Clearly, any division of a 2-sided area by a single X-dipath with end-vertices in ∂X divides the area into two 2-sided areas. Consider any area Y in X/D which is incident with ∂X. Thus $\partial Y \subseteq \partial X \cup L_0 \cup R_0$. Since P starts from ∂X and finishes in s_0, the only possibility of Y not being split from the rest of X by a single dipath is if it has t_0 as a terminal corner. In this case Y must be 4-sided if, moreover, it has t a as a terminal corner; otherwise Y can only be 2-sided. Thus, Y is either 2-sided or 4-sided. Suppose instead that $Y \in X/D$ is not incident with ∂X. Then $\partial Y \subseteq L_1 \cup R_1$. Since the boundary of Y is contained in the union of two dipaths, and since we have no cycles, Y has to be 2-sided. Thus follows (i) and hence the lemma. ■

Our next concern is to divide 2- and 4-sided areas into strictly smaller 2-sided areas. The first step in doing this is to prove two technical lemmas that will be used repeatedly.

Lemma 13 *Let $F \cup C$ be a planar acyclic digraph embedded into a closed disc Δ. Suppose the following conditions are satisfied:*

(i) F is a forest of rooted trees and $C \subseteq V(F)^2$.

(ii) The boundary of Δ divides into two segments: one containing all the roots of F, and one containing all the leaves of F. Denote by $l_0, \ldots, l_n$ be leaves of F as they occur in a clock-wise walk along the segment containing the leaves. Moreover, denote by P_i the dipath in F to l_i starting in the root of the tree containing l_i.

(iii) Given any arc $(v, w) \in C$, if $(v, w) \in V(P_i) \times V(P_j)$ then $i < j$.

(iv) No two arcs in C have a common heads or common tails.

Then there is a forest of anti-rooted trees (directed trees where all the arcs are directed towards some anti-root) E in the transitive closure of $F \cup C$ such that

(v) $|A(E)| \leq |C|/2$.

(vi) If w is reachable from v in $F \cup C$, there is a dipath $P_1P_2P_3$ from v to w in $F \cup C \cup E$ such that

- *P_1 is contained in $F \cup C$ and, if P_1 is not trivial, it has its last arc in C and at most one other arc in C;*
- *P_2, if non-trivial, is contained in E, and*
- *P_3 is contained in $F \cup C$ and contains at most one arc from C.*

Proof: Notice that, by conditions (i) and (ii), for all $(v, w) \in C$, there are indices i, j with $|j - i| \leq 1$ such that $(v, w) \in \mathrm{V}(P_i) \times \mathrm{V}(P_j)$. Adding (iii) gives us that if i is the maximum index with v in P_i, then $(v, w) \in \mathrm{V}(P_i) \times \mathrm{V}(P_{i+1})$.

Denote by H the sub-digraph of $F \cup C$ obtained by removing arcs (v, w) from F satisfying either

(a) $v \in V(P_i) \cap V(P_{i+1})$ and $w \in V(P_i) \setminus V(P_{i+1})$, or

(b) v is the tail of an arc in C.

Claim 1 *H is a forest of anti-rooted trees.*

Proof: Since $F \cup C$ is acyclic, H is acyclic. Hence Claim 1 follows if the maximal out-degree in H is one, i.e. if no two arcs in H have common tail. By (a) no two arcs from $H \cap F$ can have a common tail, and by (b) we cannot have an arc from $H \cap F$ with the same tail as an arc from $H \cap C$. Finally, by (iv), no two arcs from $H \cap C$ can have a common tail. □

Claim 2 *If the vertex w is reachable from the vertex v in $F \cup C$, then there is a dipath $Q_1 Q_2$ in $F \cup C \cup H$ such that Q_1 is contained in H and Q_2 is contained in F.*

Proof: Let Q_1 be the longest dipath in H starting in v and finishing in a vertex v_1 from which w is reachable in $F \cup C$. Notice that Q_1 and v_1 are uniquely defined since H is a forest of anti-rooted trees by Claim 1. Let Q_2 be a dipath in $F \cup C$ from v_1 to w containing a minimum number of arcs from C. We want to show that Q_2 contains no arcs from C.

First, let v_2 be the second vertex in Q_2. By the maximality of Q_1, we cannot have (v_1, v_2) in H. On the other hand, all arcs from C are in H, so (v_1, v_2) is in F.

Suppose that Q_2 contains an arc (v_3, v_4) from C, and denote by Q_2^- the initial segment of Q_2 finishing in v_2. Moreover, denote by i the largest index such that Q_2^- is contained in P_i. Then $(v_3, v_4) \in V(P_i) \times V(P_{i+1})$. Clearly, we cannot have v_1 in P_{i+1}; for then in Q_2 we could substitute $Q_2^-(v_3, v_4)$ with a segment of P_{i+1}, hence obtaining a dipath from v_1 to w with fewer arcs from C. Thus i is the largest index such that v_1 is contained in P_i. In particular, this implies that (v_1, v_2) cannot be excluded by (a). Thus (v_1, v_2) must be excluded by (b), implying that we have an arc (v_1, u) in C. Since i is the largest index such that v_1 is contained P_i, we have $(v_1, u) \in P_i \times P_{i+1}$. Now, v_1 is before v_2 in P_i, and (v_1, u) does not cross (v_3, v_4) in Δ, so u is before v_4 in P_{i+1}. But w is reachable from v_4, and (v_1, u) is in H since no arcs from C have been excluded from H, so this contradicts the maximality of Q_1. Thus Q_2 contains no arcs from C. □

The important consequence of Claim 2 is that in order to settle (vi), it is sufficient to consider vertices v and w where w is reachable from v in H.

