BIPARTITE DIMENSIONS AND BIPARTITE DEGREES OF GRAPHS[1]

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Abstract

A *cover* (bipartite) of a graph G is a family of complete bipartite subgraphs of G whose edges cover G's edges. G's *bipartite dimension* $d(G)$ is the minimum cardinality of a cover, and its *bipartite degree* $\eta(G)$ is the minimum over all covers of the maximum number of covering members incident to a vertex. We prove that $d(G)$ equals the Boolean interval dimension of the irreflexive complement of G, identify the 21 minimal forbidden induced subgraphs for $d \leq 2$, and investigate the forbidden graphs for $d \leq n$ that have the fewest vertices. We note that for complete graphs, $d(K_n) = \lceil \log_2 n \rceil$, $\eta(K_n) = d(K_n)$ for $n \leq 16$, and $\eta(K_n)$ is unbounded. The list of minimal forbidden induced subgraphs for $\eta \leq 2$ is infinite. We identify two infinite families in this list along with all members that have fewer than seven vertices.

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1. Introduction

This paper investigates properties of a graph G that are based on complete bipartite subgraphs whose edges cover the edges of G. We focus on two properties, referred to as bipartite dimension and bipartite degree, that have interesting connections to intersection graphs, chromatic numbers, and combinatorial optimization problems. We say more about these things after we introduce some basic terminology.

Throughout, $G = (V, E)$ denotes an undirected graph with finite nonempty vertex set V and irreflexive edge set E. The *irreflexive complement* of G is denoted by $G^c = (V, E^c)$ with

$$E^c = \{\{u, v\}: \ u, v \in V, \ u \neq v, \ \{u, v\} \notin E\} \ .$$

Subgraphs of G need not be induced, and when we mean *induced* subgraph, we say so. As usual, K_n is a complete graph on n vertices, and a complete bipartite graph $B = K_{p,q}$ has the form

$$B = (A_1 \cup A_2, \ \{\{a_1, a_2\}: \ a_i \in A_i \quad \text{for} \quad i = 1, 2\})$$

with $p, q \in \{1, 2, \ldots\}$, $|A_1| = p$, $|A_2| = q$ and $A_1 \cap A_2 = \emptyset$. The symbol $\mathcal{B}$ *always denotes a set of complete bipartite graphs with cardinality* $|\mathcal{B}|$. It is allowed to be empty. We say that $\mathcal{B}$ *covers* (is a cover of) $G = (V, E)$ if every $B \in \mathcal{B}$ is a subgraph of G and E equals the union of the edge sets of the members of $\mathcal{B}$. When $\mathcal{B}$ covers G, and $v \in V$, we let $\mathcal{B}(v)$ denote the number of complete bipartite graphs in $\mathcal{B}$ that have v as a vertex.

Our study considers structural properties of graphs from the perspective of complete bipartite subgraphs. The *bipartite dimension* $d(G)$ of G is the minimum cardinality of a cover:

$$d(G) = \min\{|\mathcal{B}|: \ \mathcal{B} \ \text{ covers } \ G\} \ .$$

We have $d(G) = 0$ if and only if $E = \emptyset$, and $d(G) = 1$ if and only if G consists of a $K_{p,q}$ and isolated vertices. The *bipartite degree* $\eta(G)$ of G is the minimum over covers $\mathcal{B}$ of the maximum of $\mathcal{B}(v)$ over V:

$$\eta(G) = \min_{\{\mathcal{B}: \ \mathcal{B} \text{ covers } G\}} \ \max_V \mathcal{B}(v) \ .$$

We have $\eta(G) = 0$ if and only if $E = \emptyset$, and $\eta(G) = 1$ if and only if $E \neq \emptyset$ and every connected component of G is a $K_{p,q}$ or an isolated vertex. Clearly, $\eta(G) \leq d(G)$. Disjoint copies of $K_2 = K_{1,1}$ show that d can be arbitrarily large when $\eta = 1$. An open problem discussed later asks whether $\eta(K_n) = d(K_n)$ for all n.

Our investigation of bipartite dimension and degree is motivated by works on Boolean algebraic forms associated with graphs and combinatorial optimization problems [3, 4, 6, 7, 10], by intersection graphs [8, 9, 11] based on intervals of Boolean lattices, and by a fundamental curiosity about complete bipartite subgraph covers. The rest of this introduction outlines our results and mentions ties to Boolean structures. The next three sections and an appendix provide formal statements and proofs of the main results. Section 5 concludes the paper with a brief discussion of a few of the many open problems provoked by consideration of complete bipartite covers.

Sections 2 and 3 focus on d. Section 2 begins with a few examples, then proves that $d(G)$ equals the *Boolean interval dimension* of G^c, which is the minimum n such that each v can be assigned an interval in $(\{0,1\}^n, \leq)$ so that $\{u, v\} \in E^c$ when $u \neq v$ if and only if the intervals assigned to u and v have a nonempty intersection. An elementary characterization of $d \leq 1$ in terms of minimal forbidden induced subgraphs is noted. Three forbidden graphs are used, namely K_3 (a triangle), the 4-vertex path P_4, and two vertex-disjoint copies of K_2. This motivates a restricted edge coloring of G in which no K_3 is monochromatic and no two vertex-disjoint edges whose vertex set includes at most one other edge have the same color. We prove that the minimum number of colors needed for such an edge coloring equals $d(G)$ when G is triangle-free, but can be strictly less than $d(G)$ when G has triangles. Section 2 concludes with a minimal forbidden induced subgraph characterization of $d \leq 2$ that uses 21 graphs, seven of which have five vertices and 14 of which have six vertices.

Section 3 examines $\sigma(m)$, the maximum k such that every G with k vertices has $d(G) \leq m$. Equivalently, $\sigma(m)+1$ is the smallest $|V|$ for which some graph has bipartite dimension $m+1$. For example, $\sigma(2) = 4$ since every 4-vertex G has $d(G) \leq 2$ and some 5-vertex graphs have $d = 3$. We verify $\sigma(m)$ values for $m \leq 6$ and give linear bounds for larger m on the basis of

$$\sigma(m) + 1 \leq \sigma(m+1) \leq \sigma(m) + 2$$

and particular results like $\sigma(4) = 6$ and $\sigma(10) \leq 14$. The latter follow from the claims about d in Figure 1 and the fact that the 7-vertex graph H shown there is the *only* G with $|V| \leq 7$ for which $d(G) > 4$.

Figure 1 about here

Section 4 presents results for bipartite degree. We begin with a proof that $\eta(K_n) = d(K_n) = \lceil \log_2 n \rceil$ for $n \leq 16$, then show that η is unbounded above because $\eta(K_n) \to \infty$ as n gets large. The well-known fact that a connected graph is complete bipartite if and only if no induced subgraph is a K_3 or P_4 provides a forbidden induced subgraph characterization of $\eta \leq 1$. Graphs with $\eta \leq 2$ are referred to in [3, 4, 6, 10] as *quadratic graphs* because of their connection to Boolean quadratic forms. We recall a result from [4] which says that, unlike $d \leq 2$, the list of minimal forbidden induced subgraphs for $\eta \leq 2$ is infinite. We do not characterize this list but note a few of its infinite families. We also prove that K_5 is the only graph with $|V| \leq 5$ for which $\eta = 3$ and that the six graphs shown in Figure 2 are the only 6-vertex graphs that do not include a K_5 and have $\eta = 3$.

Figure 2 about here

2. Bipartite Dimension

The following example and lemma identify d for some familiar graphs.

Example 1. Let P_n and C_n denote the n-vertex path and n-vertex cycle respectively for $n \geq 3$. We have $d(P_n) = \lfloor n/2 \rfloor$ because $K_{1,2}$ is the largest complete bipartite subgraph of P_n. Since $C_3 = K_3$ and $C_4 = K_{2,2}$, $d(C_3) = 2$ and $d(C_4) = 1$. For $n \geq 5$, $d(C_n) = \lceil n/2 \rceil$.

Lemma 1. $d(K_n) = \lceil \log_2 n \rceil$.

Proof. The conclusion is evident if $n \leq 3$. Suppose $n = 2^{m+1}$, $m \geq 1$, and a $K_{p,q}$ is used in a cover $\mathcal{B}$ of K_n. If either p or q is not 2^m, there is a K_k in K_n with $k > 2^m$ that has no edges in the $K_{p,q}$. If $p = q = 2^m$ then the largest vertex set with no edges in the $K_{p,q}$ has 2^m members. It follows by successive splitting and recombination of the independent sets A_1 and A_2 for $B = K_{2^m,2^m}$ that $m+1$ copies of $K_{2^m,2^m}$ cover K_n and minimize $|\mathcal{B}|$. With

0 denoting $v_j \in A_1$ and 1 denoting $v_j \in A_2$, a suitable arrangement for $m = 2$ is

$$\begin{array}{r|cccccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\ \hline B_1: & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ B_2: & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ B_3: & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \qquad \mathcal{B} = \{B_1, B_2, B_3\}$$

The column under v_j in the general case of $n = 2^{m+1}$ is the binary representation in $\{0,1\}^{m+1}$ of $j-1$. Hence

$$d(K_{2^{m+1}}) = m + 1 = \log_2(2^{m+1}) \ .$$

If $2^{m+1} < n < 2^{m+2}$, then every $K_{p,q}$ in K_n misses all edges in some K_k subgraph of K_n for $k > 2^m$, and it follows that $d(K_n) = d(K_{2^{m+2}})$. ∎

Our first main result for d involves the Boolean lattice $(\{0,1\}^n, \leq)$ with partial order $\leq$ defined by

$$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \quad \text{if} \quad x_i \leq y_i \quad \text{for} \quad i = 1, \ldots, n \ .$$

The set of intervals in $(\{0,1\}^n, \leq)$ is

$$I_n = \{[x,y] : \ x, y \in \{0,1\}^n \quad \text{and} \quad x \leq y\} \ ,$$

and $[x,y] = \{z \in \{0,1\}^n : \ x \leq z \leq y\}$. We say that $G = (V,E)$ is *I_n-representable* if there is a map $f : V \to I_n$ such that, for all distinct u and v in V,

$$\{u,v\} \in E \Leftrightarrow f(u) \cap f(v) \neq \emptyset \ .$$

It is easily seen that every G is I_n-representable for some n. We refer to the minimum n for which G is I_n-representable as the *Boolean interval dimension* of G and denote it by $b(G)$. We allow $b(G) = 0$ for the 0-dimensional lattice with single point 0 and $I_0 = \{[0,0]\}$.

Theorem 1. *For every G, $b(G) = d(G^c)$.*

Proof. We show first that $d(G^c) \leq b(G)$. If $b(G) = 0$ then $G = K_m$, so $G^c = (V, \emptyset)$ and $d(G^c) = 0$. Assume henceforth that $b(G) \geq 1$. Suppose G is I_n-representable with $f : V \to I_n$ as above. Define $T : I_n \to \{0,1,2\}^n$ by

$$T[x,y] = x + y \qquad \text{(addition componentwise)} \quad .$$

Thus T maps each interval $f(u)$ into a ternary vector in $\{0,1,2\}^n$. Let

$$T(u) = Tf(u) = (Tu_1, \ldots, Tu_n) \ .$$

CLAIM. Given $u \neq v$, $\{u,v\} \in E^c$ if and only if $(Tu_i, Tv_i) \in \{(0,2),\ (2,0)\}$ for some $i \in \{1,\ldots,n\}$.

