# Eigenvalues and Expansion of Regular Graphs

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**Abstract**

The spectral method is the best currently known technique to prove lower bounds on expansion. Ramanujan graphs, which have asymptotically optimal second eigenvalue, are the best known explicit expanders. The spectral method yielded a lower bound of $k/4$ on the expansion of linear sized subsets of $k$-regular Ramanujan graphs. We improve the lower bound on the expansion of Ramanujan graphs to approximately $k/2$. Moreover, we construct a family of $k$-regular graphs with asymptotically optimal second eigenvalue and linear expansion equal to $k/2$. This shows that $k/2$ is the best bound one can obtain using the second eigenvalue method. We also show an upper bound of roughly $1+\sqrt{k-1}$ on the average degree of linear-sized induced subgraphs of Ramanujan graphs. This compares positively with the classical bound $2\sqrt{k-1}$. For a linear-sized subset $X$ and integer $t \geq 2$, we show that the set of nodes at distance $t$ from $X$ has size at least $k(k-2)(k-1)^{t-2}|X|/2$. Given a graph $H$, we exhibit a necessary and sufficient condition for the existence of a family of $k$-regular graphs having a given asymptotic bound on the second eigenvalue, and containing $H$ as an induced subgraph. As a byproduct, we obtain improved bounds on random walks on expanders, we construct selection networks (resp. extrovert graphs) of smaller size (resp. degree) than was previously known, and we construct highly expanding non-Ramanujan graphs.

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# 1 Introduction

Expander graphs are widely used in Computer Science, in areas ranging from parallel computation [2, 7, 28, 36, 41] to complexity theory and cryptography [1, 8, 20, 42]. Given an undirected $k$-regular graph $G = (V, E)$ and a subset $X$ of $V$, the expansion of $X$ is defined to be the ratio $|N(X)|/|X|$, where $N(X) = \{w \in V : \exists v \in X,\ (v, w) \in E\}$ is the set of neighbors of $X$. An $(\alpha, \beta, k, n)$-expander is a $k$-regular graph on $n$ nodes such that every subset of size at most $\alpha n$ has expansion at least $\beta$. The range of the subsets whose expansion is relevant and the magnitude of the expansion needed depends on the nature of the application.

It is known that random regular graphs are good expanders. For example, it is not hard to show that for any $\beta < k - 1$, there exists a constant $\alpha$ such that, with high probability, all the subsets of a random $k$-regular graph of size at most $\alpha n$ have expansion at least $\beta$. However, the explicit construction of expander graphs is much more difficult. The first explicit construction of an infinite family of expanders was discovered by Margulis [32] in 1973. Using deep results in group representation theory, he proved a lower bound on the expansion that exceeds 1 by an undetermined but positive factor. In 1981, Gabber and Galil [19] constructed similar graphs and gave an estimate of their expansion coefficient using techniques from harmonic analysis. Their results were improved in [5, 23] using techniques from linear algebra.

The best currently known method to calculate lower bounds on the expansion in polynomial time relies on analyzing the second eigenvalue of the graph. Since the adjacency matrix $A$ is symmetric, all its eigenvalues are real and will be denoted by $\lambda_0 \geq \lambda_1 \geq \ldots \lambda_{n-1}$. We have $\lambda_0 = k$, and $\lambda = \max(\lambda_1, |\lambda_{n-1}|) \leq k$, with equality if and only if $G$ is bipartite or not connected. Tanner [40] proved that for any subset $X$ of $V$,

$$|N(X)| \geq \frac{k^2|X|}{\lambda^2 + (k^2 - \lambda^2)|X|/n}. \tag{1}$$

Therefore, in order to get high expansion, we want to find graphs with $\lambda$ as small as possible. However, Alon and Boppana [3] (see [30, 34] for the proof) showed that, for any sequence $G_{n,k}$ of $k$-regular graphs on $n$ vertices, $\liminf \lambda(G_{n,k}) \geq 2\sqrt{k-1}$ as $n$ goes to infinity. Therefore, even for sparse sets, the best expansion coefficient we can obtain by applying Tanner's result is $k^2/4(k-1) \approx k/4$. This bound is achieved by Ramanujan graphs. By definition, a Ramanujan graph is a connected $k$-regular graph whose eigenvalues $\neq \pm k$ are at most $2\sqrt{k-1}$ in absolute value. Ramanujan graphs, have been explicitly constructed by [30, 33] for many pairs $(k, n)$. Fig. 1 represents a Ramanujan graph on 8 vertices. The relationship between the eigenvalues of the adjacency matrix and the expansion coefficient has also been investigated in [3, 5, 6, 14], but the bound they get, when applied to non-bipartite Ramanujan graphs and for sufficiently large $k$, is no better than Tanner's bound. Alon [3] showed that a $k$-regular graph is an expander if and only if there is a gap between $k$ and

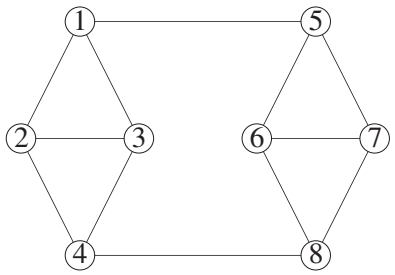


Figure 1: An $(1/2, 3/2, 3, 8)$-expander. The set of neighbors of $X = \{2, 3, 8\}$ is $\{1, 2, 3, 4, 6, 7\}$. The expansion of $X$ is 2. The eigenvalues of the adjacency matrix are $\{3, \sqrt{5}, 1, -1, -1, -1, -1, -\sqrt{5}\}$. This graph is Ramanujan since its second largest eigenvalue $\sqrt{5}$ is less than $2\sqrt{2}$.

the second largest eigenvalue. An excellent survey on expanders is found in [9]. The recent books by Sarnak [37] and Lubotzky [29] give a mathematical background on the construction of Ramanujan graphs. Other results about graph spectra are contained in [10, 15, 16].

Some applications, such as the construction of non-blocking networks in [7], required an expansion greater than $k/2$ of linear-sized subsets. Indeed, if the expansion of a subset $X$ is greater than $k/2$, a constant fraction its nodes have *unique neighbors*, i.e. neighbors adjacent to only one node in $X$. This allows the construction of a matching between $X$ and $N(X)$ in a logarithmic number of steps and using only *local computations*. This can be done by matching the nodes having unique neighbors to their neighbors, then eliminating the matched nodes from $X$ and recursing. Recently, Kelsen [27] designed a randomized local algorithm for finding such a matching in a logarithmic number of steps provided that the expansion is $\Omega(\sqrt{k \log k})$. Pippenger [35] showed that weak expanders are sufficient in applications where an expansion greater than $k/2$ was required.

We define the linear expansion of a family of $k$-regular graphs $G_n$ on $n$ vertices to be the best lower bound on the expansion of subsets of size up to $\alpha n$, where $\alpha$ is an arbitrary small positive constant, that is

$$\sup_{\alpha > 0} \inf_{n} \inf_{X} \frac{|N(X)|}{|X|},$$

where $X$ ranges over the subsets of $G_n$ of size at most $\alpha n$. Our aim is to calculate the best linear expansion one can prove using the second eigenvalue technique. We saw in the previous paragraph why an expansion greater than $k/2$ was required in some applications. We show that if $(G_n)$ is a family of $k$-regular graphs whose second largest eigenvalue is upper bounded by $\tilde{\lambda}$, the linear expansion of $(G_n)$ is at least $(k/2)\left(1 - \sqrt{1 - (4k-4)/\tilde{\lambda}^2}\right)$. In particular, the expansion of linear-sized subsets of Ramanujan graphs is lower bounded by a factor arbitrary close to $k/2$. More precisely, for any $\beta < k/2$, there exists $\alpha > 0$ such that any $k$-regular Ramanujan graph on $n$ nodes is an $(\alpha, \beta, k, n)$-expander. On the other hand, for any integer $k$ such that $k-1$ is a prime congruent to 1 modulo 4, and for any function $m$ of $n$ such that $m = o(n)$, we explicitly construct an infinite

family of $k$-regular graphs $G_n$ on $n$ vertices such that $\lambda(G_n) \leq (2 + o(1))\sqrt{k-1}$ and $G_n$ contains a subset of size $2m$ with expansion $k/2$. Since such a family has asymptotically optimal second eigenvalue, this shows that $k/2$ is essentially the best lower bound on the linear expansion one can obtain by the second eigenvalue method. However, it is still an open question whether there exists a family of Ramanujan graphs with linear expansion at most $k/2$. Our results provide an efficient way to test that the expansion of linear sized subsets of random graphs is at least $k/2 - O(k^{3/4}\log^{1/2} k)$.

We also analyze the *edge expansion* of subsets of $k$-regular graphs, defined as the ratio of the number of edges leaving the subset to its size. Our results are best expressed in terms of the average degree of the induced subgraph. The two quantities, edge expansion and average degree, obviously sum up to $k$. We show that the average degree of the induced subgraphs on linear-sized subsets of a $k$-regular graph $G$ is upper bounded by a factor arbitrary close to $1 + \tilde{\lambda}/2 + \sqrt{\tilde{\lambda}^2/4 - (k-1)}$, where $\tilde{\lambda} = \max(\lambda, 2\sqrt{k-1})$. This bound is equal to $1 + \sqrt{k-1}$ in the case of Ramanujan graphs, improving upon the previously known bound [4] of $2\sqrt{k-1}$. Our method gives lower bounds on the size of $N^t(X)$, for integer $t \geq 2$, where $N^t(X)$ is the set of nodes that can be reached from $X$ by a path of length $t$, that is,

$$\underbrace{N(N(\ldots N(}_{t \text{ times}} X)) \ldots).$$

In particular, we show that the size of $N^t(X)$ is at least $(k-1)^t|X|/2$. We also establish that if $X$ is a linear-sized subset of a Ramanujan graph, the expansion of $N(X)$ is at least $2k/3 - O(1)$. We also study the expansion of subsets of sublinear size. Using the fact that the known explicit Ramanujan graphs have high girth, we show that subsets of sublinear size of these graphs have expansion greater than $k/2$.

Our approach to lower bounding the expansion is to use the second largest eigenvalue to derive information on the structure of subgraphs induced on linear-sized subsets. Roughly speaking, we show that the largest eigenvalue of a matrix associated with an induced subgraph is upper bounded by the second largest eigenvalue of the graph. We prove this result in Section 3. We then use it in Section 4 to derive a lower bound on the expansion of $X$ and an upper bound on its average degree. We use it again in Section 5 to bound from below the expansion of $N(X)$ and the size of $N^t(X)$, for integer $t \geq 2$. In Section 6, we construct $k$-regular graphs with linear expansion $k/2$ and asymptotically optimal second largest eigenvalue, for any integer $k$ such that $k-1$ is a prime congruent to 1 modulo 4. We generalize this result as follows: given a graph $H$, we exhibit an asymptotic necessary and sufficient condition for the existence of a family of $k$-regular graphs that contains $H$ as an induced subgraph, and whose second largest eigenvalue does not exceed a given bound. In Section 7, we study the expansion properties of subsets of sublinear size of the LPS-M graphs. In particular, we show that the expansion of subsets of size at most $n^\gamma$ is at least $k-2$, if $\gamma = \gamma(k)$ is sufficiently small.

