# Interior-Point Methods for Max-Min Eigenvalue Problems

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**Abstract**

The problem of maximizing the smallest eigenvalue of a symmetric matrix subject to modifications on the main diagonal that sum to zero is important since, for example, it yields the best bounds for graph-partitioning. Current algorithms for this problem work well when the multiplicity of the minimum eigenvalue at optimality is one. However, real-world applications have multiplicity at optimality that is greater than one. For such problems, current algorithms break down quickly as the multiplicity increases. We present a primal-dual interior-point

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method for this problem. The method proves to be simple, robust and efficient. And most importantly, it is not adversely affected by high multiplicities.

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# 1 Introduction.

Consider the *max-min eigenvalue problem*:

$$\begin{array}{ll} \text{Maximize} & \lambda_{\min}(C - \operatorname{Diag}(v)) \\ \text{subject to} & e^T v = 0. \end{array} \tag{1.1}$$

Here, $C$ is an $n \times n$ real symmetric matrix, $v$ is a real $n$-vector, $\operatorname{Diag}(\cdot)$ denotes the linear operator that maps vectors to diagonal matrices, $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of its argument, and $e$ denotes the vector of all ones. There are many important applications for this problem (see e.g. [8], [9] and the references therein). For many of these applications, it is essential to have a fast algorithm for (1.1) since it has to be solved many times within the application. For example, problem (1.1) can be used to obtain excellent bounds in branch and bound codes for graph bisection (see e.g. [4] and the survey article [5]).

The objective function in (1.1) is not differentiable when the multiplicity of the smallest eigenvalue exceeds one. In fact, a singleton eigenvalue characterizes differentiability. Since the smallest eigenvalue is a concave function, subgradient approaches can be used to solve (1.1) (see, e.g., [3]). More recently, it has been shown that Newton-based algorithms with local quadratic convergence exist (see, e.g., [10]) but the local convergence depends on correctly identifying the multiplicity of the smallest eigenvalue. In some sense, high multiplicity is analogous to degeneracy in linear programming. Since interior-point methods for linear programming perform especially well on degenerate problems, one is led to consider deriving an interior-point method to solve (1.1). This is the first aim of this paper. In addition, we present computational experiments showing that interior-point methods are indeed robust in the presence of high multiplicity.

# 2 Alternate Formulation.

It is easy to see that (1.1) is equivalent to the following problem:

$$\begin{array}{ll} \text{Maximize} & \omega \\ \text{subject to} & \omega I \preceq C - \operatorname{Diag}(v) \\ & e^T v = 0. \end{array} \tag{2.1}$$

Here, the inequality ($\preceq$) is relative to the partial order induced by the cone of positive semidefinite matrices. Interior-point methods for problems involving matrix inequalities are studied in [6, 7, 1, 2, 12]. (See the latter for a historical overview.)

Now, by substituting

$$y = v + \omega e \tag{2.2}$$

for $v$, (2.1) can be reformulated as:

$$\begin{array}{ll} \text{Maximize} & \omega \\ \text{subject to} & \mathrm{Diag}(y) \preceq C \\ & e^T y = n\omega. \end{array} \tag{2.3}$$

Finally, we can think of the last equation as a defining equation for $\omega$ and use it to eliminate $\omega$ from the problem:

$$\begin{array}{ll} \text{Maximize} & \frac{1}{n} e^T y \\ \text{subject to} & \mathrm{Diag}(y) \preceq C. \end{array} \tag{2.4}$$

# 3 Interior-Point Method for Positive Semidefinite Programming.

Problem (2.4) is an example of a larger class of *positive semidefinite programs*:

$$\begin{array}{ll} \text{Maximize} & b^T y \\ \text{subject to} & \mathrm{Diag}(y) \preceq C. \end{array} \tag{3.1}$$