Let B_i denote the complete bipartite graph that has independent sets $A_{i1} = \{u : Tu_i = 0\}$ and $A_{i2} = \{u : Tu_i = 2\}$ when neither of these sets is empty. It then follows immediately from the Claim that the set of defined B_i covers G^c, hence that $d(G^c) \leq b(G)$.

Proof of Claim. Given $u \neq v$ let $f(u) = [x^1, y^1]$, $f(v) = [x^2, y^2]$. Suppose $\{u,v\} \in E$. Then, since $f(u) \cap f(v) \neq \emptyset$, we have $x^j \leq z \leq y^j$ for $j = 1,2$ for some $z \in \{0,1\}^n$. For such a z,

$$\begin{aligned} z_i = 0 &\Rightarrow x_i^1 = x_i^2 = 0 \Rightarrow \max\{Tu_i, Tv_i\} \leq 1, \\ z_i = 1 &\Rightarrow y_i^1 = y_i^2 = 1 \Rightarrow \min\{Tu_i, Tv_i\} \geq 1, \end{aligned}$$

so no i has $(Tu_i, Tv_i) \in \{(0,2),(2,0)\}$. Conversely, suppose no i has $(Tu_i, Tv_i) \in \{(0,2),(2,0)\}$, so for every i, (Tu_i, Tv_i) is one of $(0,0),(1,0),(0,1),(1,1),(1,2),(2,1)$ and $(2,2)$. Observe that

$$(x_i^1, y_i^1) = \begin{cases} (0,0) & \text{if} \quad Tu_i = 0 \\ (0,1) & \text{if} \quad Tu_i = 1 \\ (1,1) & \text{if} \quad Tu_i = 2 \end{cases}$$

and similarly for (x_i^2, y_i^2) and Tv_i. Define $z \in \{0,1\}^n$ by

$$\begin{aligned} z_i &= 0 \quad \text{if} \quad Tu_i + Tv_i \leq 2 \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

It follows that $x^j \leq z \leq y^j$ for $j = 1,2$, so $f(u) \cap f(v) \neq \emptyset$ and $\{u,v\} \in E$. This completes the proof that $\{u,v\} \in E \Leftrightarrow (Tu_i, Tv_i) \in \{(0,2),(2,0)\}$ for no i, which is equivalent to the Claim. $\square$

We now show that $b(G^c) \leq d(G)$. If $d(G) = 0$ then $E = \emptyset$, $G^c = K_m$ and $b(G^c) = 0$. Assume henceforth that $d(G) \geq 1$. Suppose $\mathcal{B} = \{B_1, \ldots, B_n\}$ covers G, $|\mathcal{B}| = n$. We reverse the process of the first part of the proof under the bijection T. Let A_{i1} and A_{i2} be B_i's independent sets. Define $T(u) = (Tu_1, \ldots, Tu_n)$ in $\{0,1,2\}^n$ for each $u \in V$ by

$$Tu_i = \begin{cases} 0 & \text{if} \quad u \in A_{i1} \\ 1 & \text{if} \quad u \notin A_{i1} \cup A_{i2} \\ 2 & \text{if} \quad u \in A_{i2} \ . \end{cases}$$

Set $f(u) = T^{-1}(T(u))$. Thus, when $f(u) = [x, y]$, $(x_i, y_i) = (0,0)$ if $Tu_i = 0, = (0,1)$ if $Tu_i = 1$, and $= (1,1)$ if $Tu_i = 2$. It then follows that, when $u \neq v$,

$$\{u, v\} \in E \Leftrightarrow [(Tu_i, Tv_i) \in \{(0,2),(2,0)\} \text{ for some } i] \Leftrightarrow f(u) \cap f(v) = \emptyset \ .$$

Thus $\{u, v\} \in E^c \Leftrightarrow f(u) \cap f(v) \neq \emptyset$, so G^c is I_n-representable. It follows that $b(G^c) \leq d(G)$. ■

Our next result gives the minimal forbidden induced subgraphs for $b(G) \leq 1$ and $d(G) \leq 1$. We precede its statement by recalling a reduction procedure that simplifies the analysis of intersection representations of graphs.

Given $G = (V, E)$, define the binary relation $\approx$ on V by

$$\begin{aligned} u \approx v \quad \text{if} \quad u = v \quad \text{or} \quad [\ & u \neq v, \{u, v\} \in E \text{ and, for all } x \in V \backslash \{u, v\}, \\ & \{u, x\} \in E \Leftrightarrow \{v, x\} \in E] \ . \end{aligned}$$

It is easily seen that $\approx$ is an equivalence relation. Let $V/\approx$ be the set of equivalence classes in V determined by $\approx$, denote by $[u]$ the class that contains u, and observe that if $[u] \neq [v]$ then either

$$\begin{aligned} &\{x, y\} \in E \quad \text{for all} \quad x \in [u] \quad \text{and all} \quad y \in [v], \text{ or} \\ &\{x, y\} \notin E \quad \text{for all} \quad x \in [u] \quad \text{and all} \quad y \in [v] \ . \end{aligned}$$

It follows that E' defined on $V/\approx$ by

$$\{[u], [v]\} \in E' \quad \text{if} \quad [u] \neq [v] \quad \text{and} \quad \{u, v\} \in E$$

is unambiguous and that $G/\approx$, defined as $(V/\approx, E')$, is a graph. We refer to $G/\approx$ as a *reduced graph* and note that any graph G is a reduced graph if and only if $G = G/\approx$ under identification of singleton equivalence classes with their vertices. For an arbitrary graph G, the assignment of the same interval in I_n to all vertices in $[u]$ for each $[u] \in V/\approx$ shows that $G/\approx$ is I_n-representable if and only if G is I_n-representable. It follows that

$$b(G) = b(G/\approx)$$

and that the sets of graphs that are minimal forbidden induced subgraphs for $b(G) \leq k$ and $b(G/\approx) \leq k$ are identical. Figure 3 notes all reduced graphs with three or four vertices.

Figure 3 about here

Lemma 2. *$b(G) \leq 1$ if and only if no induced subgraph of G is a K_3^c, P_4 or C_4. $d(G) \leq 1$ if and only if no induced subgraph of G is a K_3, P_4 or C_4^c.*

Proof. The three special graphs forbidden for $d \leq 1$ are the irreflexive complements of those for $b \leq 1$, so the second half of the lemma follows from the first half and Theorem 1.

We prove the first half. Assume without loss of generality that G is reduced. Since $I_1 = \{[0,0],[0,1],[1,1]\}$, its intersection graph is a P_3. Therefore $b(G) \leq 1$ if and only if G is an induced subgraph of P_3. The only G with $|V| \leq 3$ that is not an induced subgraph of P_3 is the independent 3-set K_3^c. The only G's with $|V| = 4$ that do not have an induced K_3^c are P_4 and C_4: see Figure 3. Hence K_3^c, P_4 and C_4 are the minimal forbidden induced subgraphs for $b(G) \leq 1$ that have fewer than five vertices.

The proof is completed by noting that every reduced G with $|V| \geq 5$ has K_3^c, P_4 or C_4 as an induced subgraph. Assume that $|V| \geq 5$. Since a reduced graph has at most one vertex with edges to all others, we can assume that $|V| \geq 4$ and no vertex has edges to all others. We suppose that G has no induced K_3^c, P_4 or C_4 and obtain a contradiction.

Fix $u \neq v$ with $\{u,v\} \notin E$. To avoid an induced K_3^c, the other vertices partition into

$$\begin{aligned} V_1 &= \{x : \{x,u\} \in E,\ \{x,v\} \notin E\} \\ V_2 &= \{x : \{x,u\}, \{x,v\} \in E\} \\ V_3 &= \{x : \{x,u\} \notin E,\ \{x,v\} \in E\}\ . \end{aligned}$$

Each nonempty V_j is a clique, else we get an induced C_4 from V_2 (and u and v) or an induced K_3^c from $V_1 \cup V_3$. If $V_2 \neq \emptyset$, avoidance of P_4 forces edges from everything in V_2 to everything in $V_1 \cup V_3$, which contradicts our assumption that no vertex has edges to all others. If $V_2 = \emptyset$, avoidance of P_4 implies no edges between V_1 and V_3, and this contradicts the assumption that G is reduced. ∎

The second half of Lemma 2 suggests the possibility of characterizing $d(G)$ as the chromatic number of a class of restricted edge colorings of G that prohibit monochromatic K_3's and avoid other simple color combinations that prevent the construction of a cover $\mathcal{B}$ such that all edges of the same color appear in a single $B \in \mathcal{B}$. We note shortly that this possibility is elusive for the general case. However, it has an elegant realization for the important subclass of triangle-free graphs.

As a basis for discussion, we say that an edge coloring $c : E \to \{1, 2, \ldots, k\}$ for $G = (V, E)$ is *simply-restricted* if no induced K_3 is monochromatic and the vertex-disjoint edges in an induced P_4 or C_4^c have different colors: see the top row of Figure 4.

Figure 4 about here

We denote the chromatic number of a simply-restricted edge coloring of G by $\chi_s(G)$. It is 0 if $E = \emptyset$ and is otherwise the minimum k for which G has a simply-restricted $c : E \to \{1, 2, \ldots, k\}$. We have $\chi_s(G) \le d(G)$, for if $\mathcal{B} = \{B_1, \ldots, B_m\}$ covers G then c is simply-restricted when $c(e)$ is defined as the smallest j for which B_j includes $e \in E$.

Theorem 2. *$d(G) = \chi_s(G)$ for every triangle-free G.*

Remark. The theorem has a nice vertex-coloring version. Given $G = (V, E)$, let $G_E = (E, \mathcal{E})$ with an edge in $\mathcal{E}$ between distinct $e, e' \in E$ if $e \cap e' = \emptyset$ and the two lie in an induced P_4 or C_4^c of G. If G is triangle-free, then $d(G)$ is the vertex chromatic number of G_E.

Proof of Theorem 2. The conclusion is obvious if $E = \emptyset$, so assume that $E \neq \emptyset$. We noted above that $\chi_s(G) \le d(G)$. To prove the converse, let c be a simply-restricted edge coloring of K_3-free G onto $\{1, \ldots, m\}$. Let $E_j = \{e \in E : c(e) = j\}$.

We show that E_1 is included in the edge set of a complete bipartite subgraph of G. This is obvious if $|E_1| = 1$. Suppose E_1 has two edges, e_1 and e_2. If they share a vertex then they form a $P_3 = K_{1,2}$. If they are vertex-disjoint, our restrictions imply that they are "opposite" edges of a $C_4 = K_{2,2}$. When $|E_1| \ge 3$, we add vertices involved with E_1 sequentially, verifying after each addition that all E_1 edges between vertices used thus far are among the edges of a subgraph $K_{p,q}$ of G. Suppose this is true to a point at which we have such a $K_{p,q}$ and that $e = \{u, v\}$ with $c(e) = 1$ is not yet in the construction because at least one of u and v is new. Possibilities for this are illustrated in the second row of Figure 4.