As a byproduct of our techniques, we obtain improved results on random walks on expanders. More precisely, in Section 8, we give an upper bound on the probability that a random walk stays inside a given subset of an expander. This property is often used in complexity theory and cryptography. Several papers [1, 20, 22] have addressed this problem, but our result is stronger. Section 9 contains other applications of our results. We adapt a result in [36] to build explicit selection networks of asymptotic size $(3 + o(1))n \log_2 n$, improving on the bound $6n \log_2 n$ that was previously known. As defined in [36], a selection network is a network of comparators that classifies a set of $n$ numbers, where $n$ is even, into two subsets of $n/2$ numbers such that any element in the first set is smaller than any element in the second set. Our results also show that Ramanujan graphs of degree at least 7 are extrovert graphs. We say that a graph is extrovert if all its linear-sized subsets contain a constant fraction of extrovert nodes; a node is extrovert with respect to a subset if more than half of its neighbors are outside the subset. Such graphs have been used [13] to solve the token distribution problem. Classical results [4] require the degree to be at least 15. Finally, we show how to construct, from any family of $k$-regular Ramanujan graphs with high girth, a family of $k(k-1)^t$-regular non-Ramanujan graphs with linear expansion at least $k(k-2)(k-1)^{t-1}/2$, for any integer $t \geq 1$.

Some of the results in this paper have appeared in an extended abstract form in [24, 25, 26].

## 2 Notation, definitions, and background

We first fix some notation and give basic results. We also state elementary facts in linear algebra, whose proof can be found in introductory books in linear algebra, such as [31, 39]. Throughout the paper, $G = (V, E)$ will denote an undirected graph on a set $V$ of vertices. Let $L^2(V)$ denote the set of real valued functions on $V$ and $L_0^2(V) = \{f \in L^2(V);\ \sum_{v \in V} f(v) = 0\}$. As usual, we define the scalar product of two vectors $f$ and $g$ of $L^2(V)$ by

$$f \cdot g = \sum_{v \in V} f(v)g(v),$$

and the euclidean norm of a vector $f$ by $||f|| = \sqrt{f \cdot f}$. We denote the adjacency matrix of $G$ by $A_G$, or simply by $A$ if there is no risk of confusion. The matrix $A$ is the 0-1 $n \times n$ matrix whose $(i, j)$ entry is equal to 1 if and only $(i, j) \in E$. It defines a linear operator in $L^2(V)$ that maps every vector $f \in L^2(V)$ to the vector $Af$ defined by

$$(Af)(v) = \sum_{(v,w) \in E} f(w). \tag{2}$$

This operator is selfadjoint since $\forall f, g \in L^2(V)$ we have

$$(Af) \cdot g = f \cdot (Ag) = \sum_{(v,w) \in E} f(v)g(w). \tag{3}$$

We remind the reader that an operator $B$ in a vector space $H$ is said to be selfadjoint if for all vectors $f$ and $g$ in $H$, $Bf \cdot g = f \cdot Bg$. For any matrix $M$ with real eigenvalues, we denote by $\lambda_i(M)$ the $i$-th largest eigenvalue of $M$. Since the adjacency matrix of any graph $G$ is real and symmetric, all its eigenvalues are real. We will denote $\lambda_i(A_G)$ by $\lambda_i(G)$, or simply by $\lambda_i$. The same remark holds for $\lambda(G) = \max(\lambda_1(G), |\lambda_{n-1}(G)|)$. For any subset $W$ of $V$, we denote by $\chi_W$ the characteristic vector of $W$:

$$\chi_W(v) = \begin{cases} 1 & \text{if } v \in W \\ 0 & \text{otherwise} \end{cases}$$

We denote the adjacency matrix of the graph induced on $W$ by $A_W$, the real number $\lambda_i(A_W)$ by $\lambda_i(W)$, and the set of nodes at distance at most $l$ from $W$ by $B_l(W)$. The support of a vector $f \in L^2(V)$ is defined to be the set of nodes $v$ for which $f(v) \neq 0$. For the rest of this section, we assume that $G$ is $k$-regular.

**Fact 1** *For any vector $g \in L^2(V)$, the sum of the coordinates of $Ag$ is equal to $k$ times the sum of the coordinates of $g$.*

**Proof** This follows simply by summing the two sides of Eq. 2 for all vertices $v$. Each term $f(w)$ in the right-hand side will be counted $k$ times. ■

**Fact 2** *The vector space $L_0^2(V)$ is invariant under $A$.* ■

**Fact 3** *[21, page 314] A selfadjoint operator in a finite dimensional space is diagonalizable in an orthonormal basis.* ■

**Fact 4** *If $B$ is a selfadjoint operator in a vector space $L$, then*

$$\lambda_0(B) = \max_{g \in L - \{0\}} \frac{g \cdot Bg}{||g||^2}.$$

■

Clearly, the vector $\chi_V$ is an eigenvector of $A$ with eigenvalue $k$. This eigenvalue is simple and is the largest in absolute value [10, page 14]. Denote by $A_{|L_0^2(V)}$ the restriction of $A$ to $L_0^2(V)$, i.e. the operator in $L_0^2(V)$ that coincides with $A$ on $L_0^2(V)$. The operator $A_{|L_0^2(V)}$ is diagonalizable since it is selfadjoint (Fact 3), and so the eigenvalues of $A$ consist of $k$ and the eigenvalues of $A_{|L_0^2(V)}$. Therefore, the eigenvalues of $A_{|L_0^2(V)}$ are $\lambda_1, \ldots, \lambda_{n-1}$. In particular, the largest eigenvalue of $A_{|L_0^2(V)}$ is equal to $\lambda_1$.

**Fact 5** *For any $g \in L_0^2(V)$, we have $g \cdot Ag \leq \lambda_1(G)||g||^2$.* ■

**Fact 6** *The eigenvalues of a graph are the union of the eigenvalues of its connected components.* ■

We will use the following elementary results:

**Fact 7 (Pythagorean theorem)** *If $g$ and $h$ are two orthogonal vectors, then $||g+h||^2 = ||g||^2 + ||h||^2$.* ■

**Fact 8 (Cauchy-Schwartz inequality)** *[21, page 277] For any two vectors $g$ and $h$, we have $|g \cdot h| \leq ||g||\ ||h||$.* ■

The following three results follow from the classic theory of matrices with nonnegative entries [38]. For simplicity, we state them in the symmetric case. For two column vectors $g$ and $h$, we say that $g \leq h$ if every coordinate of $g$ is at most its corresponding coordinate in $h$. We say that a matrix $B$ with nonnegative entries is irreducible if its underlying (weighted) graph is connected.

**Fact 9** *[38, page 28, ex. 1.12] If a real symmetric matrix has only nonnegative entries, its largest eigenvalue is nonnegative and has a corresponding eigenvector with nonnegative components. This eigenvalue is largest in absolute value.* ■

**Fact 10** *[38, page 22] If a real symmetric matrix with nonnegative entries is irreducible, then its largest eigenvalue is positive and has a corresponding eigenvector with positive components.* ■

**Fact 11** *If a real symmetric matrix $B$ has only nonnegative entries, and $s$ is a vector with positive components such that $Bs \leq \gamma s$, then the largest eigenvalue of $B$ is at most $\gamma$. This property still holds if only the off-diagonal entries of $B$ are assumed to be nonnegative [18].* ■

**Fact 12** *The average degree of a graph is no greater than its largest eigenvalue.*

**Proof** By Fact 4, we have $\chi_V \cdot A\chi_V / ||\chi_V||^2 \leq \lambda_0(A)$, where $V$ is the vertex set of the graph and $A$ is its adjacency matrix. The left-hand side is equal to the average degree. ■

**Fact 13** *The largest eigenvalue of a graph is no less than the largest eigenvalue of any induced subgraph [12, page 156].* ■

**Fact 14** *The function that associates to a symmetric matrix its largest eigenvalue is subadditive.*

**Proof** This follows immediately from Fact 4. ■

Given a graph $H$, the *cover graph* $H'$ of $H$ is the graph defined on $V' = V \times \{0,1\}$ and where $((u,l),(v,m)) \in V' \times V'$ is an edge if and only $(u,v) \in E$ and $l \neq m$.

**Fact 15** *The eigenvalues of $H'$ consist of the eigenvalues of $H$ and their negated values.*

**Proof** The graph $H'$ is the product of the graph $H$ with $K_2$, and so its eigenvalues consist of the products of the eigenvalues of $H$ with the eigenvalues $\{1,-1\}$ of $K_2$. ■

# 3 Main lemma

In this section, we prove the main lemma (Theorem 1) that we will use later to derive lower bounds on the expansion.

**Lemma 1** *If $G = (V, E)$ is $k$-regular on $n$ vertices, then for any $f \in L^2(V)$, we have*

$$f \cdot Af \leq \lambda_1(G)||f||^2 + \frac{k - \lambda_1(G)}{n}\Big(\sum_{v \in V} f(v)\Big)^2$$

**Proof** First note that Lemma 1 reduces to Fact 5 if $f \in L_0^2(V)$. To prove that it holds for any $f$, we decompose $f$ as the sum of a constant vector and a vector in $L_0^2(V)$. Let $\overline{f} = (f \cdot \chi_V/n)\chi_V$ be the orthogonal projection of $f$ on the subspace spanned by the constant vector $\chi_V$. Then $f_0 = f - \overline{f}$ is the orthogonal projection of $f$ on $L_0^2(V)$. By linearity, $Af = A\overline{f} + Af_0 = k\overline{f} + Af_0$, and so $f \cdot Af = k||\overline{f}||^2 + f_0 \cdot Af_0$ since $\overline{f} \cdot Af_0 = A\overline{f} \cdot f_0 = k\overline{f} \cdot f_0 = 0$. By Fact 5 and the Pythagorean theorem, we have

$$f_0 \cdot Af_0 \leq \lambda_1(G)||f_0||^2 = \lambda_1(G)(||f||^2 - ||\overline{f}||^2).$$

Therefore, $f \cdot Af \leq \lambda_1(G)||f||^2 + (k - \lambda_1(G))||\overline{f}||^2$. We conclude the proof by noting that

$$||\overline{f}||^2 = \frac{(f \cdot \chi_V)^2}{n^2}||\chi_V||^2 = \Big(\sum_{v \in V} f(v)\Big)^2/n.$$

■

**Lemma 2** *For any subset $W$ of a $k$-regular graph $G$, we have $\lambda_0(W) \leq \lambda_1(G) + (k - \lambda_1(G))|W|/n$.*

**Proof** Let $g$ be any element of $L^2(W)$. Consider the vector $f \in L^2(V)$ that coincides with $g$ on $L^2(W)$ and is null on $V - W$. By Eq. 3, we see that $g \cdot A_W g = f \cdot A_G f$. By applying Lemma 1 to $f$ and using the fact that $g$ and $f$ have the same norm and the same sum of coordinates, we have

$$g \cdot A_W g \leq \lambda_1(G)||g||^2 + \frac{k - \lambda_1(G)}{n}\Big(\sum_{v \in W} g(v)\Big)^2 \tag{4}$$

On the other hand, using the Cauchy-Schwartz inequality,

$$\Big(\sum_{v \in W} g(v)\Big)^2 = |\chi_W \cdot g|^2 \leq |W|\ ||g||^2$$

Combining this with Eq. 4 shows that $g \cdot A_W g \leq (\lambda_1(G) + (k - \lambda_1(G))|W|/n)||g||^2$. We conclude using Fact 4. ■

A similar relation [25, Section 4.1] holds between $\lambda_i(W)$ and $\lambda_{i+1}(G)$, for $1 \leq i \leq |W| - 1$. Lemma 2 shows that the largest eigenvalue of subgraphs induced on small linear-sized subsets is

upper bounded by about $\lambda_1(G)$. This bound is optimal for Ramanujan graphs, even for subsets of sublinear size. Lemma 2 already gives a restriction on the structure of induced subgraphs. Together with Fact 12, it implies that the average degree of the induced subgraph on $W$ is at most $\lambda_1(G) + (k - \lambda_1(G))|W|/n$, which is roughly $\lambda_1(G)$ for small linear-sized subsets, or for subsets of sublinear size. To obtain a stronger restriction on the induced subgraph on a linear-sized subset $X$, we will apply Lemma 2 to the set $B_l(X)$. We start by comparing the largest eigenvalue of the subgraph induced on $B_l(X)$ to the matrix of a weighted graph associated with $X$.

