Affine-Scaling Trajectories Associated with a Semi-Infinite Linear Program

Robert J. Vanderbei

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Rutgers University
New Brunswick, NJ 08903 November 8, 1993

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Abstract

Semi-infinite linear programs often arise as the limit of a sequence of approximating linear programs. Hence, studying the behavior of extensions of linear programming algorithms to semi-infinite problems can yield valuable insight into the behavior of the underlying linear programming algorithm when the number of constraints or the number of variables is very large. In this paper, we study the behavior of the affine-scaling algorithm on a particular semi-infinite linear programming problem. We show that the continuous trajectories converge to the optimal solution but that, for any strictly positive step, there are starting points for which the discrete algorithm converges to nonoptimal boundary points.

Key words: semi-infinite programming, affine-scaling algorithms, interior point methods, linear programming theory, linear programming algorithms,

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1 Introduction.

Semi-infinite linear programming offers a natural framework from which to investigate the behavior of linear programming algorithms when either the number of variables or the number of constraints is very large. In two recent papers, [1], [2], Ferris and Philpott have extended the affine-scaling algorithm for linear programming to semi-infinite programming. Their papers focus on problem formulation coupled with some particular applications for which they provide computational results.

At about the same time, Powell [3] provided an example of a semi-infinite linear programming problem on which Karmarkar's algorithm fails to converge to the optimal solution.

Subsequently, Todd [4] studied the formulation of semi-infinite programming algorithms for a variety of currently popular interior-point methods. Tunçel and Todd have then taken this work further in [5] by analyzing potential functions and search directions associated with semi-infinite linear programs. They provide the first polynomial-iteration interior-point algorithms for semi-infinite programming.

In this paper, we study the behavior of the affine-scaling algorithm when applied to the semi-infinite linear program studied by Powell. We shall show that the continuous trajectories converge to the optimal solution but that for every positive step length (measured as a fraction of the distance to the edge of the feasible set) there are starting points from which the discrete algorithm does not converge to the optimal solution. This result sheds light on the empirical observation that, for very large linear programs, interior point methods are sensitive to the heuristic for selecting a starting point.

Powell's semi-infinite linear program is:

$$\begin{array}{ll} \text{Minimize} & x_2 \\ \text{subject to} & x_1 \cos\theta + x_2 \sin\theta \le 1, \quad 0 \le \theta \le 2\pi. \end{array} \tag{1.1}$$

The feasible region is the closed disk of radius one centered at the origin. The optimal solution is $x^* = (0, -1)$. To see what the affine-scaling algorithm is for such a problem, we approximate this semi-infinite linear program by an ordinary linear program obtained by considering a large but finite number of constraints. So, let $\theta_i = 2\pi i/m$ and consider the following approximating linear program:

$$
\begin{array}{rl}
\text{Minimize} & x_2 \\
\text{subject to} & x_1 \cos\theta_i + x_2 \sin\theta_i \le 1, \quad 1 \le i \le m.
\end{array}
\tag{1.2}
$$

The affine-scaling algorithm applied to this problem can be described as follows (see, e.g., [7] for a derivation of the inequality form of the algorithm from the standard form). Let

$$
w(\theta) = 1 - x_1 \cos\theta - x_2 \sin\theta \tag{1.3}
$$

and

$$
\Delta x = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{i=1}^{m} \frac{\cos^2\theta_i}{w^2(\theta_i)} & \displaystyle\sum_{i=1}^{m} \frac{\cos\theta_i \sin\theta_i}{w^2(\theta_i)} \\ \displaystyle\sum_{i=1}^{m} \frac{\cos\theta_i \sin\theta_i}{w^2(\theta_i)} & \displaystyle\sum_{i=1}^{m} \frac{\sin^2\theta_i}{w^2(\theta_i)} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{1.4}
$$