Color the arcs in C with red and blue depending on whether the dipath in H from the head to the relevant anti-root contains an odd or an even number of arcs from C. Thus the arcs from C in any dipath in H have alternating colors. Without loss of generality we may assume that red is the rarest color. Let E be the sub-digraph of the transitive closure of H consisting of the arcs (v, w) such that v and w both are red heads but the dipath in H from v to w contains no other red heads.

Clearly E inherits from H that it is a forest of anti-rooted trees. Since red was the rarest color, we have $|V(E)| \leq |C|/2$, and since E is a forest $|\mathrm{A}(E)| < |\mathrm{V}(E)|$, so (v) follows.

Also, from the construction of E it follows that if a red head w is reachable from a red head v in H then w is reachable from v in E.

Consider a dipath P in H from a vertex v to a vertex w. If there are no red heads in P then P contains at most one blue and no red arcs (only arcs from C are colored), so (vi) is satisfied with $P_1 = P_2 = (v)$ and $P_3 = P$. Thus we may assume that P contains some red heads. Let v' be the first red head in P and let w' be the last read haed in P. Then (iv) is satisfied with P_1 the segment of P from v to v', P_2 the dipath in E from v' to w', and P_3 the segment of P from w' to w. This completes the proof of the lemma. ■

Lemma 14 *Denote by $\delta_{14}(n)$ the smallest number such that a diameter of at most $\delta_{14}(n)$ can be achieved by matched shortcutting for any acyclic digraph $F \cup C$ with n vertices if it satisfies the following conditions:*

(i) F is a forest of rooted trees embedded on a bounded disc Δ.

(ii) The boundary of Δ divides into two segments: one containing all the roots of F, and one containing all the leaves of F.

(iii) $C \subseteq V(F)^2$, and any arc in C can be drawn in Δ such that it only intersects F in its end-points.

Then $\delta_{14}(n) = O(\alpha(n, n))$.

Proof: Notice that for all $(v, w) \in C$, there are indices i, j such that $(v, w) \in \mathrm{V}(P_i) \times \mathrm{V}(P_j)$ and $|j - i| \leq 1$. Moreover, if, above, i and j cannot be chosen equal, they are unique. Without loss of generality, we may assume that C is minimal in the sense that no arc can be deleted without affecting the transitive closure of $F \cup C$.

For $s = -1, 0, 1$, denote by C_s the set of arcs in C such that if (i, j) is a closest pair of indices with $(v, w) \in \mathrm{V}(P_i) \times \mathrm{V}(P_j)$ then $s = j - i$. By our previous remarks, C_{-1}, C_0, C_{-1} form a partitioning of C.

We claim that $C_0 = \emptyset$. Suppose that C_0 contained an arc (v, w). There would then be a dipath P_i containing both v and w. Then (v, w) is contained in the transitive closure of P_i since we have no dicycles. This contradicts our assumption that C is minimal, so we may conclude that, indeed, $C_0 = \emptyset$.

Another easy consequence of our hypothesis that $F \cup C$ is acyclic is that if we have a dipath from a vertex v to a vertex w, then there must be such a dipath contained either in $F \cup C_{-1}$ or in $F \cup C_{+1}$. We want to apply Lemma 13 to these two digraphs.

Fix $s \in \{-1, +1\}$. Observe that all arcs of C_s can be drawn simultaneously in Δ without any crossings; for an obstruction to this should be two arcs $(v_0, w_0), (v_1, w_1) \in C \cap (\mathrm{V}(P_i) \times \mathrm{V}(P_{i+1}))$ such that v_0 was before v_1 in P_i and w_1 was before w_0 in P_{i+1}. But then (v_1, w_1) is in the transitive closure of $P_i \cup \{(v_0, w_0)\} \cup P_{i+1}$, contradicting the minimality of C. The same kind of argument shows that no two arcs of C_s can have common head or common tail. Thus we may apply Lemma 13 to $F \cup C_s$ (Strictly speaking, for $F \cup C_{-1}$, we apply Lemma 13 to the mirror image). Denote the anti-rooted forest provided by Lemma 13 by E_s.

Let S_F be an optimal matched shortcutting of F, and for $s = -1, +1$, let Let S_s be an optimal matched shortcutting of E_s. By Theorem 5, each of these shortcuttings achieves a diameter of at most $\delta_5(n) = O(\alpha(n, n))$. Set $S = S_F \cup E_{-1} \cup S_{-1} \cup E_{+1} \cup S_{+1}$. By condition (v) in Lemma 13, we have $|E_s| < |C_s|/2$. Moreover, $|S_F| \leq |F|$ and $|S_s| \leq |E_s|$, so S is a matched shortcutting of $F \cup C_{-1} \cup C_{+1} = F \cup C$. Finally, by condition (vi) in Lemma 13, we get that

$$\mathrm{diam}(F \cup C \cup S) \leq 5\delta_5(n) + 3 = O(\alpha(n, n)),$$

completing the proof. ∎

The machinary is now in place for dividing 2-sided areas into 2-sided proper subareas.