In this section, we will derive a primal-dual interior-point algorithm for (3.1). To this end, we need to identify the dual of (3.1). Duality theory for linear programs over general convex cones has been extensively studied for many years. In our particular case, the theory is elementary and we include the details below. We begin by introducing slacks:

$$\begin{array}{ll} \text{Maximize} & b^T y \\ \text{subject to} & \mathrm{Diag}(y) + Z = C \\ & Z \succeq 0. \end{array} \tag{3.2}$$

By analogy with terminology from the interior-point literature, we refer to this problem as the *dual problem*. The domain for the variable $Z$ is the space of symmetric matrices. The natural inner product in this space is the trace of the matrix product. Hence, the Lagrangian for (3.2) is

$$L(X, y, Z) = b^T y + \operatorname{tr}\{(C - \operatorname{Diag}(y) - Z)X\}. \tag{3.3}$$

Problem (3.2) is the max-min problem associated with this Lagrangian:

$$\max_{y; Z \succeq 0} \min_{X} L(X, y, Z). \tag{3.4}$$

(Indeed, the minimization over $X$ yields minus infinity unless $C - \operatorname{Diag}(y) - Z = 0$.) The dual of (3.2) is obtained by interchanging the max and the min:

$$\min_{X} \max_{y; Z \succeq 0} L(X, y, Z). \tag{3.5}$$

To evaluate the maximization, we first maximize over $Z \succeq 0$. This maximization yields plus infinity unless $X \succeq 0$, in which case the maximum occurs when $Z$ satisfies complementarity:

$$\operatorname{tr}(ZX) = 0. \tag{3.6}$$

Next we maximize over $y$ by differentiating with respect to $y$ and setting the derivative to zero. Using the adjoint identity

$$\operatorname{tr}(\operatorname{Diag}(y)X) = y^T \operatorname{diag}(X) \tag{3.7}$$

(here, $\operatorname{diag}(\cdot)$ denotes the linear operator from the space of matrices to the space of vectors that extracts the main diagonal of its matrix argument; it is the adjoint of the linear operator $\operatorname{Diag}(\cdot)$), we see that the optimization over $y$ imposes the constraint

$$\operatorname{diag}(X) = b. \tag{3.8}$$

Now, (3.6) and (3.8) allow us to simplify the Lagrangian and we arrive at the following form for the *primal problem* associated with (3.2):

$$\begin{array}{ll} \text{Minimize} & \operatorname{tr}(CX) \\ \text{subject to} & \operatorname{diag}(X) = b \\ & X \succeq 0. \end{array} \tag{3.9}$$

The primal/dual pair of problems satisfy weak duality:

**Theorem 1** *If $X$ is primal feasible and $(y, Z)$ is dual feasible, then*

$$b^T y \leq tr(CX).$$

**Proof.** Using primal and dual feasibility and the adjoint identity (3.7), we see that

$$\begin{aligned} y^T b &= y^T \mathrm{diag}(X) \\ &= \mathrm{tr}\{\mathrm{Diag}(y) X\} \\ &= \mathrm{tr}\{(C - Z)X\} \\ &= \mathrm{tr}(CX) - \mathrm{tr}(ZX). \end{aligned}$$

Now, since $Z$ and $X$ are both positive semidefinite, we see that

$$b^T y \leq \mathrm{tr}(CX)$$

and the duality gap is simply $\mathrm{tr}(ZX)$. □

Strong duality holds as well, since the Slater constraint qualification is trivially satisfied, see e.g. [14].

## 3.1 Derivation of the Interior-Point Algorithm

Primal-dual methods for (3.2) are derived by first introducing an associated *barrier problem*:

$$\begin{array}{ll} \text{Maximize} & b^T y + \mu \log \det Z \\ \text{subject to} & \mathrm{Diag}(y) + Z = C. \end{array} \tag{3.10}$$

Here $\mu$ is a positive real number called the *barrier parameter*. For each $\mu > 0$, there is a corresponding Lagrangian:

$$L_\mu(X, y, Z) = b^T y + \mu \log \det Z + \mathrm{tr}\{(C - \mathrm{Diag}(y) - Z)X\}. \tag{3.11}$$