Then, starting from an arbitrary feasible point x, the affine-scaling algorithm iteratively updates x by moving in the direction of Δx a certain fraction of the way to the boundary of the polytope:

$$
x \leftarrow x + \gamma \frac{\Delta x}{\tau(x, \Delta x)}, \tag{1.5}
$$

where $0 < \gamma < 1$ ($\gamma = 1$ represents moving all the way to the boundary of the polytope in one step) and

$$
\tau(x, \Delta x) = \max_i \left(\frac{\Delta x_1 \cos\theta_i + \Delta x_2 \sin\theta_i}{w(\theta_i)} \right).
$$

Since $\tau(x, \Delta x)$ is positively homogeneous of degree one in Δx, it follows that Δx could be multiplied by an arbitrary scale factor and the algorithm remains unchanged. Hence, multiplying the matrix in (1.4) by $1/m$ and letting m tend to infinity, we obtain the following formula for the step direction in the semi-infinite linear programming problem:

$$
\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \displaystyle\int_0^{2\pi} \frac{\cos^2\theta}{w^2(\theta)} d\theta & \displaystyle\int_0^{2\pi} \frac{\cos\theta \sin\theta}{w^2(\theta)} d\theta \\ \displaystyle\int_0^{2\pi} \frac{\cos\theta \sin\theta}{w^2(\theta)} d\theta & \displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{w^2(\theta)} d\theta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{1.6}
$$

Also, the limiting formula for $\tau(x, \Delta x)$ is

$$\tau(x, \Delta x) = \max_{0 \le \theta \le 2\pi} \left(\frac{\Delta x_1 \cos\theta + \Delta x_2 \sin\theta}{w(\theta)} \right).$$

We will be interested in two types of trajectories associated with this field of step directions. The *continuous trajectories* are the solutions (with any feasible starting point) to the following differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}, \tag{1.7}$$

where $x = (x_1, x_2)$ is a function of a real parameter t and the dots represent differentiation with respect to this parameter. For the continuous trajectories, we will be interested in $\lim_{t\to\infty} x(t)$. We will prove that this limit exists and coincides with the optimal solution $(0, -1)$.

The second type of trajectories are the *discrete trajectories.* A discrete trajectory is a sequence $x^{(k)}$ obtained by iterative application of

$$x^{(k)} = x^{(k-1)} + \gamma \frac{\Delta x^{(k-1)}}{\tau\left(x^{(k-1)}, \Delta x^{(k-1)}\right)}.$$

In this case, we shall be interested in the limiting behavior of the sequence $x^{(k)}$. We shall show that for any step length there are certain starting points from which this sequence does not converge to the optimal solution.

In the next section we derive explicit formulas for the components of Δx.

2 Evaluation of the Step Direction

Let r and α denote the radial and angular polar coordinates of the current point $x = (x_1, x_2)$. Then, $x_1 = r\cos\alpha$, $x_2 = r\sin\alpha$, and $w(\theta)$ becomes

$$w(\theta) = 1 - r\cos(\theta - \alpha).$$

Changing the integration variable from θ to $\theta - \alpha$ and invoking the formulas for the sine and cosine of the sum of two angles, we see that

$$\int_0^{2\pi} \frac{\cos^2\theta}{w^2(\theta)} d\theta \quad = \quad c(r)\cos^2\alpha + s(r)\sin^2\alpha$$

$$
\begin{aligned}
\int_0^{2\pi} \frac{\cos\theta \sin\theta}{w^2(\theta)} d\theta &= (c(r) - s(r)) \cos\alpha \sin\alpha \\
\int_0^{2\pi} \frac{\sin^2\theta}{w^2(\theta)} d\theta &= s(r)\cos^2\alpha + c(r)\sin^2\alpha,
\end{aligned}
$$

where

$$
\begin{aligned}
c(r) &= \int_0^{2\pi} \frac{\cos^2\theta}{(1 - r\cos\theta)^2} d\theta \\
s(r) &= \int_0^{2\pi} \frac{\sin^2\theta}{(1 - r\cos\theta)^2} d\theta.
\end{aligned}
$$

Substituting these expressions into (1.6), we get

$$
\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} c(r)\cos^2\alpha + s(r)\sin^2\alpha & (c(r) - s(r))\cos\alpha\sin\alpha \\ (c(r) - s(r))\cos\alpha\sin\alpha & s(r)\cos^2\alpha + c(r)\sin^2\alpha \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix}.
$$