Lemma 15 *Denote by $\delta_{15}(n)$ the least number such that for any 2-sided area X with n vertices, all reachable from ∂X, there is a division D with a division shortcutting S satisfying the following conditions*

(i) All areas Y in X/D which contain vertices are 2-sided and strictly contained in X.

(ii) iib-diam$(D, X, S) \leq \delta_{15}(n)$.

Then $\delta_{15}(n) = O(\alpha(n, n))$.

Proof: Let Y be the area in $X/\mathcal{G}^+(X)$ that in some left side has the last arc of the left side of X. Let F be a forest of rooted trees in $\mathcal{G}^+(X)$ satisfying the following conditions:

(a) $F \cap \partial X$ is the set of roots in F.

(b) All vertices in $\partial Y|\mathcal{V}(X)$ are in F.

(c) All leaves of F are in $\partial Y|\mathcal{V}(X)$.

Such a forest F exists since all vertices in X are reachable from ∂X. Let B be the set of arcs in ∂Y that are neither in ∂X, nor in F. The construction is illustrated in Figure 3. Set $D = F \cup B$. From (a) and (b) it follows that $D \cup \partial X$ is connected, and hence that D is a division of X.

Claim 1 *All areas in $X/D \setminus \{Y\}$ are 2-sided.*

Proof: By (a) we have that $F \cup \partial X$ is connected, so F is a division of X. Let $y_1, \ldots, y_n$ be the arcs in B. Since F is a forest, there is only one area in X/F. Notice that by the Jordan curve Theorem, the single area in X/F is cut by y_i into two sub-areas. Denote by Y_i the sub-area not containing Y. Since $y_1, \ldots, y_n$ are all in ∂Y, no $y_j \neq y_i$ can cross Y_i. Thus $Y_i \in X/D$. Since X/F only contains one area, if $n > 0$, all areas in X/D must have an arc y_i in its boundary. No arc of a division can be in the boundary of more than two sub-areas, and by construction y_i has Y on the one side and Y_i on the other side. Thus we may conclude that $X/D = \{Y, Y_1, \ldots, Y_n\}$.

We want to show that Y_i is 2-sided for all i. By (c) all leaves of F are in ∂Y, and Y occupies the terminal corner of X if it is not the root of some tree in F, so the only possibility of a terminal corner for Y_i is the head of y_i. On the other hand, it is clear that the head of y_i has to be a terminal corner, so indeed Y_i is 2-sided. □

Claim 2 *There is a matched shortcutting S of* CrossExt$(D, X)|\mathcal{V}(X)$ *achieving a diameter in $O(\alpha(n, n) \log n)$.*

Proof: Set $F^- = F|\mathcal{V}(X)$ and $C^- = (B \cup \bigcup_{X_i \in X/D} \text{Cross}(X_i))|\mathcal{V}(X)$, then $F^- \cup C^- =$ CrossExt$(D,X)|\mathcal{V}(X)$. We want to apply Lemma 14. As the closed disc Δ we chose the one in X containing $\mathcal{G}(X)$ whose boundary coincides with $\mathcal{G}(X)$. Then F^- is already embedded in Δ. Also, the arcs from $B|\mathcal{V}(X)$ are drawn in Δ so that they only intersect F in their end points. For arcs $(v, w) \in \bigcup_{X_i \in X/D} \mathrm{Cross}(X_i)|\mathcal{V}(X)$, as drawing, we can use the embedding into Δ of any X_i-dipath from v to w. Thus Lemma 14 applies to $F^- \cup C^-$, and hence we get a matched shortcutting of $\mathrm{CrossExt}(D, X)|\mathcal{V}(X)$ achieving a diameter of at most $\delta_{14}(n) = O(\alpha(n, n))$. □

Now the result follows, for by Lemmas 9 and 10 we can extend S to a division shortcutting S' such that iib-diam$(D, X, S') \in O(\alpha(n, n))$, as desired. ■

Our final sub-goal in proving Key Lemma 8 is to divide 4-sided areas into 2-sided areas.

Lemma 16 *Denote by $\delta_{16}(n)$ the minimum number such that for any 4-sided area X with n vertices, all reachable from ∂X, there is a division D with a division shortcutting S satisfying the following conditions:*

(i) Any area Y in X/D is either 2-sided or 4-sided. If Y is 4-sided, for one of the source corners s, there is no non-trivial Y-dipath from a side incident with s to a side incident with the opposing source corner.

(ii) iib-diam$(D, X, S) \leq \delta_{16}(n)$.

Then $\delta_{16}(n) = O(\alpha(n, n))$.

Proof: We will prove the result by finding a division D satisfying (i) and which is the union of at most two arc-disjoint dipaths, for then the result follows as a trivial application of Lemma 11.