Suppose the left diagram applies. If $A_1 = \{u\}$, we have $K_{1,q+1}$ and go to the next step. Suppose $|A_1| \ge 2$. Then for each $w \in A_1 \backslash \{u\}$ there is an E_1 edge from w to a vertex $y \in A_2$ and an edge (not necessarily E_1) from u to y. Our restrictions require an edge between w and v and forbid edges between u and w and between v and y. Hence we have a $K_{p,q+1}$ that includes all E_1 edges between vertices used thus far.

Suppose the right diagram of row 2 applies. Fix $w \in A_1$ and let $z \in A_2$ have $\{w, z\} \in E_1$. Then $\{u, v, w, z\}$ has exactly two more G edges, which for definiteness we can assume are

$\{u,z\}$ and $\{w,v\}$. Suppose $\{x,y\} \in E_1$ for $x \in A_1$ and $y \in A_2$. If $x \neq w$, there is no edge between x and u, else $\{x,z,u\}$ forms a K_3. Similarly, if $y \neq z$ then there is no edge between y and v. It follows that $\{x,v\}$, $\{y,u\} \in E$. Since every $x \in A_1$ has an E_1 edge with something in A_2, and every $y \in A_2$ has an E_1 edge with something in A_1, we conclude that the addition of u and v creates a $K_{p+1,q+1}$ subgraph of G that includes all E_1 edges between vertices used thus far.

The process for E_1 terminates when all vertices for E_1 are covered. We then have a complete bipartite subgraph B_1 of G that includes all edges in E_1. Similar constructions for each $j > 1$ yield complete bipartite subgraphs of G, say $B_2, \ldots, B_m$ that include all edges in $E_2, \ldots, E_m$ respectively. Hence $\mathcal{B} = \{B_1, \ldots, B_m\}$ covers G, so $d(G) \leq \chi_s(G)$. ∎

Difficulties in extending the approach of Theorem 2 to general graphs are indicated in the last two rows of Figure 4, where $\chi_s(G) < d(G)$. The edge coloring in each satisfies the restrictions of the top row, so both colorings are simply-restricted. Additional restrictions on colors that would increase the minimum number of colors needed for those cases are shown to the right of the graphs. If these restrictions on induced subgraphs are added to those of the top row, the proof for $\chi \leq d$ that precedes Theorem 2 remains valid. Hence the new restrictions are necessary for $\chi = d$, but it is probably false that they are sufficient when joined to the others. The situation is substantially complicated by the presence of triangles, and we do not pursue the matter here.

We conclude this section with the minimal forbidden induced subgraph characterizations of $b(G) \leq 2$ and $d(G) \leq 2$. The appendix outlines the proof for $b \leq 2$ using the intersection graph $G(I_2)$ for I_2 and reduced G's. The result for $d \leq 2$ follows from that for $b \leq 2$ and Theorem 1.

Figure 5 pictures $G(I_2)$ and the minimal forbidden induced subgraphs for $b \leq 2$ and $d \leq 2$. Those for d are the irreflexive complements of those for b. Their designations, (5.1) through (6.14), apply to both b and d. When needed, we use the b or d designation to avoid ambiguity. It is straightforward to check that no graph for $b \leq 2$ is an induced subgraph of $G(I_2)$, that none of the 5-vertex graphs is an induced subgraph of any of the 6-vertex graphs, and that all *proper* induced reduced subgraphs of the $b \leq 2$ graphs *are* induced subgraphs of $G(I_2)$. Hence all 21 are indeed minimal forbidden induced subgraphs of G for $b(G) \leq 2$. The proof in the appendix shows this for the 21 and then proves that there are no others.

Figure 5 about here

Theorem 3. *For every G, $b(G) \leq 2$ if and only if none of the 21 graphs for $b \leq 2$ on Figure 5 is an induced subgraph of G, and $d(G) \leq 2$ if and only if none of the 21 graphs for $d \leq 2$ on Figure 5 is an induced subgraph of G.*

Since the intersection graph $G(I_n)$ for I_n is finite, for every n the list of minimal forbidden induced subgraphs for $d \leq n$ or $b \leq n$ is finite. We make no attempt to identify the lists explicitly for $n \geq 3$, but in the next section we consider the minimum number of vertices $\sigma(n)+1$ that yield a forbidden induced graph for $d \leq n$, and we identify all such graphs for $n = 3$ and $n = 4$. As mentioned earlier, Section 4 notes that the list of minimal forbidden induced subgraphs for bipartite degree $\eta \leq 2$ is infinite.

3. Minimum Forbidden Graphs

We begin our examination of $\sigma(m) = \max\{k :$ every G with k vertices has $d(G) \leq m\}$ with a few easy observations.

Lemma 3. *$\sigma(1) = 2$, $\sigma(2) = 4$ and $\sigma(3) = 5$.*

Proof. $\sigma(1) = 2$ follows from Lemma 2, and $\sigma(2) = 4$ from Theorem 3. Each of (5.1)–(5.7) for $d \leq 2$ on Figure 5 can be covered by three complete bipartite graphs, so $\sigma(3) \geq 5$. The 6-vertex graph obtained by removing vertex 5 from the left graph on Figure 1 has $d = 4$, so $\sigma(3) = 5$. ∎

Lemma 4. *$\sigma(m)+1 \leq \sigma(m+1) \leq \sigma(m)+2$ for all $m \geq 1$.*

Proof. Let $\sigma(m) = k$. All G with $|V| \leq k$ can be covered by a $\mathcal{B}$ with $|\mathcal{B}| \leq m$. Hence every G with $|V| \leq k+1$ can be covered with $|\mathcal{B}| \leq m+1$ since $K_{1,q}$ can be used at one vertex. Therefore $\sigma(m+1) \geq k+1$. The smallest G for which $d(G) > m$ has $k+1$ vertices. Add a disjoint $K_{1,1}$ to such a G to get G' with $|V'| = k+3$ and $d(G') > m+1$. Then $\sigma(m+1) \leq k+2$. ∎

We verify the claims of Figure 1 before we turn to the main results of the section.

Lemma 5. *The left graph of Figure* 1 *has bipartite dimension* 5*, and the right graph has bipartite dimension* 11*.*

Proof. Let H denote the left graph on Figure 1. The only way to cover edge 35 is by a *star* ($K_{1,q}$) centered at 3 or 5. Assume without loss of generality that 3 is used. When 3 or some of its edges is removed, what remains of H has bipartite dimension 4, so $d(H) = 5$.

The right graph on Figure 1 uses two copies of H and a central vertex 8. It has several *squares* ($K_{2,2}$'s) but no subgraph $K_{p,q}$ with $\min\{p, q\} \geq 2$ and $p + q \geq 5$. No square has vertices from both copies of H. Some squares use 8, namely 8613, 8635, 8 10 15 13 and 8 10 13 11. Each of these covers only two H edges.

Edge $\{1, 15\}$ must use a star in $\mathcal{B}$ centered at 1 or 15. By symmetry we can assume that the 1-centered star (edges to $3, 4, 6, 15$) is in $\mathcal{B}$. There is no advantage then to using square 8613 since its remaining two uncovered edges can be covered in other ways. If square 8635 is used in $\mathcal{B}$, we have $\{3, 8\}$ uncovered. Consequently, there is no more efficient arrangement beyond the original 1-centered star than to use stars centered at 3 and 8. We still need three more members of $\mathcal{B}$ to cover the top H, and five for the bottom H, for 11 altogether. ∎

The following main result provides a focal point for the rest of the section. Again, H is the 7-vertex graph of Figure 1.

Theorem 4. *H is the only graph with $|V| \leq 7$ that has bipartite dimension* 5*.*

We defer the proof of Theorem 4 to the end of the section.

Corollary 1. *$\sigma(4) = 6$, $\sigma(5) = 8$ and $\sigma(6) = 9$.*

Proof. $\sigma(4) = 6$ follows from Theorem 4 and prior remarks. By Lemma 4, $\sigma(5) \in \{7, 8\}$. The smallest forbidden graph for $d = 6$ has more than eight vertices, for if $d(G) = 6$ for a G with $|V| = 8$ then every seven-vertex induced subgraph of G would be a copy of H, and that is clearly impossible. Hence $\sigma(5) = 8$. By Lemma 4, $\sigma(6) \in \{9, 10\}$, and since $d = 7$ for H and a disjoint triangle, $\sigma(6) = 9$. ∎

The second part of Lemma 5 says that $\sigma(10) \leq 14$. This, Lemma 4, and $\sigma(6) = 9$ imply that $[\sigma(7), \sigma(8), \sigma(9), \sigma(10)]$ is a strictly increasing subsequence of $10, 11, 12, 13, 14$. General bounds are given by the next corollary.

Corollary 2. *$m + 3 \leq \sigma(m) \leq \lceil \frac{15}{11} m + 1 \rceil$, for all $m \geq 10$.*

Proof. The lower bound is implied by $\sigma(6) = 9$ and Lemma 4. The upper bound follows from Lemmas 4 and 5. Since k disjoint copies of the right graph on Figure 1 give a G with $|V| = 15k$ and $d = 11k$, and those k copies plus disjoint H give a G with $|V| = 15k + 7$ and $d = 11k + 5$, we have

$$\begin{aligned} \sigma(11k-1) &\leq 15k-1 \\ \sigma(11k+4) &\leq 15k+6 \\ \sigma(11k+10) &\leq 15k+14 \ . \end{aligned}$$

Upper bounds on $\sigma(m)$ for $11k-1 < m < 11k+4$ and $11k+4 < m < 11k+10$ by Lemma 4 are $15k+1$, $15k+3$, $15k+4$, and $15k+5$ for the first m interval, and $15k+8$, $15k+10$, $15k+11$, $15k+12$, and $15k+13$ for the second m interval. It is easily checked that these values are no greater than $\lceil (15/11)m + 1 \rceil$. ■

Let

$$c_0 = \inf \frac{\sigma(m)}{m} \ .$$

The use of disjoint copies of graphs with relatively small $|V|$ to d ratios shows that for every $\epsilon > 0$, $\sigma(m)/m < c_0 + \epsilon$ for all sufficiently large m. It follows that $\sigma(m)/m \to c_0$. The smallest upper bound we now have for c_0 is $15/11 = 1.3636\ldots$, but we suspect that $c_0 \leq 4/3$.

Proof of Theorem 4

Lemma 5 gives $d(H) = 5$. By Lemma 3, $\sigma(3) = 5$, so $\sigma(4) \geq 6$ by Lemma 4. Hence all six-vertex graphs have $d \leq 4$. Lemma 4 also gives $\sigma(5) \geq 7$, so if any G has $|V| \leq 7$ and $d \geq 5$ then $|V| = 7$ and $d = 5$.

Let $\mathcal{H}$ denote the set of all seven-vertex graphs with bipartite dimension 5. Theorem 4 says that $\mathcal{H} = \{H\}$. To prove this we note first that if $G \in \mathcal{H}$ then G has the following properties:

P1. G has no induced C_4;
P2. G is connected;
P3. G has no induced K_3^c;
P4. Every 6-vertex induced subgraph of G has $d = 4$;
P5. Every 5-vertex induced subgraph of G has $d = 3$.

If G has an induced $C_4 = K_{2,2}$, use this as one member of $\mathcal{B}$ and stars centered at the other three vertices to get $d \leq 4$. If G is not connected, minimal covers of components give $d \leq 4$. If G has an independent three-set, stars centered at the other four vertices give $d \leq 4$. If a six-vertex induced subgraph has $d \leq 3$, a star centered at the other vertex gives $d \leq 4$. If a five-vertex induced subgraph has $d \leq 2$, two more stars give $d \leq 4$.