Simple calculations show that the determinant of the above matrix is simply $c(r)s(r)$ which is clearly positive. Hence, it can be dropped from the formula for Δx since it affects only the length of the vector which will be scaled appropriately by $\tau(x, \Delta x)$ anyway. Therefore,

$$
\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} (c(r) - s(r))\cos\alpha\sin\alpha \\ -c(r)\cos^2\alpha - s(r)\sin^2\alpha \end{bmatrix}. \tag{2.1}
$$

In the following analysis, we shall have further occasion either to multiply or divide the step direction vector by a scale factor. This rescaling has no effect on the algorithm as long as we ensure that the factor is strictly positive. As the careful reader will see, the factors that arise are strictly positive except possibly for the case where $r = 0$. Hence, we drop from consideration those trajectories that pass through the center of the disk. That is, we rule out all trajectories whose starting point lies on the nonnegative x_2 axis.

The integrals defining $c(r)$ and $s(r)$ can be evaluated explicitly. Indeed, the usual substitution $t = \tan\theta/2$ reduces the integrands to rational functions in t which are then integrated using partial fraction expansions. The results are:

$$
\begin{aligned}
c(r) &= f(r)(r^2 - h(r)), \\
s(r) &= f(r)h(r),
\end{aligned}
$$

where

$$\begin{aligned} f(r) &= \frac{2\pi}{r^2(1-r^2)^{3/2}}, \\ h(r) &= 1 - r^2 - (1-r^2)^{3/2}. \end{aligned} \tag{2.2}$$

The function $h(r)$, which will appear frequently in what follows, vanishes at $r = 0$ and $r = 1$ and is strictly positive at all points in between.

Substituting these expressions into (2.1), replacing $\cos\alpha$ and $\sin\alpha$ with x_1/r and x_2/r respectively, and again dropping common positive factors, we see that

$$\begin{aligned} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} &= \begin{bmatrix} x_1 x_2 (r^2 - 2h(r)) \\ -x_1^2(r^2 - h(r)) - x_2^2 h(r) \end{bmatrix} \\ &= h(r) \begin{bmatrix} -2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix} + r^2 \begin{bmatrix} x_1 x_2 \\ -x_1^2 \end{bmatrix}. \end{aligned} \tag{2.3}$$

The step direction has the following properties:

Proposition 1

1. *For $r > 0$, Δx is a descent direction; i.e., $\Delta x_2 < 0$;*
2. *For $0 < r < 1$ and $x_2 < 0$, $x^T \Delta x > 0$;*
3. *On the boundary of the feasible set, the step direction is tangent to the boundary. In fact,*

$$\lim_{r \to 1} \Delta x = \begin{bmatrix} x_1 x_2 \\ -x_1^2 \end{bmatrix}.$$

Proof. Property 1 follows from the fact that $h(r) < r^2$. From (2.3) we see that $x^T \Delta x = -x_2 r^2 h(r)$ from which property 2 follows immediately. Property 3 is obvious from (2.3) and the fact that $h(1) = 0$. □

Property 3 says that when r is close to 1, the second term in (2.3) dominates the first one. Since our analysis will revolve around points close to the boundary, it seems instructive to consider the related problem where the step direction consists only of the second term. This is the subject of the next section. Then in Section 4, we will return to a detailed analysis of the trajectories associated with Δx as given above.

3 Trajectories of the Related Problem

In this section, we consider trajectories associated with the following step direction vector which was introduced at the end of the preceding section:

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ -x_1^2 \end{bmatrix} \tag{3.1}$$

(again we have dropped a positive common factor – this time r^2).

3.1 Continuous Trajectories

The continuous trajectories associated with the above step direction are the solutions to the following differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ -x_1^2 \end{bmatrix}, \tag{3.2}$$

where $x = (x_1, x_2)$ is now thought of as a function of a real parameter t. In this subsection we will prove the following theorem.