The first-order optimality conditions for the saddle point of this Lagrangian are

$$\nabla_X L_\mu = C - \mathrm{Diag}(y) - Z = 0 \tag{3.12}$$
$$\nabla_y L_\mu = b - \mathrm{diag}(X) = 0 \tag{3.13}$$
$$\nabla_Z L_\mu = \mu Z^{-1} - X = 0. \tag{3.14}$$

The strict concavity of $\log\det Z$ implies that there exists a unique solution $(X_\mu, y_\mu, Z_\mu)$ to these optimality conditions. The triple $(X_\mu, y_\mu, Z_\mu)$ is called the *central trajectory.* Given a triple $(X, y, Z)$ on the central trajectory it is easy to determine its associated $\mu$ value using (3.14):

$$\mu = \frac{\mathrm{tr}(ZX)}{n}. \tag{3.15}$$

We shall use this expression to associate $\mu$ values with triples $(X, y, Z)$ even when these triples don't belong to the central trajectory.

Our interior-point algorithm is derived as follows. We start with a triple $(X, y, Z)$ for which $X$ and $Z$ are positive definite but which is otherwise arbitrary. From this point we estimate the current $\mu$ value using (3.15) and divide it by ten:

$$\mu = \frac{\mathrm{tr}(ZX)}{10n}.$$

(Experience from linear programming indicates that this simple heuristic performs very well, even though it does not guarantee monotonic decrease in $\mu$, see [13].) We next attempt to find step directions $(\Delta X, \Delta y, \Delta Z)$ such that the new triple $(X + \Delta X, y + \Delta y, Z + \Delta Z)$ lies on the central trajectory at this value of $\mu$. However, since not all the defining equations, (3.12)–(3.14), are linear, it is not possible to solve this system directly. In fact, only (3.14) is nonlinear. It can be written in four equivalent forms, each form giving rise to a different linearization. The four equivalent forms are:

$$\mu Z^{-1} - X = 0, \tag{3.16}$$

$$\mu X^{-1} - Z = 0, \tag{3.17}$$

$$\mu I - Z^{1/2} X Z^{1/2} = 0, \tag{3.18}$$

and

$$\mu I - X^{1/2} Z X^{1/2} = 0. \tag{3.19}$$

(In the linear programming case, $X$ and $Z$ are diagonal matrices and therefore the last two forms are the same and don't really involve square roots.) We rule out the last two forms, since they involve matrix square roots. Of the first two, both are reasonable but it turns out to be better to linearize (3.17).

(Replacing (3.14) with (3.17) results in the equations for the first-order optimality conditions from an analogous primal barrier problem.) Hence, we use

$$(X + \Delta X)^{-1} = X^{-1} - X^{-1}\Delta X X^{-1} + o((\Delta X)^2),$$

to arrive at the following linear system for $(\Delta X, \Delta y, \Delta Z)$:

$$\begin{aligned} C - \mathrm{Diag}(y + \Delta y) - Z - \Delta Z &= 0 \qquad (3.20)\\ b - \mathrm{diag}(X + \Delta X) &= 0 \qquad (3.21)\\ \mu(X^{-1} - X^{-1}\Delta X X^{-1}) - Z - \Delta Z &= 0. \qquad (3.22) \end{aligned}$$

This linear system can now be solved for $(\Delta X, \Delta y, \Delta Z)$. Indeed, first we solve (3.20) for $\Delta Z$ (in terms of $\Delta y$):

$$\Delta Z = C - \mathrm{Diag}(y) - Z - \mathrm{Diag}(\Delta y) \qquad (3.23)$$

and substitute this expression into (3.22) to get

$$\mu X^{-1}\Delta X X^{-1} - \mathrm{Diag}(\Delta y) = \mu X^{-1} - \Sigma, \qquad (3.24)$$

where

$$\Sigma = C - \mathrm{Diag}(y).$$

Then, we solve (3.24) for $\Delta X$ (in terms of $\Delta y$):

$$\Delta X = \frac{1}{\mu} X(\mathrm{Diag}(\Delta y) - \Sigma)X + X. \qquad (3.25)$$