Let $\Delta_X$ be the diagonal matrix indexed by the vertices of $X$ and whose entry $(v, v)$ is equal to the degree of $v$ in the subgraph induced on $X$. We will simply denote by $\Delta_X(v)$ the diagonal entry $\Delta_X(v, v)$. For $\theta' > 0$ and integer $l$, let

$$M_X^{\theta',l} = A_X + \frac{1}{\sqrt{k-1}} \frac{\sinh(l\theta')}{\sinh((l+1)\theta')}(kI - \Delta_X).$$

The matrix $M_X^{\theta',l}$ can be regarded as the matrix of the weighted graph that is induced on $X$ and has in addition a loop of weight $(k-1)^{-1/2}(\sinh(l\theta')/\sinh((l+1)\theta'))(k - \Delta_X(v))$ on each node $v$ of $X$.

**Lemma 3** *Suppose $G = (V, E)$ is $k$-regular. Let $l$ be a positive integer, $X$ a non-empty subset of $V$, $\theta'$ a positive real number, and $\lambda' = 2\sqrt{k-1}\cosh\theta'$. If $\lambda_0(B_l(X)) \leq \lambda'$, then $\lambda_0(M_X^{\theta',l}) \leq \lambda'$.*

**Proof** Let $W = B_l(X)$. The idea behind the proof is as follows: given an eigenvector of $M_X^{\theta',l}$ corresponding to its largest eigenvalue, we will extend it to $W$ so that it becomes roughly an eigenvector of $A_W$. If the largest eigenvalue of $M_X^{\theta}$ were too big, we would get a large eigenvalue for $A_W$, contradicting the fact that $\lambda_0(W) \leq \lambda'$.

We define the sequence $r_i$ as follows:

$$r_i = \frac{\sinh((l+1-i)\theta')(k-1)^{-i/2}}{\sinh((l+1)\theta')}.$$

The sequence $r_i$ is strictly positive and decreasing for $0 \leq i \leq l$, and

$$\lambda' r_i = r_{i-1} + (k-1)r_{i+1} \quad (5)$$

$$\lambda' r_l = r_{l-1}, \quad (6)$$

Eq. 6 follows from Eq. 5 and the equality $r_{l+1} = 0$. Note that the sequence $r_i$ behaves like $e^{-\theta' i}(k-1)^{-i/2}$ for $0 \leq i \leq l+1$, but it vanishes at $l+1$. Let $f \in L^2(X)$ be an eigenvector of the matrix $M_X^{\theta',l}$ corresponding to its largest eigenvalue. Since all the entries of $M_X^{\theta',l}$ are nonnegative, we can assume that all the entries of $f$ are nonnegative (Fact 9.) We extend the vector $f$ to $W$ by setting

$$f(v) = \max_{u \in X}(f(u)r_{d(v,u)}),$$

for $v \in W - X$, where $d(v,u)$ denote the distance (length of shortest path) between node $v$ and node $u$. Note that $f$ is positive on $W$ since every node in $W$ is at distance at most $l$ from some node in $X$.

**Claim 1** *For any $v \in W - X$, we have $(A_W f)(v) \geq \lambda' f(v)$.*

**Proof** Let $u \in X$ be such that $f(v) = f(u) r_{d(v,u)}$. Note that $i = d(v,u)$ is at most $l$ since $f$ is positive and $r_j \leq 0$ for $j \geq l+1$. Let $v_1$ be the first node on a shortest path from $v$ to $u$. Since the distance between $v_1$ and $u$ is $i-1$, we have $v_1 \in W$ and

$$f(v_1) \geq f(u) r_{i-1}. \tag{7}$$

We now distinguish two cases:

- **Case 1.** $i = l$. Combining Eqs. 7 and 6, we get in this case

  $$(A_W f)(v) \geq f(v_1) \geq f(u) r_{l-1} = \lambda' f(u) r_l = \lambda' f(v),$$

  as required. The first equation is a consequence of the positivity of $f$.

- **Case 2.** $i < l$. In this case, the $k$ neighbors of $v$ are at distance at most $i+1 \leq l$ from $u$. In particular, all these neighbors belong to $W$. Moreover, by monotonicity of the sequence $r$, it follows that the value of $f$ on each of these neighbors is at least $f(u) r_{i+1}$. Using again Eq. 7, we have

  $$\begin{aligned} (A_W f)(v) &\geq f(v_1) + (k-1) f(u) r_{i+1} \\ &\geq f(u)(r_{i-1} + (k-1) r_{i+1}) \\ &= \lambda' f(u) r_i \\ &= \lambda' f(v). \end{aligned}$$

■

We now establish a lower bound on $(A_W f)(v)$, for $v \in X$.

**Claim 2** *For any $v \in X$, we have $(A_W f)(v) \geq \lambda_0(M_X^{\theta',l}) f(v)$.*

**Proof** It follows from the definition of $f$ that for any $v \in X$, the value of $f$ on each of the $k - \Delta_X(v)$ neighbors of $v$ in $W - X$ is at least $f(v) r(1)$. Therefore,

$$(A_W f)(v) \geq (A_X f)(v) + (k - \Delta_X(v)) f(v) r(1) = (M_X^{\theta',l} f)(v) = \lambda_0(M_X^{\theta',l}) f(v).$$

The second equation follows from the expression of $r(1)$ and definition of $M_X^{\theta',l}$. ■

Since $\lambda_0(W) \leq \lambda'$, we have $(A_W f) \cdot f \leq \lambda' ||f||^2$, by Fact 4. Assume for contradiction that $\lambda' < \lambda_0(M_X^{\theta',l})$. By considering a node in $X$ where $f$ is nonzero and using Claim 1, Claim 2 and the fact that $f$ is nonnegative, we conclude that $(A_W f) \cdot f > \lambda' ||f||^2$, leading to a contradiction. This concludes the proof. ■

For $\theta \geq 0$, define

$$M_X^\theta = A_X + \frac{1}{\sqrt{k-1}e^\theta}(kI - \Delta_X).$$

The matrix $M_X^\theta$ can be regarded as the matrix of the weighted graph that is induced on $X$ and has in addition a loop of weight $(k-1)^{-1/2}e^{-\theta}(k - \Delta_X(v))$ on each node $v$ of $X$. For any graph $H$, we can define the matrix $M_H^\theta$ similarly to the definition of $M_X^\theta$.

**Theorem 1** *Suppose $G = (V, E)$ is $k$-regular, and let $\tilde{\lambda} = \max(\lambda_1(G), 2\sqrt{k-1}) = 2\sqrt{k-1}\cosh\theta$, where $\theta \geq 0$. For any non-empty subset $X$ of $V$ of size at most $k^{-1/\epsilon}|V|$, we have*

$$\lambda_0(M_X^\theta) \leq \tilde{\lambda}(1 + O(\epsilon)),$$

*where the constant behind the $O$ is a small absolute constant.*

**Proof** Let $l = \lfloor 1/2\epsilon \rfloor$ and let $W$ be the set of nodes at distance at most $l$ from $X$. A simple calculation shows that $|W| \leq 3k^l|X| \leq 3k^{-1/(2\epsilon)}n$. It follows from Lemma 2 that $\lambda_0(W) \leq \lambda'$, where $\lambda' = \tilde{\lambda} + 3k^{1-1/(2\epsilon)} = 2\sqrt{k-1}\cosh\theta'$ and $\theta' > \theta \geq 0$. A straightforward calculation shows that $\lambda' = \tilde{\lambda}(1 + O(\epsilon))$ and $\cosh\theta' - \cosh\theta = O(k^{1/2-1/(2\epsilon)}) = O(\epsilon^2)$. We will use the following results to show that the matrix $M_X^{\theta',l}$ is approximated by $M_X^\theta$.

**Claim 3** *For $x \geq y \geq 0$, we have $(x-y)^2 \leq 2(\cosh x - \cosh y)$.*

**Proof** We apply Taylor's expansion formula. Since the derivative of $\cosh x$ is $\sinh x$ and vice-versa, we have

$$\cosh x = \cosh y + (x-y)\sinh y + \frac{(x-y)^2}{2}\cosh\gamma,$$

where $y \leq \gamma \leq x$. The claim follows immediately. ■

**Claim 4**

$$\frac{l}{l+1}e^{-\theta'} \leq \frac{\sinh(l\theta')}{\sinh((l+1)\theta')} \leq e^{-\theta'} \tag{8}$$

**Proof** Since $\sinh((l+1)\theta') \geq (l+1)\sinh\theta'$, we have

$$e^{-\theta'} - \frac{\sinh(l\theta')}{\sinh((l+1)\theta')} = \frac{e^{-l\theta'} - e^{-(l+2)\theta'}}{e^{(l+1)\theta'} - e^{-(l+1)\theta'}} = e^{-(l+1)\theta'}\frac{\sinh\theta'}{\sinh((l+1)\theta')} \leq \frac{e^{-\theta'}}{l+1}.$$

■

It follows from Claim 3 that $\theta' - \theta = O(\epsilon)$. On the other hand, Eq. 8 implies that

$$\frac{\sinh(l\theta')}{\sinh((l+1)\theta')} = e^{-\theta'}(1 + O(1/l)) = e^{-\theta}(1 + O(\epsilon)).$$

Therefore, all entries of the matrix $M_X^{\theta} - M_X^{\theta',l}$ are $O(\sqrt{k-1}\epsilon)$. Since it is diagonal, it follows that its largest eigenvalue is $O(\sqrt{k-1}\epsilon)$. By Fact 14 and Lemma 3, we have

$$\begin{aligned} \lambda_0(M_X^{\theta}) &\leq \lambda_0(M_X^{\theta',l}) + O(\sqrt{k-1}\epsilon) \\ &\leq \lambda' + O(\tilde{\lambda}\epsilon) \\ &= \tilde{\lambda}(1 + O(\epsilon)) \end{aligned}$$

■

**Remark 1** The only place where we used in the proof the fact that $\lambda_1(G)$ is the second eigenvalue of $G$ was in conjunction with Lemma 2 to upper bound $\lambda_0(W)$. In particular, if $\lambda^*$ is a real number such that for any subset $W$ of $V$, we have $\lambda_0(W) \leq \lambda^* + 2k|W|/|V|$, then Theorem 1 remains valid if $\lambda_1(G)$ is replaced by $\lambda^*$.

# 4 Lower bounds on the expansion and on the average degree

We will derive lower bounds on the expansion by applying Theorem 1 to the union of $X$ and $N(X)$, after reducing to the case where the graph is bipartite and $X$ is on one side of the partition. The idea is that if the expansion of $X$ is small, a node in $N(X)$ will be adjacent to many nodes in $X$, in average. This implies that the largest eigenvalue of the weighted matrix associated to the subgraph induced on $X \cup N(X)$ is large, contradicting with Theorem 1. We also use Theorem 1 to derive upper bounds on the average degree of induced subgraphs.