Theorem 2 *Starting from any $(x_1(0), x_2(0))$ with $x_1(0) \neq 0$, the trajectory $x(t)$ converges to $(0, -r)$ where $r = \sqrt{x_1(0)^2 + x_2(0)^2}$.*

Proof. This differential equation is best studied using polar coordinates. Since $r = \sqrt{x_1^2 + x_2^2}$, we see that

$$\dot{r} = \frac{(x_1\dot{x}_1 + x_2\dot{x}_2)}{r} = \frac{x_1^2 x_2 - x_2 x_1^2}{r} = 0.$$

Therefore the radius remains constant (this is obvious since the "velocity" vector is everywhere orthogonal to the position vector). To obtain a differential equation for the angular component α, we differentiate the defining relation $\tan\alpha = x_2/x_1$:

$$\begin{aligned} \sec^2\alpha \;\; \dot{\alpha} &= \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2} \\ &= \frac{-x_1 x_1^2 - x_2 x_1 x_2}{x_1^2} \\ &= -\frac{r}{\cos\alpha}. \end{aligned}$$

Hence,

$$\dot{\alpha} = -r \cos \alpha,$$

and so

$$\int_{\alpha(0)}^{\alpha(t)} \sec \alpha \, d\alpha = -rt.$$

Integrating the secant function and exponentiating the result, we get

$$|\sec \alpha(t) + \tan \alpha(t)| = |\sec \alpha(0) + \tan \alpha(0)| e^{-rt}.$$

Now, the function

$$\phi(\alpha) = \sec \alpha + \tan \alpha$$

is a strictly increasing function from $(-3\pi/2, \pi/2)$ onto $(-\infty, \infty)$ and $\phi(\alpha) = 0$ at and only at $\alpha = -\pi/2$. Therefore,

$$\lim_{t \to \infty} \alpha(t) = -\pi/2$$

and for $\alpha(0) > -\pi/2$, $\alpha(t)$ is strictly decreasing and for $\alpha(0) < -\pi/2$, $\alpha(t)$ is strictly increasing. □

3.2 Discrete Trajectories

In this subsection, we prove that the discrete trajectories do not converge to the same point to which the continuous trajectories converge.

Theorem 3 *For any $0 < \gamma < 1$ and for any starting point $x^{(0)}$ lying in the interior of the feasible region, the sequence $x^{(k)}$ converges to a point on the boundary. There exist starting points from which the limiting polar angle is not $-\pi/2$.*

Proof. The discrete trajectories are obtained by taking a step a fraction γ of the way to the boundary along the direction vector

$$\Delta x = \begin{bmatrix} x_1 x_2 \\ -x_1^2 \end{bmatrix}.$$

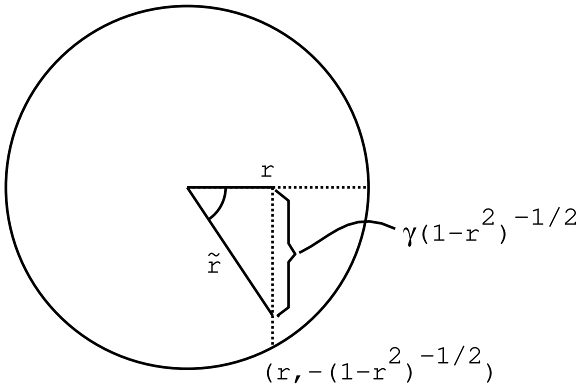

Figure 1: One Step of the Discrete Trajectory

If the current point has polar coordinates (r, α), then simple trigonometry shows (see Figure 1) that the next iterate will have polar coordinates $\tilde{r}$ and $\tilde{\alpha}$ given by

$$\tilde{r}^2 = r^2 + \gamma^2(1-r^2) = (1-\gamma^2)r^2 + \gamma^2$$

and

$$\tilde{\alpha} = \alpha \pm \tan^{-1}\frac{\gamma\sqrt{1-r^2}}{r}$$

where the plus/minus in the formula for $\tilde{\alpha}$ is a plus if the current point is in the left half-plane and a minus if it is in the right half-plane. If we let (r_k, α_k) denote the polar coordinates at the kth iterate, then we get that

