set of poles of a canonical rotation group R_c. $|P| = 0$ iff $R_c = \{1\}$; $|P| = 2$ iff R_c is a line group; and $|P| > 2$ iff R_c is a point group.*

Proof :

Since $|P| = 0$, by Definition 3.37, for all $p \in S_0$ there exists no $r \neq 1$ such that $r(p) = p$. Then for all $p \in S_0, |R_c^p| = 1 \Leftrightarrow$ By Lemma 3.38 $R_c = \cup R_c^p = \{1\}$.

When $|P| = 2$, let $P = \{p_1, p_2\}$, $p_1 \neq p_2$. By the definition of poles, there exists at least one nontrivial rotation $r_1 \in R_c$ such that $r_1(p_1) = p_1$. Then $p_1, p_1' \in \bar{r}_1$ (Figure 7). If $p_2 \neq p_1'$ then $|P| = |\{p_1, p_2, p_1' ...\}| > 2$, a contradiction. Therefore $p_2 = p_1'$. Hence every $r \in R_c$ rotates about the line determined by p_1 and p_2, i.e. $R_c^{p_1} = R_c^{p_2} = R_c$ is a line group. Now, if R_c is a line group then $\mathcal{F}_{R_c}$ is a line passing through s_0. For all $r \in R_c, \bar{r} = \mathcal{F}_{R_c}$, since otherwise there would be a $r \in R_c, x \in \mathcal{F}_{R_c}$ such that $r(x) \neq x$, a contradiction of the

definition of fixed point set. Therefore only two points on S_0 are left fixed by members of R_c and so $|P| = 2$.

When $|\mathcal{P}_R| > 2$, by Corollary 3.21 and the proof above, R cannot be a line group $\Leftrightarrow$ R is a point group. □

Obviously, this proposition is also true for any rotation subgroup of $\mathcal{E}^+$ that is not a canonical rotation group (Definition 3.17 and Proposition 3.6).

Proposition 3.41 *If R_1, R_2 are rotation line groups about same axis then* $\gcd^*(|R_1|, |R_2|) = |R_1 \cap R_2|$ *where*

$$\gcd^*(m,n) = \begin{cases} \gcd(m,n) & \text{if } m,n \text{ both are integers} \\ n & \text{if } m = \infty \\ m & \text{if } n = \infty \\ \infty & \text{if } a = b = \infty \end{cases}$$

Proof :

If one of R_1, R_2 has an infinite order, w.l.o.g. let $R_1 \cong SO(2)$. Clearly R_2 is a subgroup of R_1 and $R_1 \cap R_2 = R_2$. Thus $|R_1 \cap R_2| = |R_2| = \gcd^*(|R_1|, |R_2|)$.

If R_1, R_2 are both finite rotation groups, let $|R_1| = m, |R_2| = n, \gcd(m,n) = k$. There exists a cyclic group C_k such that $C_k \subseteq R_1 \cap R_2$ (p.63, theorem 5.3 [8]). Let $R_1 \cap R_2 = C_q$, a cyclic group of order q. By Lagrange's theorem, $k|q$. Thus there exists an integer c such that $q = ck$. Since also $q|m$ and $q|n$, we have $ck|m, ck|n$. However, k is the largest common divisor of m and n, so c must be 1. Therefore $q = k = \gcd(m,n)$. Hence $|R_1 \cap R_2| = \gcd^*(|R_1|, |R_2|)$. □

Corollary 3.42 $\gcd^*(|R_1|, |R_2|) = 1$ *iff* $R_1 \cap R_2 = \{1\}$

3.3 Translational Characteristic Invariants for Translation Subgroups

In this section we define translational invariants, which provide a geometric characterization for translation subgroups of $\mathcal{E}^+$.

Definition 3.43 *The translational invariant $\mathcal{T}_T$ of a group $T \subseteq \mathbf{T}^3$ is defined to be the T-orbit of the origin s_0: $\mathcal{T}_T = T(s_0) = \{t(s_0) | t \in T\}$.*

Clearly, $\forall t \in T, t(\mathcal{T}_T) = t(T(s_0)) = T(s_0) = \mathcal{T}_T$.

Definition 3.44 *If $\mathcal{T}_T$ is a manifold then T is called a* **continuous translation subgroup**. *Otherwise, T is called a* **discrete translation subgroup**.

Besides manifolds, $\mathcal{T}_T$ can be a one, two or three dimensional lattice, a set of parallel lines, or a set of parallel planes in $\mathbf{R}^3$.

Proposition 3.45 *Let T_1, T_2 be translation subgroups of $\mathcal{E}^+$. If $\mathcal{T}_{T_1} = \mathcal{T}_{T_2}$ then $T_1 = T_2$.*

Proof :

Let $\mathcal{T}_{T_1} = \mathcal{T}_{T_2} = \mathcal{T}$. For all $t_1 \in T_1$ there exists $x \in \mathcal{T}$ such that $x = t_1(s_0) \in \mathcal{T}$ (Definition 3.43 and Definition 3.3). There also exists $t_2 \in T_2$ such that $t_2(s_0) = x = t_1(s_0)$. $t_1 = t_2 \in T_2$ (Proposition 3.28) , thus $T_1 \subseteq T_2$. Similarly, $T_2 \subseteq T_1$. Therefore we conclude $T_1 = T_2$. □

We can thus define a *bijection* $\beta_t : \{T\} \to \{T(s_0)\}$ which maps each translation subgroup to its unique translational invariant.

Proposition 3.46 *The translational invariant of the intersection of* T_1 *and* T_2 *is the intersection of their translational invariants, i.e.* $\mathcal{T}_{T_1 \cap T_2} = \mathcal{T}_{T_1} \cap \mathcal{T}_{T_2}$

Proof :

If $x \in \mathcal{T}_{T_1 \cap T_2} = (T_1 \cap T_2)(s_0)$ then there exists $t \in T_1 \cap T_2$ such that $x = t(s_0) \in T_1(s_0) \cap T_2(s_0) = \mathcal{T}_{T_1} \cap \mathcal{T}_{T_2}$. Therefore $\mathcal{T}_{T_1 \cap T_2} \subseteq \mathcal{T}_{T_1} \cap \mathcal{T}_{T_2}$.

If $x \in \mathcal{T}_{T_1} \cap \mathcal{T}_{T_2}$ then $x \in T_1(s_0) \cap T_2(s_0)$. Thus $x = t_1(s_0) = t_2(s_0)$ where $t_1 \in T_1, t_2 \in T_2$. By Proposition 3.28, $t_1 = t_2 \in T_1 \cap T_2$. So $x \in (T_1 \cap T_2)(s_0) = \mathcal{T}_{T_1 \cap T_2}$. Therefore $\mathcal{T}_{T_1} \cap \mathcal{T}_{T_2} \subseteq \mathcal{T}_{T_1 \cap T_2}$.

So we conclude $\mathcal{T}_{T_1 \cap T_2} = \mathcal{T}_{T_1} \cap \mathcal{T}_{T_2}$. □

3.4 Rotational Characteristic Invariants for Rotation Subgroups

Now let us define the rotational invariants, which provide a geometric characterization for rotation subgroups of $\mathcal{E}^+$.

Let $R = tR_c t^{-1}$ be a rotation subgroup of $\mathcal{E}^+$, where $t \in \mathbf{T}^3, R_c \subseteq SO(3)$.

Definition 3.47 *The* **pole-set** $\mathcal{P}_R$ *of* R *is defined to be a set of pairs* (p, n) *where* p *is a pole and* $n \in \mathcal{N} \cup \{\infty\}$, *i.e.*

$$\mathcal{P}_R = \{(p, n) | p \in S_0, n = |R_c^p| > 1\}$$

where $|R_c^p|$ *is the order of the stabilizer subgroup of* R_c *at* p.

R_c^p is a *line group* (Definition 3.18) since it fixes at least two distinct points p and s_0. Obviously, $\mathcal{P}_R = \mathcal{P}_{R_c}$. Hence all rotation subgroups R that are conjugate to each other via translations have the same pole-set $\mathcal{P}_R$, as is proved in the following:

Proposition 3.48 *Let* $R_1 = t_1 R_{c_1} t_1^{-1}, R_2 = t_2 R_{c_2} t_2^{-1}$. $\mathcal{P}_{R_1} = \mathcal{P}_{R_2} \Leftrightarrow \exists t \in \mathbf{T}^3, R_1 = tR_2 t^{-1}$.

Proof :

($\Rightarrow$) By Definition 3.47 $\mathcal{P}_{R_1} = \mathcal{P}_{R_{c_1}} = \mathcal{P}_{R_2} = \mathcal{P}_{R_{c_2}}$. For all $(p, n) \in \mathcal{P}_{R_{c_1}}, (p, n) \in \mathcal{P}_{R_{c_2}}$. That is to say that $R_{c_1}^p = R_{c_2}^p$ for all the poles p of R_{c_1}, R_{c_2}, since the stabilizer groups $R_{c_1}^p, R_{c_2}^p$ have the same axis of rotation and the same order n. Therefore $R_{c_1} = \cup R_{c_1}^p = \cup R_{c_2}^p = R_{c_2}$ (Lemma 3.38). Rewrite $R_{c_1} = R_{c_2}$ in terms of R_1, R_2 we have $t_1^{-1} R_1 t_1 = t_2^{-1} R_2 t_2 \Rightarrow R_1 = tR_2 t^{-1}$ where $t = t_1 t_2^{-1} \in \mathbf{T}^3$.

($\Leftarrow$) Since $R_1 = tR_2 t^{-1} = tt_2 R_{c_2} t_2^{-1} t^{-1}$, by Definition 3.47, $\mathcal{P}_{R_1} = \mathcal{P}_{R_{c_2}} = \mathcal{P}_{R_2}$. □

Proposition 3.49 *Let R_1, R_2 be rotation subgroups. $R_1 = tR_2t^{-1}$ for some $t \in \mathbf{T}^3 \Leftrightarrow \mathcal{P}_{R_1} = \mathcal{P}_{R_2}$ and $\mathcal{F}_{R_1} = t\mathcal{F}_{R_2}$.*

Proof:

($\Rightarrow$) Given some $t \in \mathbf{T}^3, R_1 = tR_2t^{-1}$, by Proposition 3.6 $\mathcal{F}_{R_1} = t\mathcal{F}_{R_2}$, and from Proposition 3.48 $\mathcal{P}_{R_1} = \mathcal{P}_{R_2}$.

($\Leftarrow$) Since $\mathcal{P}_{R_1} = \mathcal{P}_{R_2}$, there exists a translation t' such that $R_1 = t'R_2t'^{-1}$ (Proposition 3.48). From Proposition 3.6 we have $\mathcal{F}_{R_1} = t'(\mathcal{F}_{R_2})$. For each $r_1 \in R_1$ there exists $r_2 \in R_2$ such that $r_1 = t'r_2t'^{-1} \Rightarrow \bar{r}_1 \| \bar{r}_2$ and $\theta_{r_1} = \theta_{r_2}$ (Proposition 3.30).

If R_1, R_2 are line groups, then $\mathcal{F}_{R_1} \| \mathcal{F}_{R_2}$. For all $r_1 \in R_1, r_2 \in R_2, t(\bar{r}_2) = \bar{r}_1$. By Corollary 3.31 $r_1 = tr_2t^{-1}$. Thus $R_1 \subseteq tR_2t^{-1}$. Similarly, $tR_2t^{-1} \subseteq R_1$. Therefore $R_1 = tR_2t^{-1}$.

If R_1, R_2 are point groups then $t = t'$ (Proposition 3.28). Therefore $R_1 = tR_2t^{-1}$. □

We see from this Proposition that both $\mathcal{P}_R$ and $\mathcal{F}_R$ are needed to characterize a rotation group. $\mathcal{P}_R$ tells the different orders of the rotations in R and $\mathcal{F}_R$ tells where in $\mathbf{R}^3$ the rotation axes reside, so different rotation groups that are conjugate via a translation can be distinguished. We thus have the following definition:

Definition 3.50 *The* **rotational characteristic invariant of a rotation group** *R is the pair $(\mathcal{F}_R, \mathcal{P}_R)$, where $\mathcal{F}_R$ is the fixed point set of R (Definition 3.5) and $\mathcal{P}_R$ is the pole-set of R (Definition 3.47).*

Examples of rotational characteristic invariants of $SO(3)$, of the rotational symmetry group of a finite cylinder R_{cyl}, and of the symmetry group of a square-based pyramid C_4 are shown in Figure 8. For $SO(3)$, $\mathcal{P}_{SO(3)}$ is the whole of S_0. Each point on S_0 is a pole whose stabilizer group has an infinite order. The pole-set of R_{cyl} consists of the "north pole", the "south pole" and the "equator" of S_0. For C_4, only two points of S_0 are poles. They come from the intersection of the rotation axis, which happens to be $\mathcal{F}_{C_4}$, and S_0. All the possible rotations of a rotation group can be obtained from this representation by re-centering the unit sphere S_0, which carries the poles of R, at any point of $\mathcal{F}_R$ (Proposition 3.6 and 3.7).

Since the stabilizer group R^p at a pole p is a line group, when $|R^p| = n$, there are $n-1$ nontrivial rotations about the axis determined by the pair of antipodal poles $(p, -p)$.

We now prove that $\mathcal{P}_R$ and $\mathcal{F}_R$ characterize the rotation group R completely.

Proposition 3.51 *Let R_1, R_2 be rotation groups. $\mathcal{F}_{R_1} = \mathcal{F}_{R_2}$ and $\mathcal{P}_{R_1} = \mathcal{P}_{R_2} \Rightarrow R_1 = R_2$.*

Proof:

This follows from Proposition 3.49. □

This proposition proves that there exists a bijection β_r which maps the set of rotation groups $\{R\}$ to the set of rotational invariants $\{(\mathcal{F}_R, \mathcal{P}_R)\}$, i.e. $\beta_r : \{R\} \rightarrow \{(\mathcal{F}_R, \mathcal{P}_R)\}$.