**Theorem 2** *If $G = (V, E)$ is $k$-regular and $\tilde{\lambda} = \max(\lambda_1(G), 2\sqrt{k-1})$, then for any non-empty subset $X$ of $V$ of size at most $k^{-1/\epsilon}|V|$,*

$$\frac{|N(X)|}{|X|} \geq \frac{k}{2}\left(1 - \sqrt{1 - \frac{4k-4}{\tilde{\lambda}^2}}\right)(1 - O(\epsilon)),$$

*where the constant behind the $O$ is a small absolute constant.*

**Proof** We first show how to reduce the problem to the case where the graph $G$ is bipartite and $X$ is on one side of the partition. Consider the cover graph $G'$ of $G$, as defined in Section 3. We will show that Remark 1 applies to $\lambda^* = \lambda_1(G)$. Indeed, let $W$ be a subset of $V'$, let $W_p \subseteq V$ be its projection on $V$, i.e. the set of nodes $u$ of $V$ such that $(u,0) \in W$ or $(u,1) \in W$. Let $W^* = W_p \times \{0,1\}$. Since $W \subseteq W^*$, it follows from Fact 13 that $\lambda_0(W) \leq \lambda_0(W^*)$. (We remind the reader that $\lambda_0(W)$ (resp. $\lambda_0(W^*)$) is the largest eigenvalue of the subgraph of $G'$ induced on $W$ (resp. $W^*$), and $\lambda_0(W_p)$ is the largest eigenvalue of the subgraph of $G$ induced on $W_p$.)

On the other hand, the subgraph of $G'$ induced on $W^*$ is the graph cover of the subgraph of $G$ induced on $W_p$. It follows from Fact 15 that $\lambda_0(W^*)$ is the largest eigenvalue in absolute value of the subgraph of $G$ induced on $W_p$. Hence, by Fact 9, we have $\lambda_0(W^*) = \lambda_0(W_p)$. Using Lemma 2, it follows that

$$\lambda_0(W) \leq \lambda_0(W_p) \leq \lambda_1(G) + k\frac{|W_p|}{|V|} \leq \lambda_1(G) + 2k\frac{|W|}{|V'|},$$

as required by Remark 1. Note that we cannot apply directly Theorem 1 because, by Fact 15, we have $\lambda_1(G') = \lambda(G)$, which may be different from $\lambda_1(G)$.

Let $Y$ be the subset of $V'$ equal to $X \times \{0\}$. Denote the adjacency matrix of $G'$ by $A'$, and the set of neighbors of $Y$ in $G'$ by $N'(Y)$. Note that $|Y| = |X|$ and $|N'(Y)| = |N(X)|$.

By applying Remark 1 to the graph $G'$, the set of vertices $Y \cup N'(Y)$, and $\epsilon' = 2\epsilon$, we see that the largest eigenvalue of the matrix $M' = M^{\theta}_{Y \cup N'(Y)}$ is at most $\tilde{\lambda}(1 + O(\epsilon))$ ($M' = M^{\theta}_H$, where $H$ is the subgraph of $G'$ induced on $Y \cup N'(Y)$.) Now, consider the function $f \in L^2(Y \cup N'(Y))$ defined by $f = k\chi_Y + \tilde{\lambda}\chi_{N'(Y)}$. We have (Fact 4)

$$M'f \cdot f \leq \tilde{\lambda}(1 + O(\epsilon))||f||^2. \tag{9}$$

The left-hand side is the sum of two terms. The first is equal to $A'f \cdot f$, and the second corresponds to the weighted self-loops. By Eq. 3, and since the two sets $Y$ and $N'(Y)$ are independent in $G'$ and there are $k|Y|$ undirected edges between $Y$ and $N'(Y)$, we have $A'f \cdot f = 2\tilde{\lambda}k^2|Y|$. To calculate the second term, we observe that the loops have no weight on $Y$ and have average weight $e^{-\theta}(k-1)^{-1/2}(k - k|Y|/|N'(Y)|)$ on $N'(Y)$. This is because a vertex in $N'(Y)$ has in average $k|Y|/|N'(Y)|$ neighbors in $Y$. Hence, the second term is equal to $e^{-\theta}(k-1)^{-1/2}(k|N'(Y)| - k|Y|)\tilde{\lambda}^2$.

By expanding the two sides of Eq. 9, we see that it reduces to

$$2\tilde{\lambda}k^2|Y| + \frac{\tilde{\lambda}^2}{\sqrt{k-1}e^{\theta}}k(|N'(Y)| - |Y|) \leq \tilde{\lambda}(1 + O(\epsilon))(k^2|Y| + \tilde{\lambda}^2|N'(Y)|).$$

By dividing the two sides by $\tilde{\lambda}$ and replacing $|Y|$ by $|X|$, $|N'(Y)|$ by $|N(X)|$, and $\tilde{\lambda}$ by $2\sqrt{k-1}\cosh\theta$ in the left-hand side, we get

$$k|X|(k(1-O(\epsilon)) - 2e^{-\theta}\cosh\theta) \leq |N(X)|(\tilde{\lambda}^2 - 2ke^{-\theta}\cosh\theta)(1+O(\epsilon)). \tag{10}$$

The factor $1-O(\epsilon)$ in the left-hand side gets absorbed in the right-hand side since $e^{-\theta}\cosh\theta \leq 1$. On the other hand,

$$\begin{aligned}
\tilde{\lambda}^2 - 2ke^{-\theta}\cosh\theta &= 2((2k-2)\cosh\theta - ke^{-\theta})\cosh\theta \\
&= 2(k(2\cosh\theta - e^{-\theta}) - 2\cosh\theta)\cosh\theta \\
&= 2(ke^{\theta} - 2\cosh\theta)\cosh\theta.
\end{aligned}$$

Eq. 10 therefore reduces to:

$$\frac{|N(X)|}{|X|} \geq \frac{k}{2e^{\theta}\cosh\theta}(1-O(\epsilon)).$$

It remains to replace the terms in $\theta$ with a function of $\tilde{\lambda}$. We do this using the formula $e^{-\theta} = \cosh\theta - \sqrt{\cosh^2\theta - 1}$, which leads to

$$\frac{1}{e^{\theta}\cosh\theta} = \frac{e^{-\theta}}{\cosh\theta} = 1 - \sqrt{1 - \frac{1}{\cosh^2\theta}} = 1 - \sqrt{1 - \frac{4k-4}{\tilde{\lambda}^2}}.$$

This concludes the proof. ■

**Theorem 3** *If $G=(V,E)$ is $k$-regular and $\tilde{\lambda} = \max(\lambda_1(G), 2\sqrt{k-1})$, then for any non-empty subset $X$ of $V$ of size at most $k^{-1/\epsilon}|V|$, the average degree $\sigma$ of the subgraph of $G$ induced on $X$ is at most*

$$\left(1 + \frac{\tilde{\lambda}}{2} + \sqrt{\frac{\tilde{\lambda}^2}{4} - (k-1)}\right)(1+O(\epsilon)),$$

*where the constant behind the $O$ is a small absolute constant.*

**Proof** We use the same notations as in Theorem 1. As noted before, the matrix $M_X^{\theta}$ can be regarded as the matrix of the weighted graph on $X$ that is induced on $X$ and has in addition a loop of weight $(k-1)^{-1/2}e^{-\theta}(k - \Delta_X(v))$ on each node $v$ of $X$. By applying Theorem 1 and Fact 12 to this graph, we get

$$\sigma + \frac{k-\sigma}{\sqrt{k-1}e^{\theta}} \leq 2\sqrt{k-1}(1+O(\epsilon))\cosh\theta,$$

which implies

$$\begin{aligned}
\sigma &\leq \frac{2(k-1)e^{\theta}\cosh\theta - k}{\sqrt{k-1}e^{\theta} - 1}(1+O(\epsilon)) \\
&= (\sqrt{k-1}e^{\theta} + 1)(1+O(\epsilon)).
\end{aligned}$$

We conclude by noting that $e^\theta = \cosh\theta + \sinh\theta = \cosh\theta + \sqrt{\cosh^2\theta - 1}$. ■

# 5 Lower bounds on the expansion of $N(X)$ and on $|N^t(X)|$, for $t \geq 2$.

We again apply the same techniques as in Section 4 to get lower bounds on the size of $N^t(X)$, for $t \geq 2$, and on the expansion of $N(X)$. We start with the case $t = 2$.

**Theorem 4** *If $G = (V, E)$ is a $k$-regular graph such that $\lambda_1(G) \leq 2\sqrt{k-1}$, then for any non-empty subset $X$ of $V$ of size at most $k^{-1/\epsilon}|V|$,*

$$\frac{|N^2(X)|}{|X|} \geq (1 + k(k-2)/2)(1 - O(\epsilon)),$$

*where the constant behind the $O$ is a small absolute constant.*

**Proof** As in the proof of Theorem 2, we can assume without loss of generality that $G$ is bipartite and that $X$ is on one side of the partition. Let $f \in L^2(N(X) \cup N^2(X))$ be the function defined by $f = k\chi_X + 2\sqrt{k-1}\chi_{N(X)} + 2\chi_{N^2(X)-X}$. By applying Remark 1 to the set $N(X) \cup N^2(X)$, we have

$$M'f \cdot f \leq 2\sqrt{k-1}(1 + O(\epsilon))||f||^2, \tag{11}$$

where $M' = M^0_{N(X)\cup N^2(X)}$. As in Section 4, we can calculate $M'f \cdot f$ by using Eq. 3 and the fact that there are $k|X|$ edges between $X$ and $N(X)$, and $k|N(X)| - k|X|$ edges between $N(X)$ and $N^2(X) - X$. This yields

$$M'f \cdot f = 4k^2\sqrt{k-1}|X| + 8k\sqrt{k-1}(|N(X)| - |X|) + \frac{4k}{\sqrt{k-1}}(|N^2(X)| - |N(X)|). \tag{12}$$

On the other hand,

$$||f||^2 = k^2|X| + 4(k-1)|N(X)| + 4(|N^2(X)| - |X|). \tag{13}$$

By combining Eqs. 11, 12 and 13, and after some simplifications and factoring out the term $k - 2$, we get

$$(k-1)(k-2)|X| + 2|N(X)| \leq 2|N^2(X)|(1 + O(\epsilon)). \tag{14}$$

(A technical point: the expansion coefficients of $X$ and of $N(X)$ are both $\Theta(k)$, as can be seen directly using Theorem 2. Therefore, the three terms in the expression of $M'f \cdot f$ are proportional to each other. The same holds for $||f||^2$. This allows us to make simplifications without worrying about the multiplicative term $1 + O(\epsilon)$.)