Let L_1, R_1, L_2, R_2 be sides of X such that $L_1 R_2^{-1} L_2 R_1^{-1}$ is a closed curve clock-wise around X. Denote by s_i the source corner incident with L_i and R_i. Choose an X-dipath P_1 from $L_1 \cup R_1$ to the earliest possible vertex in R_2. In the extreme case where P_1 finishes in s_2 (See Figure 4a), P_1 divides X into two 2-sided areas, end hence we are done by setting $D = P_1$. We may therefore assume that we are not in this extreme case. Then, in general, P_1 divides X into one 2-sided and one 4-sided area. (See Figure 4b). However, if P_1 is trivial, i.e. if $P_1 = (t)$ where t is the terminal corner incident with L_1 and R_2, there is no 2-sided area. The 4-sided area X' has L_2 as a left side and s_2 as an incident source corner. With L_2 and s_2 fixed, define L'_1, s'_1, R'_1, R'_2 for X' as L_1, s_1, R_1, R_2 was defined for X. Then P_1 is a final segment of L'_1, then R'_1 is a final segment of R_1, and R'_2 is an initial segment of R_2. By our choice of P_1, there cannot be a non-trivial dipath from $L'_1 \cup R'_1$ to R'_2. Now, choose an X'-dipath P_2 from $L'_1 \cup R'_1$ to the earliest possible vertex in L'_2. Since P_2 cannot reach $s_2 \in R_2$, as above, it divides X' into one 4-sided area X'' together with at most one 2-sided area. Clearly, there cannot by any non-trivial dipaths from the sides of X'' incident with the source corner s''_2 opposing s_2 to the the sides incident with s_2. Thus (i) is satisfied by setting $D = P_1 \cup P_2$, and this implies the result since P_1 and P_2 are arc-disjoint. ■

Lemma 17 *Denote by $\delta_{17}(n)$ the least number with the following property: if X is a 4-sided area with n vertices, all reachable from ∂X, and if, moreover, for one of the source corners s, there is no non-trivial X-dipath from a side incident s to a side incident with the opposing source corner, then there is a division D of X with a division shortcutting S such that* *(i) All areas Y in X/D which contains vertices are 2-sided.*

(ii) iib-diam$(D, X, S) \leq \delta_{17}(n)$.

Then $\delta_{17}(n) = O(\alpha(n, n))$.

Proof: Basically we just repeat the proof of Lemma 15 twice. Let L_1, R_1, L_2, R_2 be sides of X such that $L_1 R_2^{-1} L_2 R_1^{-1}$ is a closed curve clock-wise around X and such that, moreover, there is no proper X-dipath from $L_1 \cup R_1$ to $L_2 \cup R_2$. Let G_1 be the union of all X-dipaths starting in $L_1 \cup R_1$. Denote by Y_1 the unique area in X/G_1 that has L_2 contained in a left side and R_2 contained in a right side. As in the proof of Lemma 15, let F_1 be a a sub-digraph of G_1 such that F_1 is a forest of rooted trees, such that $F \cap \partial X$ is the set of roots in F, such that all vertices in $\partial Y_1|\mathcal{V}(X)$ are in F_1, and such that all leaves of F_1 are in $\partial Y_1|\mathcal{V}(X)$. Denote by B_1 the set of arcs in ∂Y_1 but not in ∂X, and set $D_1 = F_1 \cup B_1$. As in the proof of Lemma 15 we observe that all areas apart from Y_1 in X/D_1 are 2-sided areas. Moreover, with C_1 denoting the crossovers for these 2-sided areas, applying Lemma 14, we get a matched shortcutting S_1 of $(D_1 \cup C_1)|\mathcal{V}(X)$ achieving a diameter in $O(\alpha(n, n))$.

Now, set $G_2 = \mathcal{G}^+(Y_1)$. Then G_2 is the sub-digraph of $\mathcal{G}^+(X)$ containing all arcs not in G_1. Since all vertices in X are reachable from ∂X, all vertices in G_2 are reachable by dipaths in G_2 from the sides L_2 and R_2. Denote by Y_2 the unique area in X/G_2 that has L_1 contained in a left side and R_1 contained in a right side. Then $X \setminus \overline{Y_2} \subseteq Y_1$. As above we construct a division $D_2 \subseteq G_2$ such that all areas apart from Y_2 in X/D_2 are 2-sided areas. Moreover, with C_2 denoting the crossovers for these 2-sided areas, we get a matched shortcutting S_2 of $(D_2 \cup C_2)|\mathcal{V}(X)$ achieving a diameter in $O(\alpha(n, n))$.

Consider the division $D = D_1 \cup D_2$. The areas from X/D that are not contained in $Y_1 \cap Y_2$ are exactly the areas in $(X/D_1 \setminus \{Y_1\}) \cup (X/D_2 \setminus \{Y_2\})$. Hence any such area is 2-sided. On the other hand, by definition $Y_1 \cap Y_2$ contains no vertices, so none of the areas from X/D contained in $Y_1 \cap Y_2$ can contain any vertices, so the division D satisfies (i).