Theorem 3 shows that (5.1)–(5.7) for $d \leq 2$ in Figure 5 are the only five-vertex graphs with $d = 3$. Hence these seven graphs exhaust the possibilities for P5. We enhance them with a sixth vertex to discover all possibilities for P4.

Lemma 6. *A six-vertex graph has bipartite dimension* 4 *if and only if it is one of* (6.15) – (6.19) *on Figure* 6.

Figure 6 about here

Proof. It is easily checked that each of (6.15)–(6.19) has $d = 4$. The extensions of (5.1) through (5.7), listed by the vertices with edges to vertex 6 (top of Figure 6), that yield graphs (6.15)–(6.19) are as follows:

$$\begin{aligned}
&(5.1): \quad 123 \rightarrow (6.18)\\
&(5.3): \quad 12 \rightarrow (6.17),\ 123 \rightarrow (6.19)\\
&(5.4): \quad 12 \rightarrow (6.16),\ 123 \rightarrow (6.17),\ 145 \rightarrow (6.18)\\
&(5.5): \quad 12 \rightarrow (6.15),\ 123 \rightarrow (6.16),\ 1234 \rightarrow (6.17),\ 12345 \rightarrow (6.19)\\
&(5.6): \quad 12 \rightarrow (6.19)\\
&(5.7): \quad 12 \rightarrow (6.17),\ 15 \rightarrow (6.18)\ .
\end{aligned}$$

Here is a summary of how other connections with vertex 6 are covered by $\mathcal{B}$ with $|\mathcal{B}| = 3$. We use S_i to denote the star centered at i, $Q_{ij,ab}$ to denote the square with corners $iajb$, and $K_{A,B}$ to denote the complete bipartite graph of all edges between A and B.

(5.1). All but vertex 6 with edges to 123 use three stars.

(5.2). $1\,[S_1;\ Q_{23,45};\ Q_{24,35}]$; $12\,[K_{13,245};\ K_{14,235};\ K_{12,3456}]$; $123\,[K_{14,235};\ K_{12,3456};\ K_{13,2456}]$; $1234\,[K_{14,2356};\ K_{12,3456};\ K_{13,2456}]$; $12345\,[Q_{13,24};\ K_{56,1234};\ K_{125,346}]$.

(5.3). Others with one or two edges for 6 use three stars. $124\,[S_1;\ S_5;\ Q_{24,36}]$; $134\,[S_1;\ S_3;\ S_4]$; $1234\,[S_2;\ Q_{14,36};\ Q_{34,56}]$; $1245\,[S_1;\ S_5;\ Q_{24,36}]$; $12345\,[S_5;\ Q_{13,26};\ K_{36,1245}]$.

(5.4). Others with one or two edges to 6 use three stars. $124\,[S_1;\ S_5;\ Q_{24,36}]$; $134\,[S_4;\ S_5;\ Q_{13,24}]$; $234\,[S_2;\ S_3;\ S_4]$; $245\,[S_2;\ S_4;\ Q_{36,45}]$; $345\,[S_2;\ Q_{36,45};\ Q_{35,46}]$; $1234\,[S_1;\ Q_{24,36};\ Q_{34,56}]$; $1245\,[S_1;\ S_4;\ Q_{25,36}]$; $1345\,[S_4;\ S_5;\ Q_{13,26}]$; $2345\,[S_2;\ Q_{34,56};\ Q_{35,46}]$; $12345\,[S_4;\ Q_{13,26};\ K_{36,245}]$.

(5.5). 1 and 3 $[S_1;\ S_3;\ S_5]$; 13 and 134 $[S_1;\ S_3;\ S_4]$; 34 $[S_1;\ S_3;\ Q_{34,56}]$; 345 and 1345 $[S_1;\ Q_{34,56};\ Q_{35,46}]$.

(5.6). 1 $[S_1;\ Q_{23,45};\ Q_{24,35}]$; 2 and 3 and 34 $[S_2;\ S_3;\ S_4]$; 13 and 134 $[S_4;\ S_5;\ Q_{13,26}]$; 23 $[S_2;\ S_4;\ Q_{23,56}]$; 123 $[S_1;\ Q_{24,35};\ K_{23,456}]$; 234 $[S_2;\ S_4;\ K_{56,234}]$; 345 and 2345 $[S_2;\ Q_{34,56};\ Q_{35,46}]$; 1234 $[S_3;\ Q_{14,26};\ K_{56,234}]$; 1345 $[S_3;\ S_4;\ K_{26,1345}]$; 12345 $[S_3;\ Q_{24,56};\ K_{26,1345}]$.

(5.7). Others with one or two edges to 6, or with three edges to 6 as 134, 234 or 235, use three stars. 123 $[S_5;\ Q_{13,26};\ Q_{23,46}]$; 125 $[S_1;\ S_3;\ K_{25,346}]$; 135 $[S_4;\ S_5;\ Q_{13,26}]$; 345 and 2345 $[S_2;\ Q_{34,56};\ Q_{35,46}]$; 1234 $[Q_{13,26};\ Q_{25,34};\ Q_{24,36}]$; 1235 $[S_3;\ Q_{13,26};\ Q_{25,46}]$; 1345 $[S_3;\ S_5;\ Q_{14,26}]$; 12345 $[S_3;\ Q_{14,26};\ K_{25,346}]$. ■

The proof of Theorem 4 is completed when we repeat the process of the preceding proof by adding vertex 7 to graphs (6.15)–(6.19). We invoke P1 and P3 when they are useful. It will be seen that additions to (6.15) and (6.19) produce no graph with $d = 5$. The other three, which are the six-vertex induced subgraphs of H, yield only H for $d = 5$.

(6.15). Because $\sigma(2) = 4$, every enhancement of (6.15) with vertex 7 has $d \le 4$: $d \le 2$ for 7 and the left triangle, and $d \le 2$ for 7 and the right triangle.

(6.16). To avoid coverage by four stars, assume that 7 has edges to both a and b, or to both f and g. Take $\{a, 7\} \in E$ and $\{b, 7\} \in E$ for definiteness. To avoid a K_3^c, 7 must have an edge to c or to f (or g). Suppose $\{c, 7\} \in E$. We then get coverage by two $K_{2,2}$'s on the left (with 7) and two stars, e.g. S_e and S_f. Hence, to try to force $d = 5$, add an edge from 7 to e or 7 to f. If 7 to f, we also need 7 to e to avoid an induced C_4. If 7 to e, and perhaps 7 to f, we get coverage by four complete bipartite subgraphs as before. So the only way to avoid $d = 4$ is to have 7 going to all vertices of (6.16). But then the whole is covered by $S_f, S_g, Q_{a7,bc}$ and $K_{c7,abe}$.

Alternatively, suppose 7 has an edge to f along with edges to a and b, but no edge to c. Then $\{g, 7\} \in E$ to avoid K_3^c. At this point we have a copy of H. If we then add an edge to c or e from 7, we require 7 to be saturated and have the $|\mathcal{B}| = 4$ coverage at the end of the preceding paragraph.

(6.17). To avoid coverage by four stars, we need edges from 7 to a, b and c, or to f and g. Suppose 7 has edges to a, b and c but no others. Then two $K_{2,2}$'s cover $\{a, b, c, 7\}$, and $d \le 4$. We therefore add an edge from 7 to e, or to f and e (avoiding C_4), but not g. Both cases have $d \le 4$ with the two $K_{2,2}$'s and S_e and S_f. So we try to force $d = 5$ by having edges from 7 to all other vertices. However, this is covered by $S_a, S_b, Q_{g7,ef}$ and $K_{e7,bcfg}$.

Alternatively, suppose at the outset that 7 has edges to f and g. To avoid an induced K_3^c, we also need an edge from 7 to a or e. If to a, but no others, we have H. If to e but no others, we get coverage by two $K_{2,2}$'s for $\{e, f, g, 7\}$ and two stars for the left part. If 7 has edges to both a and e, we also need $\{b, 7\}$, $\{c, 7\} \in E$ to avoid an induced C_4, and therefore have the situation at the end of the preceding paragraph. If 7 has edges to one or both of b and c, and to only one of a and e, the latter must be e (else an induced C_4), and again we have coverage by two $K_{2,2}$'s and two stars.

(6.18). We get coverage by four stars if 7 has no edge to c or e, so assume $\{c, 7\} \in E$. If $\{e, 7\} \in E$ also, then 7 has edges to f and g to avoid an induced C_4, and in this case $S_a, S_b, Q_{7e,fg}$ and $K_{7f,ceg}$ cover G. So assume that $\{e, 7\} \notin E$. Then, to avoid K_3^c, $\{a, 7\} \in E$, and either b or f (or g) has an edge to 7. If the latter is $\{b, 7\} \in E$ and neither $\{f, 7\}$ nor $\{g, 7\}$ is in E, we have H. If 7 goes exactly to c, a, b and f, then $S_a, S_b, Q_{cf,g7}$ and $Q_{ce,fg}$ cover the graph. If 7 goes only to c, a and f, the same coverage obtains. If 7 goes to c, a, f and g, S_a, S_e and two $K_{2,2}$'s for $\{c, f, g, 7\}$ cover G. Finally, if 7 goes to all other vertices except e, G is covered by $S_b, S_e, Q_{cg,f7}$ and $K_{c7,afg}$.

(6.19). To avoid coverage by four stars, 7 must have edges to f and g or to a, b, and c. Suppose 7 has edges to a, b and c, so $\{a, b, c, 7\}$ is a K_4, which is covered by two $K_{2,2}$'s. To avoid complete coverage by two additional stars, we need $\{f, 7\} \in E$, hence $\{e, 7\} \in E$ to avoid an induced C_4. But then S_e and S_f complete the coverage unless $\{g, 7\} \in E$, so assume that 7 goes to all other vertices. Then G is covered by $S_a, S_b, Q_{g7,ef}$ and $K_{e7,bcfg}$.

Alternatively, suppose at the outset that 7 has edges to f and g. If there are no other edges besides perhaps $\{e, 7\}$, then $d \leq 4$, so assume without loss of generality that $\{a, 7\} \in E$. This forces $\{e, 7\} \in E$ to avoid an induced C_4. Then, whether 7 has edges to b or c, G is covered by $S_b, S_c, Q_{f7,eg}$ and $K_{e7,afg}$. ∎

4. Bipartite Degree

The bipartite degree $\eta(G)$ of $G = (V, E)$ is the minimum, over all covers $\mathcal{B}$ of G, of the maximum over V of the number $\mathcal{B}(v)$ of members of $\mathcal{B}$ that have v as a vertex. We have $\eta(G) \leq d(G)$, and it appears that most graphs have $\eta(G) < d(G)$. For example, only three of the 21 graphs for d on Figure 5 have $\eta = 3$, namely K_5, (6.4) and (6.14).

It is clear that $\eta(G) = d(G)$ if and only if every cover of G has some one vertex in at least $d(G)$ members of the cover, but it seems hard to characterize equality in any simple way. We suspect, however, that equality holds for complete graphs and support this by the following result.