Now, substituting this expression for $\Delta X$ into (3.21), we get the following expression for $\Delta y$:

$$\frac{1}{\mu}\mathrm{diag}(X\mathrm{Diag}(\Delta y)X) = b + \frac{1}{\mu}\mathrm{diag}(X\Sigma X) - 2\mathrm{diag}(X). \qquad (3.26)$$

Although $\Delta y$ seems to be buried in the left-hand side, it turns out that this left-hand side simplifies nicely. Indeed, if $W$ is any symmetric matrix and $\xi$ is any vector, we see that

$$(\mathrm{diag}(W\mathrm{Diag}(\xi)W))_i = \sum_j w_{ij}\xi_j w_{ji} = \sum_j w_{ij}^2 \xi_j.$$

Hence,

$$\mathrm{diag}(W\mathrm{Diag}(\xi)W) = (W \circ W)\xi,$$

where $W \circ W$ denotes the Hadamard (or entrywise) product of $W$ with itself. Applying this identity to (3.26), we see that

$$\Delta y = \mu(X \circ X)^{-1}\left(b + \frac{1}{\mu}\mathrm{diag}(X\Sigma X) - 2\mathrm{diag}(X)\right). \qquad (3.27)$$

To summarize, we solve for the triple $(\Delta X, \Delta y, \Delta Z)$ by first solving (3.27) for $\Delta y$ and then substituting this into (3.25) to solve for $\Delta X$ and finally substituting that into (3.23) to solve for $\Delta Z$.

Having determined the desired triple, $(\Delta X, \Delta y, \Delta Z)$, of step directions, we would step to the new triple $(X+\Delta X, y+\Delta y, Z+\Delta Z)$ except that it might violate the positive definiteness property required of the two matrices. Hence, we perform a line search to find constants $\alpha_p$ and $\alpha_d$ such that $X + \alpha_p\Delta X$ and $Z + \alpha_d\Delta Z$ are positive definite. Given $\alpha_p$ and $\alpha_d$, we step to the new point

$$\begin{array}{c} X + \alpha_p\Delta X \\ y + \alpha_d\Delta y \\ Z + \alpha_d\Delta Z \end{array}$$

and repeat. The algorithm continues until the current triple $(X, y, Z)$ satisfies primal feasibility, dual feasibility and the duality gap is sufficiently small. This completes the description of our interior-point algorithm.

Note that complementary slackness at optimality implies that $ZX = 0$. Dual feasibility then implies that the eigenvectors for the optimal eigenvalue are found as the columns of $X$. If strict complementary slackness holds, i.e. $\mathrm{rank}(X) + \mathrm{rank}(Z) = n$, then $\mathrm{rank}(X)$ equals the multiplicity of the optimal eigenvalue.

# 4 Implementation.

We implemented the algorithm presented in the previous section as a C program. In this section, we describe some of the implementation details.

## 4.1 Starting Points.

We experimented with a few choices for starting points. The simplest is to set $X$ and $Z$ to some constant times the identity matrix and to set $y$ to zero. From our limited experience this choice seems to work fine. However, if the components of $b$ are all strictly positive, it is trivial to initialize $X$ so that the primal is feasible. Indeed, one can simply set

$$X = \mathrm{Diag}(b).$$

(Of course, the primal problem is feasible if and only if $b$ is nonnegative.) With relatively little work, it is also easy to initialize $y$ and $Z$ so that the dual is feasible. Indeed, we let

$$y = (\underline{\lambda} - 1)e$$

where $\underline{\lambda}$ denotes a lower bound on the smallest eigenvalue of $C$, and

$$Z = C - \mathrm{Diag}(y).$$

We used

$$\underline{\lambda} = \min(10\lambda_{\min}(C), 0).$$

Although the algorithm requires neither primal nor dual feasibility for the initial solution, we found that starting with a feasible solution works very well. The experiments described in the next section use feasible starting points.