Using again the inequality $|N(X)| \geq k|X|(1-O(\epsilon))/2$ and combining it with Eq. 14 yields the desired lower bound on $|N^2(X)|$. ■

Theorem 4 can be generalized as follows:

**Theorem 5** *If $G=(V,E)$ is a $k$-regular graph such that $\lambda_1(G) \leq 2\sqrt{k-1}$, then for any integer $t \geq 2$ and any non-empty subset $X$ of $V$ of size at most $k^{-t-1/\epsilon}|V|$,*

$$\frac{|N^t(X) - N^{t-2}(X)|}{|X|} \geq \frac{k(k-2)(k-1)^{t-2}}{2}(1-O(t\epsilon)),$$

*where the constant behind the $O$ is a small absolute constant.*

**Proof** Since $N^{t-2}(X) \subseteq N^t(X)$, we have $|N^t(X) - N^{t-2}(X)| = |N^t(X)| - |N^{t-2}(X)|$. Hence, as before, we can reduce the problem to the case where the graph is bipartite and $X$ is on one side of the partition. Let $M' = M^0_{N^{t-1}(X) \cup N^t(X)}$, and $f \in L^2(N^{t-1}(X) \cup N^t(X))$ be the function defined by

$$f = k\chi_X + 2\sqrt{k-1}\chi_{N(X)} + 2\sum_{i=2}^{t}(k-1)^{1-i/2}\chi_{N^i(X)-N^{i-2}(X)}.$$

As before,

$$M'f \cdot f \leq 2\sqrt{k-1}(1+O(\epsilon))||f||^2.$$

There are $k|N^t(X)| - k|N^{t-1}(X)|$ edges between $N^t(X) - N^{t-2}(X)$ and $N^{t+1}(X) - N^{t-1}(X)$. Indeed, if $e(A,B) = |E| \cap (A \times B)$ is the number of edges whose first endpoint is in $A$ and second endpoint in $B$, then

$$\begin{aligned} e(N^t(X) - N^{t-2}(X), N^{t+1}(X) - N^{t-1}(X)) &= e(N^t(X), N^{t+1}(X) - N^{t-1}(X)) \\ &= e(N^t(X), N^{t+1}(X)) - e(N^t(X), N^{t-1}(X)) \\ &= k|N^t(X)| - k|N^{t-1}(X)|, \end{aligned}$$

as required. As a consequence,

$$M'f \cdot f = 2\sqrt{k-1}\left(2k^2|X| + 4k\sum_{i=1}^{t-1}\frac{|N^i(X)| - |N^{i-1}(X)|}{(k-1)^{i-1}} + \frac{2k}{(k-1)^{t-1}}(|N^t(X)| - |N^{t-1}(X)|)\right),$$

and

$$||f||^2 = k^2|X| + 4(k-1)|N(X)| + 4\sum_{i=2}^{t}\frac{|N^i(X)| - |N^{i-2}(X)|}{(k-1)^{i-2}}.$$

But

$$\sum_{i=2}^{t}\frac{|N^i(X)| - |N^{i-2}(X)|}{(k-1)^{i-2}} = \sum_{i=2}^{t}\frac{|N^i(X)| - |N^{i-1}(X)|}{(k-1)^{i-2}} + \sum_{i=1}^{t-1}\frac{|N^i(X)| - |N^{i-1}(X)|}{(k-1)^{i-1}}$$

$$= \frac{|N^t(X)| - |N^{t-1}(X)|}{(k-1)^{t-2}} + |N(X)| - |X| + k\sum_{i=2}^{t-1} \frac{|N^i(X)| - |N^{i-1}(X)|}{(k-1)^{i-1}},$$

and so

$$||f||^2 = (k^2-4)|X| + 4k|N(X)| + 4\frac{|N^t(X)| - |N^{t-1}(X)|}{(k-1)^{t-2}} + 4k\sum_{i=2}^{t-1} \frac{|N^i(X)| - |N^{i-1}(X)|}{(k-1)^{i-1}}.$$

After some simplifications, we conclude that

$$(k-2)(k-1)^{t-1}|X| + 2|N^{t-1}(X)| \leq 2|N^t(X)|(1+O(t\epsilon)). \tag{15}$$

The same inequality holds if $t$ is replaced by $t-1$. Theorem 5 follows immediately. By applying Eq. 15 for $t, t-1, \ldots, 2$, we see that $|N^t(X)| \geq ((k-1)^t + 1)(1 - O(t^2\epsilon))|X|/2$. ■

**Theorem 6** *If $G = (V, E)$ is a $k$-regular graph such that $\lambda_1(G) \leq 2\sqrt{k-1}$, then for any non-empty subset $X$ of $V$ of size at most $k^{-1/\epsilon}|V|$,*

$$\frac{|N^2(X)|}{|N(X)|} \geq (2k/3 + O(1))(1 + O(\epsilon)),$$

*where the constants behind the $O$ are small absolute constants.*

**Proof** We use the same notations and similar techniques as in Theorem 4, but we use the function $f \in L^2(N(X) \cup N^2(X))$ defined $f = k\chi_X + 2\sqrt{k}\chi_{N(X)} + 3\chi_{N^2(X)-X}$. We have

$$M'f \cdot f \leq 2\sqrt{k-1}(1 + O(\epsilon))||f||^2. \tag{16}$$

A simple calculation shows that

$$\begin{aligned} M'f \cdot f &= 4k^{5/2}|X| + 12k^{3/2}(|N(X)| - |X|) + \frac{9k}{\sqrt{k-1}}(|N^2(X)| - |N(X)|) \\ &= (4k^{5/2}|X| + 12k^{3/2}|N(X| + 9\sqrt{k}|N^2(X)|)(1 + O(1/k)). \end{aligned}$$

The last equality follows from the fact that the expansion coefficients of $X$ and of $N(X)$ are both $\Theta(k)$, as can be seen directly using Theorem 2. On the other hand,

$$||f||^2 = k^2|X| + 4k|N(X)| + 9(|N^2(X)| - |X|).$$

By combining Eqs. 16 with the last equations and doing some simplifications, we get

$$2k^2|X| + 4k|N(X)| \leq 9(1 + O(1/k))(1 + O(\epsilon))|N^2(X)|.$$

We conclude by noting that $|N(X)| \leq k|X|$. ■

# 6 A family of almost Ramanujan graphs with expansion $k/2$

In this section, we show that no technique based on the second eigenvalue method can yield a lower bound on the expansion better than $k/2$. We do so by constructing an explicit family of $k$-regular graphs $G_n$ containing subsets of sublinear size having expansion $k/2$, and such that $\lambda(G_n) = (2 + o(1))\sqrt{k-1}$. This leads us to study the following question: given a graph $H$ and real number $\overline{\lambda}$, is there an infinite family of graphs containing $H$ as a subgraph and whose second eigenvalue is at most $\overline{\lambda}(1 + o(1))$? We give a necessary and sufficient condition on $H$ for the existence of such a family. We start with a lemma that will be used throughout this section.

**Lemma 4** *Consider a graph on a vertex set $W$, a subset $X$ of $W$, and $s \in L^2(W)$. Let $X_i$ be the set of nodes at distance $i$ from $X$. Assume the following conditions hold:*

1. *For $h-1 \leq i, j \leq h$, all nodes in $X_i$ have the same number of neighbors in $X_j$.*
2. *The vector $s$ is constant on $X_{h-1}$ and $X_h$.*
3. *$s$ has positive components and $As \leq \mu s$ on $B_{h-1}(X)$, where $\mu$ is a positive real number.*

*Then for any $g \in L^2(W)$ such that $|Ag(u)| = \mu|g(u)|$ for $u \in B_{h-1}(X)$, we have*

$$\frac{\sum_{v \in X_h} g(v)^2}{\sum_{v \in X_h} s(v)^2} \geq \frac{\sum_{v \in X_{h-1}} g(v)^2}{\sum_{v \in X_{h-1}} s(v)^2}.$$

**Proof** Let $P, P_{h-1}$ and $P_h$ be the projections on the sets $B_{h-1}(X), X_{h-1}$ and $X_h$ respectively. We need to show that $||P_h g||^2/||P_h s||^2 \geq ||P_{h-1} g||^2/||P_{h-1} s||^2$. Let $A_h = (P + P_h)A(P + P_h)$. The operator $A_h$ corresponds in some sense to the adjacency matrix of the subgraph induced on $B_h(X)$, but it acts on $L^2(W)$. By Conditions 1 and 2, there exist positive coefficients $\alpha$, $\beta$ and $\gamma$ such that $P_h A_h s = \gamma P_h s$ and $A_h P_h s = \alpha P_h s + \beta P_{h-1} s$. By hypothesis, we have $A_h s \leq \mu P s + \gamma P_h s$. Premultiplying both sides of this equation by $P$ and $A_h$ yields successively

$$\begin{aligned} PA_h s &\leq \mu P s \\ &= \mu s - \mu P_h s, \\ A_h P A_h s &\leq \mu A_h s - \mu(\alpha P_h s + \beta P_{h-1} s) \\ &\leq \mu^2 P s + \mu(\gamma - \alpha) P_h s - \mu\beta P_{h-1} s. \end{aligned}$$

But the matrix $A_h P A_h - \mu^2 P - \mu(\gamma - \alpha)P_h + \mu\beta P_{h-1}$ has only nonnegative entries off its diagonal, and so its largest eigenvalue is 0 since $s$ is positive (Fact 11.) The quadratic form associated to this

matrix is therefore semi-definite negative (Fact 4), and so

$$A_h P A_h g \cdot g \leq \mu^2 P g \cdot g + \mu(\gamma - \alpha) P_h g \cdot g - \mu\beta P_{h-1} g \cdot g. \tag{17}$$

Since both $A_h$ and $P$ are selfadjoint and since $P^2 = P$, the left-hand side of Eq. 17 is equal to $||PA_h g||^2$. We can rewrite Eq. 17 as follows:

$$||PA_h g||^2 \leq \mu^2 ||Pg||^2 + \mu(\gamma - \alpha)||P_h g||^2 - \mu\beta ||P_{h-1} g||^2.$$

But $||PA_h g|| = \mu ||Pg||$ by hypothesis, and so

$$(\gamma - \alpha)||P_h g||^2 \geq \beta ||P_{h-1} g||^2. \tag{18}$$

On the other hand, since $A_h$ and $P_h$ are selfadjoint, we have $A_h P_h s \cdot s = P_h A_h s \cdot s$, and so $\alpha||P_h s||^2 + \beta||P_{h-1} s||^2 = \gamma||P_h s||^2$. Comparing this with Eq. 18 concludes the proof. ■

**Remark 2** The only time where we used Conditions 1 and 2 in the lemma was to prove the existence of $\alpha$, $\beta$ and $\gamma$ such that $P_h A_h s = \gamma P_h s$ and $A_h P_h s = \alpha P_h s + \beta P_{h-1} s$. Therefore, the conclusion of Lemma 4 still holds if Conditions 1 and 2 are replaced by the existence of $\alpha$, $\beta$ and $\gamma$ satisfying the above equations.

**Theorem 7** *For any integer $k$ such that $k-1$ is prime, we can explicitly construct an infinite family of $k$-regular graphs $G_n$ on $n$ vertices whose linear expansion is $k/2$ and such that $\lambda_1(G_n) \leq 2\sqrt{k-1}(1 + 2\log^2 \log n / \log_k^2 n)$.*

**Proof** We construct the family $(G_n)$ by altering the known constructions of explicit Ramanujan graphs, so that the expansion of $(G_n)$ is $k/2$. Roughly speaking, we show that an eigenvector of $G_n$ associated to a large eigenvalue corresponds to an eigenvector of the original graph with the same eigenvalue, and thereby conclude that $G_n$ is almost Ramanujan.

