Minors and the Chromatic Index of r-graphs

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Abstract. Let T_2 be the graph obtained from the Petersen graph by first deleting a vertex and then contracting an edge incident to a vertex of degree two. We give a simple characterization of the graphs that contain no minor isomorphic to T_2. This characterization is used to show that if every planar r-graph is r-edge colourable, then every r-graph with no minor isomorphic to T_2 is r-edge colourable.

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1 Introduction

We consider lopless graphs $G = (V, E)$ with vertex set V and edge set E. A *minor* (respectively *subgraph*; respectively *induced subgraph*) of G is any graph obtained from G by deleting vertices and contracting or deleting edges (respectively deleting vertices and edges; respectively deleting vertices). For a graph H, G is said to be *H-free* if no minor of G is isomorphic to H. For $S \subset V(G)$, we denote with $\delta_G(S)$, or simply $\delta(S)$, the set of edges of G with exactly one end in S. An r-graph is an r-regular graph in which $|\delta(S)| \geq r$ for every subset of vertices S with odd cardinality.

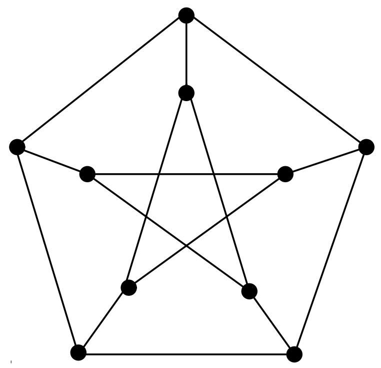


Figure 1: The Petersen graph

The subject of this paper is the following 2 conjectures.

Conjecture 1.1 *Every planar r-graph is r-edge colourable.*

Conjecture 1.2 *Every r-graph with no Petersen graph (see Figure 1) as a minor is r-edge colourable.*

Historically, Conjectures 1.1 and 1.2 date back to the famous four colour problem. Consider the special case when $r = 3$. Then Conjecture 1.1 is Tait's [12] equivalent formulation of the four colour problem and Conjecture 1.2 is Tutte's [14] generalization of the same problem. Now to obtain Conjectures 1.1 and 1.2 in their general form we need only introduce the following conjecture of Seymour [9]: when $r \geq 4$, every r-graph G has a matching M such that $G - M$ is an $(r-1)$-graph. This together with Conjecture 1.2 would imply the following conjecture attributed by Goddyn to Lovász.

Conjecture 1.3 *The perfect matchings of a P_{10}-free graph form a Hilbert basis.*

The set $\mathcal{M}$ of perfect matchings of a graph G forms a *Hilbert basis* if each integer vector in the $cone(\{\chi^M : M \in \mathcal{M}\})$ is expressable as a nonnegative integer combination of incidence vectors of perfect matchings.

There is a more general (than 1.1) conjecture of Seymour [10] that also contains the four colour problem as a special case. Denote by Δ the maximum degree of a graph G and let $\eta'(G) = \max\{\{\Delta, \{2|E(S)|/(|S|-1) : S \subset V(G), |S| = 2k+1, k \geq 1\}\}$, where $E(S) = \{vw \in E(G) : v, w \in S\}$. Sometimes $\eta'(G)$ is called the *fractional chromatic index* of G. It follows from the work of Edmonds [2] that $\eta'(G) = r$ when G is an r-graph and so Theorem 1.1 would imply that for r-regular graphs, fractional r-edge colourability implies that G is r-edge colourable. In the case that G is not r-regular things become more complex although there is a simple description of the fractionally 3-edge colourable graphs: a 2-connected graph G satisfies $\eta'(G) = 3$ if and only if it arises by making k subdivisions of edges in a 3-graph, where k is any integer other than one. Grötszch has conjectured the following:

Conjecture 1.4 (Grötzsch) *For a planar graph G, $\eta'(G) = 3$ implies G is 3-edge colourable.*

This conjecture would imply that the 3-edge colourability of a planar graph could be determined by finding its fractional chromatic index (which amounts to solving a linear program). Seymour has conjectured that such a "fractional determines integral" property holds for all planar graphs:

Conjecture 1.5 (Seymour) *Every planar graph G is $\lceil \eta'(G) \rceil$-edge colourable.*