$$\begin{aligned} r_k^2 &= (1-\gamma^2)r_{k-1}^2 + \gamma^2 \\ &= (1-\gamma^2)\left((1-\gamma^2)r_{k-2}^2 + \gamma^2\right) + \gamma^2 \\ &= \cdots = \\ &= 1 - (1-\gamma^2)^k(1-r_0^2) \end{aligned}$$

and

$$\alpha_k = \alpha_0 + \sum_{j=0}^{k-1} \epsilon_j \tan^{-1}\frac{\gamma\sqrt{1-r_j^2}}{r_j}$$

$$= \quad \alpha_0 + \sum_{j=0}^{k-1} \epsilon_j \tan^{-1} \gamma \sqrt{\frac{(1-\gamma^2)^j(1-r_0^2)}{1-(1-\gamma^2)^j(1-r_0^2)}}, \tag{3.3}$$

where the ϵ_j's are ± 1's depending on the position of the jth iterate. From these explicit formulas, we see that

$$\lim_{k\to\infty} r_k = 1$$

and that the limit $\alpha_\infty = \lim_k \alpha_k$ exists since the sum in (3.3) converges absolutely (the summand being bounded by $\gamma(1-\gamma^2)^{j/2}\sqrt{1-r_0^2}/r_0$). In fact, we can estimate the deviation of α_∞ from α_0 as follows:

$$\begin{aligned}
|\alpha_\infty - \alpha_0| &\leq \sum_j \tan^{-1} \gamma \sqrt{\frac{(1-\gamma^2)^j(1-r_0^2)}{1-(1-\gamma^2)^j(1-r_0^2)}} \\
&\leq \sum_j \tan^{-1} \frac{\gamma(1-\gamma^2)^{j/2}\sqrt{1-r_0^2}}{r_0} \\
&\leq \sum_j \frac{\gamma(1-\gamma^2)^{j/2}\sqrt{1-r_0^2}}{r_0} \\
&= \frac{\gamma}{1-\sqrt{1-\gamma^2}} \frac{\sqrt{1-r_0^2}}{r_0}.
\end{aligned}$$

For any fixed γ and $\alpha_0 \neq -\pi/2$, it is possible to pick r_0 sufficiently close to one so that

$$\frac{\sqrt{1-r_0^2}}{r_0} < |\alpha_0 + \pi/2| \frac{1-\sqrt{1-\gamma^2}}{\gamma}$$

and for such starting points it is impossible for α_∞ to equal $-\pi/2$. $\square$

4 Affine-Scaling Trajectories

Now we return to an analysis of the affine-scaling trajectories whose field of step directions is given by (2.3).

4.1 Continuous Trajectories

The continuous trajectories are the solutions to the following differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = h(r) \begin{bmatrix} -2x_1x_2 \\ x_1^2 - x_2^2 \end{bmatrix} + r^2 \begin{bmatrix} x_1x_2 \\ -x_1^2 \end{bmatrix}.$$

Theorem 4 *Starting from any feasible* $(x_1(0), x_2(0))$ *not lying on the non-negative* x_2 *axis, the trajectory* $x(t)$ *converges to the optimal point* $(0, -1)$.

Proof. Property 1 of Proposition 1 implies that x_2 is strictly decreasing. From $r = \sqrt{x_1^2 + x_2^2}$, we see that

$$\begin{aligned} \dot{r} &= \frac{(x_1\dot{x}_1 + x_2\dot{x}_2)}{r} \\ &= \frac{x^T\dot{x}}{r} \\ &= -x_2 r h(r) \\ &= -r^2 \sin\alpha\ h(r), \end{aligned} \tag{4.1}$$

and from $\tan\alpha = x_2/x_1$, we get

$$\begin{aligned} \sec^2\alpha\ \dot{\alpha} &= \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2} \\ &= \frac{rh(r) - r^3}{\cos\alpha}. \end{aligned}$$