## 4.2 Line Search.

We experimented with a few line searches. The simplest one seems to work the best. Given a positive definite matrix $A$ and a step direction $\Delta A$, the problem is to find an $\alpha$ between 0 and 1 such that

$$A + \alpha \Delta A$$

is positive definite. We used a geometric reduction algorithm. That is, we start with $\alpha = 1$ and we keep multiplying $\alpha$ by a fixed reduction parameter, which we set to 0.8, until we arrive at a value of $\alpha$ for which $A + \alpha\Delta A$ is positive semidefinite. Then, we multiply $\alpha$ by 0.95 for extra security (unless $\alpha = 1$, in which case we leave it alone).

To test whether a symmetric matrix is positive semidefinite, we perform an $LDL^T$ factorization and check whether all the entries of the diagonal matrix $D$ are nonnegative.

## 4.3 Stopping Rule.

The algorithm is terminated when $X$ is feasible for the primal, $(y, Z)$ is feasible for the dual, and the primal and dual objective values agree to a specified number of figures. Of course, starting with feasible initial solutions, the algorithm produces feasible solutions at every iteration, so the feasibility requirement is automatic in that case. In our experiments, we used six figures of agreement as our stopping criteria.

## 4.4 Symmetry.

In an early implementation of the algorithm, we stored symmetric arrays as full two-dimensional arrays (i.e., every off diagonal entry was stored twice). This implementation used twice as much memory and ran two times more slowly than our current implementation which stores off-diagonal elements only once. It is interesting to note that severe numerical difficulties arose when we allowed small numerical imprecision to violate symmetry. Adding to the code periodic resymmetrization of the symmetric matrices improved the robustness tremendously.

# 5 Numerical Experiments.

We compared our interior-point method to the Bundle Trust (FORTRAN) code described in ([11]) for minimizing convex functions.

The experiments described in this section were all performed on a Silicon Graphics Indigo workstation (R4000). This workstation uses RISC technology and implements IEEE standard arithmetic. It is rated at about 10 MFLOPS. For all experiments, we compiled with the `-O` flag for full compiler optimization. Simple benchmarks indicate that the C and FORTRAN compilers generate executable codes having similar performance.

| $n$ | Time (MM:SS.S) BT | IP | multiplicity of opt. e-value |
|---|---|---|---|
| 20 | 1.5 | 0.3 | 6 |
| 30 | 8.6 | 1.1 | 4 |
| 50 | 2:34.9 | 10.0 | 4 |
| 100 | >40:00.0 | 1:19.6 | 5 |

Table 1: Statistics for randomly generated graph partitioning problems. BT refers to the Bundle Trust method and IP refers to our Interior-Point method.

## 5.1 Problems from Graph Partitioning.

One of the main applications of the max-min eigenvalue problem is to obtain bounds in the graph partitioning problem (see [4]). For such problems, the matrix $C$ is the incidence matrix of an undirected graph. We generated a few random incidence matrices and compared our interior-point method against the bundle trust method. The results are summarized in Table 1. For these problems, the interior-point method is clearly superior.

The last column shows the multiplicity of the minimum eigenvalue at optimality. Experience with the bundle trust method indicates that it works well when the multiplicity is close to one but that it runs into trouble quickly as the multiplicity increases.

## 5.2 Problems with Known Multiplicity.

To test the code on problem instances that exhibit given multiplicity at the optimal solution, we developed a special generator which we now describe.

First, we generate $b$. To generate max-min eigenvalue problems (2.4), we take $b = e/n$ whereas to generate positive semidefinite programs we generate the elements of $b$ uniformly on some interval of the nonnegative half-line (the primal problem is clearly infeasible if any component of $b$ is negative). For the experiments described below, we used $b = e/n$.