This conjecture appears to require substantially more than the veracity of the four colour conjecture. It was only recently shown, for instance, that the result holds for series-parallel graphs (see [11]). The smallest example of a graph not being $\lceil \eta'(G) \rceil$-edge colourable is the Petersen graph minus a vertex, denoted by T_3. This can also be described as $K_{3,3}$ with the edges of a perfect matching subdivided once. It was shown in [4], that every T_3-free graph G with $\eta'(G) = 3$ is 3-edge colourable if Conjecture 1.4 holds (and that any T_3 3-graph is 3-edge colourable). This motivates us to propose the following strengthening of Conjecture 1.5 which would characterize the largest possible minor-closed class of graphs with the fractional-determines-integral property.

Conjecture 1.6 *Every T_3-free graph is $\lceil \eta'(G) \rceil$-edge colourable.*

We mention that Conjectures 1.5 and 1.6 also deal with the structure of the matching polyhedron of the corresponding graphs. The results of Edmonds [2] and Baum and Trotter Jr [1] imply that every replication (i.e., adding multiple copies of edges) H of a subgraph of G is $\lceil \eta'(H) \rceil$-edge colourable if and only if the matching polyhedron of G has the *integer decomposition property* (a polyhedron P has the integer decomposition property if, for each integer k and each integral vector $z \in kP(:= \{kx : x \in P\})$, z can be expressed as the sum of k integral vectors in P; in particular P has integral extreme points). Thus 1.6 could be stated as: the matching polyhedron of every T_3-free graph has the integer decomposition property.

The present paper attempts inroads to Conjectures 1.1 and 1.6 by considering the class of graphs with no minor isomorphic to the graph obtained from T_3 by suppressing a degree two vertex. This graph is denoted by T_2 (see Figure 3) and note that it is obtained from $K_{3,3}$ by subdividing two independent edges. We give a decomposition theorem for T_2-free graphs based on vertex 1- and 2- cuts and replicating degree three vertices (see Theorem 3.4). The decomposition is subsequently used to show that if Conjecture 1.1 is true, then every T_2-free graph is r-edge colourable (see Theorem 4.1).

2 A Decomposition of T_2-Free Graphs

In this section we state the main decomposition result concerning the structure of T_2-free graphs. Clearly, a graph is T_2-free if and only if its underlying simple graph is T_2-free. Thus we restrict ourselves to simple graphs.

We note next that if G is a T_2-free graph and v is a degree three vertex of G, then replacing v by a stable set of vertices with the same neighbourhood as v is again a T_2-free graph. Moreover, if G is a planar graph (and therefore abviously T_2-free), we claim that by adding a single new vertex v' adjacent to $N(v) \cup \{v\}$, also results in a T_2-free graph. To see this, observe that if G has a T_2 minor H, then $\pi(H)$ has to contain v and v'. Moreover, in $\pi(H)$, one of v, v' has degree two and they are either adjacent or they are both adjacent to a vertex of degree three. But then, it is easy to see that we can construct a minor $H' \cong K_{3,3}$ whose expansion does not contain one of v, v'. This clearly contradicts the fact that G is planar.

The above discussion essentially describes the decomposition of T_2-free graphs. Since composing along 1 and 2-cuts preserves T_2-freeness, this yields a decomposition procedure for the class of T_2-free graphs. We now introduce two classes of graphs in order to give the precise statement of the theorem.

Let F be a graph. Two vertices v, v' of F are *mates* if in $G - vv'$ they are both of degree three and $N(v) = N(v')$. Let I be a stable set of degree three vertices of G. Let G be a graph obtained from F by adding for each $v \in I$ either a mate v' of v and the edge vv' or by adding a non-empty set of mutually non-adjacent mates of v. Let M be the set of mate vertices so added. We say that G is a *mate graph* of F and write $G = [F, I, M]$. Note that every graph can be construed as a mate graph of some graph since $G = [G, \emptyset, \emptyset]$.

We shall denote by $\mathcal{M}$ the class of mate graphs of planar graphs.