Given two rotation groups R_1 and R_2, we are looking for a computational method to construct both the pole-set of $R_1 \cap R_2$ i.e. to obtain $\mathcal{P}_{R_1 \cap R_2}$ from $\mathcal{P}_{R_1}$ and $\mathcal{P}_{R_2}$; and the fixed point set of $R_1 \cap R_2$. In the following Proposition we prove that the pole-set of the intersection of two rotation groups can be constructed by intersecting the pole-sets of each rotation group

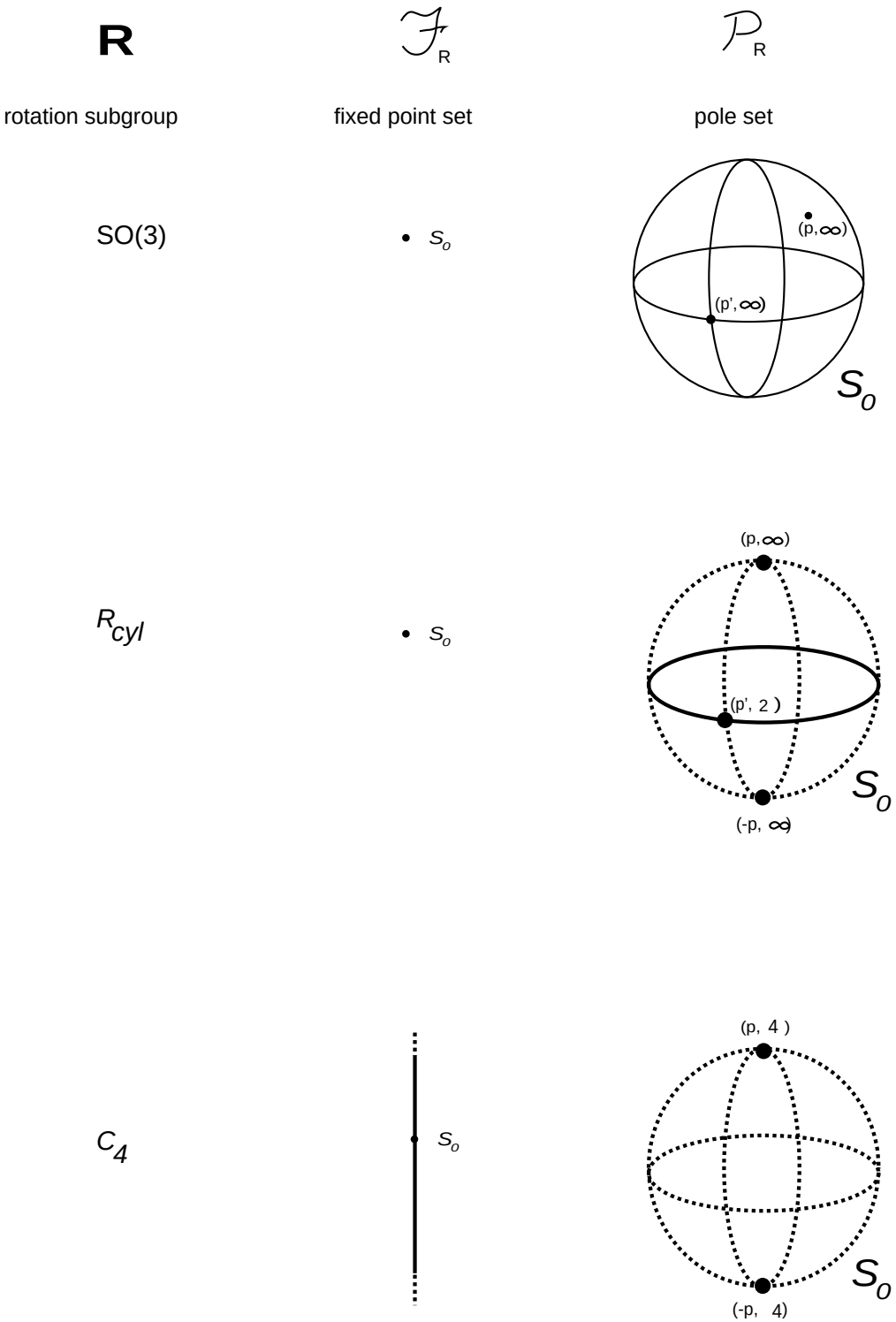

Figure 8: Examples for the fixed point set $\mathcal{F}_R$ and pole-set $\mathcal{P}_R$ of some rotation subgroups R.

where $\mathcal{P}_{R_i}$ *is the pole-set of group* R_i, $i = 1$ *or* 2;
$R_i^x \subseteq gSO(2)g^{-1}$ *is the* stabilizer *subgroup (a line group) of* R_i *at* x;
$\mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2} = \{(p,n) | \gcd^*(|R^p_{c_1}|, |R^p_{c_2}|) = n > 1\}$;
$\mathcal{P}_{R_1} \cap^*_{\mathcal{L}_0} \mathcal{P}_{R_2} = \{(p,n) | (p,n) \in \mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2}, p \in \mathcal{L}_0\}$;
$\mathcal{L}$ *is the line going through both* $\mathcal{F}_{R_1}$ *and* $\mathcal{F}_{R_2}$, *two distinct points;*
$\mathcal{L}_0 \| \mathcal{L}$ *and* $s_0 \in \mathcal{L}_0$;
$\mathcal{L}_{p_1,p_2}$ *is a line formed by a pair of antipodal poles* p_1 *and* p_2;
$t_{\mathcal{F}}$ *is a translation such that* $t_{\mathcal{F}}(s_0) \in \mathcal{F}$.

in a particular way. The reason why a standard set intersection of two pole-sets is inadequate can be illustrated by a counter example: the pole-sets of the cyclic groups C_6 and C_9 are: $\{((0,0,1),6),(0,0,-1),6)\}$, and $\{((0,0,1),9),((0,0,-1),9)\}$, respectively. An intersection of the two pole-sets will result in an empty set. The correct pole-set that characterizes this group intersection is $\{((0,0,1),3),((0,0,-1),3)\}$. This is reflected in $\gcd^*(|C_6|,|C_9|) = \gcd(6,9) = 3$ (For definition of $\gcd^*$ see Proposition 3.41), so the intersection should have the original poles but associated with order 3. For the cyclic groups C_3 and C_4 their pole-set are: $\{((0,0,1),3),((0,0,-1),3)\}, \{((0,0,1),4),((0,0,-1),4)\}$. Since C_3 and C_4 as rotation groups do not have any common element other than the identity, the resulted pole-set of their intersection should be the empty set. This is determined by the fact that $\gcd^*(|C_3|,|C_4|) = \gcd(3,4) = 1$ which corresponds to the identity group (Corollary 3.42). Therefore we define $\cap^*$ as the following intersection operation on two pole-sets:

$\mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2} = \{(p,n) | \gcd^*(|R^p_{c_1}|, |R^p_{c_2}|) = n > 1\}$

Note, $\gcd^*(|R^p_{c_1}|, |R^p_{c_2}|) = n > 1$ implies $(p, |R^p_{c_1}|) \in \mathcal{P}_{R_1}$ and $(p, |R^p_{c_2}|) \in \mathcal{P}_{R_2}$.

Proposition 3.52 *Let* $R = R_1 \cap R_2$ *and* $\mathcal{F} = \mathcal{F}_{R_1} \cap \mathcal{F}_{R_2}$ *Then*

$$\mathcal{P}_R = \begin{cases} \mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2} & \text{if } \mathcal{F} \neq \emptyset \\ \mathcal{P}_{R_1} \cap^*_{\mathcal{L}_0} \mathcal{P}_{R_2} & \text{if } \mathcal{F} = \emptyset, \text{ and } R_1, R_2 \text{ are both point groups.} \\ \emptyset & \text{Otherwise} \end{cases}$$

and

$$\mathcal{F}_R = \begin{cases} \mathbf{R}^3 & \text{if } |\mathcal{P}_R| = 0 \\ t_{\mathcal{F}}(\mathcal{L}_{p_1,p_2}) & \text{if } \mathcal{F} \neq \emptyset \text{ and } |\mathcal{P}_R| = 2 \\ \mathcal{F} & \text{if } \mathcal{F} \neq \emptyset \text{ and } |\mathcal{P}_R| > 2 \\ \mathcal{L} & \text{if } \mathcal{F} = \emptyset \text{ and } |\mathcal{P}_R| = 2 \end{cases}$$

Proof :
Case 1. Assume $\mathcal{F} \neq \emptyset$.

For all nontrivial rotations $r \in R = R_1 \cap R_2$, all $x \in \mathcal{F} = \mathcal{F}_{R_1} \cap \mathcal{F}_{R_2}$, $r(x) = x$ (Definition 3.5). Thus $\mathcal{F} \subseteq \mathcal{F}_R$.

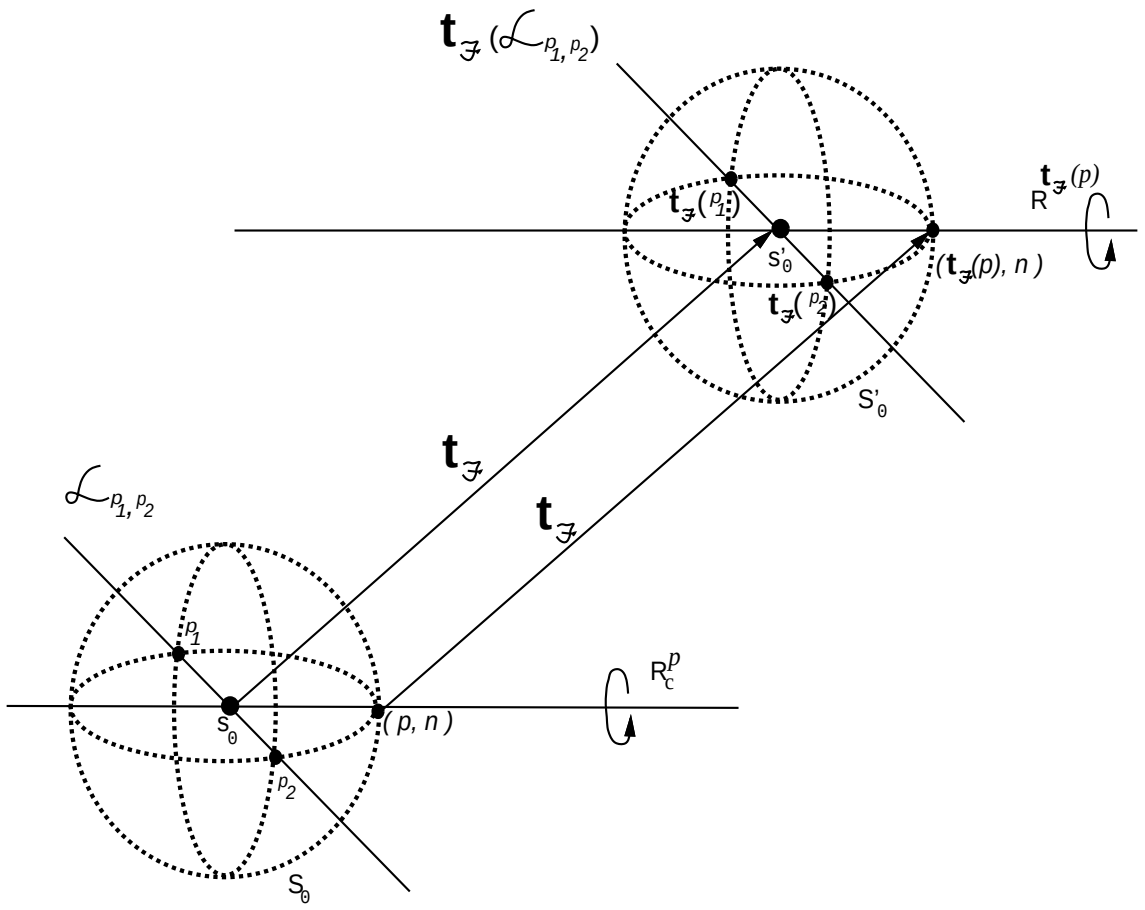

Figure 9: When $\mathcal{F} \neq \emptyset$, $R = R_1 \cap R_2 = t_{\mathcal{F}} R_{c_1} t_{\mathcal{F}}^{-1} \cap t_{\mathcal{F}} R_{c_2} t_{\mathcal{F}}^{-1} = t_{\mathcal{F}}(R_{c_1} \cap R_{c_2}) t_{\mathcal{F}}^{-1}$.

Now let us choose a point $s'_0 \in \mathcal{F}$ and some $t_{\mathcal{F}} \in T^3$ such that $s'_0 = t_{\mathcal{F}}(s_0)$. Then we have $R_1 = t_{\mathcal{F}} R_{c_1} t_{\mathcal{F}}^{-1}$ and $R_2 = t_{\mathcal{F}} R_{c_2} t_{\mathcal{F}}^{-1}$ (Proposition 3.49). Thus $R = R_1 \cap R_2 = t_{\mathcal{F}} R_{c_1} t_{\mathcal{F}}^{-1} \cap t_{\mathcal{F}} R_{c_2} t_{\mathcal{F}}^{-1} = t_{\mathcal{F}}(R_{c_1} \cap R_{c_2}) t_{\mathcal{F}}^{-1} = t_{\mathcal{F}} R_c t_{\mathcal{F}}^{-1}$. This means that it is equivalent to discuss the intersection of pole-sets on S_0 or, alternatively, on S'_0 which is the unit sphere S_0 being translated and re-centered at s'_0 (Figure 9).

(a). To find $\mathcal{P}_R$

If $(p, n) \in \mathcal{P}_R$ then $|R_c^p| = n > 1$. From Proposition 3.7 and Proposition 3.8, $|R_c^p| = |R^{t_{\mathcal{F}}(p)}| = |R_1^{t_{\mathcal{F}}(p)} \cap R_2^{t_{\mathcal{F}}(p)}| = \gcd^*(|R_1^{t_{\mathcal{F}}(p)}|, |R_2^{t_{\mathcal{F}}(p)}|) = n > 1$ (Proposition 3.41). Using Proposition 3.7 once more, $\gcd^*(|R_{c_1}^p|, |R_{c_2}^p|) = n > 1$. Then $(p, n) \in \mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2}$ (Definition of $\mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2}$). Therefore $\mathcal{P}_R \subseteq \mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2}$.

If $(p, n) \in \mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2}$, then simply following the above arguments in reverse order one reaches the conclusion $(p, n) \in \mathcal{P}_R$. Thus $\mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2} \subseteq \mathcal{P}_R$.

This completes the proof that $\mathcal{P}_R = \mathcal{P}_{R_1} \cap^* \mathcal{P}_{R_2}$.

(b). To find $\mathcal{F}_R$

If $|\mathcal{P}_R| = 0$ then R is the identity group (Proposition 3.40). By Definition 3.5, $\mathcal{F}_R = \mathbf{R}^3$.

This is true for both $\mathcal{F} = \emptyset$ and $\mathcal{F} \neq \emptyset$.

If $|\mathcal{P}_R| = 2$ then R is a line group (Proposition 3.40). Thus $\mathcal{P}_R = \{(p_1, n), (p_2, n)\}$. $\mathcal{F}_{R_c}$ is a line containing p_1, p_2, and s_0. That is $\mathcal{F}_{R_c} = \mathcal{L}_{p_1,p_2}$. By Proposition 3.6 $\mathcal{F} = t_{\mathcal{F}}(\mathcal{F}_{R_c}) = t_{\mathcal{F}}(\mathcal{L}_{p_1,p_2})$ (Figure 9).

If $|\mathcal{P}_R| > 2$ then R is a point group (Proposition 3.40). Since R is a point group, $\mathcal{F}_R$ is a single point set. Then $\mathcal{F} \neq \emptyset$ and $\mathcal{F} \subseteq \mathcal{F}_R \Rightarrow \mathcal{F} = \mathcal{F}_R$.