Lemma 7. *$\eta(K_n) = d(K_n)$ for all $n \leq 16$.*

Proof. It is easily seen that $\eta(K_n) = d(K_n)$ for $n \leq 4$. By Lemma 1, $d(K_n) = 3$ for $5 \leq n \leq 8$ and $d(K_n) = 4$ for $9 \leq n \leq 16$. It therefore suffices to show that $\eta(K_5) = 3$ and $\eta(K_9) = 4$.

Suppose $n = 5$. If a $K_{2,3}$ is used as one member of $\mathcal{B}$ for K_5, it leaves a triangle completely uncovered, one of whose vertices must be in two complete bipartite graphs that cover the triangle. Since every vertex is involved in the $K_{2,3}$, some vertex is in at least three members of $\mathcal{B}$. Suppose $K_{2,3}$ is not used for $\mathcal{B}$, but a square ($K_{2,2}$) is used. If $\eta \leq 2$ then the two other edges from a corner of the square must be only in the same $B \in \mathcal{B}$. However, the end vertices of those edges complete a triangle, and the edge between them must be in another $B' \in \mathcal{B}$, contradicting $\eta(K_5) \leq 2$. Hence we cannot have $\eta \leq 2$ unless $\mathcal{B}$ uses only stars, but in that case $\eta \leq 2$ is also impossible. Hence $\eta(K_5) = d(K_5) = 3$.

Assume henceforth that $n = 9$. We suppose that $\eta(K_9) \leq 3$ and will obtain a contradiction. For convenience, we assign a different color to (the edges and vertices in) each member of $\mathcal{B}$. It can be assumed that $\mathcal{B}$ uses no $K_{p,q}$ with $\max\{p, q\} \geq 5$, else it leaves a completely uncovered monochromatic 5-vertex set and, by the preceding analysis, one of these five must have three other colors in any cover.

Suppose $\mathcal{B}$ has a red $K_{p,4}$, $p \leq 4$. This leaves a completely uncovered five-set $\{1, 2, 3, 4, 5\}$, the first four vertices of which are red. To have $\eta \leq 3$ with those four, we need a blue $K_{2,2}$, say with $A_1 = \{1, 2\}$ and $A_2 = \{3, 4\}$ for definiteness, and either a green $K_{2,2}$ with $A'_1 = \{1, 3\}$ and $A'_2 = \{2, 4\}$, or a green edge between 1 and 2 and a yellow edge between 3 and 4. However, vertex 5 then forces one of 1 through 4 to have a fourth color. For example, if the two $K_{2,2}$'s are used for $\{1, 2, 3, 4\}$, each can be extended to a $K_{2,3}$, and then 5 will have two blue edges and two green edges to $\{1, 2, 3, 4\}$. However, two of these must coincide since all $A_i \cap A'_j \neq \emptyset$, and the open edge to 5 must use a fourth color.

The preceding analysis to get $\eta(K_9) \leq 3$ shows that we cannot have a monochromatic 4-set with no edges of the same color between those four vertices. This forces the situation at the top of Figure 7 in which three edges from vertex 1 have one color (r = red) and another three have another color (b = blue). If there are no red or blue edges between

$\{2,3,4\}$ and $\{5,6,7\}$, we can regard (r or b) as one color and have a monochromatic 6-set with no internal edges of the same color, thus forcing $\eta \geq 4$ overall. Suppose for definiteness that there is a red edge from 4 to 5. Then, to complete a $K_{2,3}$ in red, we also need red edges from 5 to 2 and 3. Since this leaves an (r or b) set $\{2,3,4,6,7\}$ with no internal edges, $\eta \geq 4$ follows from $\eta(K_5) = 3$. Suppose we add more red and blue edges to the preceding construction either by making red edges from 6 to $\{2,3,4\}$ for a red $K_{3,3}$ or by adding blue edges from 4 to $\{5,6,7\}$: see Figure 7. In the first case, 5 and 6 are both red and blue, and 7 is blue. Then edges for triangle $\{5,6,7\}$ will force 5 or 6 to have two new colors, contradicting $\eta \leq 3$.

Figure 7 about here

Suppose then that there is no $K_{3,3}$ within $\{1,2,\ldots,7\}$ but there are red and blue $K_{2,3}$'s as on the right of Figure 7. We then have $\{2,3,6,7\}$ as an (r or b) set with no internal edges. If one of vertices 8 and 9 is not also involved with red or blue, we have the situation analyzed for $K_{p,4}$ and contradict $\eta \leq 3$. So suppose that 8 has red edges to $\{2,3,4\}$, and 9 has blue edges to $\{5,6,7\}$. Since edge $\{8,9\}$ can be neither red, blue nor green, let it be yellow (y), as shown on the bottom of Figure 7. At this point, 8 has colors r, g, y, and 9 has b, g, y. Suppose $\{5,8\}$ is green, so $\{5,9\}$ is also green ($K_{2,2}$). To prevent four colors at vertex 5, we also need $\{5,6\}$ and $\{5,7\}$ green. But then $\{6,7,8,9\}$ is a green set with no internal green edges, and this contradicts $\eta \leq 3$. So we must have $\{5,8\}$ and, by analogy, $\{4,9\}$ yellow. Then 4 and 5 each has colors r, b and y. It follows that, since we have exhausted the use of red and blue in $K_{3,3}$'s, edges $\{4,2\}, \{4,3\}, \{5,6\}$ and $\{5,7\}$ must be yellow. However, this gives a yellow $K_{4,4}$ with $A_1 = \{2,3,5,9\}$ and $A_2 = \{4,6,7,8\}$, for our final contradiction to $\eta(K_9) \leq 3$. ∎

Although we do not know whether $\eta(K_n) = d(K_n)$ for all n, complete graphs provide an easy proof that η is unbounded.

Theorem 5. $\eta(K_n) \to \infty$.

Proof. Suppose otherwise, so $\max \eta(K_n) = k < \infty$. Let $n_0 = \min\{n : \eta(K_n) = k\}$. Then coverage of K_N for $N \gg n_0$ by k $B_i \in \mathcal{B}$ forces at least one B_i to be a $K_{p,q}$ with $q \geq n_0$, and since $\eta(K_q) = k$ we reach the conclusion that $\eta(K_N) \geq k+1$. ∎

We conclude our discussion of bipartite degree with comments on minimal forbidden induced subgraphs for the quadratic case of $\eta \le 2$. We identify all such graphs for $|V| \le 6$ and then note two infinite minimal forbidden families.

Lemma 8. *K_5 is the only five-vertex graph with $\eta(G) > 2$.*

Proof. Let $V = \{1,2,3,4,5\}$. Suppose G has a $K_{2,3}$ as a subgraph, say with $A_1 = \{1,2,3\}$ and $A_2 = \{4,5\}$. The potential edges uncovered by $K_{2,3}$ are those in triangle $\{1,2,3\}$ and $\{4,5\}$. A $K_{1,2}$ or $K_{1,1}$ for $\{1,2,3\}$ covers two or one of its edges, if present, and a vertex-disjoint $K_{1,1}$ covers $\{4,5\}$. Hence $\eta \le 2$ as long as at least one of $12, 23$ and 13 is not an edge of G. It follows by appropriate vertex labeling that $\eta(G) \le 2$ if $|E|$ is 8 or 9. Moreover, $\eta \le 2$ if $|E|$ is 6 or 7 and G has a $K_{2,3}$.

The only cases for $G \ne K_5$ that remain have $|E| \le 7$ and no $K_{2,3}$ subgraph in G. Suppose G has no $K_{2,3}$ and $|E| = 7$. There are exactly two such graphs. One has degree sequence $(4,3,3,3,1)$ and appears as (5.6) for d on Figure 5. The other has degree sequence $(4,3,3,2,2)$ and is the same as (5.7) with an additional edge from the top vertex to the bottom-left vertex. Each has $\eta = 2$. (The latter uses overlapping $K_{2,2}$'s.) We omit the straightforward analysis for $|E| \le 6$. ∎

Theorem 6. *There are exactly six six-vertex graphs with $\eta(G) > 2$ that have no induced K_5. They are described in Figure 2.*

Proof. The proof has three parts. Part 1 verifies that the six graphs of Figure 2 have $\eta > 2$. Part 2 notes that all others for d with six vertices on Figure 5 have $\eta \le 2$. Part 3 shows that all one-vertex extensions of (5.1) and (5.3)–(5.7) for d on Figure 5 (except those of Figure 2) have $\eta \le 2$. Because all six-vertex graphs with $d > 2$ are given by (5.1)–(5.7), their one-vertex extensions, and (6.1)–(6.14), the three parts and Lemma 8 complete the proof of Theorem 6.

Part 1

We use the notation of the proof of Lemma 6 for stars, squares and other complete bipartite graphs and associate a color with each B_j proposed for $\mathcal{B}$, as in the proof of

Lemma 7. In addition, $S_{i,A}$ denotes the i-centered star with edges from i to precisely the vertices in A. We try to get $\eta \leq 2$ and obtain contradictions.

(6.4). Consider the induced subgraph on $\{1,2,3,4,5\}$, with vertices labeled as on Figure 2. This can be covered for $\eta \leq 2$ in only three basic ways: by a red $Q_{13,24}$ and blue $Q_{14,35}$; by a red $Q_{13,24}$, a blue $S_{1,35}$ and a green $S_{4,5}$; or by three stars such as $S_{1,235}$, $S_{4,15}$ and $S_{3,24}$. Suppose $Q_{13,24}$ and $Q_{14,35}$. Then edge 16 is red or blue, and we can assume red for definiteness. This also requires a red 36 for $K_{13,246}$. Then 46 must be blue, which forces a blue $K_{14,356}$. Then 26 can't be red and it can't be blue (else a blue $K_{124,356}$, but 25 is not in E), so it must be a third color, and this forces three colors at vertex 6.

Suppose next that $\{1,2,3,4,5\}$ is covered by a red $Q_{13,24}$, a blue $S_{1,35}$ and a green $S_{4,5}$. Suppose 16 is red, which gives a red $K_{13,246}$. Then 46 can't be red, so it must be green, for a green $S_{4,56}$. Edge 56 can't be green (else a green triangle) or blue (the vertex 6 is tricolored), so it must be another color, which forces 5 to be tricolored. Suppose then that 16 is not red. Then it is blue for a blue $S_{1,356}$. Then 36 can't be red (else 16 is also red) or blue (else a blue triangle), so vertex 3 must be tricolored.

Suppose finally for (6.4) that we have a red $S_{1,235}$, a blue $S_{4,15}$ and a green $S_{3,24}$. If 16 is red, then 26 and 36 must be green for $\eta \leq 2$, but this gives a green triangle. If 16 is not red, it must be blue for a blue $Q_{46,15}$. Then 46 must be green, which forces a green $Q_{36,24}$. This leaves edge 36 uncovered. It cannot be green, hence must be red, but then 6 is tricolored.

(6.20). Suppose $\mathcal{B}$ uses a square, say a red $Q_{13,24}$. It must then have a blue $S_{1,356}$ if $\eta \leq 2$. Assume edge 56 is green. This would then require green 45 and 62, but path 4562 is not part of a complete bipartite subgraph. Suppose that $\mathcal{B}$ uses only stars. If a star involves an outer edge and one to the center, say 23 and 21, then 13 and 34 must be in a different star, and edge 14 forces vertex 1 to be tricolored. If no star uses an outer edge and one to the center, it is obvious that $\eta \leq 2$ is impossible.