A graph $G \in \mathcal{K}$ if either

(a) has a stable set I such that $G - I$ is a subgraph of K_4, or

(b) has a stable set I consisting of pairwise non-adjacent replications of at most two vertices (respectively of a single vertex) such that, $G - I$ is a subgraph of K_5 (respectively of K_6) (see figure 2).

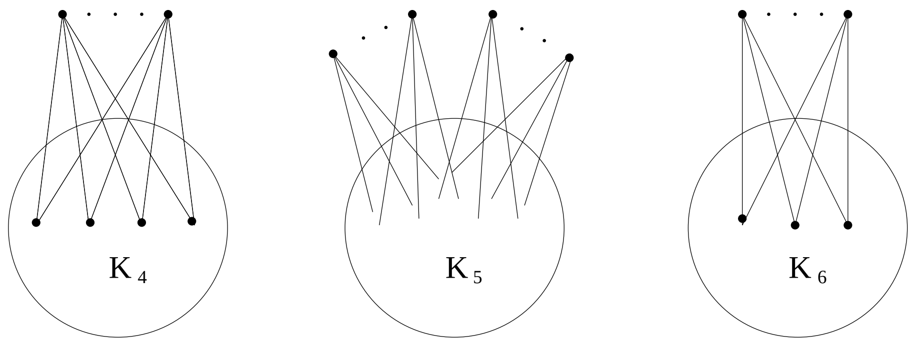


Figure 2: Maximal members of $\mathcal{K}$

Theorem 3.4 *Let G be a 3-connected graph, $|V(G)| \geq 9$. G is T_2-free if and only if $G \in \mathcal{M} \cup \mathcal{K}$.*

Robertson [8] and Kelmans [3] have independently obtained decomposition results for the related class of triangle-free graphs with no $M(4)$ minor. (A *Möbius 4-ladder*, $M(4)$, is a circuit $(v_0, \cdots, v_7)$ with the edges $v_i v_{i+4}$, $i = 0, \cdots, 3$.) Kelmans has shown that for 3-connected graphs G with no triangles such that, for every 3-cut S, $G - S$ has exactly two components one of which is a single vertex, then G does not have a Möbious 4-ladder as a minor if and only if it is planar or a bipartite graph whose one side of the bipartition has at most four vertices. This result has also been obtained by Robertson who in addition showed how G can be decomposed along 3-cuts so that G is $M(4)$-free if and only if (both) the resulting graphs are $M(4)$-free. The statement of Theorem 3.4 is distinct from these results in two ways. The first is that it considers T_2 which is a proper subgraph of the Möbius 4-ladder and hence a more restrictive excluded minor but on the other hand it does not disallow triangles. The other distinction is that it considers only a special type of 3-cut reduction (those arising from degree three replications). This is because for our purposes we ideally

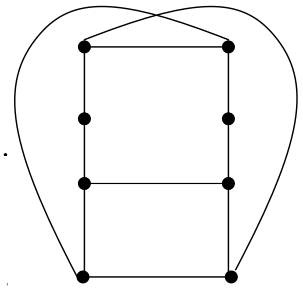


Figure 3: The graph T_2

seek operations which can be used to reduce associated edge-colouring problems. It turns out that degree three replications can be used in this way for the edge-colouring problems on r-graphs (see Theorem 4.1). Such a decomposition for T_3-free 3-graphs is simple and is given in [4]. For general graphs, however, finding such colouring-preserving operations seems to be a difficult problem; it was only recently shown, for example, how to reduce edge-colouring problems (i.e., with a view to resolving Conjecture 1.5) along vertex 2-cuts [6].

We close with few remarks about minors and notation. If H is a maximum degree three minor of a graph G, then G contains a subdivision of H as a subgraph. Thus, there exists a one-to-one mapping f which makes correspond with each vertex v of H a vertex $f(v)$ of G, and a one-to-one mapping π which makes correspond with each edge $e = uv$ of H a path $\pi(e)$ of G connecting $f(u)$ and $f(v)$. Moreover, for any two edges y_1y_2, x_1x_2, $V(\pi(y_1y_2)) \cap V(\pi(x_1x_2)) = \{f(y_1), f(y_2)\} \cap \{f(x_1), f(x_2)\}$.