Case 2. Assume $\mathcal{F} = \emptyset$.

(a). If one of R_1, R_2 is a line group, say R_1:

For all nontrivial rotations $r_1 \in R_1$, and all points $x \in \mathcal{F}_{R_2}, r_1(x) \neq x$. This is because a rotation only leaves points on its axis of rotation pointwise fixed. By the definition of the fixed point set (Definition 3.5), no such r_1 is a member of R_2. Therefore $R_1 \cap R_2 = \{1\} \Rightarrow \mathcal{P}_{R_1 \cap R_2} = \emptyset$. (We just completed the *otherwise* part for $\mathcal{P}_R$).

(b). If R_1, R_2 are both point groups:

For all nontrivial rotations r in R, r fixes both point $\mathcal{F}_{R_1}$ and $\mathcal{F}_{R_2}$. Then $r(\mathcal{L}) = \mathcal{L}$ where $\mathcal{L}$ is the line determined by the two distinct points $\mathcal{F}_{R_1}$ and $\mathcal{F}_{R_2}$ (Figure 10).

Therefore R will either be a line group or the identity group $\{1\}$. This restricts $\mathcal{P}_R$ to be those poles located on $\mathcal{L}$ only, i.e. the poles resulted from the intersection of $\mathcal{L}$ and S_0'.

For all $(p, n) \in \mathcal{P}_{R_c}$, $|R_c^p| = n > 1$. Then $|R_c^p| = |R^{t(p)}| = \gcd^*(|R_1^{t(p)}|, |R_2^{t(p)}|) = n > 1$. Because $R_1^{t(p)}$ fixes $\mathcal{L}$, $R_1^{t(p)} = R_1^{tt'(p)} = R_1^{t_1(p)}$ and $R_2^{t(p)} = R_2^{tt''(p)} = R_2^{t_2(p)}$ (Figure 10). Then $\gcd^*(|R_1^{t_1(p)}|, |R_2^{t_2(p)}|) = \gcd^*(|R_{c_1}^p|, |R_{c_2}^p|) = n > 1$.

Since $s_0 \in \mathcal{F}_{R_c} = t^{-1}(\mathcal{F}_R) = t^{-1}(\mathcal{L}) = \mathcal{L}_0$ and $p \in \mathcal{F}_{R_c} = \mathcal{L}_0$, $p \in \mathcal{L}_0$. Thus $(p, n) \in \mathcal{P}_{R_1} \cap^*_{\mathcal{L}_0} \mathcal{P}_{R_2}$. Therefore $\mathcal{P}_R \subseteq (\mathcal{P}_{R_1} \cap^*_{\mathcal{L}_0} \mathcal{P}_{R_2})$.

For all $(p, n) \in \mathcal{P}_{R_1} \cap^*_{\mathcal{L}_0} \mathcal{P}_{R_2}$, following the above argument in reverse order one finds $(p, n) \in \mathcal{P}_R$. Thus $(\mathcal{P}_{R_1} \cap^*_{\mathcal{L}_0} \mathcal{P}_{R_2}) \subseteq \mathcal{P}_R$.

Therefore $\mathcal{P}_R = \mathcal{P}_{R_1} \cap^*_{\mathcal{L}_0} \mathcal{P}_{R_2}$ as claimed. □

This is a very important proposition which demonstrates the interplay between algebra and geometry. Compared to translations, rotations vary with respect to position in Euclidean space. This proposition shows that the varieties of rotations can be captured by simple and static elements, $\mathcal{P}$ and $\mathcal{F}$, such that the intersection computation of any rotation groups in $\mathbf{R}^3$ can be performed at the origin s_0. This is the key leading to a constructive algorithm.

3.5 TR Subgroups of $\mathcal{E}^+$

Now comes to the point that translational invariants and rotational invariants are combined to provide the geometric characterization for **TR** subgroups of $\mathcal{E}^+$.

Definition 3.53 *A subgroup G of $\mathcal{E}^+$ is a* **TR** *group iff $G = TR = \{tr | t \in T, r \in R\}$, where T is a translation subgroup and R is a rotation subgroup of $\mathcal{E}^+$.*

In the following, let $G = TR$ be a **TR** group.

Lemma 3.54 $G \cap \mathbf{T}^3 = T$.

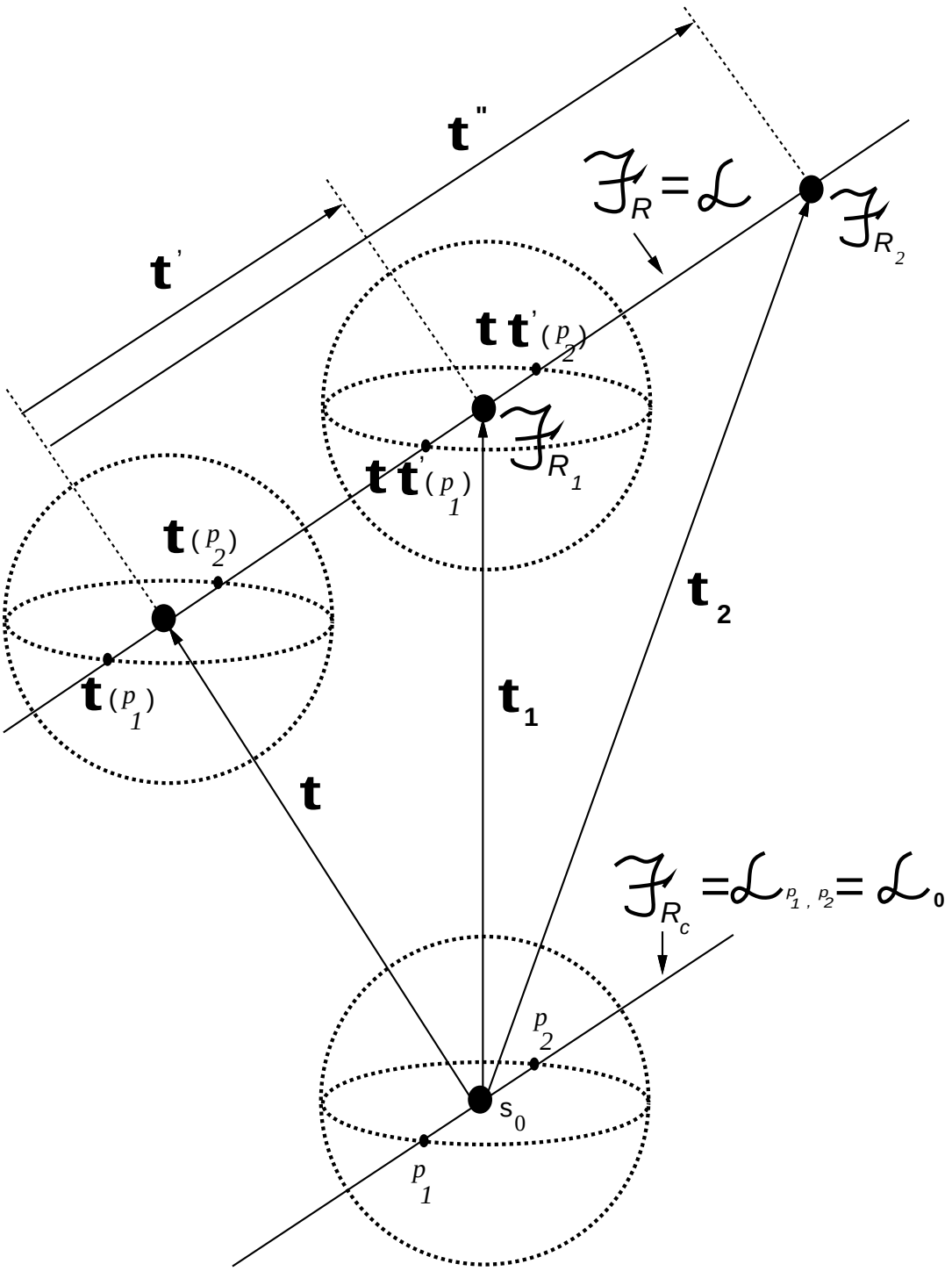

Figure 10: $R = R_1 \cap R_2$ is a line group. All the rotations in R fix both points $\mathcal{F}_{R_1}$ and $\mathcal{F}_{R_2}$. Thus they fix the line $\mathcal{L}$. So $\mathcal{F}_R = \mathcal{L}$.

Proof :

If $g \in G \cap \mathbf{T}^3$ then $g = tr$ must be a translation, with $t \in T, r \in R$. Then $t^{-1}g = r$ should be a translation also. However, for $r \in R$ to be both a rotation and a translation, there is only one possibility i.e. $r = 1$. Therefore $g = t \in T$. So $G \cap \mathbf{T}^3 \subseteq T$.

If $g \in T$ then g is a translation and $g \in G$, i.e. $g \in G \cap \mathbf{T}^3$. So $T \subseteq G \cap \mathbf{T}^3$.

Therefore $G \cap \mathbf{T}^3 = T$. □

Lemma 3.55 *T is a normal subgroup of G, i.e. $T \lhd G$.*

Proof :

For all $r \in R$ and all $t \in T, rtr^{-1} \in T$, because rtr^{-1} is a translation (Proposition 3.24) and it is in G (Lemma 3.54). Then $rTr^{-1} \subseteq T$. For all $g = tr \in G$, since T^3 is an abelian group, $gTg^{-1} = trTr^{-1}t^{-1} = rTr^{-1} \subseteq T$. By the Definition of normal subgroup of a group, T is a normal subgroup of G. □

Proposition 3.56 *G is the semidirect product $T \times_\theta R$ where $\theta(r) : t \mapsto rtr^{-1}$.*

Proof :

Since T is a normal subgroup of G, $T \cap R = \{1\}$ and $G = TR$, G is a semidirect product of T and R ([19] p. 415). □

Corollary 3.57 *For all $t \in T, TR = TtRt^{-1}$.*

Proof :

This proof uses the previous results namely $T \lhd G$ and $G = T \times_\theta R$.

For all $r \in R$ and for all $t \in T$ there exists $t' \in T$, such that $trt^{-1} = tt'r = t'tr \in TR$, i.e. $tr = t'^{-1}trt^{-1} \in TtRt^{-1}$. Therefore $\forall t \in T, TR \subseteq TtRt^{-1}$. Now for all $t', t \in T$ there exists $t'' \in T$ such that $t'trt^{-1} = t'tt''r \in TR$. Therefore $TtRt^{-1} \subseteq TR$. So we have $TtRt^{-1} = TR$. □

Proposition 3.58 *If $G = T_1R_1 = T_2R_2$ then $T_1 = T_2 = T$ and $R_1 = tR_2t^{-1}$ for some $t \in \mathbf{T}^3$.*

Proof :

By Lemma 3.54, $G \cap \mathbf{T}^3 = T_1 = T_2 = T$.

From Proposition 3.34 we have $R_1 = t_1R_{c_1}t_1^{-1}, R_2 = t_2R_{c_2}t_2{}^{-1}$ where R_{c_1}, R_{c_2} are subgroups of $SO(3)$, $t_1, t_2 \in \mathbf{T}^3$. Since $TR_1 = TR_2 \Rightarrow Tt_1R_{c_1}t_1{}^{-1} = Tt_2R_{c_2}t_2{}^{-1}$. For all $r_{c_1} \in R_{c_1}$ there exist $r_{c_2} \in R_{c_2}, t, t' \in T$ such that $tt_1r_{c_1}t_1^{-1} = t't_2r_{c_2}t_2^{-1} \Rightarrow tt_1r_{c_1}t_1^{-1}r_{c_1}^{-1}r_{c_1} = t't_2r_{c_2}t_2^{-1}r_{c_2}^{-1}r_{c_2}$. Since both $r_{c_1}t_1r_{c_1}^{-1}, r_{c_2}t_2r_{c_2}^{-1} \in \mathbf{T}^3$ (Proposition 3.24), by Proposition 3.23 we can equate the rotational elements of both sides $r_{c_1} = r_{c_2} \in R_{c_2}$. Thus $R_{c_1} \subseteq R_{c_2}$. Similarly, $R_{c_2} \subseteq R_{c_1}$. Therefore $R_{c_1} = R_{c_2} \Rightarrow \mathcal{P}_{R_{c_1}} = \mathcal{P}_{R_{c_2}}$ (Proposition 3.49). By Proposition 3.48 we have $\mathcal{P}_{R_1} = \mathcal{P}_{R_2}$ and $\exists t \in \mathbf{T}^3$ such that $R_1 = tR_2t^{-1}$. □

This proposition states that the denotation of a **TR** group $G = TR$ is unique up to the conjugation of its rotation subgroup R by a translation t. The question is whether this t has to be in T?

In the next few sections we shall see that if T is a continuous translation subgroup then $t \in T$, otherwise t is not necessarily in T but in a form derived from the generator(s) of T.

3.5.1 Translational Characteristic Invariant for TR subgroups

Since T is unique in G (Proposition 3.58), one can describe all the translations in G by characterizing T only. Therefore we have the following definition:

Definition 3.59 *If $G = TR$ is a* **TR** *group then the* **translational invariant** $\mathcal{T}_G$ *of G is defined to be the T-orbit of the origin s_0, i.e.*

$$\mathcal{T}_G = T(s_0) = \{t(s_0) | t \in T\}$$

Given a pair of T and R, whether they form a **TR** group? The following proposition, also called the **TR−restriction**, describes how rotations of G act on its translational invariant.

Proposition 3.60 *Let $G = TR$ and $R = tR_ct^{-1}$ where $t \in \mathbf{T}^3, R_c \subseteq SO(3)$. $G = TR$ is a group $\Leftrightarrow^{(1)} \forall r \in R, r(t(\mathcal{T}_G)) = t(\mathcal{T}_G) \Leftrightarrow^{(2)} \forall r_c \in R_c, r_c(\mathcal{T}_G) = \mathcal{T}_G$.*

Proof :

For each $r \in R$ there exists $r_c \in R_c$ such that $r = tr_ct^{-1}$. Then $r(t(s_0)) = tr_ct^{-1}(t(s_0)) = tr_c(s_0) = t(s_0)$.

($\Rightarrow^{(1)}$) If G is a **TR** group then for all $r \in R$,

$$\begin{aligned}
rt(\mathcal{T}_G) &= r(t(T(s_0)) && \textit{Definition 3.59} \\
&= rT(t(s_0)) && \textit{commutativity of translations} \\
&= rTr^{-1}(t(s_0)) && \textit{since } t(s_0) = r(t(s_0)) \\
&= Tt(s_0) && \textit{Since } T \lhd G \\
&= t(T(s_0)) && \textit{commutativity of translations} \\
&= t(\mathcal{T}_G) && \textit{Definition 3.59.}
\end{aligned}$$

($\Rightarrow^{(2)}$) If $\forall r \in R, r(t(\mathcal{T}_G)) = t(\mathcal{T}_G)$ then $\exists r_c \in R_c$ such that $r(t\mathcal{T}_G) = tr_ct^{-1}(t\mathcal{T}_G) = tr_c(\mathcal{T}_G) = t(\mathcal{T}_G) \Rightarrow r_c(\mathcal{T}_G) = \mathcal{T}_G$.