Set $S = S_1 \cup S_2$. Notice that $(D_1 \cup C_1)$ and $(D_2 \cup C_2)$ are arc disjoint and that $\mathrm{CrossExt}(D, X) = (D_1 \cup C_1) \cup (D_2 \cup C_2)$ since no area from X/D in $Y_1 \cap Y_2$ has any crossover. Hence S is a matched shortcutting of $\mathrm{CrossExt}(D, X)|\mathcal{V}(X)$. By construction any dipath in $\mathcal{G}^+(X) = G_1 \cup G_2$ that starts in G_1 will stay in G_1. Hence a dipath starting in G_2 will either stay in G_2, or go to G_1 where it will stay. This property of G_1 and G_2 is inherited by $D_1 \cup C_1$ and $D_2 \cup C_2$, so

$$\mathrm{diam}((D \cup S)|\mathcal{V}(X)) \leq \mathrm{diam}((D_1 \cup C_1)|\mathcal{V}(X)) + \mathrm{diam}((D_2 \cup C_2)|\mathcal{V}(X)) \leq 2\delta_{14}(n) = O(\alpha(n, n)).$$

Now the result follows by Lemmas 9 and 10. ∎

Proof of Key Lemma 8: Let X be any 2-sided area containing n vertices, all reachable from ∂X. By Lemma 7 we may nest the divisions and division shortcuttings provided by Lemmas 12, 15, 16, and 17 so that we get a division D of X with a division shortcutting S satisfying the following conditions:

(i) All areas in X/D containing vertices are 2-sided and contain at most $n/2$ vertices.

(ii) ib-diam$(D, X, S) \leq \delta_{12}(n) + \max\{\delta_{15}(n), \delta_{16}(n) + \delta_{17}(n)\} = O(\alpha(n, n))$
ii-diam$(D, X, S) \leq \delta_{12}(n) + 2\max\{\delta_{15}(n), \delta_{16}(n) + \delta_{17}(n)\} = O(\alpha(n, n))$.

■

5 Planar digraphs

In this section first we will settle Conjecture 1 for all acyclic planar digraphs. From there we jump directly to all planar digraphs using a result from [15].

Lemma 18 *Let $\delta_{18}(p)$ be the minimal number with the following property: if G is a plane acyclic digraph with p vertices, all of which are reachable from the exterior boundary of G, then there is a matched shortcutting achieving a diameter of at most $\delta_{18}(p)$. Then $\delta_{18}(p) = O(\alpha(p,p)(\log p)^2)$*

Proof: Without loss of generality, we may assume that G is connected. By induction on p we will prove that there is a matched shortcutting S of G achieving

$$\text{diam}(G, S) \leq (\log' p)(4\,\delta_6(p) + 1)$$

Here $\log'$ is as defined at page 9. The lemma will then follow by Theorem 6 which says that $\delta_6(p) = O(\alpha(p,p)\log p)$. The statement of the induction is trivial if $p = 0$, so assume that $p > 0$.

We shall use the notation that if G_0 and G_1 are sub-digraphs of G then $G_0 - G_1 = \mathrm{G}(\mathrm{A}(G_0) \setminus \mathrm{A}(G_1))$. Notice that if G_0 is connected and does not consist of an isolated vertex then $(G_0 - G_1) \cup G_1 = G_0 \cup G_1$.

Let c be a closed curve clock-wise around a simply connected open subset X of the plane such that G is contained in $X \cup c$, and such that c contains exactly the vertices from the exterior region of G vertices with indegree 0. We call the vertices in c *source vertices*. Thus the set of source vertices is the minimum set from which all vertices in G can be reached. For any vertex v in c, by $G(v)$ we denote the sub-digraph of G induced by the vertices reachable from v. Notice that $G(v)$ is a rooted acyclic planar digraph.

Claim 1 *There is a strictly increasing sequence of sub-digraphs of G*

$$(\emptyset, \emptyset) = H_0 \subset H_1 \subset \cdots \subset H_m = H$$

satisfying the following properties:

(a) H_i is connected and the union of H_{i-1} and at most two digraphs $G(v)$.

(b) Any dipath in G starting in H_i is contained in H_i.

(c) *Any dipath in H starting in $H_i \setminus \mathrm{V}(H_{i-1})$ is contained in $H_i \setminus \mathrm{V}(H_{i-2})$. For the case that $i < 2$, we define $H_{-2} = H_{-1} = (\emptyset, \emptyset)$.*

(d) *No connected component in $G \setminus \mathrm{V}(H)$ contains more than $p/2$ vertices.*

PROOF: We will give a concrete construction of the H_is satisfying some additional properties:

(e) For each $i < m$, there is a unique connected component C_i of $G - H_i$ which contains more than $p/2$ vertices outside H_i. For $i = m$, there is no such component.

(f) There is no dipath in G from C_i to H_{i-1}, and $H_{i+1} - H_i$ is contained in C_i.

Notice that (e) and (f) implies (c) and (d). Consider the components of c after removal of the source vertices in H_i. By c_i we denote the component containing the source vertices of C_i. Now, the H_is are constructed as follows.