It is convenient to assume that the vertices of a minor H have the same labels as the corresponding vertices of G. In other words, for $v \in V(H)$, $f(v) = v$. Thus, if U is a subset of $V(G)$, and H is a cubic graph with $V(H) = U$, then H is a minor of G if for each edge uv of H there corresponds a (u, v)-path $\pi(uv)$ of G, called the *expansion* of uv, such that, for any two edges x_1x_2, y_1y_2 of H, $V(\pi(x_1x_2) \cap \pi(y_1y_2)) = \{x_1, x_2\} \cap \{y_1, y_2\}$.

Let H be a minor of a graph G. An edge e of H is said to be *short* if $\pi(e)$ has exactly one edge; otherwise e is *long*. For a subgraph F of H we denote by $\pi(F)$ the graph $\cup_{e \in F} \pi(e)$. A subgraph L of G is *H-external*, if its edge set is contained in $E - \pi(H)$. When L is a path with ends v and w and $V(L) \cup V(\pi(H) = \{v, w\})$, then L is also called an *H-external path* of G. The *attachment* vertices of L are the elements of $V(L) \cap V(\pi(H))$. The *attachment edges* of L are the edges of H whose expansions contain an attachment vertex of L.

3 Proof of the Decomposition Theorem

Recall that $\mathcal{M}$ denotes the class of mate graphs of planar graphs. We commence with a characterization of the class $\mathcal{M}$, but first we verify the following simple property of these graphs.

3.0.1 *$\mathcal{M}$ is closed under taking subgraphs.*

Proof. Let $G = [F, I, M]$, where F is a planar graph, and let $e \in E(G)$. Note that if e is not incident to any vertex of I or M, then $G - e = [F - e, I, M]$. So suppose that e is incident to a vertex v' which, without loss of generality, is not in I but is a mate of a vertex v in I. Firstly, if $e = vv'$, then, in $G - e$, v and v' are non-adjacent mates and so $G - e = [F, I, M]$. So we assume that $e \neq vv'$. Let x_1, x_2, x_3 be the neighbours of V in $G - vv'$. We may assume that $e = v'x_1$. Let F' be the subgraph of G with $V(F') = V(F) \cup \{v'\}$ and $E(F') = E(F) \cup (\delta(v') \backslash \{v'x_1\})$. Clearly, $G - e = [F', I', M']$, where $I' = I$ or $I \backslash \{v\}$ and $M' = M \backslash \{v'\}$. But since v has degree three in F, x_2, x_3 lie on the boundary of a face of F, and so F' is also planar. It follows that $G - e \in \mathcal{M}$. $\square$

Lemma 3.1 *Let G be a 3-connected graph. $G \in \mathcal{M}$ if and only if every $K_{3,3}$ minor H of G has two mate vertices of G.*

Proof. Suppose that $G \in \mathcal{M}$ and let H be a $K_{3,3}$ minor of G. Clearly, $\pi(H) \in \mathcal{M}$, and therefore is obtained from a planar graph by replicating a stable set I'. If G is obtained from a planar graph by replicating some stable set I, then the proof of (3.0.1) indicates that I' can be chosen to be a subset of I. Since I' can only contain degree three vertices, we conclude that H has two mate vertices of G.

Conversely, write $G = [F, I, M]$, where F has as few vertices as possible. We shall show that F is a planar graph.

Suppose this is not true. Then by Kuratowski's Theorem [5], F contains a subdivsion of a graph H which is isomorphic to K_5 or $K_{3,3}$. Suppose first that $H \cong K_5$. Since each $v \in V(H)$ has degree four in H, $v \notin I$. Hence $F \neq H$ unless $M = \emptyset$ which contradicts the minimality of $|V(F)|$ since K_5 itself has a pair of adjacent mates. So let $y \in V(G) \backslash (V(H) \cup M)$. G is 3-connected and thus, by the fan lemma (a well-known corollary of Menger's Theorem [7]), there are internally disjoint paths P_{yy_i}, $1 \leq i \leq 3$, from y to distinct vertices of $V(H)$. It can be seen that each P_{yy_i} may be chosen so as to not contain any pair of mate vertices. With such a choice of the paths P_{yy_i}, $\pi(H) \cup_{i=1}^{3} P_{yy_i}$ is contained in F and contains the expansion of a $K_{3,3}$ minor of G.