($\Leftarrow^{(1)}$) Since for all $r \in R, r(t(\mathcal{T}_G)) = t(\mathcal{T}_G)$ then $rt(T(s_0)) = tT(s_0) \Rightarrow T(t(s_0)) = rT(t(s_0)) = rTr^{-1}(t(s_0))$. For all $t_r \in rTr^{-1}, \exists t' \in T$ such that $t_r(t(s_0)) = t'(t(s_0))$. By Proposition 3.28 $t_r = t' \Rightarrow rTr^{-1} \subseteq T$. Similarly, we can prove $T \subseteq rTr^{-1}$. Therefore $rTr^{-1} = T$. It can be easily verified that TR forms a group, and T is its normal subgroup. By Proposition 3.56 $G = TR$ is a **TR** group.

($\Leftarrow^{(2)}$) If $\forall r_c \in R_c, r_c(\mathcal{T}_G) = \mathcal{T}_G$ then $r_c(\mathcal{T}_G) = t^{-1}rt(\mathcal{T}_G) = \mathcal{T}_G \Rightarrow r(\mathcal{T})) = t(\mathcal{T}_G)$. □

Corollary 3.61 *If $G = TR$ is a* **TR** *group and $R = tR_ct^{-1}$, $R_c \subseteq SO(3), t \in \mathbf{T}^3$ then $\mathcal{F}_R \cap t(\mathcal{T}_G) \neq \emptyset$.*

Proof :

From the proof of Lemma 3.60 we have $t(s_0) \in \mathcal{F}_R$, since $s_0 \in \mathcal{T}_G \Rightarrow t(s_0) \in t(\mathcal{T}_G) \Rightarrow \mathcal{F}_R \cap t(\mathcal{T}_G) \neq \emptyset$. □

The choices for $\mathcal{T}_G$ can be enumerated: when T is continuous, $\mathcal{T}_G$ include $\mathbf{R}3$, planes, lines, and s_0; when T is non-continuous, $\mathcal{T}_G$ can be one, two and three dimension lattices[7], parallel lines, and parallel planes.

When $\mathcal{T}_G$ is a one or two dimensional discrete grid (lattice), saying a rotation r is parallel or perpendicular to such a $\mathcal{T}_G$ means that $\bar{r}$ is parallel or perpendicular to the minimal manifold containing $\mathcal{T}_G$.

Corollary 3.62 *If $G = TR$ is a* **TR** *group, T is continuous, $R = t_c R_c t_c^{-1}$, and $g \in G$ is a nontrivial rotation, then $\bar{g} \cap t_c(\mathcal{T}_G) \neq \emptyset$.*

Proof:

Since $g = t_g r$ for some $t_g \in T, r \in R$ and g is a rotation in G, then

1) $\bar{g} \| \bar{r}$ and $\theta_g = \theta_r$ (Corollary 3.31).

2) By the **TR**−restriction $r(t_c(\mathcal{T}_G)) = t_c(\mathcal{T}_G)$. Since $t_g \in T$, $g(t_c(\mathcal{T}_G)) = t_g r(t_c(\mathcal{T}_G)) = t_g t_c(\mathcal{T}_G) = t_c t_g(\mathcal{T}_G) = t_c(\mathcal{T}_G)$. Since T is continuous, $\mathcal{T}_G$ is $\mathbf{R}^3$, a plane, a line or a point. In every case if $\bar{g} \cap t_c(\mathcal{T}_G) = \emptyset$ then g is a trivial rotation. Therefore $\bar{g} \cap t_c(\mathcal{T}_G) \neq \emptyset$. □

Proposition 3.63 *If $G = TR$ is a* **TR** *group, T is continuous, and $g \in G$ is a nontrivial rotation, then $g \in tRt^{-1}$, for some $t \in T$.*

Proof:

If $\mathcal{T}_G$ is $\mathbf{R}^3$ or s_0 then $g \in tRt^{-1}$. Let us look into each case of nontrivial $\mathcal{T}_G$ and r:

- $\mathcal{T}_G$ is a line and $\bar{r} \| \mathcal{T}_G$. Since $\bar{g} \| \bar{r}, \bar{g} \| \mathcal{T}_G$. Since $g(t_c \mathcal{T}_G) = t_c(\mathcal{T}_G), \bar{g} = t_c(\mathcal{T}_G)$. Therefore $\bar{r} = \bar{g} \Rightarrow r = g \in R$.

- $\mathcal{T}_G$ is a line or a plane and $\bar{r} \perp \mathcal{T}_G$. Then $\bar{g} \perp \mathcal{T}_G$. Since $\bar{g} \cap t_c(\mathcal{T}_G) \neq \emptyset$. and T acts on $t_c(\mathcal{T}_G)$ transitively, there always exists $t \in T$ such that $\bar{g} = t(\bar{r})$. By Corollary 3.31 $g = trt^{-1} \in tRt^{-1}$.

- $\mathcal{T}_G$ is a plane and $\bar{r} \| \mathcal{T}_G$. So $\bar{g} \| \mathcal{T}_G$. Since $g(t_c \mathcal{T}_G) = t_c(\mathcal{T}_G)$ and $\mathcal{T}_G$ is continuous, it must be true that $\bar{g} \subset t_c(\mathcal{T}_G)$, and also $\bar{r} \subset t_c(\mathcal{T}_G)$. Since T is continuous and T acts on $t_c(\mathcal{T}_G)$ transitively, there exists $t \in T$ such that $\bar{g} = t(\bar{r})$. By Corollary 3.31 $g = trt^{-1} \in tRt^{-1}$. □

This proposition proves that when T is continuous any rotation in a **TR** group G must belong to some rotation group R' where $G = TR'$. In another words, the set $\{trt^{-1} | t \in T, r \in R\}$ contains *all* the possible rotations in G.

When T is a discrete translation group, there are cases where g is a rotation in G, but $g \notin R_i$ for any rotation subgroup R_i such that $G = TR_i$. Therefore it is not sufficient to

[7] Although the three dimensional discrete translation group satisfies all the previous definitions and properties for **TR** groups, from now on we exclude it in our proofs since it is a special case for our purposes.

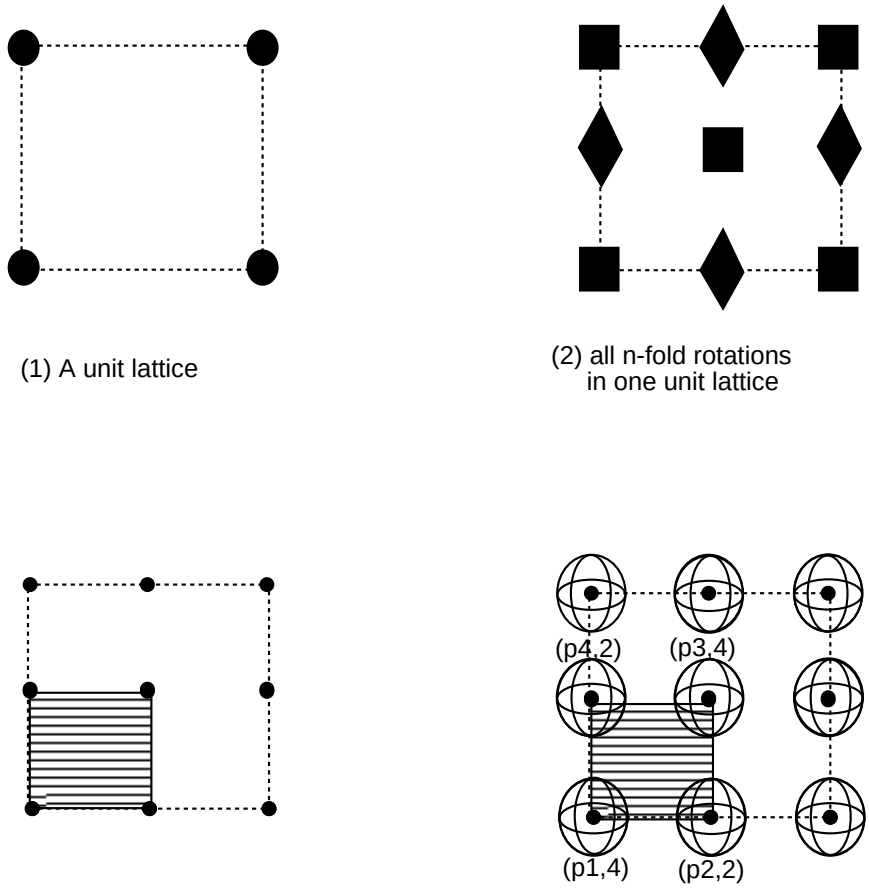

Figure 11: The Crystallographic group $p4$ characterized by $\mathcal{F}_{\bar{R}}$ and $\mathcal{P}_{\bar{R}}$.

characterize R_i only. For example, the Crystallographic group p_4, the symmetry group of a two-dimensional lattice generated by a square lattice unit (Figure 11, where a diamond shaped symbol denotes a 2-fold rotation symmetry, a square denotes a 4-fold rotation), the two-fold rotations about the mid-point of each edge of a unit lattice are in none of the rotation subgroups R_i of $G = TR_i$, rather, they are resulted from the products of distinct R_is.

Proposition 3.64 *Given $G = TR_1 = TR_2$, if T is a continuous translation group then there exists $t \in T$ such that $R_1 = tR_2t^{-1}$.*

Proof :

If $T \cong \mathbf{T}^3$ then any translation t such that $R_1 = tR_2t^{-1}$, $t \in T$. If $T = \{1\}$ then $R_1 = R_2$. Now we consider $T \cong \mathbf{T}^1$ and $T \cong \mathbf{T}^2$ only.

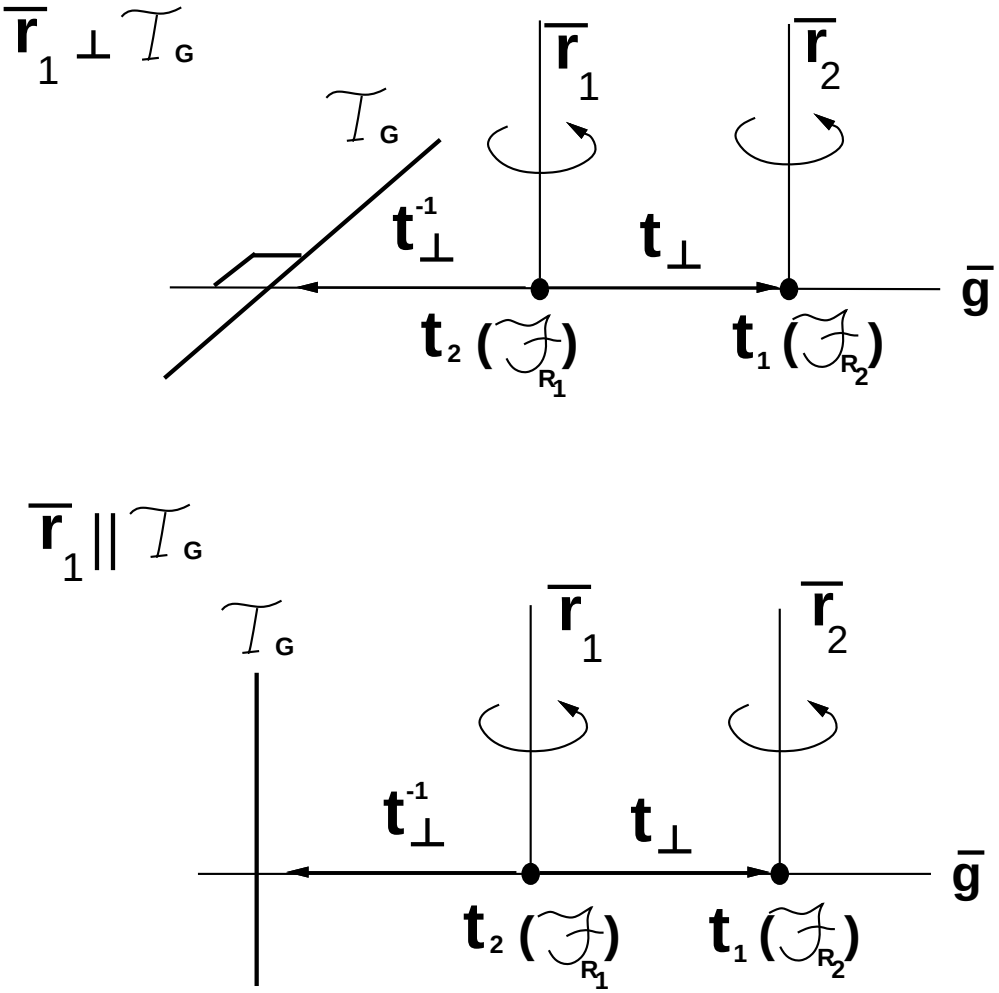

Figure 12: When R_1, R_2 are both point groups and $\bar{g} \perp \mathcal{T}_G$.

From Proposition 3.63, we know that if $g \in G$ is a nontrivial rotation then $g \in t_1 R_1 t_1^{-1}$ and $g \in t_2 R_2 t_2^{-1}$ where $t_1, t_2 \in T$. Since by Proposition 3.58 there exists $t \in \mathbf{T}^3$ such that $R_1 = t R_2 t^{-1}$, $\mathcal{F}_{R_1}, \mathcal{F}_{R_2}$ are either both lines or both points of $\mathbf{R}^3$.

If R_1, R_2 are both line groups then $\bar{g} = \mathcal{F}_{t_1 R_1 t_1^{-1}} = \mathcal{F}_{t_2 R_2 t_2^{-1}} = t_1(\mathcal{F}_{R_1}) = t_2(\mathcal{F}_{R_2})$. Thus $\mathcal{F}_{R_1} = t_1^{-1} t_2(\mathcal{F}_{R_2})$ and so $R_1 = t_1^{-1} t_2 R_2 t_2^{-1} t_1 = t R_2 t^{-1}$ (Proposition 3.51).

If R_1, R_2 are point groups then $t_1(\mathcal{F}_{R_1}) \in \bar{g}$ and $t_2(\mathcal{F}_{R_2}) \in \bar{g}$. By the **TR** -restriction there are only two choices for $\bar{g}$, i.e. $\bar{g} \| \mathcal{T}_G$ or $\bar{g} \perp \mathcal{T}_G$:

When $\bar{g} \| \mathcal{T}_G$, since T is continuous there exists $t' \in T$ such that $\mathcal{F}_{t_1 R_1 t_1^{-1}} = t'(\mathcal{F}_{t_2 R_2 t_2^{-1}})$. Therefore $t_1(\mathcal{F}_{R_1}) = t' t_2(\mathcal{F}_{R_2}) \Rightarrow R_1 = t R_2 t^{-1}, t = t_1^{-1} t' t_2 \in T$ (Proposition 3.51).