From [30] and [33], we know that we can explicitly construct an infinite family of bipartite Ramanujan graphs $(F_n)$ on $n$ vertices whose girth is $(4/3 + o(1)) \log_{k-1} n$. Let $F_n = (V, E)$ be an element of the family, $u \in V$ a vertex of $F_n$ and $l = \lfloor c(F_n)/2 \rfloor - 2$. Let $u_1, \ldots, u_k$ be the neighbors of $u$ and let $v_1, \ldots, v_k$ be $k$ vertices at distance two from $u$ such that $(u_i, v_i) \in E$. The subgraph of $F_n$ induced on $B_{l+1}(\{u\})$ is a tree rooted at $u$ since it is connected and contains no cycle. Let $u'$ and $v'$ be two elements not belonging to $V$. Consider the $k$-regular graph $G_{n+2} = (V', E')$, where $V' = V \cup \{u', v'\}$ and $E' = E \cup \bigcup_{i=1}^{k} \{(u', u_i), (u_i, u'), (v', v_i), (v_i, v')\} - \bigcup_{i=1}^{k} \{(u_i, v_i), (v_i, u_i)\}$. Fig. 2 shows the graph $G_{n+2}$ in the neighborhood of $u$ in the case $k = 3$. For shorthand, we denote $A_{F_n}$ by $A$, $A_{G_{n+2}}$ by $A'$ and $\lambda_1(A')$ by $\lambda'$. We need to show that $\lambda' \leq (2 + o(1))\sqrt{k-1}$. Assume that

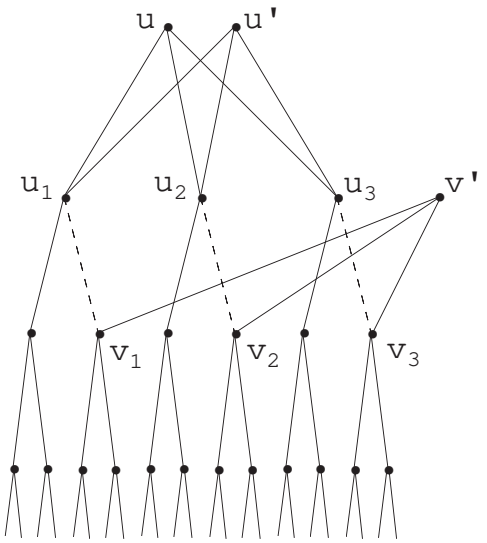


Figure 2: The graph $G_{n+2}$ in the neighborhood of $u$ in the case $k = 3$. The dotted edges are those belonging to $E - E'$.

$\lambda' > 2\sqrt{k-1}$ (otherwise we are done), and let $\lambda' = 2\sqrt{k-1}\cosh\theta'$, with $\theta' > 0$. Let $g \in L_0^2(V')$ be an eigenvector corresponding to $\lambda'$.

We outline informally the basic ideas of the proof. Roughly speaking, we will show that the values that $g$ takes on the nodes $u', v', u_i, v_i$ are small compared to $||g||$. Since $g \in L_0^2(V')$, we conclude that $g$ almost belongs to $L_0^2(V)$. Fact 5 then implies that $g \cdot Ag \leq (2 + o(1))\sqrt{k-1}||g||^2$. But, since $g(u')$, $g(v')$, $g(u_i)$ and $g(v_i)$ are small, the scalar product $g \cdot Ag$ is close to $g \cdot A'g = \lambda'||g||^2$, and so $\lambda' \leq (2 + o(1))\sqrt{k-1}$.

Since $u$ and $u'$ have the same neighbors in $G_{n+2}$ and $\lambda' \neq 0$, we have $g(u) = g(u')$. Let $f$ be the element of $L^2(V)$ that coincides with $g$ on $V$. By Eq. 3, we have

$$\begin{aligned} \lambda'||g||^2 &= g \cdot A'g \\ &= f \cdot Af - 2\sum_{i=1}^{k} g(u_i)g(v_i) + 2\sum_{i=1}^{k} g(u')g(u_i) + 2\sum_{i=1}^{k} g(v')g(v_i). \end{aligned} \tag{19}$$

We upper bound $-2g(u_i)g(v_i)$ by $g(u_i)^2 + g(v_i)^2$. On the other hand, using the Cauchy-Schwartz inequality and the equality $(A'g)(u') = \lambda' g(u')$, we have

$$\sum_{i=1}^{k} g(u')g(u_i) = \lambda' g(u')^2 = \frac{1}{\lambda'}\Big(\sum_{i=1}^{k} g(u_i)\Big)^2 \leq \frac{k}{\lambda'}\sum_{i=1}^{k} g(u_i)^2.$$

A similar relation holds for $v'$. Combining this with Eq. 19, we get

$$\lambda' ||g||^2 \leq f \cdot Af + (1 + \frac{2k}{\lambda'}) \sum_{i=1}^{k} (g(u_i)^2 + g(v_i)^2). \tag{20}$$

Next, we bound the term $f \cdot Af$. Note that $\sum_{w \in V} f(w) = -g(u') - g(v')$ since $g \in L_0^2(V')$. Using Lemma 1, we get

$$\begin{aligned} f \cdot Af &\leq \lambda_1(A)||f||^2 + \frac{k}{n}(g(u') + g(v'))^2 \\ &\leq 2\sqrt{k-1}(||g||^2 - g(u')^2 - g(v')^2) + \frac{2k}{n}(g(u')^2 + g(v')^2) \\ &\leq 2\sqrt{k-1}||g||^2, \end{aligned}$$

for sufficiently large $n$. Combining this with Eq. 20 and noting that $1 + 2k/\lambda' \leq 4\sqrt{k-1}$, we obtain

$$\lambda' ||g||^2 \leq 2\sqrt{k-1}\big(||g||^2 + 2\sum_{i=1}^{k} (g(u_i)^2 + g(v_i)^2)\big). \tag{21}$$

We now compare $g(u_i)$ and $g(v_i)$ to $||g||$. Let $s$ be the function on $V'$ defined by $s(u) = s(u') = k$ and, for $v \in V' - \{u, u'\}$ at distance $i$ from $u$,

$$s(v) = 2(k-1)^{1-i/2} \cosh i\theta'.$$

An easy calculation shows that the function $s$ verifies the conditions of Lemma 4 for the graph $G'$, with $X = \{u, u'\}$, $\mu = \lambda'$, and for any integer $h$ between 1 and $l+1$. Therefore, the sequence $||P_h g||^2/||P_h s||^2$ is an increasing function of $h$. Hence

$$\begin{aligned} ||g||^2 &\geq \sum_{v \in X_{l+1}} g(v)^2 \\ &\geq \frac{\sum_{v \in X_{l+1}} s(v)^2}{\sum_{v \in X_1} s(v)^2} \sum_{v \in X_1} g(v)^2 \\ &= \frac{k-2}{k-1} \frac{\cosh^2(l+1)\theta'}{\cosh^2 \theta'} \sum_{i=1}^{k} g(u_i)^2 \\ &\geq \frac{1}{2} \cosh^2(l\theta') \sum_{i=1}^{k} g(u_i)^2, \end{aligned}$$

The last equation follows from the inequality $\cosh(l+1)\theta' \geq \cosh\theta' \cosh l\theta'$. Similarly, we can prove the inequality $||g||^2 \geq \frac{1}{2}\cosh^2(l\theta') \sum_{i=1}^{k} g(v_i)^2$. Combining this with Eq. 21 yields

$$\cosh\theta' \leq 1 + \frac{8}{\cosh^2 l\theta'}. \tag{22}$$

Using the inequalities $\cosh\theta' \geq 1+\theta'^2/2$ and $\cosh l\theta' \geq e^{l\theta'}/2$, we see from Eq. 22 that $\theta' = O(e^{-l\theta'})$, and so $\log\theta' \leq -l\theta' + O(1)$. Hence $\theta' \leq (\log l)/l$ for sufficiently large $n$. Finally,

$$\lambda' = 2\sqrt{k-1}\cosh\theta' = 2\sqrt{k-1}(1+\frac{\theta'^2}{2}(1+o(1))) \leq 2\sqrt{k-1}(1+2\frac{\log^2\log n}{\log_k^2 n}),$$

since $l \geq (2/3-o(1))\log_k n$. Theorem 2 implies that the linear expansion of the family $(G_n)$ is at least $k/2$. Since the subset $\{u,u'\}$ has $k$ neighbors, this bound is tight. ■

If $k-1$ is a prime congruent to 1 modulo 4, we know from [30] that there exists an infinite family of non-bipartite $k$-regular Ramanujan graphs with girth at least $(2/3+o(1))\log_{k-1} n$. By doing the same construction as in Theorem 7, we obtain $k$-regular graphs whose second largest eigenvalue *in absolute value* is $(2+o(1))\sqrt{k-1}$ and linear expansion $k/2$. Moreover, by adjoining nodes at regions of the graph at sufficiently large distance from each other, we can construct for any $m = m(n) = o(n)$ a family of $k$-regular graphs whose second largest eigenvalue in absolute value is $(2+o(1))\sqrt{k-1}$ and containing a subset of size $2m$ with expansion $k/2$. The proofs of these two statements are very similar to the proof of Theorem 7. The following theorem generalizes Theorem 7.

**Theorem 8** *Let $k$ be an integer such that $k-1$ is a prime congruent to 1 modulo 4, and $\overline{\lambda}$ a real number no smaller than $2\sqrt{k-1}$. Let $\overline{\lambda} = 2\sqrt{k-1}\cosh\theta$, where $\theta \geq 0$. Consider a graph $H$ of maximum degree at most $k$ and such that $\lambda_0(M_H^\theta) \leq \overline{\lambda}$. There exists an explicitly constructible infinite family $G_n$ of $k$-regular graphs containing $H$ as an induced subgraph and such that $\lambda(G_n) \leq (1+o(1))\overline{\lambda}$.*

**Proof** The proof is similar to the proof of Theorem 7. Let $X$ be the vertex set of $H$. Consider an explicit family of non-bipartite Ramanujan [30] graphs $(F_n)$ on $n$ vertices whose girth is $(2/3+o(1))\log_{k-1} n$. Let $F_n = (V,E)$ be an element of the family. Consider a set of $|X|$ vertices of $F_n$ at distance at least $l+4$ from each other, where $l = 1+(\log_{k-1}(n/|X|))/4$. We can find such set greedily if $n$ is large enough. Identify the vertices of $H$ with this set of vertices. For each such vertex $v$, delete $\Delta_H(v)$ edges of $E$ adjacent to it. Since $\sum_{v\in H}\Delta_H(v)$ is even, we obtain an even number of vertices of degree $k-1$ as a result of the edge deletion process. We add a perfect matching between these vertices. The graph $G_n$ we obtain after these operations is $k$-regular and has $V$ as vertex set. The subgraph of $G_n$ induced on $X$ is identical to $H$. Moreover, the subgraph of $G_n$ induced on $B_{l+1}(X)$ has no loops other than those inside $X$. Denote by $E'$ be the set of edges of $G_n$, by $A'$ its adjacency matrix, and by $A$ the adjacency matrix of $F_n$. Without loss of generality, we will assume that the second largest eigenvalue in absolute value of $G_n$ is positive (the case where it is negative can be treated similarly) and greater than $\overline{\lambda}$. We denote it by $\lambda' = 2\sqrt{k-1}\cosh\theta'$, with $\theta' > \theta$. Let $g \in L_0^2(V)$ be an eigenvector of $A'$ corresponding to $\lambda'$. As before, we need to show that the value

of $g$ on the set of vertices adjacent to edges in $E \Delta E'$ is small compared to $||g||$. We have

$$\begin{aligned} \lambda' ||g||^2 &= g \cdot A'g \\ &= g \cdot Ag + \sum_{(u,v) \in E' - E} g(u)g(v) - \sum_{(u,v) \in E - E'} g(u)g(v) \\ &\leq 2\sqrt{k-1}||g||^2 + \frac{1}{2} \sum_{(u,v) \in E \Delta E'} (g(u)^2 + g(v)^2). \end{aligned} \tag{23}$$