So we assume that $H \cong K_{3,3}$. By hypothesis, H has two mate vertices of G, x and y that necessarily belong to the same colour class B of H. Note that this implies that the edges $\delta(x) \cup \delta(y)$ of H are short and for every $v \in V(H) \backslash B$, $v \notin I$. Now if $xy \in E(G)$, then neither x nor y belong to I or M and we can include x in I and y

in M, leading to a contradiction. If $xy \notin E(G)$, then by our choice of M, one of x, y is in I and we can include the other vertex in M, again a contradiction. It follows that $G \in \mathcal{M}$. □

Since T_2 has no mate vertices we have the following.

Corollary 3.2 *The graphs in $\mathcal{M}$ are T_2-free.*

Consider the graphs R and Q_0 of figure 4. R is obtained from the cube by adding an edge between two vertices at distance three, and Q_0 obtained from R by contracting an edge incident to two vertices of degree three. It is routine to verify the following.

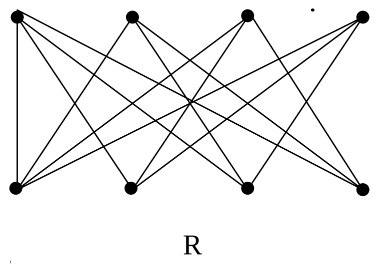
3.2.1 *The graph resulting from subdividing any edge of R has T_2 as a minor.*

3.2.2 *If we apply one of the following operations on Q_0, then we obtain a graph that either has T_2 as a minor or has R as a subgraph.*

(a) *Subdivide an edge of $Q_0 - z$.*

(b) *Subdivide an edge e incident to z and join the newly created vertex with $v \in V(Q_0)$, $v \neq v_4$ when $e = zv_0$ and $v \neq v_5$ when $e = zv_1$.*

(c) *Subdivide two edges of Q_0 incident to z and join the newly created vertices.*



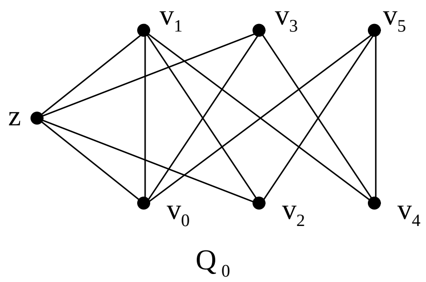


Figure 4: The graphs R and Q_0

Let Q_1 be obtained from Q_0 by subdividing with x_2 the edge zv_1 and by adding the edge x_2v_5. Let Q_2 be obtained from Q_1 by subdividing with x_3 the edge zv_0 and by adding the edge x_3v_4. Note that the graphs R, Q_0, Q_1 and Q_2 are T_2-free but are not mate graphs of planar graphs.

Lemma 3.3 *Let G be a 3-connected, T_2-free graph with at least nine vertices. If $G \notin \mathcal{M}$, then it contains a subgraph L isomorphic to one of R, Q_0, Q_1, Q_2 and no proper subdivision of L.*

Proof. We first show that G contains a subdivision of one of R, Q_0, Q_1, Q_2.

By Lemma 3.1, G has a minor $H \cong K_{3,3}$ that has no pair of mate vertices of G. Moreover, H has at least one long edge. Indeed, if this is not true, then by the fan lemma, there exist internally disjoint paths P_{yy_i}, $1 \le i \le 3$, from y to distinct vertices of H. Our assumption on the edges of H implies that $\{y_1, y_2, y_3\}$ is a colour class B of H and each P_{yy_i} is an edge. Moreover, if y' is any other vertex of $v(G)\backslash V(H)$, then y, y' and the vertices of $H - B$ are mutually disjoint. It follows that $G \in \mathcal{M}$, a contradiction.

Label the colour classes of H by $\{v_0, v_2, v_4\}$, $\{v_1, v_3, v_5\}$. Since G is 3-connected and has at least one long edge e, there exists an H-external path P_{zz_1} of G with z being an internal vertex in the expansion of e. In addition, P_{zz_1} and e can be selected so that the second attachment edge f of P_{zz_1} is different from e. If now G does not have T_2 as a minor, by symmetry, we may assume that $e = v_3v_2$ and $f = v_3v_0$.