When $\bar{g} \perp \mathcal{T}_G$, assuming there exists $t_\perp \perp \mathcal{T}_G$ and $t_\perp \neq 1$ such that $t_1(\mathcal{F}_{R_1}) = t_\perp t_2(\mathcal{F}_{R_2})$ (Figure 12). Obviously $t_\perp \notin T$. Let $R_1' = t_1 R_1 t_1^{-1}, R_2' = t_2 R_2 t_2^{-1}$. For each nontrivial rotation $r_1 \in R_1'$ there exists $r_2 \in R_2'$ such that $r_1 = t_\perp r_2 t_\perp^{-1} \Rightarrow r_1 r_2^{-1} = t_\perp (r_2 t_\perp^{-1} r_2^{-1})$. Thus

both sides are tranlations in T (Lemma 3.54). If $\mathcal{T}_G$ is a line and $r_1 \perp \mathcal{T}_G$ (since R_1, R_2 are point groups), or $\mathcal{T}_G$ is a plane where r_1 has to be $r_1 \| \mathcal{T}_G$, then $\theta_{r_1} = \theta_{R_2}$ has to be π (TR-restriction). Then $t_\perp(r_2 t_\perp^{-1} r_2^{-1}) = 2t_\perp \notin T$, a contradiction. If $\mathcal{T}_G$ is a line and $r_1 \| \mathcal{T}_G$ then $t_\perp(r_2 t_\perp^{-1} r_2^{-1}) \perp T$ and thus $\notin T$ unless $\theta_{r_1} = \theta_{r_2} = 0$, a contradiction. Therefore, $r_\perp = 1$ and so $t_1(\mathcal{F}_{R_1}) = t_2(\mathcal{F}_{R_2}) \Rightarrow \mathcal{F}_{R_1} = t_1^{-1} t_2(\mathcal{F}_{R_2})$. By Proposition 3.51 $R_1 = t_1^{-1} t_2 R_2 t_2^{-1} t_1 = t R_2 t^{-1}$ where $t = t_1^{-1} t_2 \in T$. □

3.5.2 Crystallographic Groups

The observation in the previous section on discrete **TR** groups calls for a more elaborate characterization for the rotations in Crystallographic groups. Fortunately, such groups have been carefully studied (see [11, 24, 25]). A *Frieze* is a pattern which is invariant under all multiples of just one translation. The associated symmetry groups (the set of all isometries which map the pattern onto itself [24]) are commonly called **frieze groups**. Patterns that are invariant under linear combinations of two linearly independent translations repeat at regular intervals in two directions and their symmetry groups are often termed **wallpaper groups** or plane groups. Figure 13 taken from [24] illustrates the unit lattices of seventeen two dimensional Crystallographic groups. Similarly, if there are three independent translations, the associated symmetry groups are called **space groups.** These three types of groups form the basic set of Crystallographic groups. Due to the famous *Crystallographic restriction*, the only possible n-fold rotational symmetries in Crystallographic groups are when $n = 1, 2, 3, 4, 6$. Only a *finite* number of essentially distinct Crystallographic groups exist. There are *seven* frieze groups in a plane, *seventeen* wallpaper groups, and 230 space groups [8, 9].

Since we are interested in those symmetry groups that act on a pattern in the three dimension Euclidean space, the behavior of the Frieze groups and the plane groups should be re-examined in 3-space. These groups have been traditionally studied on a static plane as planar mixed polarity groups or black-white groups [25]. **TR** groups only include those Crystallographic groups which can be represented as a semidirect product of a translation subgroup T and a rotation subgroup R. Those Crystallographic groups with reflections, glide-reflections and glide-rotations as generators are excluded. Table 4 gives all the **TR** Crystallographic groups from an exhaustive counting of the $7 + 17 = 24$ one-dimensional two-sided infinite Crystallographic groups and $17 + 46 = 63$ two-dimensional two-sided infinite Crystallographic groups[8]. We have omitted those black-white groups which have the same color on both sides because they have the principal-plane reflection symmetries as their generators. Therefore the total number of one and two dimensional **TR** Crystallographic groups is $4 + 12 = 16$. Space Crystallographic groups contribute to the **TR** groups 12 more besides the 12 plane ones, giving a total of 24 (Table 5).

In [24] seventeen plane Crystallographic groups are discussed. Figure 13 from [24] shows the complete symmetries for each *primitive cell* (lattice unit or unit cell) of the seventeen possible one-sided plane patterns. Usually a primitive cell is chosen with centers of highest

[8]Note, three out of the four one-dimensional **TR** Crystallographic groups in Table 4 have the same labelings as some members of the two-dimensional **TR** Crystallographic groups, however they are **different** groups.

Figure 13: The seventeen two-dimensional Crystallographic groups

Table 4: The **TR** subgroups in one and two dimensional Crystallographic Groups

Crystallographic Groups	Freize Group ($1D$)	Wallpaper Group ($2D$)
Number of Polar ones (on a fixed plane)	7	17
The **TR** ones	$p1, p112$	$p1, p2, p3, p4, p6$
Mixed polar ones (on a flipping plane)	17	46
The **TR** ones	$p121, p211, p222$	$p121, c121, p222, c222$ $p422, p312, p321, p622$
Notes	$p121 = p112$	$p2$ has two forms $p112 = p121$
Total in **TR**	4	$5 + 7 = 12$

Table 5: The **TR** subgroups in three dimensional Crystallographic Groups

Crystallographic Group	Space groups ($3D$)
Total Number	230
The **TR** ones	$P432, P23, B112, F23, F222$ $I222, I23, I4, I422, R3, F432, I432$
Notes	plus all the 2D ones by adding one translation $\perp$ to the plane
Total in **TR**	24

order of rotation at the vertices. A set of figures similar to Figure 13 can be given for the Freize groups as well [26]. All the possible rotations can be represented explicitly in a primitive cell and then translated all over the space/plane/line. Furthermore, rotations in each primitive cell can be generated by a *generating region* (also called asymmetric unit, fundamental region or fundamental domain) which can be even smaller than the primitive cell itself. A generating region is the smallest region of the plane (line) whose images under the full symmetry group of the lattice cover the whole plane (line).

For example, Figure 11 shows that only a quarter of the primitive cell is chosen as the generating region of $p4$, one of the wallpaper groups. In order to capture all the rotations in a **TR** Crystallographic group $G = TR$ we need to make those rotations of G which neither belong to R nor to any conjugate of R through $t \in T$ explicit. Since we use a fixed point set and a pole-set to represent a rotation group, one obvious way to achieve this is to extend the pole-set $\mathcal{P}_R$ and fixed point set $\mathcal{F}_R$ for a single rotation group R to the smallest set of pole-sets and fixed points for a set of rotation subgroups in the generating region of the fixed point sets $\mathcal{F}_{all}$ of all the rotation subgroups in **TR**.

According to the convention of primitive cell in a Crystallographic group $G = TR$, R always appears at a corner of a unit lattice. Let $\tilde{R}$ denote the set of rotation subgroups in the generating region of which R is located at the lower left corner:

$$\mathcal{F}_{\tilde{R}} = \{\mathcal{F}_{R_1}, ..., \mathcal{F}_{R_n}\}, \mathcal{P}_{\tilde{R}} = \{\mathcal{P}_{R_1}, ..., \mathcal{P}_{R_n}\}$$

The essence of number n here is that $\mathcal{F}_{\tilde{R}}$ contains those and only those points (lines) which are not in the same T orbit of $\mathcal{F}_{all} = T(\mathcal{F}_{\tilde{R}})$. In the one dimensional case there are only two distinct T orbits $n = 2$. There are four distinct T orbits in the generating region of $p4 = TR$ therefore $\mathcal{F}_{\tilde{R}}$ contains four fixed point sets and $\mathcal{P}_{\tilde{R}}$ contains four pole-sets correspondingly (Figure 11). Since the number of Crystallographic groups is finite and their $\mathcal{F}_{all}$ are known, $\mathcal{F}_{\tilde{R}}$ can be determined constructively.

Proposition 3.65 *If $g \in G = TR$ is a nontrivial rotation and T is discrete then $g \in t\tilde{R}t^{-1}$ for some $t \in T$.*

Proof :

If $g \in G = TR$ is a rotation then $\bar{g} \cap T(\mathcal{F}_{\tilde{R}}) \neq \emptyset$ (by the definition of $T(\mathcal{F}_{\tilde{R}})$). Then there exists at least one $t \in T$ such that $\bar{g} \cap t(\mathcal{F}_{\tilde{R}}) \neq \emptyset$. If $g \notin t\tilde{R}t^{-1}$ then $g \notin G$, a contradiction. Therefore $g \in t\tilde{R}t^{-1}$. □

This proposition is a paraphrase of proposition 3.63, except this one is for T is non-continuous. As a generalization of the two propositions, we can state

Proposition 3.66 *If $g \in G = TR$ is a nontrivial rotation then $g \in tRt^{-1}$ for some $t \in T$.*

Here R should be understood as the smallest set of rotation subgroups to generate the whole set of rotations in **TR**. When T is continuous R shrank into one single rotation subgroup. From now on we drop $\tilde{}$ for R.

Definition 3.67 *A group T* **acts on a discrete set of points D transitively in terms of** $\mathcal{F}$ *means that if for $S_1, S_2 \subset D$, there exist $t_1, t_2 \in \mathbf{T}^3$ such that $S_1 = t_1(\mathcal{F}), S_2 = t_2(\mathcal{F})$, then there exists $t \in T$ such that $S_1 = t(S_2)$.*

Let $t' = 1$ or

$$t' = \sum \frac{t_i}{n}. \tag{4}$$

Here $\sum$ denotes the collective composition of translations, t_i represents the discrete generator(s) of T, and $n = 2$ or 3 depending on which of the 16 **TR** Crystallographic groups together with the parallel-line and parallel-plane groups is under consideration.

Lemma 3.68 *If $G = TR$, and T is discrete, then $\mathcal{F}_{\bar{R}}$ and t' can be so chosen such that the group $< t_i, t' >$ acts on $\mathcal{F}_{all} = T(\mathcal{F}_{\bar{R}})$ transitively in terms of $\mathcal{F}_{\bar{R}}$.*

Lemma 3.69 *If $G = T_1R_1 = T_2R_2$ and T is discrete then $R_1 = tt'R_2(tt')^{-1}$ where $t \in T$ and t' is the translation defined in formular 4.*

Proof :

Using the above fact, there exist $t \in T$ and t' defined in equation 4 such that $\mathcal{F}_{\bar{R}_1} = tt'(\mathcal{F}_{\bar{R}_2}) \Rightarrow \mathcal{F}_{R_1} = \mathcal{F}_{R_2}$. Thus $R_1 = tt'R_2(tt')^{-1}$. □

Lemma 3.70 *If $G = TR$ and T is discrete then $T(\mathcal{F}_{\bar{R}}) = T(t'(\mathcal{F}_{\bar{R}}))$.*

Proof :

When $t' = 1$, $T(\mathcal{F}_{\bar{R}}) = T(t'(\mathcal{F}_{\bar{R}}))$.

Suppose T is a one dimensional discrete translation group with generator t_0. G has $\mathcal{F}_{\bar{R}} = \{a, b\}$ for some $\{a, b\}$ for which $a = t'(b)$ where $t' = \frac{1}{2}t_0$. Hence $t'^2 \in T$. The T orbit of the fixed point set is $T(\mathcal{F}_{\bar{R}}) = T(\{a, b\}) = T(\{a, t'(a)\})$. Here $\{T(a)\}$ and $\{T(b)\} = \{t'T(a)\}$ are two disjoint T orbits. Now, $T(t'(\mathcal{F}_{\bar{R}}) = T(\{t'(a), t'(b)\}) = \{t'T(a)\} \cup \{Tt't'(a)\} = \{t'T(a)\} \cup \{T(a)\} = T(\{t'(a), a\}) = T(\{b, a\}) = T(\mathcal{F}_{\bar{R}})$.

When T is a two dimensional discrete translation group the proof can be carried out in a similar manner. For some of these groups t' is always the identity. For example, $p6 = TR_1 = TR_2$ then $R_1 = tR_2t^{-1}, t \in T$. For others such as $p3$ and $p4$, there are cases $t' \neq 1$, and there can be more than one form ($p3$ has $n = 3$ and there are two different forms of t'). For each form of t', it can be shown that $T(\mathcal{F}_{\bar{R}}) = T(t'(\mathcal{F}_{\bar{R}}))$. □

Proposition 3.71 *If $G = TR_1 = TR_2$ then $T(\mathcal{F}_{R_1}) = T(\mathcal{F}_{R_2})$.*

Proof :

When T is continuous, there exists $t \in T$, such that $R_1 = tR_2t^{-1}$ (Proposition 3.64). By Proposition 3.6, $\mathcal{F}_{R_1} = t\mathcal{F}_{R_2} \Rightarrow T(\mathcal{F}_{R_1}) = T(t\mathcal{F}_{R_2}) = T(\mathcal{F}_{R_2})$.

When T is discrete, $\mathcal{F}_{R_1}$ represents the fixed point sets in the *generating region* denoted by R_1, as does $\mathcal{F}_{R_2}$. By Lemma 3.69 and Proposition 3.6, $\mathcal{F}_{\bar{R}_1} = tt'(\mathcal{F}_{\bar{R}_2})$ where $t \in T$. Then $T(tt'(\mathcal{F}_{\bar{R}_2})) = T(t'(\mathcal{F}_{\bar{R}_2})) = T(\mathcal{F}_{\bar{R}_2})$ (Lemma 3.70). Therefore $T(\mathcal{F}_{\bar{R}_1}) = T(\mathcal{F}_{\bar{R}_2})$. □

This leads us to the following definition.

3.5.3 Rotational Characteristic Invariants for TR subgroups

Definition 3.72 *The* **rotational invariant of a TR group** $G = TR$ *is defined to be the pair*

$$(\mathcal{F}_G, \mathcal{P}_G) = (T(\mathcal{F}_R), \mathcal{P}_R)$$

When T is a discrete group, $\mathcal{F}_R$ and $\mathcal{P}_R$ should be understood as the **fixed point set** and **pole-sets** of the rotation subgroups in the generating region denoted by R. This interpretation is actually a more general concept since when T becomes continuous, i.e. the distance between the discrete lattice points becomes infinitely small, a pole structure, usually bearing several pole-pierced unit spheres, shrinks into one unit sphere at the origin. In another words, the continuous case is the limit of discrete cases.