Given $b$, we generate $C$ as follows. First, we generate an $n \times m$ random matrix $A$ and apply row scaling to make all squared row norms equal to the corresponding elements of $b$. That is,

$$\operatorname{diag}(AA^T) = b. \tag{5.1}$$

We denote the columns of $A$ by $v_1, \ldots, v_m$. We then construct $n-m$ additional random $n$-vectors $v_{m+1}, \ldots, v_n$ and apply Gram-Schmidt orthonormalization to $v_1, \ldots, v_n$ to produce an $n \times n$ orthogonal matrix $Q$ whose first $m$ columns span the same space as $v_1, \ldots, v_m$. Finally, we set

$$C = Q\Lambda Q^T, \tag{5.2}$$

where $\Lambda$ is a diagonal matrix whose first $k \geq m$ entries are all set to $\lambda_{\min}$ (which is a constant that can be chosen arbitrarily – we used $-5$) and the remaining diagonal entries are generated uniformly on some interval that is strictly larger than $\lambda_{\min}$. For such a matrix $C$, we claim that

$$\begin{aligned} X &= AA^T \\ y &= \lambda_{\min} e \\ Z &= C - \operatorname{Diag}(y) \end{aligned}$$

is optimal for (3.2). Indeed, it follows from (5.1) that $X$ is feasible for the primal and it is clear from (5.2) that $(y, Z)$ is feasible for the dual. Finally, optimality follows from the absense of a duality gap:

$$\operatorname{tr}(ZX) = \operatorname{tr}\{(C - \lambda_{\min} I)AA^T\} = 0$$

The last equality follows from the fact that the columns of $A$ are eigenvectors of $C$ associated with the minimal eigenvalue.

Table 2 shows the comparison between the bundle trust method and our interior-point method when the optimal eigenvalue is a singleton ($k = 1$). For these problems, the bundle trust method is three to four times faster. However, this situation never arises in practice. Indeed, for $k = 1$ in our construction above, we see that we are requiring the vector of all ones to be a minimal eigenvector of $C$. This is almost a necessary and sufficient condition for obtaining a singleton and it is clearly an unlikely event in real applications.

Table 3 shows comparisons for higher multiplicities. Here the results look much better for the interior-point method. In fact, it is clear that the bundle trust method completely breaks down rather rapidly as the multiplicity increases.

| $n$ | $m$ | $k$ | Time (MM:SS.S) BT | IP |
|---|---|---|---|---|
| 10 | 1 | 1 | 0.0 | 0.0 |
| 20 | 1 | 1 | 0.1 | 0.3 |
| 30 | 1 | 1 | 0.3 | 1.0 |
| 50 | 1 | 1 | 1.2 | 4.2 |
| 100 | 1 | 1 | 12.6 | 35.6 |
| 200 | 1 | 1 | 1:56.9 | 6:17.4 |

Table 2: Statistics for problems with multiplicity 1. BT refers to the Bundle Trust method and IP refers to our Interior-Point method.

| $n$ | $m$ | $k$ | Time (MM:SS.S) BT | IP | Comments |
|---|---|---|---|---|---|
| 10 | 3 | 3 | 0.0 | 0.0 | |
| 10 | 3 | 5 | 0.0 | 0.0 | |
| 10 | 5 | 5 | 0.2 | 0.0 | |
| 20 | 3 | 3 | 0.4 | 0.4 | |
| 20 | 5 | 5 | 2.8 | 0.3 | |
| 20 | 5 | 12 | 2.8 | 0.3 | |
| 20 | 8 | 8 | 2.7 | 0.3 | |
| 20 | 12 | 12 | 3.6 | 0.3 | |
| 30 | 3 | 3 | 1.5 | 1.0 | |
| 30 | 3 | 6 | 1.5 | 0.9 | |
| 30 | 6 | 6 | 18.2 | 0.8 | |
| 30 | 10 | 10 | 4.0 | 1.0 | Num. trouble in BT |
| 50 | 5 | 5 | $>$20:00.0 | 4.3 | 5 sig. fig. in BT |
| 100 | 3 | 3 | 18.7 | 33.9 | |
| 100 | 6 | 6 | $>$15:00.0 | 36.9 | 5 sig. fig. in BT |
| 500 | 50 | 50 | - | 2:02:47.0 | No attempt at BT |

Table 3: Statistics for problems with higher built-in multiplicity. BT refers to the Bundle Trust method and IP refers to our Interior-Point method.

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