It is now straightforward to verify that the edges of $H - v_3$ are short. Moreover, since G is 3-connected and H has no mate vertices of G, there exists an H-external path P_{z_2v} with $v = v_1$ or v_5 and $z_2 \notin V(\pi(H - v_3))$. Without loss of generality, $v = v_1$. Let H' be obtained from $\pi(H)$ by contracting all edges not incident to a vertex of $V(H) \cup \{z, z_1, z_2\}$ and then adding the edges zz_1, vz_2. Depending on how z_1 can be chosen, we distinguish two cases.

1. $z_1 \neq v_0$. If $z_2 \neq z$, then one side of the bipartition of the degree three vertices of a T_2 minor of G is given by $\{z, v_1, v_5\}$. The other can be selected from $\{v_2, v_4, z_2\}$, $\{v_0, v_2, z_2\}$, $\{v_0, v_3, v_5\}$ and $\{v_0, v_4, z_2\}$, respectively, when $z_2 \in \pi(v_0z_1) \cup \pi(zz_1)$, $z_2 \in \pi(v_3, v_4) \cup \pi(zv_3)$, $z_2 \in \pi(z_1v_3)$ and $z_2 \in \pi(zv_2)$. Also, when $z_2 = z$, the required bipartition is given by $\{v_0, v_4, z\}$, $\{v_1, v_3, v_5\}$.

2. $z_1 = v_0$. Suppose that $z_2 \notin \pi(v_3z)$. If z_2 is an internal vertex of $\pi(v_3v_4)$ or $\pi(v_0v_3)$, the bipartition of the degree three vertices of a T_2 minor of G is given respectively by $\{v_1, v_3, v_5\}$, $\{v_0, v_2, z_2\}$ or $\{v_0, v_2, v_3\}$, $\{z, z_2, v_5\}$. Similarly, if z_2 is an internal vertex of $\pi(zv_0)$ or $\pi(zv_2)$, the required bipartition is given respectively by $\{v_2, v_4, z_2\}$, $\{z, v_1, v_5\}$ or $\{v_1, v_5, z\}$, $\{v_0, v_4, z_2\}$.

   Consider now the possibility of z_2 being a vertex of $\pi(zv_3)$. If $z_2 \neq z$ or v_3, then H' is a bipartite graph isomorphic to R. If $z_2 = z$ or v_3, then H' is an isomorphic to Q minor of G.

To conclude, consider H' and suppose that it is T_2-free. If all of its edges are short, then the lemma holds. So assume that e is a long edge of H'. By (3.2.1) and (3.2.2), $H' \cong Q_0$ and $e = zv_1$ or zv_0, say, $e = zv_1$. Now G has an H' external path whose one end, call it, x_2 is an internal vertex of zv_1. By (3.2.2), the other must be v_5. Now $H' + x_2v_4 \cong Q_1$. Moreover, since x_2 and v_2 are mates in $H' + x_1v_5$ and all the edges incident to v_2 are short, then the edges incident to x_2 are also short. Again, if all edges of $H' + x_2v_4$ are short, the lemma holds. Otherwise, zv_0 is the long edge of $H' + x_2x_4$. Repetition now of the above argument, with zv_0, yields a minor and subgraph of G which is isomorphic to Q_2, as required. $\square$

We are now ready to prove the main theorem.

Theorem 3.4 *Let G be a 3-connected graph, $|V(G)| \geq 9$. G is T_2-free if and only if $G \in \mathcal{M} \cup \mathcal{K}$.*

Proof. It is easy to verify that the graphs in $\mathcal{K}$ are T_2-free. Moreover, by Corollary 3.2, the graphs in $\mathcal{M}$ are T_2-free. Thus, if $G \in \mathcal{M} \cup \mathcal{K}$, then G is T_2-free.