Using Remark 2, we will show that the second term in the last equation is small compared to $||g||^2$. Assume for the moment that $H$ is connected. Let $s \in L^2(X)$ be an eigenvector of the matrix $M_H^\theta$ corresponding to its largest eigenvalue. We can assume without loss of generality that $s$ takes only positive values (Fact 10). For $v \in B_{l+1}(X) - X$ (the set of nodes in $V - X$ at distance at most $l+1$ from $X$ in $G_n$), there exists a unique point $u$ in $X$ that minimizes the distance (in $G_n$) $d(u,v)$. We extend the domain of $s$ to $B_{l+1}(X) - X$ by setting $s(v) = \alpha_i s(u)$, where $i = d(u,v)$ and

$$\alpha_i = (k-1)^{-i/2}(e^{(i-1)\theta'}(e^{-\theta} - e^{-\theta'}) + e^{-i\theta'}).$$

We assign to $s$ an arbitrary positive value on $V - B_{l+1}(X)$. Let us now verify that the conditions of Remark 2 apply for the graph $G_n$, the set $X$, $\mu = \lambda'$ and any $h$ between 2 and $l+1$. First, we verify the existence of $\alpha$, $\beta$ and $\gamma$ as defined in Remark 2. Using the same notations as in Remark 2, we have $P_h A_h s = (\alpha_{h-1}/\alpha_h) P_h s$. This is because any node $v$ in $X_h$ has a unique neighbor $v'$ in $X_{h-1}$, and $s(v') = (\alpha_{h-1}/\alpha_h)s(v)$. Similarly, $A_h P_h s = (k-1)(\alpha_h/\alpha_{h-1})P_{h-1}s$. It remains to verify that $A's \leq \lambda' s$. Observe that the sequence $\alpha_i$ satisfies the following equations:

$$\alpha_0 \geq 1 \tag{24}$$

$$\alpha_1 = (k-1)^{-1/2} e^{-\theta} \tag{25}$$

$$\lambda' \alpha_i = \alpha_{i-1} + (k-1)\alpha_{i+1}, \quad \text{for } i \geq 1. \tag{26}$$

If $v \in X$, we have

$$\begin{aligned} (A's)(v) &= (A_X s)(v) + (k - \Delta_X(v))s(v)\alpha_1 \\ &= (M_X^\theta s)(v) \\ &= \overline{\lambda} s(v) \\ &\leq \lambda' s(v). \end{aligned} \tag{27}$$

The inequality $(A's)(v) \leq \lambda' s(v)$ for $v \in B_l(X) - X$ follows from Eqs. 24 and 26. Indeed, if $v \in X_1 = N(X) - X$, and $u$ its neighbor in $X$, we have

$$\begin{aligned}(A's)(v) &= s(u) + (k-1)s(u)\alpha_2 \\ &\leq (\alpha_0 + (k-1)\alpha_2)s(u) \\ &= \lambda'\alpha_1 s(u) \\ &= \lambda' s(v).\end{aligned}$$

A similar calculation shows that $(A's)(v) = \lambda' s(v)$ for $v \in B_l(X) - (X \cup N(X))$.

By applying Remark 2 for $2 \leq h \leq l+1$, we get

$$\begin{aligned}||g||^2 &\geq \frac{\sum_{v \in X_{l+1}} s(v)^2}{\sum_{v \in X_1} s(v)^2} \sum_{v \in X_1} g(v)^2 \qquad (28) \\ &= (k-1)^l \frac{{\alpha_{l+1}}^2}{{\alpha_1}^2} \sum_{v \in X_1} g(v)^2 \\ &\geq (k-1)^l \frac{(k-1)^{-(l+1)} e^{2l\theta'} (e^{-\theta} - e^{-\theta'})^2}{(k-1)^{-1} e^{-2\theta}} \sum_{v \in X_1} g(v)^2 \\ &= e^{2l\theta'} (1 - e^{\theta - \theta'})^2 \sum_{v \in X_1} g(v)^2 \\ &\geq e^{2(l-1)\theta'} (\theta' - \theta)^2 \sum_{v \in X_1} g(v)^2.\end{aligned}$$

The last equation follows from the inequality $e^{\theta'} - e^{\theta} \geq (\theta' - \theta)$, whose proof is similar to the proof of Claim 3.

We now need to show that $g(v)^2$ is upper bounded by $\sum_{v \in X_1} g(v)^2$ times a multiplicative constant, for any $v \in X$. We proceed by induction on the distance from $v$ to $X_1$. As in Lemma 4, we denote by $P$ the projection on $X$, by $P_1$ the projection on $X_1$, and by $A_1$ the matrix $(P+P_1)A'(P+P_1)$. Let $s' = (P + P_1)s$ be the projection of $s$ on $X \cup N(X)$. We have

$$A_1 s' \leq \overline{\lambda} s'. \qquad (29)$$

Indeed, for $v \in X$, we have $(A_1 s')(v) = (A's')(v) = \overline{\lambda} s'(v)$ by Eq. 27. On the other hand, for $v \in X_1$, we have $(A_1 s')(v) = s'(v)/\alpha_1 = \sqrt{k-1} e^{\theta} s'(v) \leq \overline{\lambda} s'(v)$, since $e^{\theta} \leq 2\cosh\theta$. Premultiplying both sides of Eq. 29 by $P$ and $A_1$ yields successively:

$$\begin{aligned}PA_1 s' &\leq \overline{\lambda} s' - \overline{\lambda} P_1 s' \\ A_1 P A_1 s' &\leq \overline{\lambda}^2 s' - \overline{\lambda} A_1 P_1 s'.\end{aligned}$$

Therefore, $\overline{\lambda}^2 s' - A_1 P A_1 s' \geq \overline{\lambda} A_1 P_1 s' \geq P_{N(X_1)\cap X} s'$, where $P_{N(X_1)\cap X}$ is the projection on the set of vertices in $X$ adjacent to $X_1$. To prove the last inequality, consider a node $u$ in $N(X_1) \cap X$, and let $v \in N(\{u\}) \cap X_1$. Then $(P_{N(X_1)\cap X} s')(u) = s'(u)$, and

$$(A_1 P_1 s')(u) = \sum_{(u,w)\in E} (P_1 s')(w) \geq (P_1 s')(v) = s'(v) = \alpha_1 s'(u).$$

Finally, $\overline{\lambda}\alpha_1 \geq 1$. Hence

$$\overline{\lambda}^2 (P + P_1)s - A_1 P A_1 s \geq P_{N(X_1)\cap X} s.$$

As in Lemma 4, we conclude that

$$\begin{aligned} ||P_{N(X_1)\cap X} g||^2 &\leq \overline{\lambda}^2 ||(P + P_1)g||^2 - ||PA_1 g||^2 \\ &= \overline{\lambda}^2 ||Pg||^2 + \overline{\lambda}^2 ||P_1 g||^2 - \lambda'^2 ||Pg||^2 \\ &\leq \overline{\lambda}^2 ||P_1 g||^2 \\ &= \overline{\lambda}^2 \sum_{v\in X_1} g(v)^2. \end{aligned}$$

Now, let us consider the set of nodes in $X$ at distance $i$ from $X_1$, for $2 \leq i \leq \log_{k-1} |X|$. By a similar reasoning, we can show that the ratio of the norms of the restriction of $g$ to two successive levels is at most $\overline{\lambda}$. Therefore, the squared norm of the restriction of $g$ to level $i$ is at most

$$\overline{\lambda}^{2i} \sum_{v\in X_1} g(v)^2 \leq |X|^{2\log_{k-1}\overline{\lambda}} \sum_{v\in X_1} g(v)^2 \leq |X|^4 \sum_{v\in X_1} g(v)^2.$$

Combining this with Eq. 28 shows that, for all $u \in X$,

$$||g||^2 |X|^4 \geq e^{2(l-1)\theta'} (\theta' - \theta)^2 g(u)^2. \tag{30}$$

The same relation holds for any vertex in the graph. We have assumed so far that $H$ is connected. If it is not, then each of its components $H_i$ satisfies the condition $\lambda_0(M^\theta_{H_i}) \leq \overline{\lambda}$, by Fact 6. Therefore, Eq. 30 still holds if $X$ is replaced by the vertex set of $H_i$, for each vertex of $H_i$. But this gives an even stronger relation than Eq. 30.

Combining Eq. 30 with Eq. 23 and noting that $E \Delta E'$ contains at most $2k|X|$ undirected edges yields

$$\lambda' ||g||^2 \leq 2\sqrt{k-1}||g||^2 + \frac{2k|X|^5 e^{-2(l-1)\theta'}}{(\theta' - \theta)^2} ||g||^2,$$

from which we deduce that

$$\cosh\theta' \leq 1 + \frac{e^{-2(l-1)\theta'}}{(\theta' - \theta)^2} k|X|^5.$$

Assume now that $\theta > 0$. Using the relations $\cosh\theta' \geq 1 + \theta'^2/2 \geq 1 + \theta^2/2$, we conclude that

$$\theta' - \theta \leq 2k|X|^3 e^{-(l-1)\theta'}/\theta \leq 2k|X|^3 e^{-(l-1)\theta}/\theta.$$

This implies that $\theta' \leq \theta + o(1)$ as $n$ goes to infinity, and therefore $\lambda' \leq \overline{\lambda}(1 + o(1))$. We can get a more precise on the upper bound as follows: since $\cosh\theta' \leq \cosh\theta + 2(\theta' - \theta)\sinh\theta$ for $\theta' \approx \theta$, we deduce after some calculations that $\lambda' \leq \overline{\lambda}(1 + 4e^{-(l-1)\theta}k|X|^3(\tanh\theta)/\theta)$. Replacing $l$ by its value, we get

$$\lambda' \leq \overline{\lambda}(1 + 2\frac{\tanh\theta}{\theta}k|X|(|X|/n)^{\theta/(4\log(k-1))}). \tag{31}$$

This achieves the proof in the case $\theta > 0$. Suppose now that $\theta = 0$. For any $\epsilon > 0$, we have $\lambda_0(M_H^\epsilon) \leq \overline{\lambda} = 2\sqrt{k-1}\cosh\epsilon$. Therefore, by the above results, we have $\lambda' \leq \overline{\lambda}(1+\epsilon) = 2(1+\epsilon)\sqrt{k-1}\cosh\epsilon$, for sufficiently large $n$. Hence, $\lambda' \leq (2 + o(1))\sqrt{k-1}$ as $n$ goes to infinity, as required.

■

We now combine Theorems 1 and 8 to get the following result:

**Theorem 9** *Let $H$ be a graph, $k$ an integer such that $k-1$ is a prime congruent to 1 modulo 4, and $\overline{\lambda}$ a real number no smaller than $2\sqrt{k-1}$. Let $\overline{\lambda} = 2\sqrt{k-1}\cosh\theta$, where $\theta \geq 0$. The following conditions are equivalent:*

1. *There exists an infinite family $G_n$ of $k$-regular graphs containing $H$ as an induced subgraph and such that $\lambda(G_n) \leq (1 + o(1))\overline{\lambda}$.*
2. *There exists an infinite family $G_n$ of $k$-regular graphs containing $H$ as an induced subgraph and such that $\lambda_1(G_n) \leq (1 + o(1))\overline{\lambda}$.*
3. *The maximum degree of $H$ is at most $k$, and $\lambda_0(M_H^\theta) \leq \overline{\lambda}$.*

**Proof** Condition 1 clearly implies Condition 2. Condition 2 implies Condition 3 by Theorem 1. Indeed, it implies that $\lambda_0(M_H^{(1+o(1))\theta}) \leq (1 + o(1))\overline{\lambda}$. As $n$ goes to infinity, we get Condition 3 by continuity. Condition 3 implies Condition 1 by Theorem 8. ■

# 7 Expansion of subsets of sublinear size

The graphs constructed in [30, 33] besides being Ramanujan, have other properties such as high girth. In particular, the bipartite Ramanujan graphs constructed in [30, 33] have girth $(4/3 - o(1))\log_{k-1} n$. We will use this property to get high expansion for subsets of sublinear size.