First, we analyze α. Since

$$\dot{\alpha} = -(r^3 - rh(r))\cos\alpha,$$

we can separate variables and integrate to get

$$|\sec\alpha(t) + \tan\alpha(t)| = |\sec\alpha(0) + \tan\alpha(0)| e^{-\int_0^t r^3 - rh(r) ds}.$$

It suffices to assume that $x_2(0) = 0$ and $x_1(0) \neq 0$. Then, from (4.1), it follows that $\dot{r} > 0$ for $t > 0$ and therefore, $r(t) > r(0)$ for all $t > 0$. Also, from the definition (2.2) of $h(r)$, it is not hard to show that the map $H(r) = r^3 - rh(r)$

is positive and increasing for $0 < r \leq 1$. Hence, $H(r(t)) > H(r(0))$ for $t > 0$ and so

$$|\sec\alpha(t) + \tan\alpha(t)| \leq |\sec\alpha(0) + \tan\alpha(0)| e^{-(r(0)^3 - r(0)h(r(0)))t}.$$

Repeating the argument at the end of the proof of Theorem 2, we now see that

$$\lim_{t\to\infty} \alpha(t) = -\pi/2.$$

Next, we turn our attention to $r(t)$. Even though $\sin\alpha(0) = 0$ for $x_2(0) = 0$, the above analysis shows that $\sin\alpha(t) < 0$ for all $t > 0$. Since, we are only interested in studying the limiting behavior of $r(t)$, it suffices (by reparameterizing the trajectory) to assume now that $\sin\alpha(0) < 0$. Then, (4.1) shows that

$$cr^2 h(r) \leq \dot{r} \leq r^2 h(r),$$

where $c = -\sin\alpha(0) > 0$. To analyze $r(t)$, it is convenient to introduce a related function

$$u(t) = 1 - r^2(t).$$

Then,

$$\begin{aligned} \dot{u} &= -2r\dot{r} \\ &= 2r^3 \sin\alpha h(r) \\ &= 2(1-u)^{3/2} u(1-u^{1/2}) \sin\alpha. \end{aligned}$$

Hence,

$$-2cu \leq \dot{u} \leq -2u,$$

where

$$c = (1-u(0))^{3/2}(1-u^{1/2}(0)).$$

Integrating, we get

$$|u(0)|e^{-2ct} \leq |u(t)| \leq |u(0)|e^{-2t}.$$

Therefore, $\lim u(t) = 0$, which is the same thing as

$$\lim r(t) = 1.$$

□

4.2 Discrete Trajectories

In this subsection, we consider the discrete trajectories of the affine-scaling algorithm.

Theorem 5 *There exist starting points from which the sequence* $x^{(k)}$ *does not converge to the optimal point* $(0, -1)$.

Proof. We consider starting points lying in the lower half-plane. Consider one step of the algorithm. Let x denote the current point and let $\tilde{x}$ denote the next point. Following the tradition of scaling Δx as convenience dictates, let us assume that Δx is scaled so that $x + \Delta x$ is a point on the boundary of the feasible set. Then $(x + \Delta x)^T(x + \Delta x) = 1$. That is,

$$\|x\|^2 + 2x^T\Delta x + \|\Delta x\|^2 = 1. \tag{4.2}$$

Now, $\tilde{x} = x + \gamma\Delta x$ and so using (4.2) to eliminate $\|\Delta x\|^2$, we see that

$$\begin{aligned} \|\tilde{x}\|^2 &= \|x\|^2 + 2\gamma x^T\Delta x + \gamma^2\|\Delta x\|^2 \\ &= (1-\gamma^2)\|x\|^2 + \gamma^2 + 2\gamma(1-\gamma)x^T\Delta x. \end{aligned} \tag{4.3}$$

But for x in the lower half-plane, $x^T\Delta x > 0$ and so

$$\|\tilde{x}\|^2 \geq (1-\gamma^2)\|x\|^2 + \gamma^2.$$

Let r_n denote the polar coordinate of the nth point in the discrete trajectory. Then since all of these points belong to the lower half-plane, we have

$$\begin{aligned} r_n^2 &\geq (1-\gamma^2)r_{n-1}^2 + \gamma^2 \\ &\geq \cdots \geq \\ &\geq 1 - (1-\gamma^2)^n(1-r_0^2). \end{aligned}$$