Conversely, suppose that $G \notin \mathcal{M}$. By Lemma 3.3, G has a subgraph L isomorphic to one of R, Q_0, Q_1, Q_2, but not proper subdivision of L. Since G is 3-connected, by the Fan Lemma, there exist internally disjoint paths P_{yy_i}, $1 \leq i \leq 3$, from y to distinct vertices of L. If $L \cong R$, then by (3.2.1), $N(y)$ is a subset of a colour class B of L and each P_{yy_i} is an edge. Moreover, for any vertex y' of $V(G)\backslash V(L)$, $N(y') \subseteq B$ for otherwise we can obtain a T_2-minor whose one colour class contains y' and two vertices in $N(y)$ and the other three vertices in $N(y')$.

So, we may assume that G does not contain R as a subgraph. If $L \cong Q_0$, then, by (3.2.2), $N(y)$ is a subset of a colour class of $L - z$. Now if $y' \in V(G)\backslash(V(L) \cup \{y\}$, then $N(y') = N(y)$, for otherwise a T_2 minor of G can be formed from the bipartition $\{y, y', z\}$, $\{v_0, v_1, v_3\}$. If $L \cong Q_1$, then again by (3.2.2), y can only be a non-adjacent mate of either v_2 or v_4 and v_2, x_2 are non-adjacent. Finally, when $L \cong Q_2$, y can only be a non-adjacent mate of v_2 or v_3. In all cases, $G \in \mathcal{K}$, as required. $\square$

A *1-cut reduction* of a graph G simply involves splitting of G along a cut vertex. Call a 2-cut S of a graph G *proper*, if either $G - S$ has at least three components or it has two components each one of them non-trivial. A *proper 2-cut reduction* involves splitting the graph along a proper 2-cut S and augmenting each non-trivial connected component of the resulting graph by a path of length two joining the vertices of S.

Theorem 3.4 allows us to describe in general T_2-free graphs. Note first that a graph G is T_2-free if and only if each connected component of the graph obtained from a 1- or a proper 2-cut reduction on G is T_2-free. Moreover, mate graphs of planar graphs are T_2-free. Thus, a T_2-free graph can be reduced by 1- and proper 2-cuts to a mate graph of a planar graph, or a subdivision of a 3-connected graph in $\mathcal{K}$, or a subdivision of a 3-connected graph with at most eight vertices. We remark that it is possible and relatively easy to determine which subdivisions of $\mathcal{K}$ are T_2-free. However, their description turns out to be rather technical and we omit it.

4 Mate Reduction and Colouring

We now apply the above results to prove the following.

Theorem 4.1 *If every planar r-graph is r-edge colourable, then every T_2-free r-graph is r-edge colourable.*

Proof. We first consider graphs with a small number of vertices.

4.1.1 *Every r-graph G with at most eight vertices is r-edge colourable*

Proof. If G has an r-edge cut $\delta(S)$ such that, $|S|$, $|V(G) - S| \geq 2$ and S has odd cardinality, then we can contract respectively S and $G - S$ to a single vertex and reduce our problem to colouring two graphs with fewer than G vertices. Thus we may assume that G has no such cut. It suffices now to show that that G has a matching M such that $G - M$ is an $(r-1)$-graph. By Tutte's Theorem [13], G has a perfect matching M. We claim that $G - M$ is an $(r-1)$-graph. Indeed, consider any $S \subset V(G)$ of odd cardinality and note that $|\delta_G(S) \cap M| \leq 3$ and $|\delta_G(S)| > r$ imply that $|\delta_{G'}(S)| \geq |\delta_G(S)| - 3 \geq r - 2$. It follows that $|\delta_{G'}(S)| \geq r - 1$ since $|\delta_G(S)|$ and r have the same parity and $|\delta_{G'}(S)|$ and r have different parity. □

We proceed now with the proof of the theorem. Let G be a T_2-free r-graph. If G is planar, then, by hypothesis, it can be coloured as required. Thus we assume that G is not planar and that any graph with fewer vertices than G that meets the hypothesis, is r-edge colourable. We may also assume that G is connected.