**Proposition 1** *Let $G = (V, E)$ be a $k$-regular graph of girth $(4/3 + o(1))\log_{k-1} n$. For any subset $X$ of $V$ of size at most $n^\alpha$, the average degree of the subgraph induced on $X$ is at most $2 + 2(k - 1)^{(3+o(1))\alpha/2}$.*

**Proof** Let $\sigma$ be the average degree of the subgraph induced on $X$. A classical result in graph theory [11, p. 126] shows that the girth of this subgraph is at most $2\lceil \log_{\lfloor \sigma/2 \rfloor} |X| \rceil + 1 \leq 2\lceil \alpha \log_{\lfloor \sigma/2 \rfloor} n \rceil + 1$. On the other hand, it is at least $(4/3 + o(1)) \log_{k-1} n$. Comparing these two inequalities yields the desired bound on $\sigma$. ■

Note that $\alpha$ must be less than $1/3$ for the bound in Proposition 7 to beat the bound in Theorem 3, when the graph is Ramanujan.

**Theorem 10** *Let $G = (V, E)$ be a $k$-regular graph of girth $(4/3 + o(1)) \log_{k-1} n$. For any subset $X$ of $V$ of size at most $n^\alpha$,*

$$\frac{|N(X)|}{|X|} \geq \lfloor k - (k-1)^{(3+o(1))\alpha} \rfloor.$$

**Proof** Let $Y$ be a minimal nonempty subset of $X$ having expansion at most $\beta = |N(X)|/|X|$. If $\beta = k$, we are done. Otherwise $Y$ is not a singleton, and for any element $v$ of $Y$, the expansion of the subset $Y - \{v\}$ is greater than $\beta$. In other words,

$$|N(Y - \{v\})| > \beta |Y - \{v\}| = \beta |Y| - \beta \geq |N(Y)| - \beta.$$

Therefore, the vertex $v$ has less than $\beta$ neighbors in the complement of $N(Y - \{v\})$, and so it has at least $k - \lceil \beta \rceil + 1$ neighbors in $N(Y - \{v\})$. Hence, there are at least $k - \lceil \beta \rceil + 1$ vertices in $Y$ at distance 2 from $v$. More generally, for $l < c(G)/4$, there are at least $(k - \lceil \beta \rceil + 1)(k - \lceil \beta \rceil)^{l-1}$ vertices in $Y$ at distance $2l$ from $v$. This implies a strict lower bound of $(k - \lceil \beta \rceil)^l$ on the size of $Y$, and so

$$(k - \lceil \beta \rceil)^{(1/3+o(1))(\log_{k-1} n)} < n^\alpha.$$

Solving this inequality yields the desired bound on $\beta$. ■

As a consequence of Theorem 10, if $\alpha$ is a constant less than $1/3$, subsets of size at most $n^\alpha$ have expansion at least $k(1 - O(k^{3\alpha - 1}))$, which is greater than $k/2$ for sufficiently large $k$.

# 8 Random walks

We show that the probability that a walk stays inside a given set has an exponential decay in the length of the walk. We prove that our bound is optimal in many cases.

**Corollary 1** *If $G = (V, E)$ is $k$-regular and $W$ a subset of $V$, the fraction of walks in $G$ of length $l$ whose all vertices belong to $W$ is at most $\mu\ (\alpha + \mu - \alpha\mu)^l$, where $\alpha = \lambda_1/k$ and $\mu = |W|/n$.*

**Proof** The number of walks of length $l$ in $W$ is equal to the sum of entries of ${A_W}^l$. Hence, the fraction of walks of length $l$ in $W$ is equal to $((A_W)^l \chi_W \cdot \chi_W)/(k^l n)$. On the other hand, Fact 9 shows

that $\lambda_0(A_W)$ is the largest eigenvalue of $A_W$ *in absolute value*. Therefore, the largest eigenvalue of ${A_W}^l$ is $(\lambda_0(A_W))^l$. Using Lemma 2 and Fact 4, we conclude that

$$({A_W}^l \chi_W) \cdot \chi_W \leq (\lambda_0(A_W))^l ||\chi_W||^2 \leq \mu n \ (\lambda_1 + k\mu - \lambda_1 \mu)^l.$$

■

Define the density of a subset of vertices to be the ratio of its size to the total number of vertices. Corollary 1 is optimal in the following sense: for any rational $\alpha \in [1/2, 1]$ and any rational $\mu \in [0, 1]$, there exists an arbitrary large graph $k$-regular graph $G$ and a subset $W$ of $G$ of density $\mu$ such that $\lambda_1(G) = \alpha k$ and, for any integer $l$, the fraction of walks in $W$ of length $l$ is equal to $\mu \ (\alpha + \mu - \alpha\mu)^l$. Indeed, let $G = K_{a+2} \times K_b$, where $b = a(1-\alpha)/\alpha$ and $K_{a+2} \times K_b$ is the graph on $V = \{1, \ldots, a+2\} \times \{1, \ldots, b\}$, with $((i,j),(i',j')) \in E$ if and only if $i = i'$ XOR $j = j'$. The graph $G$ is the cartesian product of the complete graphs $K_{a+2}$ and $K_b$, and so [6] its eigenvalues are equal to the sum of the eigenvalues of $K_{a+2}$ and $K_b$. Since the eigenvalues of $K_{a+2}$ and $K_b$ are respectively $a+1, -1$ and $b-1, -1$, the eigenvalues of $G$ are $a+b, a, b-2$ and $-2$. The graph $G$ is regular of degree $a+b$, and $\lambda_1(H) = a = \alpha(a+b)$. Let $W = \{1, \ldots, a+2\} \times \{1, \ldots, \mu b\}$. The set $W$ has density $\mu$ in $V$; the subgraph induced on $W$ is isomorphic to $K_{a+2} \times K_{\mu b}$, and so it is regular of degree $a + \mu b$. Therefore, the fraction of walks in $W$ of length $l$ is equal to $\mu(a+\mu b)^l/(a+b)^l$, which is equal to the value given by Claim 1.

# 9 Other applications

We list four applications of Ths. 2, 3 and 4.

1. **Random regular graphs.** In [17], it was shown that, if $k$ is even, then for a random $k$-regular graphs $G$, we have $\lambda_1(G) \leq 2\sqrt{k-1} + O(\log k)$ with high probability. Using Theorem 2, we deduce that for a random regular graphs, we can prove with high probability *in polynomial time* that linear sized subsets (of size at most $k^{-1/\epsilon}$, where $\epsilon = k^{-1/4}$) have expansion at least

$$\frac{k}{2}\left(1 - \sqrt{1 - \frac{4k-4}{(2\sqrt{k-1} + O(\log k))^2}}\right)(1 + O(k^{-1/4})) = \frac{k}{2} - O(k^{3/4}\log^{1/2} k).$$

2. **Selection networks.** We can use Theorem 2 to build explicit selection networks of small size. A selection network is a network of comparators that classifies a set of $n$ numbers, where $n$ is even, into two subsets of $n/2$ numbers such that any element in the first set is smaller than any element in the second set. In [36], a probabilistic construction of a selection network is given using an asymptotic upper bound of $2n\log_2 n$ comparators. Also, an upper bound slightly less than $6n\log_2 n$ is shown by a deterministic construction. Using Theorem 2, we can construct

selection networks of asymptotic size $(3 + o(1))n \log_2 n$. Indeed, it is shown in [36] how to construct selection networks of size $(2 + o(1))n \log_2 n$ from expanders of degree 4 having linear expansion at least 3. The construction can be easily generalized to build selection networks of size $k(1/2 + o(1))n \log_2 n$ from expanders of degree $k$ having linear expansion at least 3. Theorem 2 shows that the linear expansion of $k$-regular Ramanujan graphs is at least $k/2$, and thus we can build explicit selection networks of size $(3 + o(1))n \log_2 n$ using 6-regular Ramanujan graphs.

3. **Extrovert graphs.** Given a graph $G = (V, E)$ and a subset $X$ of $V$, an element of $X$ is said to be extrovert if at least half of its neighbors are outside $X$. A family of graphs is called extrovert if all linear-sized subsets contain a constant fraction of extrovert nodes. Theorem 3 shows that the average degree of the nodes of a linear-sized induced subgraph of a $k$-regular Ramanujan graph is upper bounded by roughly $1 + \sqrt{k-1}$, which is less than $k/2$ for $k \geq 7$. This shows that Ramanujan graphs of degree at least 7 are extrovert graphs.

4. **Highly expanding non-Ramanujan graphs.** Let $G$ be a $k$-regular Ramanujan graph, and $t$ an integer greater than 1. We define the graph $G^t$ on the same vertex set as $G$ by joining two vertices with an edge if their distance in $G$ is $t$. The graph $G^t$ has maximum degree at most $k(k-1)^{t-1}$. The set of neighbors in $G^t$ of a subset $X$ contains $N_G^t(X) - N_G^{t-2}(X)$. Hence, by Theorem 5, subsets of $G^t$ of size at most $k^{-t-1/\epsilon}|V|$ have expansion at least $k(k-2)(k-1)^{t-2}(1 - O(t\epsilon))/2$. Assume now that the girth of $G^t$ is greater than $2t$. Then $G^t$ is regular of degree $k(k-1)^{t-1}$. As in [30, 17], we see that the adjacency matrix of $G^t$ satisfies the equality $A_{G^t} = A_G A_{G^{t-1}} - (k-1)A_{G^{t-2}}$, for $t \geq 3$. Consider the sequence $r_i$ given by $r_i = \lambda(G) r_{i-1} - (k-1) r_{i-2}$, with $r_1 = \lambda(G)$ and $r_2 = \lambda(G)^2 - k$. An eigenvector of $A_G$ corresponding to the eigenvalue $\lambda(G)$ is also an eigenvector of $G^t$ with eigenvalue $r_t$. It follows that $\lambda(G^t) \geq r_t$. Since $\lambda(G) = 2\sqrt{k-1}(1 - o(1))$, it follows that $\lambda(G^t) \geq (k-1)^{t/2-1}(t(k-2) + k)(1 - o(1))$, as $|V|$ goes to infinity. The graph $G^t$ is far from being Ramanujan if $|V|$ is large enough. Hence, from any family of $k$-regular Ramanujan graphs $G_n$ with high girth, we can construct a family of $k(k-1)^t$-regular graphs whose linear expansion is at least $k(k-2)(k-1)^{t-1}$.

   Note that by considering two disjoint copies of $G_n$, we get a family of $k$-regular non-Ramanujan graphs with the same linear expansion as $(G_n)$. However, the expansion of some subsets containing half the nodes is null. Moreover, all the eigenvalues of any element of this family, except the largest two, satisfy the Ramanujan condition. In contrast, it can be shown that a linear number of eigenvalues of $G^t$ violate the Ramanujan condition.

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