> Set $H_0 = (\emptyset, \emptyset)$. Choose an arbitrary source vertex v, and set $H_1 := G(v)$. Set $i := 1$. While C_i contains more than $p/2$ vertices, do the following:
>
> > Clearly, C_i is contained in C_{i-1}. Now, since G is connected there are vertices w in c_i from which it is possible to reach vertices in H_i. Let $w_1, \ldots, w_q$ be the sequence of such source vertices numbered in the order they appear in c_i. Denote $G(w_j)|\mathrm{V}(C_i)$ by $C_i(w_j)$. Then $C(w_j)$ cuts C_i into two parts: $L_i(w_j) = \bigcup_{k<j} C_i(w_k) \setminus \mathrm{V}(C_i(w_j))$, and $R_i(w_j) = \bigcup_{k>j} C_i(w_k) \setminus \mathrm{V}(C_i(w_j))$. Then $\mathrm{V}(C_i) = \mathrm{V}(L_i(w_j) \cup C_i(w_j) \cup R_i(w_j))$ for all w_j.
> >
> > Denote by j_1 the maximum index such that $|\mathrm{V}(L(w_{j_1}))| \leq p/2$. If $|\mathrm{V}(R(w_{j_1}))| \leq p/2$, set $j_2 := j_1$; otherwise set $j_2 := j_1 + 1$. By maximality of j_1, we have $|\mathrm{V}(R(w_{j_2}))| \leq p/2$. Finally, set $H_{i+1} = H_i \cup G(w_{j_1}) \cup G(w_{j_2})$, and $i := i + 1$.
>
> Set $m := i - 1$.

It is easy in each step to check that (a), (b), (e), and (f) are invariant under the construction. □

For all $i = 0, \ldots, m$, we will construct a matched shortcutting S_i of $H_i - H_{i-1}$. Fix i. By (a), we have that $H_i = H_{i-1} \cup G(v) \cup G(w)$ for some source vertices v and w where possibly $v = w$. By definition $G(v)$ is an acyclic planar digraph rooted in v. By (b), any dipath in G starting in H_{i-1} is contained in H_{i-1}, so $G(v) - H_{i-1}$ inherits from $G(v)$ that it is a rooted acyclic planar digraph. Moreover, we may conclude any dipath in G starting in $H_{i-1} \cup G(v)$ must be contained in $H_{i-1} \cup G(v)$. In particular, this implies that also $G(w) - (G(v) \cup H_{i-1})$ is a rooted acyclic digraph if non-empty. Let S_i be the union of an optimal matched shortcutting for $G(v) - H_{i-1}$ and an optimal matched shortcutting for $G(w) - (G(v) \cup H_{i-1})$. In either case we achieve a local diameter of at most $\delta_6(p)$. Since $G(v) - H_{i-1}$ and $G(w) - (G(v) \cup H_{i-1})$

are arc-disjoint, S_i is a matched shortcutting of $H_i - H_{i-1}$, and since there is no dipath from $G(v) - H_{i-1}$ to $G(w) - (G(v) \cup H_{i-1}$, we may conclude that $\mathrm{diam}(H_i - H_{i-1}, S_i) \leq 2\delta_6(p)$.

Clearly all the digraphs of the form $H_i - H_{i-1}$ are arc-disjoint, so $S_H = \bigcup_i S_i$ is a matched shortcutting of H. Moreover, from (c) we may conclude that $\mathrm{diam}(H, S_H) \leq 4\delta_6(p)$.

Now, apply the induction hypothesis for a matched shortcutting of each of the components of $C = G \setminus \mathrm{V}(H)$. By (d) neither contains more than $p/2$ vertices, so for each we achieve a diameter of at most $(\log' \lfloor p/2 \rfloor)(4\,\delta_6(\lfloor p/2 \rfloor) + 1)$. Denote by S_C the union of these matched shortcuttings of the components. Then $S = S_H \cup S_C$ is a matched shortcutting of G. Now, consider any dipath P in G. By (b), if P starts in H it is fully contained in H. Otherwise P starts in C. Either, it then stays within the same component of C, or at some stage it leaves this component via some arc between C and H. Having arrived at H, it stays in H. Thus, there is a dipath in $G \cup S$ with the same end-vertices as P but of length at most

$$(\log' \lfloor p/2 \rfloor)(4\,\delta_6(\lfloor p/2 \rfloor) + 1) + 1 + 4\delta_6(p) \leq (\log' p)(4\,\delta_6(p) + 1),$$

as required for the induction. ■

Proposition 19 *Let $\delta_{19}(p)$ be the minimal number such that for any acyclic planar digraph with p vertices, there is a matched shortcutting achieving a diameter of at most $\delta_{19}(p)$. Then $\delta_{19}(p) = O(\alpha(p,p)(\log p)^2)$.*

Proof: Let $\mathcal{G}$ be an arbitrary acyclic planar digraph with p vertices. Without loss of generality, we may assume that $\mathcal{G}$ is connected. As in the last section we consider a concrete embedding of $\mathcal{G}$ on some specific sphere $\mathcal{X}$ with areas etc. Given an area X, we say that X is a *sink* area if no non-trivial X-dipath finishes in the boundary, and that X is a *source* area if no non-trivial X-dipath starts in the boundary. Notice that neither sink nor source areas can have any crossovers, so a division shortcutting is just a matched shortcutting of a division.

Claim 1 *If X is a source or a sink area, there is a matched shortcutting of $\mathcal{G}^+(X)$ achieving ii-diam$(\mathcal{G}^+(X), X, S) \leq 2\,\delta_{18}(p)$ and ib-diam$(\mathcal{G}^+(X), X, S) \leq \delta_{18}(p)$.*

Recall from Lemma 18 that $\delta_{18}(p) = O(\alpha(p,p)(\log p)^2)$.