(6.21). Suppose square $Q_{14,23}$ is in $\mathcal{B}$. Then edges 23 and 35 are in another bipartite subgraph, either $S_{3,25}$ or $Q_{25,34}$. For $\eta \leq 2$, the first case requires $S_{4,56}$, in which event 5 ends up tricolored, and the second forces 4 to be tricolored. If $Q_{25,34}$ is in $\mathcal{B}$ and neither end square is in $\mathcal{B}$, $\eta \leq 2$ would force 13 and 46 to be the same color as 34, and this violates complete bipartiteness. The only other possibility for $\eta \leq 2$ is to use only stars in $\mathcal{B}$. If we use $S_{1,23}$, $\eta \leq 2$ would force triangle $\{2,3,4\}$ to be monochromatic. With $S_{2,13}$ or $S_{2,134}$, we also need $S_{3,145}$, and either 4 or 5 ends up tricolored. If we use $S_{3,12}$ or more generally $S_{3,A}$ with $1,2 \in A$, and we work left to right with other stars for $\eta \leq 2$, we encounter a problem at vertex 4.

(6.22). There are three possibilities for a 2×3 complete bipartite subgraph: $K_{34,256}$, $K_{35,246}$, and $K_{26,345}$. The first of these violates $\eta \leq 2$ at vertex 5 when 12 and 25 are assigned the same color; the second violates $\eta \leq 2$ at vertex 4 when 12 and 24 are assigned the same color; the third leaves triangle $\{3,4,5\}$ uncovered in the midst of $K_{26,345}$ and forces $\eta = 3$. Assume henceforth that $\mathcal{B}$ uses no 2×3 complete bipartite subgraph.

Up to symmetries, six squares are possible for $\mathcal{B}$: $Q_{25,34}$, $Q_{26,34}$, $Q_{26,45}$, $Q_{23,45}$, $Q_{36,45}$ and $Q_{34,56}$. For the first four of these, we encounter an impossibility for $\eta \leq 2$ when edge 12 and the other uncovered end at 2 are assigned the same color. For $Q_{36,45}$, the problem arises at vertex 4 when 25 and 45 are assigned the same color. For $Q_{34,56}$, a similar problem arises at 4 when 23 and 34 are assigned the same color. Hence $\eta \leq 2$ is impossible when $\mathcal{B}$ uses a square.

Suppose $\mathcal{B}$ uses only stars. Since K_4 cannot be covered only by stars for $\eta = 2$, and (6.22) includes a K_4, we conclude that $\eta = 3$ for (6.22).

(6.23). The possibilities for a 2×3 bipartite subgraph are the three for (6.22) plus $K_{23,145}$, which leaves triangle $\{4,5,6\}$ uncovered but has edges to 4 and 5, thus forcing 4 or 5 to be tricolored. The other three 2×3 cases encounter the same problems as before. Assume henceforth that $\mathcal{B}$ uses no 2×3 bipartite subgraph and, for the reason of the preceding paragraph, is not composed entirely of stars. It must therefore have at least one square, so we consider those cases as follows.

$Q_{15,23}$: Triangle $\{2,3,4\}$ is uncovered but has two vertices in Q.

$Q_{14,23}$: Triangle $\{2,3,5\}$ is uncovered but has two vertices in Q.

$Q_{25,34}$: Triangle $\{3,4,6\}$ is uncovered but has two vertices in Q.

$Q_{26,35}$: Triangle $\{3,4,5\}$. $Q_{26,34}$: Triangle $\{3,4,5\}$. $Q_{26,35}$: Triangle $\{3,4,5\}$. $Q_{35,46}$: Triangle $\{2,3,5\}$. $Q_{36,45}$: Triangle $\{2,4,5\}$. $Q_{34,56}$: Triangle $\{2,3,4\}$.

(6.24). If $K_{23,145}$ is used in $\mathcal{B}$, it leaves $\{4,5,6\}$ uncovered. Since (6.24) has an induced K_4, we cannot get $y \leq 2$ using only stars. $Q_{14,23}$ leaves triangle $\{2,3,5\}$ uncovered, and $Q_{23,45}$ leaves $\{1,2,3\}$ uncovered. If $Q_{25,34}$ is in $\mathcal{B}$ and 13 and 34 are assigned the same color, vertex 4 will be tricolored.

Part 2

Figure 8 shows how seven six-vertex d graphs of Figure 5 satisfy $\eta = 2$. Edges labeled j identify $B_j \in \mathcal{B}$. The other (6.k) graphs for d on Figure 5 besides (6.4) have obvious covers for $\eta \leq 2$.

Figure 8 about here

Part 3

Vertex labels from the top of Figure 6 are used to identify covers with $\eta \leq 2$ according to vertices with edges to vertex 6.

(5.1). 1 [$S_{2,136}$; $S_{4,35}$; $S_{1,5}$]; 12 [$S_{2,136}$; $S_{1,56}$; $S_{4,35}$]; 13 [$Q_{13,26}$; $S_{4,35}$; $S_{1,5}$]; 123 [$Q_{13,26}$; $S_{2,6}$; $S_{4,35}$; $S_{1,5}$]; 124 [$Q_{24,36}$; $S_{1,256}$; $S_{4,5}$]; 1234 [$Q_{13,26}$; $Q_{24,36}$; $S_{5,14}$]. 12345 is (6.20).

(5.3). 1 [$S_{1,236}$; $S_{3,245}$; $S_{4,5}$]; 3 [$S_{3,12456}$; $S_{1,2}$; $S_{4,5}$]; 12 [$Q_{12,36}$; $S_{1,2}$; $S_{3,45}$; $S_{4,5}$]; 13 [$Q_{13,26}$; $S_{3,145}$; $S_{4,5}$]; 14 [$Q_{14,36}$; $S_{3,25}$; $S_{1,2}$; $S_{4,5}$]; 123 [$Q_{13,26}$; $S_{2,6}$; $S_{3,145}$; $S_{4,5}$]; 124 [$Q_{14,36}$; $S_{2,16}$; $S_{3,25}$; $S_{4,5}$]; 134 [$Q_{14,36}$; $S_{3,256}$; $S_{1,2}$; $S_{4,5}$]; 1234 [$K_{36,124}$; $Q_{34,56}$; $S_{1,2}$]; 1245 [$K_{36,1245}$; $S_{1,2}$; $S_{4,5}$]; 12345 [$K_{36,1245}$; $S_{1,2}$; $S_{3,6}$; $S_{4,5}$].

(5.4). Single edges to 6 are routine; 12 [$S_{1,6}$; $S_{2,136}$; $S_{3,45}$; $S_{4,5}$]; 13 [$Q_{13,26}$; $S_{3,45}$; $S_{4,5}$]; 14 [$S_{1,26}$; $S_{3,245}$; $S_{4,56}$]; 23 [$S_{2,16}$; $S_{3,2456}$; $S_{4,5}$]; 24 [$Q_{24,36}$; $S_{1,2}$; $S_{5,34}$]; 34 [$Q_{34,56}$; $S_{3,24}$; $S_{1,2}$]; 45 [$Q_{36,45}$; $S_{2,13}$; $S_{4,5}$]; 123 [$Q_{13,26}$; $S_{2,6}$; $S_{3,45}$; $S_{4,5}$]; 124 [$Q_{24,36}$; $S_{1,26}$; $S_{5,34}$]; 134 [$Q_{34,56}$; $S_{1,26}$; $S_{3,24}$]; 145 [$Q_{36,45}$; $S_{1,6}$; $S_{2,13}$; $S_{4,5}$]; 234 [$Q_{24,36}$; $S_{1,2}$; $S_{3,56}$; $S_{4,5}$]; 245 [$Q_{25,36}$; $S_{1,2}$; $S_{4,356}$]; 345 [$Q_{34,56}$; $S_{4,5}$; $S_{3,26}$; $S_{1,2}$]; 1245 [$Q_{36,45}$; $S_{6,12}$; $S_{2,13}$; $S_{4,5}$]; 1345 [$Q_{13,26}$; $Q_{36,45}$; $S_{4,5}$]; 2345 [$K_{36,245}$; $S_{1,2}$; $S_{3,6}$; $S_{4,5}$]; 12345 [$K_{36,245}$; $S_{6,13}$; $S_{1,2}$; $S_{4,5}$]. 1234 is (6.21).

(5.5). Single, double and triple edges to 6 are routine; 1234 [$Q_{34,56}$; $S_{1,2}$; $S_{3,4}$; $S_{6,12}$]; 1345 [$Q_{36,45}$; $S_{1,2}$; $S_{4,5}$; $S_{6,13}$]; 12345 [$Q_{36,45}$; $S_{1,2}$; $S_{4,5}$; $S_{6,123}$].

(5.6). 1 [$Q_{25,34}$; $S_{1,6}$; $S_{2,15}$; $S_{3,4}$]; 2 [$Q_{25,34}$; $S_{2,156}$; $S_{3,4}$]; 3 [$Q_{25,34}$; $S_{2,15}$; $S_{3,46}$]; 12 [$Q_{25,34}$; $S_{1,6}$; $S_{2,156}$; $S_{3,4}$]; 13 [$Q_{13,26}$; $Q_{23,45}$; $S_{4,5}$]; 23 [$Q_{24,35}$; $S_{2,146}$; $S_{3,56}$]; 34 [$K_{34,256}$; $S_{2,15}$; $S_{3,4}$]; 123 [$K_{23,456}$; $Q_{13,26}$; $S_{4,5}$]; 134 [$Q_{14,26}$; $Q_{24,35}$; $S_{3,56}$]; 234 [$K_{34,256}$; $S_{2,156}$; $S_{3,4}$]; 1345 [$K_{34,256}$; $Q_{15,26}$; $S_{3,4}$]. 345 is (6.22), 1234 is (6.23), and 2345 and 12345 include K_5.

(5.7). Single edges to 6 are routine; 12 [$Q_{25,34}$; $S_{1,6}$; $S_{2,16}$; $S_{3,4}$]; 13 [$Q_{25,34}$; $S_{1,26}$; $S_{3,46}$]; 15 [$Q_{25,34}$; $S_{1,26}$; $S_{3,4}$; $S_{5,6}$]; 23 [$Q_{25,34}$; $S_{2,16}$; $S_{3,46}$]; 25 [$Q_{25,34}$; $S_{2,16}$; $S_{3,4}$; $S_{5,6}$];

34 $[K_{34,256};\ S_{1,2};\ S_{3,4}]$; 35 $[Q_{35,46};\ S_{2,14};\ S_{3,25}]$; 125 $[K_{25,346};\ S_{1,26};\ S_{3,4}]$; 134 $[K_{34,256};\ S_{1,26};\ S_{3,4}]$; 135 $[Q_{25,34};\ Q_{35,46};\ S_{1,26}]$; 234 $[K_{34,256};\ S_{2,16};\ S_{3,4}]$; 235 $[K_{25,346};\ S_{3,46};\ S_{1,2}]$; 345 $[K_{34,256};\ S_{1,2};\ S_{3,4};\ S_{5,6}]$; 1235 $[Q_{13,26};\ Q_{25,46};\ S_{3,45}]$; 1345 $[K_{34,256};\ S_{6,15};\ S_{1,2};\ S_{3,4}]$. 123 is (6.21), 1234 is (6.24), 2345 is (6.22), and 12345 is (6.23). ∎

Let W_n for $n \geq 3$ be the $2n$-vertex graph composed of a C_n and n other edges, called *outer edges*, one from each vertex of C_n to a degree-1 vertex. We illustrate W_5 on the left of Figure 9, covered by $K_{1,2}$'s (j stands for B_j in $\mathcal{B}$) so that $\eta(W_5) = 2$. An $\eta = 2$ cover for W_n is essentially unique up to orientation, for once a $K_{1,2}$ is assigned to an outer edge and an adjacent C_n edge we are forced to use similar $K_{1,2}$'s around the cycle.