This definition differs from that of rotational characteristic invariants for rotation groups (Definition 3.50) in that the fixed-point set is moved around in $\mathbf{R}^3$ by the translation subgroup of G. For example, the rotation symmetry group of an infinite cylinder $G_{cyl} = T_1 R_{cyl}$ has rotations about the central axis as well as 180° rotations about *any* axis perpendicular to and intersecting with the central axis. These rotations are not restricted to be around s_0 only but can 'take a ride' on the translations T in G to anywhere in $T(\mathcal{F}_R)$, its pole set structure is shown in Figure 14 (compare with Figure 8).

Next we prove the one-to-one correspondence between the **TR** groups and their characteristic invariants (Definitions 3.72 and 3.59).

Theorem 3.1 $f : \{G = TR\} \rightarrow \{(\mathcal{T}_G, (\mathcal{F}_G, \mathcal{P}_G))\}$ *is an injection.*

Proof :

Let $G_1 = T_1 R_1, G_2 = T_2 R_2$. Given $(\mathcal{T}_{G_1}, (\mathcal{F}_{G_1}, \mathcal{P}_{G_1})) = (\mathcal{T}_{G_2}, (\mathcal{F}_{G_2}, \mathcal{P}_{G_2}))$; we are going to prove that $G_1 = G_2$.

Since $\mathcal{T}_{G_1} = \mathcal{T}_{G_2}$, for all $t_1 \in T_1, t_1(\mathcal{T}_{G_2}) = \mathcal{T}_{G_2}$. By Proposition 3.29 $t_1 \in T_2$. Thus $T_1 \subseteq T_2$. Similarly, $T_2 \subseteq T_1$. Therefore $T_1 = T_2 = T$.

Since $\mathcal{P}_{R_1} = \mathcal{P}_{R_2}$, by Proposition 3.48, there exists a translation t_0 such that $R_1 = t_0 R_2 t_0^{-1}$. So $TR_1 = Tt_0 R_2 t_0^{-1} = t_0(TR_2)t_0^{-1}$. We have shown that G_1 and G_2 are at least conjugate with each other.

When T is a continuous subgroup of $\mathbf{T}^3$, $\mathcal{F}_{R_1}, \mathcal{F}_{R_2}$ are either two points or two parallel lines in $\mathbf{R}^3$. From $T(\mathcal{F}_{R_1}) = T(\mathcal{F}_{R_2})$, it follows that there exists $t \in T$ such that $\mathcal{F}_{R_1} = t(\mathcal{F}_{R_2})$. By Proposition 3.49 $R_1 = tR_2t^{-1}$. By Corollary 3.57 $G_1 = TR_1 = TtR_2t^{-1} = TR_2 = G_2$.

When T is a discrete subgroup of $\mathbf{T}^3$, however, it is not necessarily true that there exists $t \in T$ such that $\mathcal{F}_{R_1} = t(\mathcal{F}_{R_2})$ (Figure 15). Since $\mathcal{F}_{R_1} = tt'\mathcal{F}_{R_2}$ where $t \in T, t'$ is defined in expression 4, $R_1 = tt'R_2(tt')^{-1}$ (Proposition 3.49). By Lemma 3.69 and Corollary 3.57, $G_1 = TR_1 = Tt(t'R_2t'^{-1})t^{-1} = Tt'R_2t'^{-1} = TR_2 = G_2$. □

Corollary 3.73 *There exists a bijection* $\beta : \{G\} \rightarrow f(G)$.

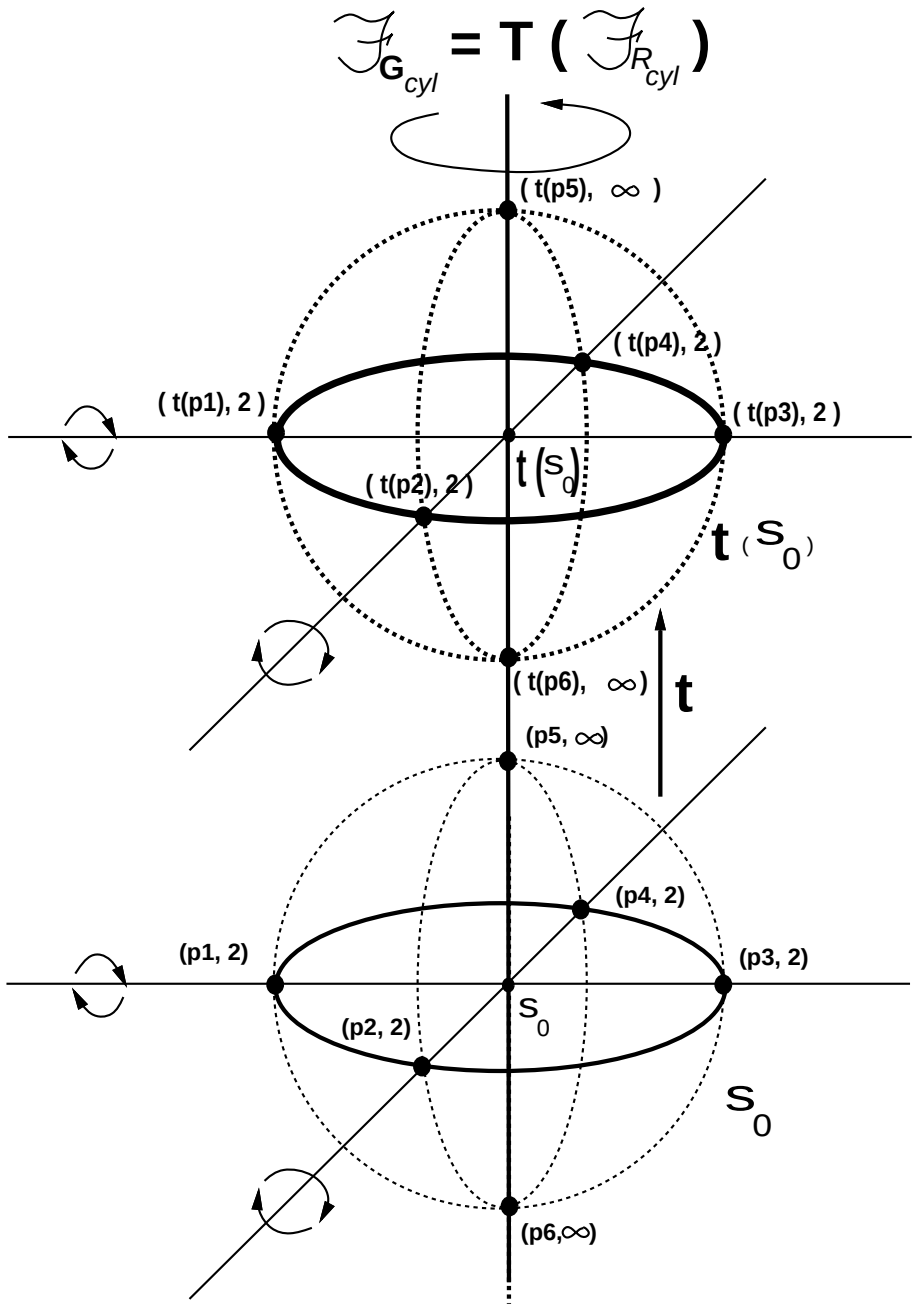

Figure 14: Rotation symmetries for an infinite cylinder: S_0 can be moved along $T(\mathcal{F}_{R_{cyl}})$.

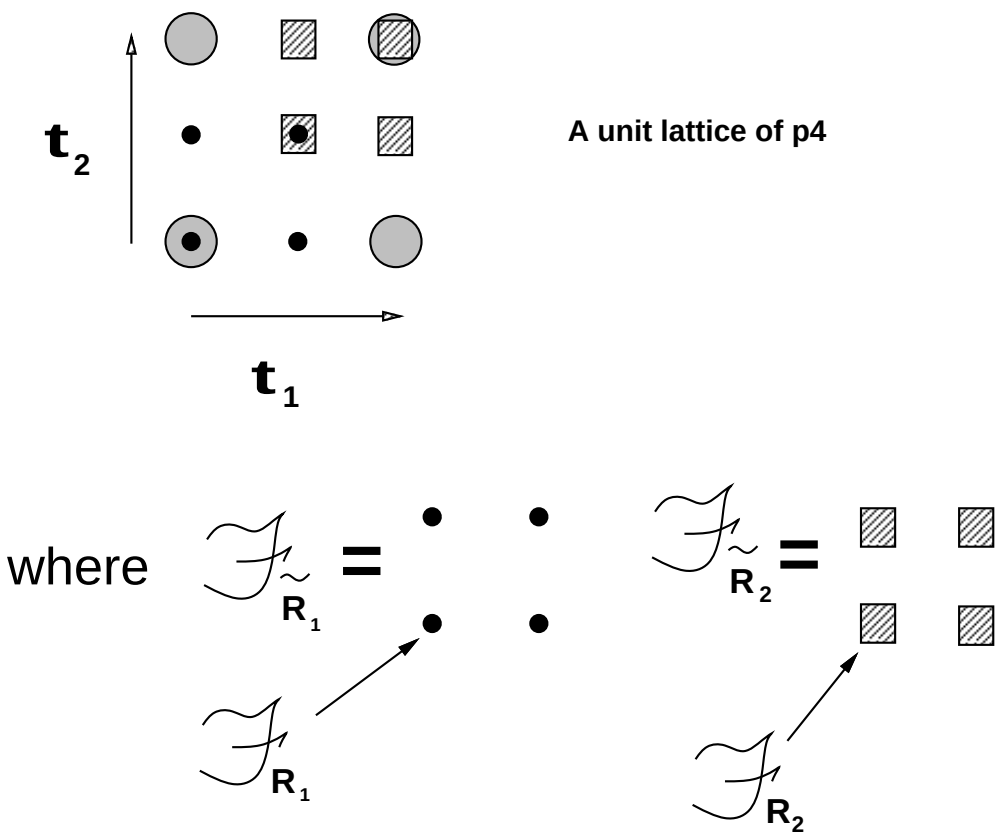

Figure 15: $G = TR_1 = TR_2$. Since $\mathcal{F}_{R_1}$ and $\mathcal{F}_{R_2}$ each denotes a set of 4 points (lines) and $T(\mathcal{F}_{R_1}) = T(\mathcal{F}_{R_2})$, but it is not necessarily true that there exists $t \in T$ such that $\mathcal{F}_{R_1} = t(\mathcal{F}_{R_2})$. Here t_1, t_2 are generators of T. One should notice that when T is discrete the rotational invariants $\mathcal{F}_{R_1}, \mathcal{F}_{R_2}$ are usually denser than the translational invariant $\mathcal{T}_G$.

Definition 3.74 *Let G_1, G_2 be* **TR** *groups. The* **distance of TR groups** *is defined as*

$$\mathbf{dist}(G_1, G_2) = \mathbf{dist}(\mathcal{F}_{G_1}, \mathcal{F}_{G_2}) = \bigwedge_{s_1 \in \mathcal{F}_{G_1}\, s_2 \in \mathcal{F}_{G_2}} \mathbf{dist}(s_1, s_2).$$

Lemma 3.75 *If $G_1 = T_1R_1, G_2 = T_2R_2$ are* **TR** *groups then there exist $t_1 \in T_1, t_2 \in T_2$ such that* $\mathbf{dist}(G_1, G_2) = \mathbf{dist}(\mathcal{F}_{t_1R_1t_1^{-1}}, \mathcal{F}_{t_2R_2t_2^{-1}})$.

Proof:

From Definition 3.74 $\mathbf{dist}(G_1, G_2) = \mathbf{dist}(T_1(\mathcal{F}_{R_1}), T_2(\mathcal{F}_{R_2}))$. Let $\mathbf{dist}(G_1, G_2) = \mathbf{dist}(s_1, s_2)$ where $s_1 \in T_1(\mathcal{F}_{R_1})$ and $s_2 \in T_2(\mathcal{F}_{R_2})$. There must exist $t_1 \in T_1, t_2 \in T_2$ such that $s_1 \in t_1(\mathcal{F}_{R_1}) = \mathcal{F}_{t_1R_1t_1^{-1}} \subseteq T_1(\mathcal{F}_{R_1}), s_2 \in t_2(\mathcal{F}_{R_2}) = \mathcal{F}_{t_2R_2t_2^{-1}} \subseteq T_2(\mathcal{F}_{R_2})$. Therefore $\mathbf{dist}(\mathcal{F}_{t_1R_1t_1^{-1}}, \mathcal{F}_{t_2R_2t_2^{-1}}) = \mathbf{dist}(G_1, G_2)$ (Lemma 3.13). □

3.6 TR Group Intersection

Theorem 3.2 *If $G_1 = T_1R_1, G_2 = T_2R_2$ and* $\mathbf{dist}(G_1, G_2) = \mathbf{dist}(R_1, R_2)$ *(By Lemma 3.75 such R_1, R_2 exist) then $G_1 \cap G_2 = \beta^{-1}(\mathcal{T}_{T_1 \cap T_2}, (\mathcal{F}_{R_1 \cap R_2}, \mathcal{P}_{R_1 \cap R_2}))$.*

Proof:

It is obvious that every $g \in (T_1 \cap T_2)(R_1 \cap R_2)$ is a member of $G_1 \cap G_2$. So we have $(T_1 \cap T_2)(R_1 \cap R_2) \subseteq G_1 \cap G_2$.

Now we prove the other inclusion $G_1 \cap G_2 \subseteq (T_1 \cap T_2)(R_1 \cap R_2)$:

A nontrivial member $g = tr$ of $G_1 \cap G_2$ can be one of the three kinds of isometries: a translation, a rotation or a screw motion, i.e. $t \| \bar{r}$.

If $g \in G_1 \cap G_2$ is a translation then $g \in T_1$ and $g \in T_2$ so $g \in T_1 \cap T_2$.

If g is a rotation then there exist $t_1 \in T_1, t_2 \in T_2$ such that $g \in t_1\tilde{R}_1t_1^{-1} \cap t_2\tilde{R}_2t_2^{-1}$ (Proposition 3.66). Then for some $r_1 \in \tilde{R}_1, r_2 \in \tilde{R}_2, g = t_1r_1t_1^{-1} = t_2r_2t_2^{-1}$. By Proposition 3.30 $\bar{g} \| \bar{r}_1 \| \bar{r}_2$ and $\theta_g = \theta_{r_1} = \theta_{r_2}$.

If one of T_1, T_2 is $\{1\}$ then $g \in R_1 \cap R_2$.

If $\mathbf{dist}(G_1, G_2) = 0$ then $\mathbf{dist}(R_1, R_2) = 0 \Rightarrow \mathcal{F}_{R_1} \cap \mathcal{F}_{R_2} \neq \emptyset \Rightarrow \bar{r}_1 \cap \bar{r}_2 \neq \emptyset$. Thus $\bar{r}_1 = \bar{r}_2 \Rightarrow r_1 = r_2 \in R_1 \cap R_2$, and so $t_1 = t_2 = t \in T_1 \cap T_2$. Therefore $g \in tR_1t^{-1} \cap tR_2t^{-1} = t(R_1 \cap R_2)t^{-1}$.