Therefore,

$$\lim_n r_n = 1.$$

For $x_2 < 0$, Property 2 of Proposition 1 implies that the change in the polar angle during one step of the algorithm cannot be larger than the analogous change derived for the related problem studied in Section 3. To see this, let $\Delta\alpha$ the change in the polar angle. Then, from (4.3) we see that

$$\cos\Delta\alpha = \frac{x^T\tilde{x}}{\|x\|\|\tilde{x}\|} = \frac{\|x\|^2 + \gamma x^T\Delta x}{\|x\|\sqrt{(1-\gamma^2)\|x\|^2 + \gamma^2 + 2\gamma(1-\gamma)x^T\Delta x}}$$

Therefore,

$$\begin{aligned}
\cos^2\Delta\alpha &= \frac{\|x\|^4 + 2\gamma\|x\|^2 x^T\Delta x + \gamma^2(x^T\Delta x)^2}{\|x\|^2[(1-\gamma^2)\|x\|^2 + \gamma^2 + 2\gamma(1-\gamma)x^T\Delta x]} \\
&\geq \frac{\|x\|^2 + 2\gamma x^T\Delta x}{(1-\gamma^2)\|x\|^2 + \gamma^2 + 2\gamma(1-\gamma)x^T\Delta x} \\
&\geq \frac{\|x\|^2 + 2\gamma(1-\gamma)x^T\Delta x}{(1-\gamma^2)\|x\|^2 + \gamma^2 + 2\gamma(1-\gamma)x^T\Delta x} \\
&\geq \frac{\|x\|^2}{(1-\gamma^2)\|x\|^2 + \gamma^2}
\end{aligned}$$

where the second inequality follows from the fact that $x^T\Delta x \geq 0$ and the third inequality follows from the fact that for any three positive numbers a, b, c, $a \leq c$ if and only if $(a+b)/(c+b) \geq a/c$. This last inequality is equivalent to

$$\tan\Delta\alpha \leq \frac{\gamma\sqrt{1-\|x\|^2}}{\|x\|}.$$

Hence,

$$|\alpha_{j+1} - \alpha_j| \leq \tan^{-1}\frac{\gamma\sqrt{1-r_j^2}}{r_j}.$$

Now, the bound on the total change in angle derived in Theorem 3 is valid here as well:

$$|\alpha_\infty - \alpha_0| \leq \frac{\gamma}{1-\sqrt{1-\gamma^2}}\frac{\sqrt{1-r_0^2}}{r_0}.$$

Hence, as for the related problem, for any fixed γ and $\alpha_0 \neq -\pi/2$, it is possible to pick r_0 sufficiently close to one so that

$$\frac{\sqrt{1-r_0^2}}{r_0} < |\alpha_0 + \pi/2|\frac{1-\sqrt{1-\gamma^2}}{\gamma}$$

and for such starting points it is impossible for α_∞ to equal $-\pi/2$. $\square$

Remarks: (1) Consider the sequence of linear programs (1.2) that we used to approximate the semi-infinite linear program (1.1). Recent results of Tsuchiya and Muramatsu [6] show that each of these approximating algorithms generate sequences that converge to optimality as long as γ is chosen to be less than or equal to 2/3. (In fact, for m not a multiple of 4, Vanderbei et.al. [9] have shown that convergence to optimality holds for all $0 < \gamma < 1$. For multiples of 4, the problems are dual degenerate but primal nondegenerate and hence the results in [8] show that for starting points lying below the x_1 axis convergence to optimality again holds for all $0 < \gamma < 1$.) Hence, there is a discontinuity in the behavior of the discrete affine-scaling algorithms as one passes from linear programs to semi-infinite programs. This is an example of the general problem of interchanging two limits and obtaining different results.

(2) In the above proof, we have not used any special properties of the affine-scaling algorithm other than those given in parts 1 and 2 of Proposition 1. Hence, the result proved here is more general than as stated.

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