Figure 9 about here

The other graphs of Figure 9 are variants of W_n. The middle graph replaces two outer edges and their degree-1 vertices by a chord between the involved C_n vertices. We refer to it as a *modified* W_n. The right graph, $W_{5,6}$, merges one outer edge of a W_5 with one outer edge of a W_6 to create a bridge between them. Except for the merged edges, W_5 and W_6 are vertex disjoint. $W_{n,m}$ for $\min\{n, m\} \geq 3$ is defined similarly. It has $2(n + m - 1)$ vertices.

It is proved in [4] that $\eta(W_{n,m}) = 3$, and every proper induced subgraph G of $W_{n,m}$ has $\eta(G) \leq 2$. Thus all $W_{n,m}$'s are minimal forbidden graphs for $\eta \leq 2$. A similar result holds for most modified W_n's.

Theorem 7. *Suppose G is a modified W_n whose chord does not create an induced C_j in G for $j \leq 4$. Then $\eta(G) = 3$, and every proper induced subgraph G' of G has $\eta(G') \leq 2$.*

Proof. Given the hypotheses, every member of a cover $\mathcal{B}$ of G must be a star. We label G's vertices as in Figure 10, identify each member of a cover with a color, and let $c(xy)$ denote the color of edge $\{x, y\}$.

Figure 10 about here

To prove that $\eta(G) = 3$, we suppose that $\eta(G) \leq 2$ and obtain a contradiction. Suppose $c(0a_1) = c(0x_1)$. Then $c(a_1b_1) = c(a_1a_2), \ldots, c(a_jb_j) = c(a_j1)$, $c(x_1y_1) = c(x_1x_2), \ldots,$

$c(x_k y_k) = c(x_k 1)$, and all these colors are different. We cannot have $c(01) \in \{c(a_j 1), c(x_k 1)\}$, and therefore vertex 1 is tricolored.

Suppose $c(0a_1) \neq c(0x_1)$. Then $c(01)$ has one of those two colors. Assume without loss of generality that $c(01) = c(0a_1)$. Then $c(a_1 b_1) = c(a_1 a_2), \ldots, c(a_j b_j) = c(a_j 1)$, and all colors thus far are different. Then $c(x_k 1) \notin \{c(01), c(a_j 1)\}$, so again vertex 1 is tricolored and $\eta(G) = 3$.

To verify that all proper induced subgraphs have $\eta \leq 2$, it suffices to show that the removal of any one vertex and its adjoining edges leaves a graph with $\eta = 2$. When vertex 0 is removed, we begin at the bottom with $c(1a_j) = c(1x_k)$ and work up each side to conclude that $\eta = 2$. The same result obviously holds when 1 is removed. When any other vertex is removed, one can assume without loss of generality that it is not adjacent to 0 and begin our coloring with $c(a_1 0) = c(x_1 0)$. Suppose x_i is removed. We proceed counterclockwise from 0 to $x_{i-1} y_{i-1}$ with no problem. Going clockwise from 0, we take $c(a_1 b_1) = c(a_1 a_2), \ldots, c(a_j b_j) = c(a_j 1)$, $c(01) = c(x_k 1)$, $c(x_k y_k) = c(x_k x_{k-1}), \ldots$ and so forth to obtain $\eta = 2$. Suppose an outer b_i vertex is removed. Proceeding clockwise as before, we come to $c(a_{i-1} b_{i-1}) = c(a_{i-1} a_i)$ and then take $c(a_i a_{i+1}) = c(a_{i+1} b_{i+1}), \ldots, c(a_j 0) = c(01)$, $c(x_k 0) = c(x_k y_k)$, $c(x_k x_{k-1}) = c(x_{k-1} y_{k-1}), \ldots, c(x_2 x_1) = c(x_1 y_1)$, so again $\eta = 2$. ∎

We have thus identified two infinite families of graphs and all graphs with six or fewer vertices that are minimal forbidden induced subgraphs for the class of quadratic graphs. There are undoubtedly many others and we encourage efforts to determine the minimal forbidden set more completely.

5. Discussion

Our study of edge coverage of a graph by complete bipartite subgraphs has focused on minimum covers and on covers that minimize the maximum number of times a vertex appears in a covering set. During the course of the paper we have left a number of loose ends and will summarize some of them here. We then conclude with related open problems suggested by the notion of complete bipartite covers.

1. What further restrictions are needed on edge colorings of G so that the chromatic number of the restricted edge colorings equals $d(G)$?

2. Determine $\sigma(m)$ precisely for $m > 6$. Does any graph with 11 vertices have bipartite dimension 8? If not, $\sigma(7) = 11$; otherwise $\sigma(7) = 10$.

3. What is the exact value of inf $[\sigma(m)/m]$?

4. Is it true for every k that some G has $d(G) = \eta(G) = k$? In particular, is $\eta(K_n) = d(K_n)$ for all n?

5. What is the minimum number of vertices of a graph that has bipartite degree 4?

6. Extend the list of minimal forbidden induced subgraphs for the class of quadratic graphs. Does the list contain a minimal forbidden graph with k vertices for every odd $k \geq 7$?

A concern of general interest not addressed in the paper is the difficulty of determining a graph's bipartite dimension or bipartite degree. Both problems seem hard.

Since the minimal forbidden class for $d \leq n$ is finite, $\tau(n)$, the *largest* k for which some minimal forbidden graph for $d \leq n$ has k vertices, is well defined. We have seen that $\tau(1) = 4$ and $\tau(2) = 6$, and it is easily seen that $\tau(n) \geq 2n + 2$. Is $\tau(n) = 2n + 2$?

Specializations of bipartite dimension and degree arise when covers are restricted. One interesting case requires all members of $\mathcal{B}$ to be stars. We refer to $d(G)$ and $\eta(G)$ thus restricted as the *stellar dimension* and *stellar degree* of G. A graph is *bistellar* if its stellar degree is two or less. Abbott and Liu [1] derives bounds on stellar dimension in terms of maximum and minimum degrees of graphs without isolated vertices, and Hammer and Simeone [10] provides extensive informative on bistellar graphs. We are not aware of additional work on stellar dimension and stellar degree.

One might also wonder about the behavior of d and η for random graphs [2, 5].

APPENDIX

This appendix outlines a proof that the 21 graphs for $b \leq 2$ in Figure 5 are the minimal forbidden induced subgraphs for $b(G) \leq 2$. The preamble to Lemma 2 shows that it suffices to show that a reduced G is not an induced subgraph of $G(I_2)$ if and only if G has an induced subgraph among those 21. Note also that a *minimal* forbidden graph has no *saturated vertex* (one with edges to all others), else its removal contradicts minimality. We therefore focus on G_2, which is $G(I_2)$ with $[00, 11]$ and its edges deleted: see Figure 11.

Figure 11 about here

We are to identify all reduced G that satisfy

(1) no vertex is saturated;
(2) G is not an induced subgraph of G_2;
(3) every proper induced reduced subgraph of G with no saturated vertex is an induced subgraph of G_2.

Every 4-vertex reduced G that satisfied (1) is an induced subgraph of G_2. For example, K_4^c uses 1357 (vertices $1, 3, 5$ and 7 on Figure 11), C_4 must use 2468, and P_4 can be done in several ways. We consider only reduced G with $|V| \geq 5$ henceforth.

The proof has two main parts. The necessity proof verifies that the 21 graphs for $b \leq 2$ on Figure 5 satisfy (1), (2) and (3). The sufficiency proof shows that there are no other minimal forbidden graphs for $b \leq 2$.

Necessity Proof

Property (1) for the 21 graphs is obvious by inspection, and (3) for (5.1)–(5.7) follows from the 4-vertex observation. Remarks on (2) for (5.1)–(5.7) follow.

(5.1). G_2 has no induced C_5.

(5.2). G_2 has no independent 5-set.

(5.3)–(5.5). Every other vertex in G_2 has exactly two edges that go to adjacent vertices in the 2468 of C_4.

(5.6)–(5.7). G_2 has no induced $K_{1,3}$. If the top vertex in (5.6) or (5.7) is removed, the central vertex is saturated.

Comments on (2) and (3) for (6.1)–(6.14) follow.

(6.1). (2) is obvious. Removal of a vertex gives P_5, which is induced in G_2 by 12467.

(6.2). But G_2 has no induced P_6. If the second vertex of (6.2) is deleted, 12457 of G_2 induce the rest.

(6.3). Up to symmetry, we induce P_5 in G_2 only with 12467. No other G_2 vertex is isolated from these. If (6.3)'s second vertex is removed, we get P_3 from 123 and two isolated vertices by 5 and 7.

(6.4). But the combination of P_4 and two isolated vertices is not induced in G_2.

(6.5). Assume without loss of generality that C_3 obtains with 234. No other vertex has an edge to 3 in G_2. If a degree-1 vertex is removed from (6.5), 12345 induce the rest.

(6.6)–(6.14). For (2), each of these has a C_3, each vertex of which has an edge to another vertex. Remarks on (3) follow.

(6.6). The bottom-middle vertex is saturated if the top vertex is removed. If the bottom-middle is deleted, 12345 gives the rest.

(6.7). If the left vertex is removed, the upper-right vertex is saturated. We do not have a reduced graph when the upper-left or lower-left vertex is removed: the remainder reduces to P_4.

(6.8). Use 24568 in G_2 when a vertex of (6.8) is deleted.

(6.9). Do likewise if the central vertex is removed. Deletion of the top vertex or a bottom vertex leaves some vertex saturated. We get a non-reduced graph when an upper corner is removed.

(6.10). Removal of the top, lower-left, upper-right or lower-right vertex leaves another saturated. If the central or upper-left vertex is removed, the rest is not reduced.

(6.11). Removal of one leaves another saturated.

(6.12). Left removal leaves another vertex saturated. Top-left corner removal arises in G_2 with 24568. Bottom-left removal gives a non-reduced graph.

(6.13). Top removal leaves P_5. In the bottom row, we have a non-reduced graph when the first, third or fourth vertex is removed. Removal of the second leaves induced 12345 in G_2. The top vertex is saturated if the right vertex is deleted.

(6.14). Top removal leaves P_4 and an isolated vertex. Removal of the right-most vertex leaves the top saturated. All other single-vertex deletions leave a non-reduced graph.

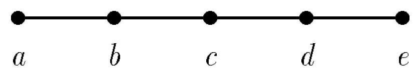
Sufficiency Proof

Our sufficiency proof has five parts. Each part identifies the minimal forbidden graphs that satisfy the part's hypotheses. We encounter the 21 graphs for $b \le 2$ on Figure 5 and no others as the minimal forbidden graphs. The parts' hypotheses are:

Part I.	G has an induced P_5.
Part II.	G has an induced C_4.
Part III.	G has an induced P_4 but not an induced P_5 or C_4.
Part IV.	G has an induced K_4^c but not an induced P_4 or C_4.
Part V.	G has no induced K_4^c, P_4 or C_4.