Proof: We will prove the result be induction on X. By symmetry we may assume that X is a sink area. Denote by H the division of X that is the union over all X-dipaths starting in ∂X. Then any area X_i in X/H must be a source area strictly contained in X, so by the induction hypothesis, there is a matched shortcutting S_i of $\mathcal{G}^+(X_i)$ such that ii-diam$(\mathcal{G}^+(X_i), X_i, S_i) \leq 2\,\delta_{18}(p)$ and ib-diam$(\mathcal{G}^+(X_i), X_i, S_i) \leq \delta_{18}(p)$. By definition H satisfies the conditions of Lemma 18, so there is a matched shortcutting S of H achieving a diameter of at most $\delta_{18}(p)$. Trivially this implies that ii-diam(H, X, S) and ib-diam(H, X, S) are at most $\delta_{18}(p)$. Notice that $\mathcal{G}^+(X) = H \cup \bigcup_i \mathcal{G}^+(X_i)$. Set $S' = S \cup \bigcup_i S_i$. Since X is a sink area and X_i is a source area contained in X, there cannot be any dipath between ∂X and X_i. Hence

$$\text{ib-diam}(\mathcal{G}^+(X), X, S') = \text{ib-diam}(H, X, S) \leq \delta_{18}(p).$$

Moreover, there cannot be a dipath from H to X_i, so

$$\begin{aligned}\text{ii-diam}(\mathcal{G}^+(X), X, S') = \max\{ & \max_i\{\text{ib-diam}(\mathcal{G}^+(X_i), X_i, S_i)\} + \text{ii-diam}(H, X, S), \\ & \text{ii-diam}(\mathcal{G}^+(X_i), X_i, S_i)\} \\ \leq 2\,\delta_{18}(p), &\end{aligned}$$

completing the induction. □

Since G is acyclic we may choose a vertex v of indegree 0. Then $\mathcal{Y} = \mathcal{X} \setminus \{v\}$ is a sink area, so there is a division shortcutting $\mathcal{S}$ of $\mathcal{G}^+(\mathcal{Y})$ achieving $\text{ii-diam}(\mathcal{G}^+(\mathcal{Y}), \mathcal{Y}, \mathcal{S}) \leq 2\,\delta_{18}(p)$ and $\text{ib-diam}(\mathcal{G}^+(\mathcal{Y}), \mathcal{Y}, \mathcal{S}) \leq \delta_{18}(p)$; but then also $\mathcal{S}$ is a matched shortcutting of $\mathcal{G} = \mathcal{G}^+(\mathcal{Y})$ achieving a diameter of at most $2\,\delta_{18}(p) \in O(\alpha(p,p)(\log p)^2)$. ■

Given any digraph G, by $G/\equiv$ we denote the result of contracting all strong components in G. Notice that $G/\equiv$ is always acyclic.

Lemma 20 (Thorup [15]) *Given a digraph G with p vertices, if some matched shortcutting of $G/\equiv$ achieves a diameter d, there is a matched shortcutting of G achieving a diameter of at most $5d+4$.*

Proof of Theorem 2: Since planarity is closed under contraction, Proposition 19 together with Lemma 20 shows us that given any planar digraph G with p vertices, there is a matched shortcutting achieving a diameter of at most $5\,\delta_{19}(p) + 4 = O(\alpha(p,p)(\log p)^2)$. This is the exact statement of Theorem 2 which we have hereby established. ■

6 Concluding remarks

We have now proved Theorem 2 which gives some evidence for Conjecture 1 by establishing it for planar digraphs. It is quite easy to generalize Theorem 2 to classes of digraphs with a fixed bounded genus, using the result from [5] that for any graph with p vertices and genus g, there is a set of $O(\sqrt{gp}\log g)$ vertices whose removal leaves a planar graph. This is particularly easy if, instead of insisting on matched shortcutting, we are willing to spend a linear number of shortcuts. Let $G = (V, A)$ be an p-vertex digraph with genus g. Partition v into two sets V_1 and V_2 such that $G|V_1$ is planar and V_2 has minimal cardinality. Let S_1 be an optimal matched shortcutting of $G|V$, and let S_2 be the set of arcs from the full transitive closure of G which goes between vertices in V_2. Set $S = S_1 \cup S_2$. Then $|S| \leq |\mathrm{A}(G|V_1)| + |V_2|^2 = O(p) + O(gp(\log g)^2) = O(p)$ for constant g, and then $\text{diam}(G \cup S) \leq 2\,\text{diam}((G|V_1) \cup S_1) + 3 = O(\alpha(p,p)(\log p)^2)$.

Further, though rather involved, it is possible to generalize Theorem 2 to classes of digraphs with an excluded minor (in the undirected sense) using the structural theorems from [13, 14]. Unfortunately an arbitrarily big but constant size clique minor seems of no use for solving the general problem for all digraphs.

Of other special classes of digraphs with known shortcuttings, Loebl [9] has shown that the case of chordal digraphs can be reduced to the planar case.