Suppose first that G is not 3-connected and note that since $V(G)$ has even cardinality, G does not have 1-cuts. Thus, $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2)$ consists of exactly two vertices. Since, G is an r-graph, the cardinalities of $V(G_1)$ and $V(G_2)$ are of the same parity. Moreover, observe that when $S := V(G_1)$, $S' := V(G_2)$ and S has odd cardinality, then $|\delta(S)| = r$, since $|\delta(S)| + |\delta(S')| \geq 2r$ and $|\delta(S)| + |\delta(S')| \leq |\delta(v)| + |\delta(u)| = 2r$. Similarly, $|\delta(S)| = r$ when $S = V(G_1) \backslash \{u\}$, $S' = V(G_2) \backslash \{v\}$ and S has odd cardinality. Thus, when G has a 2-cut one can find a subset of vertices S such that, $|S| = 2k + 1$, $k \geq 1$, and $|\delta(S)| = r$. Also, when S and $G - S$ are contracted to a single vertex, the resulting two graphs are T_2-free. It follows now, by induction, that one can r-edge colour the resulting smaller graphs and then combine the solutions found, in the obvious manner, to obtain a colouring of G.

So, we may assume that G is 3-connected. Theorem 3.4 tells us that the underlying simple graph of G is either in $\mathcal{M}$ or in $\mathcal{K}$. In the latter case, since G is an r-graph, it must have at most eight vertices and by (4.1.1), it is r-edge colourable. In the former case we will have to deal with mate vertices. We do this below.

Let $W := \{w, w'\}$ be mate vertices of G and let $X := N(w) \backslash \{w'\}$. Since G is 3-connected, $|X| = 3$ and $X = N(w') \backslash \{w\}$. Denote by x_1, x_2, x_3 the members of X and let $t = |\{e : e \text{ joins } w \text{ and } w'\}|$. Let H be the subgraph of G induced by $X \cup W$ after the edges with both ends in either X or W are removed. H is a bipartite graph and $|\delta_H(w)| = |\delta_H(w')| = r - t$. Suppose, momentarily, that $r - t$ is also the maximum degree of H. Then, it is well known that H is $(r-t)$-edge colourable. Moreover, every colour class is incident to both vertices of W. Now for every colour class $M_k = \{wx_i, w'x_j\}$, $1 \leq k \leq r - t$, we substitute in G an edge e_{M_k} joining x_i to x_j. Let G' be the graph resulting from G after each colour class of H has been considered and after deleting W. Clearly, G' is T_2-free. Suppose also that G' is an

r-graph. Then we know, from the induction hypothesis, that G' is r-edge colourable. Given any r-colouring of G' one can obtain an r-colouring of G as follows. Every edge that belongs to some colour class M_k is assigned the colour of e_{M_k}. Every edge that belongs to both G and G' is simply assigned its colour in G'. One can now verify that the colouring so obtained is a proper colouring of $G - \{e : e \text{ joins } w \text{ and } w'\}$. Moreover, the same $r - t$ colours are incident to w and w'. Hence, the remaining t colours can be used to colour $\{e : e \text{ joins } w \text{ and } w'\}$ and thus obtain an r-colouring of G.

We need to verify our assumptions regarding G' and H. First we show that the maximum degree of H is $r - t$. Let x be any element of X and let $S = W \cup \{x\}$. Then, $|\delta_G(S)| = 2r - 2t - |\delta_H(x)| + r - |\delta_H(x)|$. Moreover, since G is an r-graph, we also have that $|\delta_G(S)| \geq r$. It follows that $|\delta_H(x)| \leq r - t$, as required.

To complete the proof of the theorem we need to establish that G' is an r-graph. Let S be any odd cardinality subset of $V(G')$. If $|S \cap X| = 0$ or 1, then $|\delta_{G'}(S)| \geq r$, since $|\delta_{G'}(S)| = |\delta_G(S)|$. If $|S \cap X| = 3$, then again $|\delta_{G'}(S)| \geq r$, since $|\delta_{G'}(S)| = |\delta_G(S \cup W)|$. Thus, we may assume that $|S \cap X| = 2$. Let $\{x\} = X \backslash \{x_i, x_j\}$ and observe that in G', exactly $|\delta_H(x)|$ edges of the form e_{M_k} have one end in x and the other in $\{x_i, x_j\}$. Therefore, $|\delta_{G'}(S)| = |\delta_G(S \cup W)| - |\delta_H(x)| + |\delta_H(x)| = |\delta_G(S \cup W)| \geq r$, as required. □

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