Part I

Assume that P_5, labeled as



is an induced subgraph of G. This is induced in G_2. If G has a sixth vertex with no edge to this P_5 or with just one edge, we get forbidden (6.3) or (6.2) or (5.7). Assume henceforth that every vertex of G besides a through e has edges to at least two of these five.

Let x denote a sixth vertex. The following combinations of vertices in P_5 that have edges to x yield the noted forbidden graphs:

$$\begin{aligned}
&ac \text{ (or } ce): \ (5.4)\\
&ad: \ (5.1)\\
&ae: \ (6.1)\\
&bd: \ (5.4)\\
&abd: \ (5.4)\\
&abe: \ (5.1)\\
&acd: \ (5.3)\\
&ace: \ (5.4)\\
&abcd: \ (6.13)\\
&abce: \ (5.4).
\end{aligned}$$

The combinations for edges to x that do not immediately yield a forbidden graph are $ab, de, bc, cd, abc, cde, bcd, abde$, and $abcde$. We refer to these subsets of $\{a, b, c, d, e\}$ as *doubles, triples*, and so forth, and denote other distinct vertices of G by x, y and z. We consider cases in which other vertices of G have the same number of edges to $\{a, b, c, d, e\}$, and then look at mixed cases.

DOUBLES ONLY. If each of the doubles ab, bc, cd and de has edges to at most one other vertex, then G, after reduction, is an induced subgraph of G_2, provided that there are no edges between other vertices. (Use 12467 for P_5. If x has edges to a and b, but no others, then $x \approx a$, so we have a reduction. If y has the double bc, use 3 in G_2 for y. And so forth.) If x and y have the same double, e.g. both to ab or both to bc, and $\{x, y\} \notin E$, we get forbidden (5.6) or (5.7). Assume henceforth that $\{x, y\} \in E$ when x and y connect to the same double. If there are no edges between other vertices that go to different doubles, we have the reduced situation at the opening of this paragraph. Suppose there is an edge between x and y when x and y go to different doubles. If the doubles are ab and bc, we get forbidden (6.2), i.e. P_6, with vertices $axycde$; ab and cd produce (5.3); ab and de produce (5.1); and bc and cd yield (6.2). Other pairs are symmetric to those already considered. We omit remarks on symmetries in what follows.

TRIPLES ONLY. Triple abc by itself (x to each of a, b and c) is not forbidden since then $x \approx b$. Triple bcd by itself is not forbidden for a similar reason. If x and y have triples abc and $\{x, y\} \notin E$, we get forbidden (5.3) or (5.4). If x and y have triples bcd and $\{x, y\} \notin E$, forbidden (5.4) appears. Assume henceforth that $\{x, y\} \in E$ when x and y connect to the same triple. Suppose x goes to abc and y to bcd: if $\{x, y\} \notin E$, we get forbidden (6.7); if $\{x, y\} \in E$ and there are no other vertices, we have $x \approx b$ and $y \approx c$, so reduction yields P_5. Suppose x goes to abc and z to cde: if $\{x, z\} \in E$, we get (6.7); if $\{x, z\} \notin E$, reduction gives P_5 when there are no other vertices. Suppose x, y and z go to abc, bcd and cde respectively. To avoid a forbidden graph as just noted, we require $\{x, y\}$ and $\{y, z\}$ in E, and $\{x, z\} \notin E$. Then $x \approx b$, $y \approx c$ and $z \approx d$, so the whole reduces to P_5.

QUADRUPLES ONLY. Suppose x and y go to $abde$. If $\{x, y\} \notin E$, we have (5.3) with $axey$ and c. If $\{x, y\} \in E$ then $x \approx y$ if there are no other vertices. To avoid (5.3) with many copies of $abde$, all other vertices $(x, y, \ldots)$ form a clique, and the whole reduces to 124678 in G_2.

QUINTUPLES ONLY. If x and y go to $abcde$ and $\{x, y\} \notin E$, we have (5.5). Otherwise, all x's form a clique and are saturated.

The preceding paragraphs show that, to *avoid* a noted forbidden graph as an induced subgraph of G, we require

(4a) $\{x,y\} \notin E$ if x and y go to different doubles or to triples abc and cde;

(4b) $\{x,y\} \in E$ if x and y go to the same subset of $\{a,b,c,d,e\}$ or to contiguous triples (abc and bcd, or bcd and cde).

We expand on (4a) and (4b) by considering x and y to subsets of $\{a,b,c,d,e\}$ of different cardinalities. Forbidden graphs that arise when the stated requirement for x versus y does *not* obtain are noted for each subset pair.

(5a) $\{x,y\} \notin E$:

$ab, bcd:$ (6.7) [i.e. (6.7) arises when x to ab, y to bcd, $\{x,y\} \in E$]
$ab, cde:$ (5.4)
$bc, cde:$ (6.7)
$bc, abde:$ (5.4)
$cd, abc:$ (6.7)
$cd, abde:$ (5.4)
$de, abc:$ (5.4)
$de, bcd:$ (6.7)
$bcd, abde:$ (6.9);

(5b) $\{x,y\} \in E$:

$ab, abc:$ (6.2)	$de, cde:$ (6.2)
$ab, abde:$ (5.4)	$de, abde:$ (5.4)
$ab, abcde:$ (6.13)	$de, abcde:$ (6.13)
$bc, abc:$ (5.6)	$abc, abde:$ (5.1)
$bc, bcd:$ (5.7)	$abc, abcde:$ (5.4)
$bc, abcde:$ (5.7)	$bcd, abcde:$ (6.9)
$cd, bcd:$ (5.7)	$cde, abde:$ (5.1)
$cd, cde:$ (5.6)	$cde, abcde:$ (5.4)
$cd, abcde:$ (5.7)	$abde, adcde:$ (5.4).

When the x versus y requirement holds, the resulting graph is not reduced or it has a saturated vertex. In regard to the latter, if x goes to $abcde$, then it must have an edge to every other vertex to avoid a noted forbidden graph as an induced subgraph.

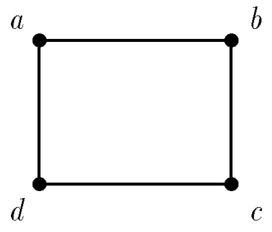
Figure 12 pictures the relationships between all other vertices when (4a)–(5b) hold. We omit others that go to $abcde$ since they are saturated. Other vertices that go to the same subset of $\{a,b,c,d,e\}$, and therefore form a clique by (4b), are shown as a single vertex.

Some of these may of course be absent, but that does not affect reductions with respect to the remainder. The reduced version of the top picture is noted in the bottom diagram to be identical to G_2, or, with absences, to an induced subgraph of G_2. Our conclusion is that if G is a reduced graph with no saturated vertex that has an induced P_5, then it is an induced subgraph of G_2 if and only if none of its induced subgraphs appear among the 21 for $b \leq 2$ on Figure 5.

Figure 12 about here

Part II

Assume that C_4, labeled as



is an induced subgraph of G. If G has a fifth vertex independent of this C_4, or with exactly one edge to the C_4, or with exactly two edges to opposite corners, then we have forbidden (5.3), (5.4) or (5.5). Assume henceforth that every vertex of G besides a through d has edges to at least two in $\{a, b, c, d\}$, and only to two if they are adjacent in the C_4. Hence we consider others with edges to $ab, bc, cd, ad, abc, bcd, acd, abd$ and $abcd$.

It is straightforward to check that if other vertices x and y go to the same subset of $\{a, b, c, d\}$, then $\{x, y\} \in E$ to avoid a forbidden graph for $b \leq 2$, namely (5.4), (5.5), (5.7) or (6.11). In addition, if x goes to ab or abc, and y goes to $abcd$, then forbidden (6.9) or (6.10) occurs unless $\{x, y\} \in E$. Other pairs require the following relationships between distinct x and y to avoid a forbidden graph:

(6a) $\{x, y\} \notin E$:

$ab, bc:$ (5.1) [i.e. (5.1) occurs when x to ab, y to $bc, \{x, y\} \in E$]
$ab, cd:$ (6.8)
$ab, bcd:$ (6.12)
$abc, acd:$ (6.10);

(6b) $\{x, y\} \in E$:

$$ab, abc : (5.4)$$
$$abc, bcd : (6.12).$$

Figure 13 shows the relationships between others that are needed to avoid a known forbidden graph. (We omit $abcd$, which is saturated, and show cliques as single vertices.) The top picture reduces to G_2, so any combination of other vertices that avoids a known forbidden graph as an induced subgraph of G yields an induced subgraph of G_2 when it is put in reduced form.

Figure 13 about here

Part III

Assume that P_4, labeled as

is an induced subgraph of G and that P_5 and C_4 are not. Then, avoiding P_5, C_4 and given forbidden graphs, namely (5.7) and (5.1), as induced in G, a fifth vertex can have edges to none of a, b, c and d (denoted by $\emptyset$), or to ab, bc, cd, abc, bcd and $abcd$. Duplicates to any of these, such as distinct x and y to $\emptyset$, must have $\{x, y\} \in E$ to avoid a forbidden graph [(6.4), (5.7), (5.6)] or C_4, and an x, y pair with y to $abcd$ must have $\{x, y\} \in E$ to avoid (6.14), (6.7) (6.6), or (6.9).

The other pairs require the following to avoid the noted configurations:

(7a) $\{x, y\} \notin E$:

$$\emptyset, ab : P_5$$
$$\emptyset, bc : (6.5)$$
$$\emptyset, abc : (5.7)$$
$$ab, bc : (6.13)$$
$$ab, cd : C_4$$
$$ab, bcd : (6.7);$$

(7b) $\{x, y\} \in E$:

$$ab, abc : (6.13)$$
$$bc, abc : (5.7)$$
$$abc, bcd : (6.7).$$

Figure 14 shows the relationships needed to avoid a known forbidden graph or C_4 or P_5 as induced in G. (Saturated *abcd* is suppressed, and cliques appear as single vertices.) The top diagram reduces to the bottom diagram, which is induced in G_2 as noted.

Figure 14 about here

Part IV

Assume that $K_4^c = (\{a, b, c, d\}, \emptyset)$ is an induced subgraph of G and that P_4 and C_4 are not. To avoid forbidden (5.2) and (5.6), every other vertex must have edges to one, two, or all four in $\{a, b, c, d\}$, and $\{x$ to ab, $\;y$ to $bc\}$ yields P_4 with or without $\{x, y\} \in E$. Thus, in avoiding a proscribed configuration, we can assume without loss of generality that x goes to a, b, c, d, ab, cd or $abcd$. Any vertex to $abcd$ must be saturated, and the others in conjunction with K_4^c are easily seen to reduce to two disjoint copies of P_3, or an induced subgraph thereof, which is accommodated in G_2 by 123567.

Part V

Assume that G has no induced K_4^c, P_4 or C_4. If it has an induced K_3^c then, apart from saturated vertices, it reduces to K_3^c, or to P_3 and an isolated vertex when we avoid proscribed configurations. If it has no induced K_3^c but an induced K_2^c, it reduces to K_2^c, or to P_3.

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