If $\mathbf{dist}(G_1, G_2) > 0$ i.e. $T_1(\mathcal{F}_{R_1}) \cap T_2(\mathcal{F}_{R_2}) = \emptyset$ then $\mathbf{dist}(R_1, R_2) > 0$. Obviously, neither of T_1, T_2 is isomorphic to $\mathbf{T}^3$ so $\mathcal{T}_G$ is either a line (one dimensional lattice) or a plane (two dimensional lattice). In order to have a nontrivial rotation in $t_1R_1t_1^{-1} \cap t_2R_2t_2^{-1}$, R_1 and R_2 both have to be point groups (recall the proof in Proposition 3.52), if one of them is a line group then the axes of the rotations have to be overlapping, this violates the condition $\mathbf{dist}(R_1, R_2) > 0$. Thus $t_1(\mathcal{F}_{R_1}) \in \bar{g}, t_2(\mathcal{F}_{R_2}) \in \bar{g}$.

- When T_1 and T_2 are both continuous, if $\bar{g} \| T_1$ then $\bar{g} \in t_{c_1}(\mathcal{T}_{G_1})$ (Corollary 3.62), where $R_1 = t_{c_1}R_{c_1}T_{c_1}^{-1}$ and $R_{c_1} \subseteq SO(3)$ (Definition 3.17). Since R_1 is a point group, $\mathcal{F}_{R_1} = t_{c_1}(s_0)$, thus $t_{c_1}(\mathcal{T}_{G_1}) = t_{c_1}(T_1(s_0)) = T_1(t_{c_1}(s_0)) = T_1(\mathcal{F}_{R_1})$. So we have

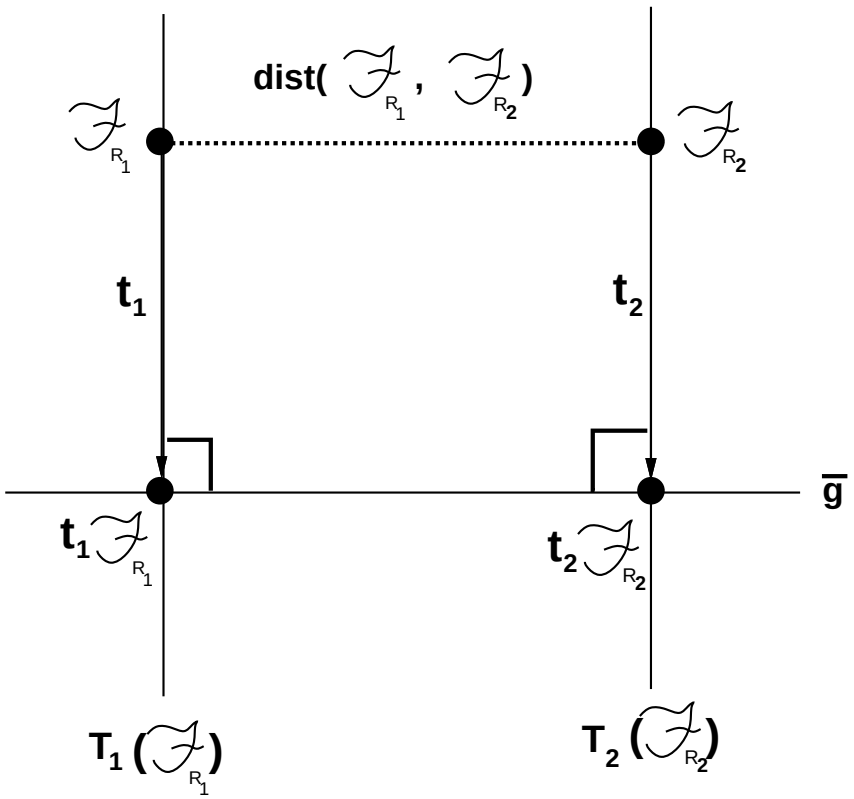

Figure 16: When R_1, R_2 are both point groups and $\bar{g} \perp \mathcal{T}_G$.

$\bar{g} \in T_1(\mathcal{F}_{R_1}) \Rightarrow t_2(\mathcal{F}_{R_2}) \in T_1(\mathcal{F}_{R_1}) \Rightarrow T_1(\mathcal{F}_{R_1}) \cap T_2(\mathcal{F}_{R_2}) \neq \emptyset \Rightarrow \mathbf{dist}(G_1, G_2) = 0$, a contradiction. Similarly, it can be shown for $\bar{g} \| T_2$. If $\bar{g} \perp T_1$ and $\bar{g} \perp T_2$ then $\bar{g} \perp T_1(\mathcal{F}_{R_1})$, $\bar{g} \perp T_2(\mathcal{F}_{R_2})$ and $\bar{g} \cap T_1(\mathcal{F}_{R_1}) = t_1(\mathcal{F}_{R_1})$, $\bar{g} \cap T_2(\mathcal{F}_{R_2}) = t_2(\mathcal{F}_{R_2})$ (Figure 16). This implies $\mathbf{dist}(T_1(\mathcal{F}_{R_1}), T_2(\mathcal{F}_{R_2})) = \mathbf{dist}(t_1(\mathcal{F}_{R_1}), t_2(\mathcal{F}_{R_2})) = \mathbf{dist}(G_1, G_2) = \mathbf{dist}(\mathcal{F}_{R_1}, \mathcal{F}_{R_2})$ (Definition 3.12) (Figure 16). It follows then $t_1 = t_2 = t \in T_1 \cap T_2 \Rightarrow g \in (T_1 \cap T_2)t(R_1 \cap R_2)t^{-1} = (T_1 \cap T_2)(R_1 \cap R_2)$ (Corollary 3.57).

- When one of T_1, T_2 is discrete, a proof similar as above can be given. The only difference is that a single rotation group R_1 or R_2 is replaced by a set of rotation groups in the generating region, namely $\tilde{R}_1$ or $\tilde{R}_2$.

- When both T_1, T_2 are discrete, for instance when both are one dimensional discrete groups, then $\mathcal{T}_{G_1}, \mathcal{T}_{G_2}$ are both one dimensional lattices. Due to Corollary 3.61 and the fact that $t_1(\mathcal{F}_{R_1}) \in \bar{g}, t_2(\mathcal{F}_{R_2}) \in \bar{g}$, there are only three possible cases (as shown in Figure 17) for g to be a nontrivial rotation:

  – case 1: $\bar{g} \| \mathcal{T}_{G_1}$ and $\bar{g} \| \mathcal{T}_{G_2}$;
  if $g \notin R_1$ then $g \notin t_1 R_1 T_1^{-1} \Rightarrow g \notin G_1 \Rightarrow g \notin G_1 \cap G_2$, a contradiction. Therefore $g \in R_1 \cap R_2$.

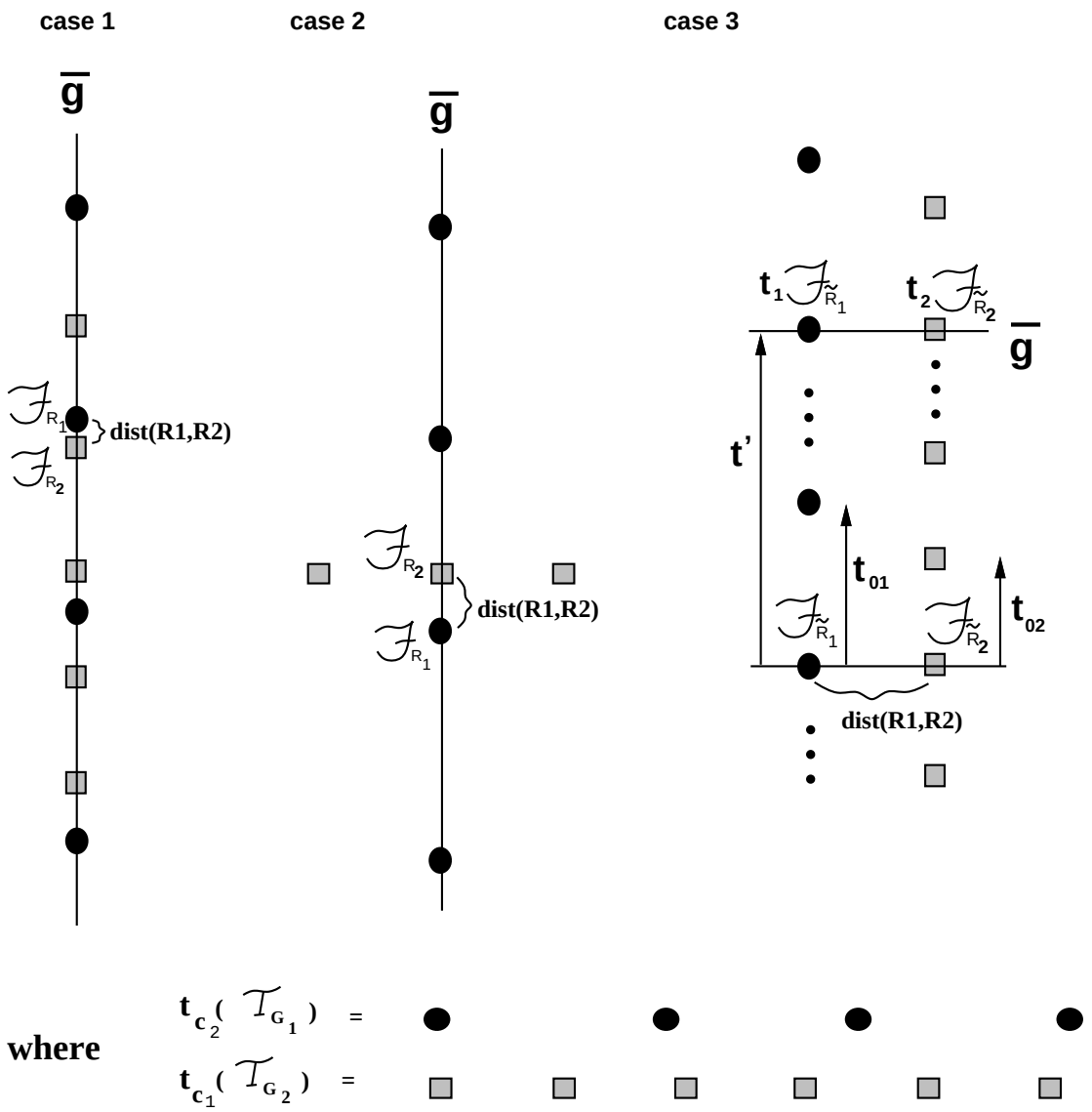

Figure 17: When T_1, T_2 are both one dimensional discrete groups.

- case 2: $\bar{g} \| \mathcal{T}_{G_1}$ but $\bar{g} \perp \mathcal{T}_{G_2}$ (or vice versa);
  same argument as above.
- case 3: $\bar{g} \perp \mathcal{T}_{G_1}$ and $\bar{g} \perp \mathcal{T}_{G_2}$;
  since $\mathbf{dist}(t_1 R_1 t_1^{-1}, t_2 R_2 t_2^{-1}) = \mathbf{dist}(R_1, R_2)$, either $t_1 = t_2 = 1$ or $t_1 = t_2 = t^n = t'$. Here n is an integer and $t = lcm(t_{01}, t_{02})$, the least common multiple for the norms of the generators t_{01}, t_{02} of T_1 and T_2 respectively, thus $t' \in T_1 \cap T_2$. Therefore $g \in t'(R_1 \cap R_2)t'^{-1} \Rightarrow g \in (T_1 \cap T_2)(R_1 \cap R_2)$.

The basic arguments are the same while the dimensions of T_1, T_2 increase.

If g is a screw motion then $g = t_1 r_1 = t_2 r_2$ where $t_1 \in T_1, r_1 \in R_1, t_2 \in T_2, r_2 \in R_2$ and $\bar{r}_1 \| t_1, \bar{r}_2 \| t_2$. Then we have $t_2^{-1} t_1 = r_2 r_1^{-1} \in \mathbf{T}^3$. Since $t_2^{-1} t_1$ is a translation perpendicular to the translation $r_2 r_1^{-1}$, for two such translations to be equal is to let both be 1. Therefore $t_2^{-1} t_1 = 1 \Rightarrow t_1 = t_2$ and $r_2 r_1^{-1} = 1 \Rightarrow r_1 = r_2$. Therefore $g \in (T_1 \cap T_2)(R_1 \cap R_2) \Rightarrow G_1 \cap G_2 \subseteq (T_1 \cap T_2)(R_1 \cap R_2)$.

Thus $G_1 \cap G_2 = (T_1 \cap T_2)(R_1 \cap R_2)$. Clearly, $\beta^{-1}(\mathcal{T}_{T_1 \cap T_2}, (\mathcal{F}_{R_1 \cap R_2}, \mathcal{P}_{R_1 \cap R_2})) = (T_1 \cap T_2)(R_1 \cap R_2)$ (Theorem 3.1, Corollary 3.73). Therefore $G_1 \cap G_2 = \beta^{-1}(\mathcal{T}_{T_1 \cap T_2}, (\mathcal{F}_{R_1 \cap R_2}, \mathcal{P}_{R_1 \cap R_2}))$. □

4 A TR Group Intersection Algorithm and Some Examples

Based on the Theorems in the previous section, we have implemented a **TR** group intersection algorithm using characteristic invariants. The diagram of the algorithm is shown in Figure 18. The current implemented version has asymptotic time complexity $\mathcal{O}(n^2) + \mathcal{O}(logM)$ where n is the number of poles, and $\mathcal{O}(logM)$ is the order of magnitude for computing the greatest common divisor for numbers a, b with $M = Max(a, b)$ [28]. The only case where n can be arbitrarily large is when both groups are dihedral groups. However this case can be taken care of by using generator and relation representations for the poles of dihedral groups.

Figure 18 illustrates the outline of the **TR** group intersection algorithm. **Note**, the subcases for computing the fixed point set $\mathcal{F}$ is not listed in Figure 18 but can be found in great detail in Proposition 3.52.

Propositions 3.34 and 3.36 establish the justification for canonical representation of each **TR** groups, which has been used in the implementation of the algorithm.

The current version of the algorithm has been implemented on SPARK worktations using programming language POP11[9], and has been applied to computing symmetry groups of the boundary models from the solid modeller PADL2 (Brown 1982) [3] in an assembly planning system called $\mathcal{KA}^3$ [15, 16, 17, 18].

[9]For portability, a C version is under implementation currently.