Finally, by simple counting it is easy to settle the conjecture for digraphs that are only a poly-logarithmic factor from being dense. More precisely, consider a digraph G with p vertices and q arcs. Make q independent random choices of arcs from the transitive closure of G. Then $|S| \leq q$, so S is a matched shortcutting of G. If for some specific diameter d, we can prove that the event that $\mathrm{diam}(G \cup S) \leq d$ has positive probability, then this will imply the existence of a matched shortcutting of G achieving a diameter of d. Consider some vertices v and w with a dipath P in G from v to w of length at least d. We say that (v, w) is an *obstruction* if the shortest dipath in $G \cup S$ from v to w is longer than d. Thus the diameter of $G \cup S$ is at most d if and only if there are no obstructions. Clearly the probability that (v, w) is an obstruction is smaller than the probability that none of the arcs in S goes from the first to the last $\lfloor d/2 \rfloor + 1$ vertices in P, and the probability of the latter event is bounded by $(1 - (\lfloor d/2 \rfloor + 1)^2/p^2)^q$. Hence the probability of the existence of some obstructing pair is bounded by $(1 - (\lfloor d/2 \rfloor + 1)^2/p^2)^q p^2 \approx \exp(-q(\lfloor d/2 \rfloor + 1)^2/p^2)p^2$ which is smaller than 1 if $-q(\lfloor d/2 \rfloor + 1)^2/p^2 + 2\ln p < 0$, hence if $d > 2\sqrt{(p^2/q)2\ln p} + 1$. We may therefore conclude that if $p^2/q = O(\log^k p)$, then there is a matched shortcutting of G achieving a diameter of $O((\log p)^{(k+1)/2})$.

Notice that the proof of Theorem 2 is constructive in the sense that with a bit of care it actually constructs the matched shortcutting of a planar digraph in pseudo-linear (linear ignoring log-factors) time. Moreover, none of the involved hidden constants are astronomic. Going through all the calculations, including those in [4, 15], they can all be proved smaller than a 1000.

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A Area related definitions

$\mathcal{X}$: a specific sphere.

$\mathcal{G}$: a specific digraph embedded into $\mathcal{X}$. *No other graph components than those of $\mathcal{G}$ are embedded into $\mathcal{X}$.* In Section 4, $\mathcal{G}$ is rooted and acyclic, and in the proof of Proposition 19, $\mathcal{G}$ is connected and acyclic.

Z: any subset of $\mathcal{X}$.

$\mathcal{V}(Z)$: the set of vertices (from $\mathcal{G}$) in Z.

$\mathcal{A}(Z)$: the set of arcs in Z.

$\mathcal{G}(Z)$: $(\mathcal{V}(Z), \mathcal{A}(Z))$.

area: region of some non-empty connected sub-digraph of $\mathcal{G}$.

X: any area.

∂X: the boundary of X.

$\overline{X}$: the closure of X.

X-dipath: dipath in $\overline{X}$ which is contained in X with the possible exception of its end-vertices.

v, w: any vertices in $\mathcal{G}$.

X-reachable: w is X-reachable from v if there is an X-dipath from v to w.

inherently X-reachable: w is inherently X-reachable from v if it is X-reachable and all dipaths in $\overline{X}$ from v to w are X-dipaths.

crossover for X: arc $(v, w) \in \mathrm{V}(\partial X)^2$ where $v \neq w$ and w is inherently X-reachable from v.

$\mathrm{Cross}(X)$: the set of all crossovers for X.

$\mathcal{G}^+(X)$: $\mathcal{G}(X \cup \mathcal{V}(\partial X))$.

division of X: sub-digraph D of $\mathcal{G}^+(X)$ such that $\partial X \cup D$ is connected.

D: any division of X.

X/D: the set of regions of $\partial X \cup D$ which are contained in X.

$\mathrm{CrossExt}(D, X)$: the extension of D with all the crossovers for areas in X/D.

S: any shortcutting of $\mathrm{CrossExt}(D, X)$.

ii-diam(D, X, S): the minimum number $d \geq 0$ such that if v and w are in $D|\mathcal{V}(X)$ and w is inherently X-reachable from v, there is a dipath in $\mathrm{CrossExt}(D, X) \cup S$ from v to w of length $\leq d$.

ib-diam(D, X, S): the minimum number $d \geq 0$ such that if one of v and w is in $D|\mathcal{V}(X)$ and the other is in ∂X, and, moreover, w is inherently X-reachable from v, then there is a dipath in $\mathrm{CrossExt}(D, X) \cup S$ from v to w of length $\leq d$.

iib-diam: $\max\{\text{ii-diam}, \text{ib-diam}\}$.

division shortcutting of (D, X): S is a division shortcutting of (D, X) if $\mathrm{Cross}(X) \subseteq \mathrm{CrossExt}(D, X) \cup S$ and, moreover, $|\mathrm{Cross}(X)| + |S| \leq |\mathrm{A}(D)| + \sum_{X_i \in X/D} |\mathrm{Cross}(X_i)|$.

left side of X: maximal dipath P in ∂X such that P is a part of the closed curve clock-wise around X. A non-trivial X-dipath P *starts (finishes) in a particular left side* L if the first (last) vertex of P is in L and walking along the closed curve clock-wise around X we are in L when the dipath P leaves towards (enters from) the right.

right side of X: analogue to left side but in the closed curve *anti*-clock-wise around X.

corner: end-vertex of a side; if the corner is the first vertex of a side the corner is a *source corner* and if it is the last vertex it is a *terminal* corner.

sink area: X is a sink area if no non-trivial X-dipath finishes in the boundary.

source area: analogue to sink area but with no non-trivial X-dipath starting in the boundary.

B Figures