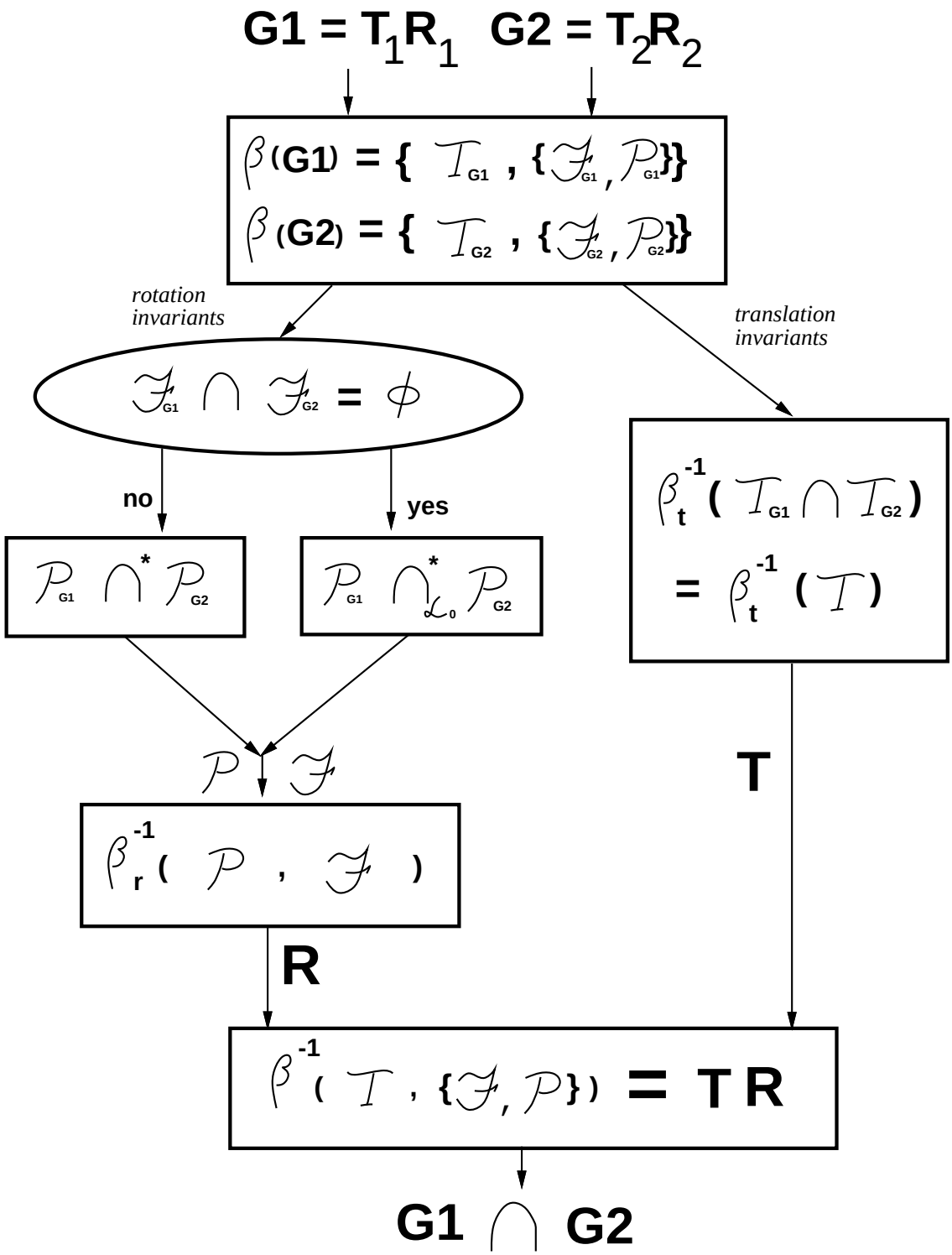

Figure 18: The outline of the **TR** group intersection algorithm

4.1 Examples

Figure 19 shows the characteristic invariants of some **TR** groups, where G_{plane} denotes the symmetry group of a unoriented planar surface while for an oriented plane the poles on the 'equator' should disappear.

Figure 20 shows one example of intersection between two **TR** groups G_{plane}, which is a conjugation of $T^2SO(2)C_2'$, and G_{cyl}, which is a conjugation of $T^1SO(2)C_2'$, where C_2' is the cyclic group of order 2 with X axis as the axis of rotation instead of Z axis. The two surfaces intersect each other as shown at the top of the figure, the cylindrical surface sits halfway above and halfway below the planar surface. The intersection of the translational invariants is a straightforward intersection between a plane and a line. Along the rotational invariants path, first of all the fixed point sets do intersect, thus it is possible for the result to be a point group. As the two pole sets have the same structure (Figure 19), except that the two are positioned with a 90^0 rotation about the center of the sphere away from each other. As shown in Figure 20 the white dots denotes the poles for G_{cyl} and the black ones for G_{plane}. All the black dots on the horizontal circle has order 2 except for the two, with order infinity, on the south and north poles; the same is true for all the white dots on the vertical circle (order 2), and the two on the left and right extremes (order infinity). The overlapping poles are six pairs as shown at the next step on the Figure. Since $\gcd^*(\infty, 2) = 2$ and $\gcd(2, 2) = 2$, each of them has order 2 (Proposition 3.52). Finally, the pair of resulted invariants determines uniquely another **TR** group G, that is the intersection of the symmetry group of the cylindrical surface and that of the planar surface shown.

5 Summary

In this article we have studied geometric denotation of a particular family of subgroups of the proper Euclidean group $\mathcal{E}^+$, namely the **TR** subgroups, and intersections of these groups. The main results we have achieved are:

a) We defined **TR** subgroups of $\mathcal{E}^+$ as $G = TR$ where T is a translation subgroup and R is a rotation subgroup. The relative independence of the translations and rotations of the **TR** group gives us the freedom to characterize translations and rotations separately. The only symmetry group we are interested that cannot be represented as a **TR** group is the symmetry group of a screw, in which translations are not independent from rotations (see $\mathcal{G}_{screw}$ in Table 1).

b) We defined characteristic invariants for **TR** groups and proved their one-to-one correspondence. This brings the geometric aspect of the **TR** groups to the foreground for computational purposes, while still preserving their algebraic nature.

c) We proved that the intersection of **TR** groups can be obtained from the calculation of their characteristic invariants, thus justifying an efficient **TR** group intersection algorithm.

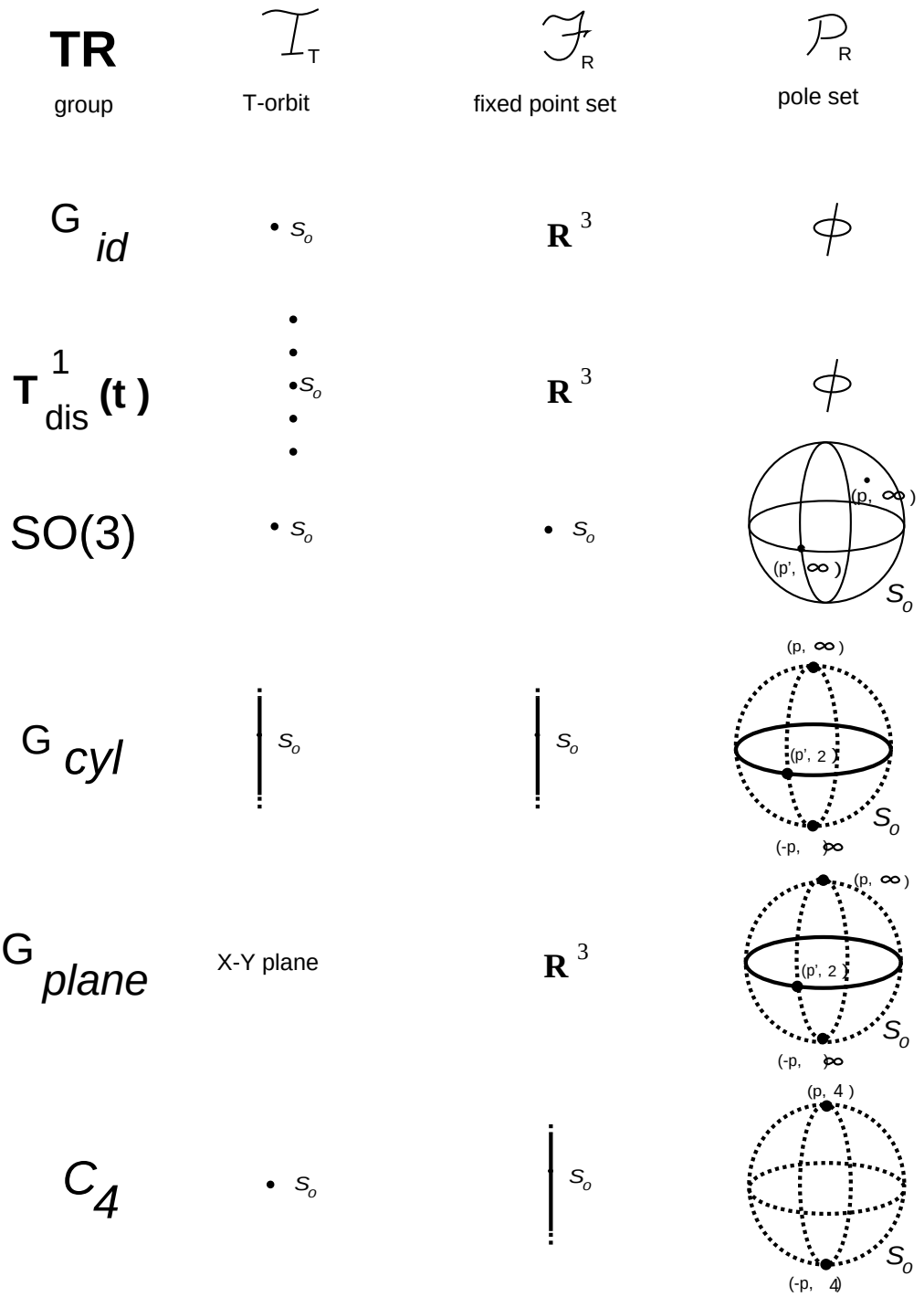

Figure 19: Characteristic Invariants of some TR groups

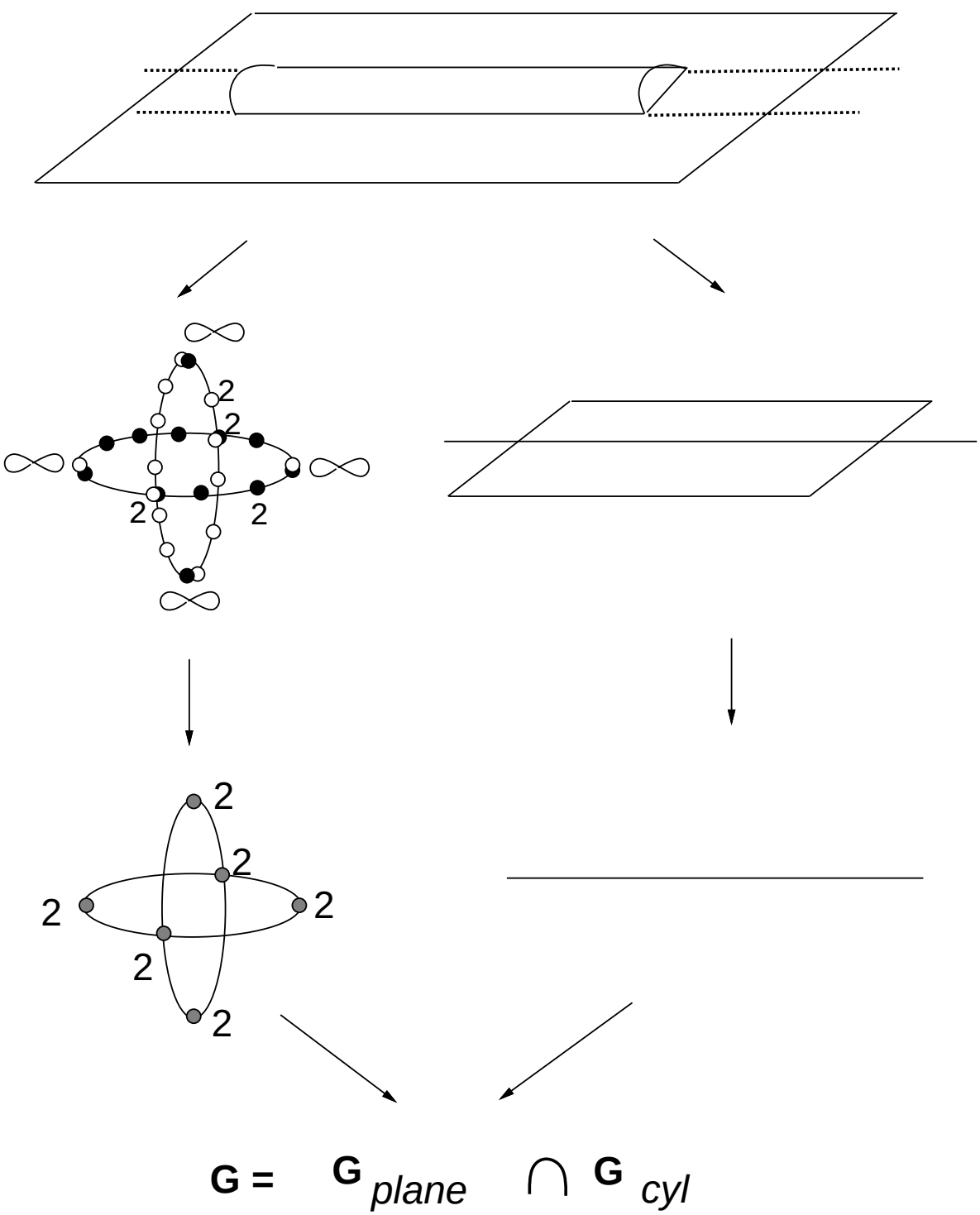

Figure 20: An example of the intersection of two **TR** groups

These results justify that using the group theoretic approach in formalizing contacts among solids is abstractive yet quantitative, general yet computationally tractable. Therefore, this approach is applicable in works involving transformations in Euclidean space such as robotics, computer graphics, computer vision and mechanical design.

Some open problems include:

- What is the stability of **TR** group intersection algorithm as it stands now with respect to the geometric computational noise?

  In our current implementation, we set a small ϵ to absorb such noises but no systematic analysis has been done.

- How will this work be related to the approximate instead of exact symmetries?

- Can characteristic invariant method be extended to handle group products?

  As pointed out at the beginning of this article that we have only dealt with a subset of the general form (formular 1 of the solid contact problem in Euclidean space, that is the surface contact (formular 2). It is known that in general a product of groups is not a group. For **TR** groups, the **TR**−restriction presented in this article provides us with a constructive way to judge from its invariants whether a product remains a **TR** group. However we do not know how to compute a product from the invariants of its subgroups.

- Given an algebraic surface how to find its symmetry group computationally, say from a polynomial of the surface?

- Given a set of algebraic surfaces and their respective symmetry groups, what is the exact algorithm to find the symmetry group of the set?

  [12, 13, 14] have only proved results for subcases: n distinct surfaces, a pair of $1-cong$ or $2-cong$ surfaces. No proved result for the general case exists yet.

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